Approximation of common fixed points for non-Lipschitzian mappings

비-Lipschitzian사상에 대한 공통 부동점의 근사

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요 약

C는 Banach 공간 X의 공집합이 아닌 유계이고 닫힌 볼록 부분집합이라 하자. 만약 $T,S : C \to C$ 가 강한 의미에서의 점근적비확대 사상이라 하면, X와 매개변수 (n_i), (α_i), (β_i)의 어떤 제약하에서 $x_1 \in C$에서 출발하여

$$x_{i+1} := \alpha_i T^{n_i}[\beta_i T^{n_i} + (1-\beta) x_i]\ (1- \alpha_i) S^{n_i} x_i, \quad x_1 \in C$$

로 반복적으로 생성된 수열 (x_n)가 T와 S의 공통인 부동점으로 약 수렴함을 밝힌다.

I. Introduction

Let X be a real Banach space, C a subset of X (not necessarily convex), and $T : C \to C$ a self-mapping of C. nonexpansive mapping. First, as the weaker definition (cf. Kirk [13]), T is said to be of *asymptotically nonexpansive type* (in brief, ANT) if for each $x \in C$, $\lim_{n\to\infty} c_n(x) = 0$, where

$$c_n(x) = 0 \vee \sup_{y \in C}(\|T^n x - T^n y\| - \|x - y\|)$$

and next, as the stronger sense, it is said to be of *strongly asymptotically nonexpansive type* (in brief, strongly ANT) if $\lim_{n\to\infty} c_n = 0$, where $c_n = \sup_{x\in C} c_n(x)$. Kirk [13] established a fixed point theorem for mappings of ANT which T^N be continuous for some $N \geq 1$. The stronger definition (in brief, called *asymptotically nonexpansive* as in [5]) requires that each iterates T^n be Lipschitzian with Lipschitz constants $L_n \to 1$ as $n \to \infty$. In this case, note that T is uniformly continuous on C. For more generalization of an averaging iteration of Schu [21], Bruck et al. [2] introduced a definition somewhere between these two: T is *asymptotically nonexpansive in the intermediate sense* provided T is uniformly continuous and of strongly ANT.

On the other hand, let C be a nonempty closed convex subset of X and $T : C \to C$ a (single-valued) nonexpansive mapping (i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x,\ y \in C$). Given a $u \in C$ and a $t \in (0,1)$, we can define a contraction $T_t : C \to C$ by

$$T_t x = tTx + (1-t)u, \quad x \in C. \tag{1}$$

Then, by Banach's contraction principle, T_t has a unique fixed point x_t in C, that is, we have

$$x_t = tTx_t + (1-t)u. \tag{2}$$

The convergence of $\{x_t\}$ as $t \to 1$ to a fixed point of T has been investigated by several authors. In fact, the strong convergence of $\{x_t\}$ as $t \to 1$ for a T of a bounded C was proved in a Hilbert space independently by Browder [1] and Halpern [10] and in a uniformly smooth Banach space by Reich [20] (cf. [9]). This result was also extended to Ishikawa iteration scheme (cf. Ishikawa [11]) by Tan and Xu [25] and very recently by Takahashi and Kim [27]. For recent progress for nonexpansive nonself-mappings, the reader is refered to [15], [24] and [29].

In this paper, we shall show how to construct (in a uniformly convex Banach space which either satisfies the Opial property or has a Fréchet differentiable norm) a common fixed point of mappings T, S which are asymptotically nonexpansive in the intermediate sense as the weak limt of a sequence $\{x_i\}$ defined by an iteration of the form

$$x_{i+1} = \alpha_i T^{n_i}[\beta_i T^{n_i} x_i + (1-\beta_i)x_i] + (1-\alpha_i)S^{n_i}x_i,$$

where $\{\alpha_i\}$ and $\{\beta_i\}$ are sequences in $(0,1)$ which are bounded away from 0 and 1, i.e., $\alpha_i, \beta_i \in [a,b]$ for some a, b with $0 < a \le b < 1$, and $\{n_i\}$ is a sequence of nonnegative integers.

II. PRELIMINARIES

Let X be a real Banach space with norm $\|\cdot\|$ and let X^* be its dual. The value of $x^* \in X^*$ at $x \in X$ will be denoted by $\langle x, x^* \rangle$. When $\{x_n\}$ is a sequence in X, then $x_n \to x$ (resp. $x_n \rightharpoonup x$, $x_n \overset{*}{\rightharpoonup} x$) will denote strong (resp. weak, weak*) convergence of the sequence $\{x_n\}$ to x.

A Banach space X is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$, where the modulus $\delta(\epsilon)$ of convexity of X is defined by

$$\delta(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \le 1,\ \|y\| \le 1,\ \|x-y\| \ge \epsilon\right\}.$$

Let $S(X) = \{x \in X : \|x\| = 1\}$. Then the norm of X is said to be *Gâteaux differentiable* (and E is said to be *smooth*) if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{3}$$

exists for each x, y in $S(X)$. It is said to be *Fréchet differentiable* if for each $x \in S(X)$, the limit in (3) is attained uniformly for $y \in S(X)$. The norm is said to be *uniformly Gâteaux differentiable* if for each $y \in S(X)$, the limit in (3) is approached uniformly for x varies over $S(X)$. Finally, it is said to be *uniformly Fréchet differentiable* (or X is said to be *uniformly smooth*) if the limit is attained uniformly for $(x, y) \in S(X) \times S(X)$.

We associate with each $x \in X$ the set

$$J_\phi(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|\|x^*\| \text{ and } \|x^*\| = \phi(\|x\|)\},$$

where $\phi : [0,\infty) \to [0,\infty)$ is a continuous and strictly increasing function with $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$. Then $J_\phi : X \to 2^{X^*}$ is said to be the *duality* mapping. Suppose that J_ϕ is single-valued. Then J_ϕ is said to be *weakly sequentially continuous* if for each $\{x_n\} \in X$ with $x_n \rightharpoonup x$, $J_\phi(x_n) \overset{*}{\rightharpoonup} J_\phi(x)$. For abbreviation, we set $J := J_\phi$. In our proof, we assume without loss of generality that J is normalized. For the relations between the duality mapping J and the above geometric properties of X, we summarize the following

Remark 2.1.

(a) If X is smooth, then the duality mapping J is single-valued and norm(strong)-to-weak* continuous.

(b) If X is uniformly smooth, it is norm-to-norm uniformly continuous on every bounded subset of X; if the norm of X has uniformly Gâteaux differentiable, then J is norm-to-weak* uniformly continuous on every bounded subset of X.

(c) The norm of X is uniformly Fréchet differentiable if and only if X^* is uniformly convex.

For more detailed properties, see [3].

A Banach space X is said to satisfy the *Opial property* [17] if for any sequence $\{x_n\}$ in X, $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$. Spaces satisfying this property include all Hilbert spaces and l^p for $1 < p < \infty$. Also it is known [7] that if X admits a weakly sequentially continuous duality mapping, then X

satisfies the Opial property. For more details of the Opial property, see also [6].

Later, Prus [19] gave the stronger Opial property, that is, we say that X satisfies the *uniform Opial property* [19] (or [18]) if for any sequence $\{y_m : m \in \mathbb{N}\}$, and any uniformly bounded sequences $\{x_{n,m} : n \in \mathbb{N}\}$ which are weakly convergent to 0

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \|x_{n,m} - y_m\| = \limsup_{m\to\infty} \limsup_{n\to\infty} \|x_{n,m}\|$$

implies $\{y_m\}$ converges to 0. It is well-known that if X is uniformly convex with the Opial property, then X satisfies the uniform Opial property.

Let X be a real Banach space, C a subset of X (not necessarily convex), and $T : C \to C$ a self-mapping of C. nonexpansive mapping. First, as the weaker definition (cf. Kirk [13]), T is said to be of *asymptotically nonexpansive type* (in brief, ANT) if for each $x \in C$, $\lim_{n\to\infty} c_n(x) = 0$, where

$$c_n(x) = 0 \vee \sup_{y\in C}(\|T^n x - T^n y\| - \|x - y\|)$$

and next, as the stronger sense, it is said to be of *strongly asymptotically nonexpansive type* (in brief, strongly ANT) if $\lim_{n\to\infty} c_n = 0$, where $c_n = \sup_{x\in C} c_n(x)$.

Recall that T is said to be *Lipschitzian* if $\exists\ L > 0$ such that $\|Tx - Ty\| \le L\|x - y\|$ for all $x, y \in C$. In particular, if $L = 1$, T is said to be *nonexpansive* and it is said to be *asymptotically* *nonexpansive* (in brief, AN) [5] if each iterate T^n is Lipschitzian with Lipschitz constants $L_n \to 1$ as $n \to \infty$. As an easy observation, we have the following

Remark 2.2. (a) all nonexpansive mappings are AN.

(b) Every AN mapping is uniformly continuous and of strongly ANT (hence, a mapping of ANT).

(c) Any mapping of strongly ANT may be non-Lipschitzian.

(d) All mappings $T : C \to C$ with the property $T^n x \to 0$ uniformly on C are of strongly ANT.

(e) For all $x \in C$, if $T^n x \in F(T) = \{z\}$ for some $n \geq 1$, T is a mapping of ANT.

For investigating the relations between the above concepts, we here give the following example.

Example 2.1.

(a) Let $C = [-1/\pi, 1/\pi] \subseteq \mathbb{R}$ and $|k| < 1$. For each $x \in C$ we define $Tx = kx \sin \frac{1}{x}$ if $x \neq 0$, and $T0 = 0$. Note that $T^n x \to 0$ uniformly on C. Hence, $T : C \to C$ is a continuous mapping of ANT which is not Lipschitzian.

(b) Let $C = [0, 1] \subseteq \mathbb{R}$ and define $Tx = \frac{1}{4}$ if $x = \frac{1}{4}, 1$, $Tx = 1$ for $x \in [0, \frac{1}{2}] \setminus \frac{1}{4}$, and $Tx = \frac{1}{2}$ for $x \in (\frac{1}{2}, 1]$. Note that for all $x \in C$, $T^n x = \frac{1}{4} \in F(T) = \{\frac{1}{4}\}$ for $n \geq 3$. Then $T : K \to K$ is a discontinuous mapping of ANT which is not nonexpansive.

(c) [16] Let $C = [0, 1] \subseteq \mathbb{R}$ and let φ be the Cantor ternary function. Define $T : K \to C$ by

$$T(x) = \begin{cases} x/2 & \text{if } 0 \le x \le 1/2, \\ \varphi((1-x)/2) & \text{if } 1/2 < x \le 1. \end{cases}$$

Note that $T^n x \to 0$ uniformly on K. Therefore, T is a discontinuous mapping of strongly ANT but not AN because φ is not Lipschizian continuous on $[0, \frac{1}{2}]$.

III. Main theorems

Schu [21] considered the averaging iteration

$$x_{i+1} = \alpha_i T^i x_i + (1 - \alpha_i)x_i$$

when $T : C \to C$ is asymptotically nonexpansive and $\{\alpha_i\}$ is a sequence in $(0, 1)$ which is bounded away from 0 and 1. Throughout this section we shall consider, instead, the more general iteration

$$x_{i+1} = \alpha_i T^{n_i} y_i + (1 - \alpha_i) S^{n_i} x_i, \tag{4}$$

$$y_i = \beta_i T^{n_i} x_i + (1 - \beta_i) x_i, \tag{5}$$

where $\{\alpha_i\}$ and $\{\beta_i\}$ are sequences in $(0, 1)$ which are bounded away from 0 and 1, i.e., $\alpha_i, \beta_i \in [a, b]$ for some a, b with $0 < a \le b < 1$, and $\{n_i\}$ is a sequence of nonnegative integers (which need not be increasing). A strictly increasing sequence $\{m_i\}$ of positive integers will be called *quasi-periodic* [2] if the sequence $\{m_{i+1} - m_i\}$ is bounded (equivalently, if there exists $b > 0$ so that any block of b consecutive positive integers must contain a term of the sequence).

We begin with the following easy observation.

Lemma 3.1 [2]. *Suppose $\{r_k\}$ is a bounded sequence of real numbers and $\{a_{k,m}\}$ is a doubly-indexed sequence of real numbers which satisfy*

$$\limsup_{k\to\infty}\limsup_{m\to\infty} a_{k,m} \leq 0, \quad r_{k+m} \leq r_k + a_{k,m} \quad \text{for each } k,m \geq 1.$$

Then $\{r_k\}$ converges to an $r \in \mathbb{R}$; if $a_{k,m}$ can be taken to be independent of k, $a_{k,m} \equiv a_m$, then $r \leq r_k$ for each k.

With a slight modification of the proof of Lemma 3.2 in [12], we also have the following:

Lemma 3.2. *Suppose X is a uniformly convex Banach space, C is a convex subset of X, and $T, S : C \to C$ are asymptotically nonexpansive in the intermediate sense with $F(T) \cap F(S) \neq \emptyset$. Put*

$$c_n = \max(0, \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x-y\|), \sup_{x,y\in C}(\|S^n x - S^n y\| - \|x-y\|)),$$

so that $\lim_{n\to\infty} c_n = 0$. Suppose that for any $x_1 \in C$, $\{x_i\}$ is generated by (4)-(5) for $i \geq 1$ and $\sum_{i=1}^{\infty} c_{n_i} < +\infty$. Then for every $w_1, w_2 \in F(T) \cap F(S)$ and $0 < t < 1$, $\lim_{i\to\infty} \|tx_i + (1-t)w_1 - w_2\|$ exists.

Proof. The proof still follows the lines of the proof in [2] and [12]. We have not assumed C is closed, but since T and S are uniformly continuous they (and their iterates) can be extended to the (norm) closure C with the same modulus of uniform continuity and the same constants c_n, so it does no harm to assume C itself is closed.

We begin with showing that for $w \in F(T) \cap F(S)$, the limit $\lim_{i\to\infty} \|x_i - w\|$ exists. From (5), since $\|y_k - w\| \le \beta_k c_{n_k} + \|x_k - w\|$, this together with (4) implies

(6)

$$\begin{aligned}\|x_{k+1} - w\| &\le \alpha_k \|T^{n_k} y_k - w\| + (1-\alpha_k)\|S^{n_k} x_k - w\| \\ &= \alpha_k \|T^{n_k} y_k - T^{n_k} w\| + (1-\alpha_k)\|S^{n_k} x_k - S^{n_k} w\| \\ &\le \alpha_k(\|y_k - w\| + c_{n_k}) + (1-\alpha_k)(c_{n_k} + \|x_k - w\|) \\ &\le \alpha_k(\|x_k - w\| + c_{n_k} + c_{n_k}\beta_k) + (1-\alpha_k)(c_{n_k} + \|x_k - w\|) \\ &\le \|x_k - w\| + c_{n_k}(1 + \alpha_k \beta_k).\end{aligned}$$

Continuing this process inductively, we have for each $k \in \mathbb{N}$,

$$\|x_{k+1} - w\| \le \|x_1 - w\| + 2\sum_{i=1}^{k} c_{n_i} < +\infty$$

and also

$$\|x_{k+m} - w\| \le \|x_k - w\| + 2\sum_{i=k}^{k+m-1} c_{n_i}. \tag{7}$$

Applying Lemma 1 with $r_k = \|x_k - w\|$ and $a_{k,m} = 2\sum_{i=k}^{k+m-1} c_{n_i}$, we see that $\lim_{i\to\infty} \|x_i - w\|$ ($\equiv r$) exists for every $w \in F(T) \cap F(S)$.

Now putting $T_i := \alpha_i T^{n_i}[\beta_i T^{n_i} + (1-\beta_i)I] + (1-\alpha_i)S^{n_i}$ (I denotes the identity mapping of X) for each $i \in \mathbb{N}$ and, for $k \ge j$, $S(k,j) := T_{k-1}T_{k-2}\cdots T_j$, it is easily seen that $x_k = S(k,j)x_j$ and $F(T_i) \supseteq F(T) \cap F(S)$. Since

$$\|T_i x - T_i y\| \le c_{n_i}(1 + \alpha_i \beta_i) + \|x - y\| \le 2c_{n_i} + \|x - y\|$$

for all $x, y \in C$, we have for $k \geq j$,

$$(8) \quad \|S(k,j)x - S(k,j)y\| \leq 2\sum_{i=j}^{k-1} c_{n_i} + \|x-y\| \quad \text{for all } x, y \in C.$$

For $w \in F(T) \cap F(S)$ and $0 < t < 1$, as in the proof of Lemma 3.2 in [12], we can obtain

$$(9) \qquad \lim_{j\to\infty} \sup_{k\geq j} \|S(k,j)[tx_j + (1-t)w] - tx_k - (1-t)w\| = 0$$

and hence the conclusion follows similarly. For more detail proof, see [12]. □

Remark 3.1. By a theorem of Kirk [13], it is easy to see that if C is bounded, and if T and S commute, then $F(T) \cap F(S) \neq \emptyset$.

Lemma 3.3 [4],[22]. *Let X be a uniformly convex Banach space, $0 < b \leq t_n \leq c < 1$ for all $n \geq 1$, $r \geq 0$. Suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are sequences of X such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$, and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Using Lemma 3.2 and 3.3, we have the following:

Theorem 3.1. *Suppose X is a uniformly convex Banach space, C is a convex subset of X, and $T, S : C \to C$ are asymptotically nonexpansive in the intermediate sense with $F(T) \cap F(S) \neq \emptyset$. Put*

$$c_n = \max(0, \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x-y\|), \sup_{x,y\in C}(\|S^n x - S^n y\| - \|x-y\|))$$

so that $\lim_{n\to\infty} c_n = 0$. *Suppose* $\{n_i\}$ *is a sequence of nonnegative integers such that*

$$\sum_{i=1}^{\infty} c_{n_i} < +\infty$$

and such that

$$\mathcal{O} = \{i : n_{i+1} = 1 + n_i\}$$

is quasi-periodic. Then for any $x_1 \in C$ *and* $\{x_i\}$ *generated by* (4)-(5) *for* $i \geq 1$, *we have* $\lim_{i\to\infty} \|x_i - Tx_i\| = 0$ *and* $\lim_{i\to\infty} \|x_i - Sx_i\| = 0$.

Proof. As in the proof of Lemma 3.2, we have for $w \in F(T) \cap F(S)$, the limit $\lim_{i\to\infty} \|x_i - w\|$ $(\equiv r)$ exists. If $r = 0$, we immediately obtain

$$\|Tx_i - x_i\| \leq \|Tx_i - w\| + \|w - x_i\| = \|Tx_i - Tw\| + \|w - x_i\|,$$

and hence by the uniform continuity of T, that $\lim_{i\to\infty} \|x_i - Tx_i\| = 0$. Similarly, we have $\lim_{i\to\infty} \|x_i - Sx_i\| = 0$. Suppose $r > 0$. Since

$$\begin{aligned} \|T^{n_i} y_i - w\| &\leq \|y_i - w\| + c_{n_i} \\ &\leq (1 + \beta_i)c_{n_i} + \|x_i - w\|, \\ &\leq (1 + b)c_{n_i} + \|x_i - w\|, \end{aligned}$$

we have $\limsup_{i\to\infty} \|T^{n_i} y_i - w\| \leq r$.Further, since $\|S^{n_i} x_i - w\| \leq c_{n_i} + \|x_i - w\|$, we have $\limsup_{i\to\infty} \|S^{n_i} x_i - w\| \leq r$. Noting that

$$\lim_{i\to\infty} \|\alpha_i(T^{n_i} y_i - w) + (1 - \alpha_i)(S^{n_i} x_i - w)\| = \lim_{i\to\infty} \|x_{i+1} - w\| = r,$$

by Lemma 3.3, we have

$$\lim_{i\to\infty} \|T^{n_i}y_i - S^{n_i}x_i\| = 0. \tag{10}$$

This is equivalent to

$$\lim_{i\to\infty} \|S^{n_i}x_i - x_{i+1}\| = 0. \tag{11}$$

On the other hand, we have, for all $i \geq 1$,

$$\begin{aligned}\|x_{i+1} - w\| &\leq \alpha_i\|T^{n_i}y_i - w\| + (1-\alpha_i)\|S^{n_i}x_i - w\| \\ &\leq \alpha_i(\|y_i - w\| + c_{n_i}) + (1-\alpha_i)(c_{n_i} + \|x_i - w\|) \\ &= \alpha_i\|y_i - w\| + c_{n_i} + (1-\alpha_i)\|x_i - w\|\end{aligned}$$

and hence

$$\frac{\|x_{i+1} - w\| - \|x_i - w\|}{\alpha_i} \leq \|y_i - w\| + \frac{c_{n_i}}{a} - \|x_i - w\|.$$

This implies immediately that

$$\begin{aligned} r \leq \liminf_{i\to\infty} \|y_i - w\| &\leq \limsup_{i\to\infty} \|y_i - w\| \\ &\leq \limsup_{i\to\infty}(\beta_i c_{n_i} + \|x_i - w\|) \\ &\leq \limsup_{i\to\infty}(b c_{n_i} + \|x_i - w\|) \\ &= \limsup_{i\to\infty} \|x_i - w\| = r\end{aligned}$$

and hence

$$\begin{aligned} r &= \lim_{i\to\infty} \|y_i - w\| \\ &= \lim_{i\to\infty} \|\beta_i(T^{n_i}x_i - w) + (1-\beta_i)(x_i - w)\|.\end{aligned}$$

Using Lemma 3.3 again, we have

$$(12) \qquad \lim_{i\to\infty} \|T^{n_i}x_i - x_i\| = 0,$$

Since $\lim_{i\to\infty} c_{n_i} = 0$, (10) and (12) yield

$$\begin{aligned}\|x_i - S^{n_i}x_i\| &\le \|x_i - T^{n_i}x_i\| + \|T^{n_i}x_i - T^{n_i}y_i\| + \|T^{n_i}y_i - S^{n_i}x_i\| \\ &\le (1+\beta_i)\|x_i - T^{n_i}x_i\| + c_{n_i} + \|T^{n_i}y_i - S^{n_i}x_i\| \\ &\le (1+b)\|x_i - T^{n_i}x_i\| + c_{n_i} + \|T^{n_i}y_i - S^{n_i}x_i\| \to 0,\end{aligned}$$

as $j \to \infty$. This with (11) implies that

$$(13) \quad \|x_{i+1}-x_i\| \le \|x_{i+1}-S^{n_i}x_i\|+\|S^{n_i}x_i-x_i\| \to 0 \qquad \text{as } j\to\infty.$$

For the remaining proof it is now possible to mimic the steps of the original argument in [2]. However, for the sake of completeness, we claim that $x_j - Tx_j \to 0$ as $j \to \infty$ through $\mathcal{O}$. Indeed, since $n_{j+1} = 1 + n_j$ for such j, we have

$$\begin{aligned}(14)\|x_j - Tx_j\| \le& \|x_j - x_{j+1}\| + \|x_{j+1} - T^{n_{j+1}}x_{j+1}\| \\ &+ \|T^{n_{j+1}}x_{j+1} - T^{n_{j+1}}x_j\| + \|TT^{n_j}x_j - Tx_j\| \\ \le& \|x_j - x_{j+1}\| + \|x_{j+1} - T^{n_{j+1}}x_{j+1}\| \\ &+ c_{n_{j+1}} + \|x_{j+1} - x_j\| + \|TT^{n_j}x_j - Tx_j\|.\end{aligned}$$

By (12)-(14) and the uniform continuity of T, we conclude that $\|x_j - Tx_j\| \to 0$ as $j \to \infty$ through $\mathcal{O}$. Similarly, replacing T in (14) by S and (12) by $\lim_{i\to\infty} \|S^{n_i}x_i - x_i\| = 0$, we have $\|x_j - Sx_j\| \to 0$ as $j \to \infty$ through $\mathcal{O}$.

But since $\mathcal{O}$ is quasi-periodic, there exists a constant $b > 0$ such that for each positive integer i we can find $j_i \in \mathcal{O}$ with $|j_i - i| \leq b$. Thus (13) and the uniform continuity of $I - T$ and $I - S$ imply that $x_i - Tx_i$ and $x_i - Sx_i$ converge to 0 as $i \to \infty$ through all of $\mathbb{N}$. □

Remark 3.2. We don't know whether Theorem 3.1 still holds in case $\{\alpha_i\}$ is a sequence in $(0, 1)$ which is bounded away from 0 and 1 and $\{\beta_i\}$ is chosen so that either $\beta_i = 0$ for all $i \geq 1$ or $\limsup_{i\to\infty} \beta_i = 1$.

As a direct observation of Theorem 3.1 in [2], we have the following:

Theorem 3.2. *Suppose a Banach space X has the uniform Opial property, C is a nonempty weakly compact subset of X, and $T, S : C \to C$ are asymptotically nonexpansive in the weak sense. If $\{x_n\}$ is a sequence in C such that $\lim_{n\to\infty} \|x_n - w\|$ exists for each common fixed point w of T and S, and if $\{x_n - T^k x_n\}$ and $\{x_n - S^k x_n\}$ are weakly convergent to 0 for each $k \geq 1$, then $\{x_n\}$ is is weakly convergent to a common fixed point of T and S.*

Proof. Our proof still follows the lines of the proof in [2]. By Opial's classical argument, it suffices to show $\omega_w(x_n) \subseteq F(T) \cap F(S)$, where $\omega_w(x_n)$ denotes the weak ω-lim set of sequence $\{x_n\}$, i.e., the set $\{w \in X : w = \text{w-}\lim_{j\to\infty} x_{n_j} \text{ for some } n_j \uparrow \infty\}$. To this end, let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup z$. Define

$$r_k = \limsup_j \|T^k x_{n_j} - z\|, \qquad a_m = \sup_{y\in C}(\|T^m y - T^m z\| - \|y - z\|).$$

Since $x_n - T^k x_n \rightharpoonup 0$ for each $k \geq 1$, we have for each $m, k \in \mathbb{N}$, $T^{k+m} x_{n_j} \rightharpoonup z$ and hence, by the Opial property,

$$(15) \qquad \begin{aligned} r_{k+m} &= \limsup_{j\to\infty} \|T^{k+m} x_{n_j} - z\| \\ &\leq \limsup_{j\to\infty} \|T^{k+m} x_{n_j} - T^m z\| \\ &\leq r_k + a_m, \end{aligned}$$

where $\limsup_{n\to\infty} a_m \leq 0$. By Lemma 1 again, therefore, $\lim_{k\to\infty} r_k := r$ exists and $r \leq r_k$ for each $k \geq 1$. Setting $x_{n_j,m} := T^m x_{n_j} - z$ and $y_m := T^m z - z$ and first taking the lim sup as $m \to \infty$ in (15), we have

$$\begin{aligned} r &\leq \limsup_{m\to\infty} \limsup_{j\to\infty} \|x_{n_j,m} - y_m\| \\ &= \limsup_{m\to\infty} \limsup_{j\to\infty} \|T^m x_{n_j} - T^m z\| \\ &= \limsup_{m\to\infty} \limsup_{j\to\infty} \|T^{k+m} x_{n_j} - T^m z\| \\ &\leq r_k \end{aligned}$$

for each $k \geq 1$ and next taking the *lim* as $k \to \infty$ this yields

$$\limsup_{m\to\infty} \limsup_{j\to\infty} \|x_{n_j,m} - y_m\| = r = \limsup_{m\to\infty} \limsup_{j\to\infty} \|x_{n_j,m}\|.$$

By the uniform Opial property, we have $\lim_{m\to\infty} T^m z = z$. Since T^N is continuous, $z \in F(T^N)$, and since

$$z = \lim_{j\to\infty} T^{jN+1} z = \lim_{j\to\infty} T T^{jN} z = Tz,$$

z is also a fixed point of T. Similarly, replacing T by S, we can prove that z is a fixed point of S. Hence $z \in F(T) \cap F(S)$. $\square$

It is known [30] that if X is uniformly convex and has the Opial property, then X has the uniform Opial property. Here, combining Theorem 3.1 and Theorem 3.2, we have the following:

Theorem 3.3. *Let X be a uniformly convex Banach space which satisfies the Opial property, C a nonempty bounded closed convex subset of X, and $T, S : C \to C$ asymptotically nonexpansive in the intermediate sense with $F(T) \cap F(S) \neq \emptyset$. Put*

$$c_n = \max(0, \sup_{x,y \in C} (\|T^n x - T^n y\| - \|x-y\|), \sup_{x,y \in C} (\|S^n x - S^n y\| - \|x-y\|)),$$

so that $\lim_{n\to\infty} c_n = 0$. Suppose $\{n_i\}$ is a sequence of nonnegative integers such that

$$\sum_{i=1}^{\infty} c_{n_i} < +\infty$$

and such that

$$\mathcal{O} = \{i : n_{i+1} = 1 + n_i\}$$

is quasi-periodic. Then the sequence $\{x_i\}$ generated by (4)-(5) with starting $x_1 \in C$ is weakly convergent to a common fixed point of T and S.

Proof. By Theorem 3.1, $\lim_{i\to\infty} \|x_i - Tx_i\| = 0$ and $\lim_{i\to\infty} \|x_i - Sx_i\| = 0$. Since T and S are uniformly continuous, we have for each $k \in \mathbb{N}$,

$$\lim_{i\to\infty} \|x_i - T^k x_i\| = 0 \quad \text{and} \quad \lim_{i\to\infty} \|x_i - S^k x_i\| = 0,$$

which in turn imply $x_i - T^k x_i \rightharpoonup 0$ and $x_i - S^k x_i \rightharpoonup 0$ respectively. The conclusion now follows from Theorem 3.2. □

Theorem 3.4. *Let X be a uniformly convex Banach space which has a Fréchet differentiable norm , C a nonempty bounded closed convex subset of X, and $T, S : C \to C$ asymptotically nonexpansive in the intermediate sense with $F(T) \cap F(S) \neq \emptyset$. If $\omega_w(x_i) \subseteq F(T) \cap \ F(S)$, then the sequence $\{x_i\}$ generated by (4)-(5) with starting $x_1 \in C$ is weakly convergent to a common fixed point of T and S.*

Proof. Using Lemma 3.2, it is easy to see that the limit $\lim_{i\to\infty}\langle x_i, J(w_1 - w_2)\rangle$ exists for all $w_1, w_2 \in F(T) \cap F(S)$ (for details, see [25] or [2]). In particular, this implies that

$$\langle p - q, J(w_1 - w_2)\rangle = 0 \qquad \text{for all } p, q \text{ in } \omega_w(x_i). \tag{14}$$

Replacing w_1 and w_2 in (14) by q and p, respectively, we have

$$0 = \langle p - q, J(q - p)\rangle = -\|p - q\|^2,$$

for all $p, q \in \omega_w(x_i)$. This proves the uniqueness of weak subsequential limits of $\{x_i\}$ and completes the proof that $\{x_i\}$ converges weakly. □

Remark 3.3. If $I - T$ (resp. $I - S$) is demiclosed at 0, i.e., for any sequence $\{x_i\}$ in C, the conditions $x_i \rightharpoonup w$ and $x_i - Tx_i \to 0$ (resp. $x_i - Sx_i \to 0$) imply $w - Tw = 0$ (resp. $w - Sw = 0$), it easily follows from Theorem 1 that $\omega_w(x_i) \subseteq F(T) \cap F(S)$.

It is well known [28] that if $T, S : C \to C$ are asymptotically nonexpansive, $I - T$ and $I - S$ are demiclosed at 0. As a direct consequence of Theorem 3 and 4, we have the following:

Corollary 3.1. *Let X be a uniformly convex Banach space which satisfies the Opial property or has a Fréchet differentiable norm, C a nonempty bounded closed convex subset of X, and $T, S : C \to C$ asymptotically nonexpansive mappings with $F(T) \cap F(S) \neq \emptyset$. Suppose $\{n_i\}$ is a sequence of nonnegative integers such that*

$$\sum_{i=1}^{\infty}(L_{n_i} - 1) < +\infty$$

and such that

$$\mathcal{O} = \{i : n_{i+1} = 1 + n_i\}$$

is quasi-periodic. Then the sequence $\{x_i\}$ generated by (4)-(5) with starting $x_1 \in C$ is weakly convergent to a common fixed point of T and S.

Remark 3.4. It is easy to see that, under the assumptions of Corollary 1, if $T, S : C \to C$ are nonexpansive, then the sequence $\{x_i\}$ generated by an iteration of the form

$$x_{i+1} = \alpha_i T[\beta_i T x_i + (1 - \beta_i) x_i] + (1 - \alpha_i) S x_i \tag{15}$$

starting $x_1 \in C$ is weakly convergent to a common fixed point of T and S, where $\{\alpha_i\}$ and $\{\beta_i\}$ are chosen so that $\alpha_i, \beta_i \in [a, b]$ for some a, b with $0 < a \leq b < 1$.

Theorem 3.5. *Under the assumptions of Theorem 3.1, if T and S have pre-compact ranges, then the sequence $\{x_i\}$ generated by (4)-(5) with starting $x_1 \in C$ is strongly convergent to a common fixed point of T and S.*

Proof. We follows the lines of the proof of Theorem 1.5 in [21]. From our assumptions it follows that $\Omega := \overline{co}\Big(\{x_1\} \cup T(C) \cup S(C)\Big)$ is a compact subset of C containing $\{x_i\}$. Hence there exists an $w \in C$ and a subsequence $\{x_{i_j}\}$ of $\{x_i\}$ which converges strongly to w. But T and S are continuous and $\lim_{i\to\infty} \|x_i - Tx_i\| = 0$ and $\lim_{i\to\infty} \|x_i - Sx_i\| = 0$ by Theorem 3.1. Thus w is a common fixed point of T and S. As in the proof of Lemma 3.2 again, we have the limt $\lim_{i\to\infty} \|x_i - w\|$ exists. Hence we have $\lim_{i\to\infty} \|x_i - w\| = 0$. □

Remark 3.4. In Theorem 1.5 of [21], Schu assumed that X is Hilbert space and that iterates T^n have Lipschitz constants $L_n \geq 1$ such that $\sum_n (L_n^2 - 1)$ converges. Even for Schu's original iteration $(n_i \equiv i)$, Theorem 3.4 is more general, since the convergence of $\sum_n (L_n^2 - 1)$ implies that of $\sum_n (L_n - 1)$, which in turn assures the convergence of our $\sum_n c_n$. We don't know whether Theorem 3.5 still remains true under the weak condition of X (that is, strict convexity) as in [27] for a nonexpansive mapping $T : C \to C$ and the sequence $\{x_i\}$ defined by (15).

Recall that a pair (T, S) of mappings $T, S : C \to C$ is said to satisfy Condition A if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$

such that

$$\frac{1}{2}(\|x - Tx\| + \|x - Sx\|) \geq f\big(d(x, F)\big)$$

for all $x \in C$, where $d(x, F) = \inf_{z \in F} \|x - z\|$ and $F := F(T) \cap F(S)$. In particular, if $T = S$, the above definition reduces to one due to [23].

Theorem 3.6. *Under the assumptions of Theorem 3.1, if a pair (T, S) of mappings $T, S : C \to C$ satisfies Condition A, then the sequence $\{x_i\}$ generated by (4)-(5) with starting $x_1 \in C$ is strongly convergent to a common fixed point of T and S.*

Proof. By Condition A, we have

$$\frac{1}{2}(\|x_i - Tx_i\| + \|x_i - Sx_i\|) \geq f\big(d(x_i, F)\big)$$

for all $i \geq 1$.

In the proof of Lemma 3.2, since $\|T_i x - T_i y\| \leq 2c_{n_i} + \|x - y\|$ for all $x, y \in C$ and $i \geq 1$, we have

$$\|x_{i+1} - z\| = \|T_i x_i - T_i z\| \leq 2c_{n_i} + \|x_i - z\| \tag{16}$$

for all $z \in F$ and so $d(x_{i+1}, F) \leq 2c_{n_i} + d(x_i, F)$ for all $i \geq 1$. By Lemma 1 (or see [25; Lemma 3.1]), the limit $\lim_{i \to \infty} d(x_i, F)$ exists. We shall claim that

$$\lim_{i \to \infty} d(x_i, F) = 0.$$

To this end, if not, i.e., $d := \lim_{i \to \infty} d(x_i, F) > 0$, then we can choose a $k \in \mathbb{N}$ such that for all $i \geq k$,

$$0 < \frac{d}{2} < d(x_i, F).$$

Then it follows from Condition (A) and Theorem 1 that

$$0 < f(\frac{d}{2}) \leq f\big(d(x_i, F)\big) \leq \frac{1}{2}(\|x_i - Tx_i\| + \|x_i - Sx_i\|) \to 0$$

as $i \to \infty$. This is a contradiction, which shows that $d = 0$. We can thus choose a subsequence $\{x_{i_j}\}$ of $\{x_i\}$ such that

$$\|x_{i_j} - z_j\| \leq 2^{-j}$$

for all $j \geq 1$ and some sequence $\{z_j\}$ in F. Replacing i and z in (16) by i_j and z_j, respectively, we have

$$\begin{aligned}\|x_{i_j+1} - z_j\| &\leq 2c_{n_{i_j}} + \|x_{i_j} - z_j\| \\ &\leq 2c_{n_{i_j}} + 2^{-j},\end{aligned}$$

and hence

$$\begin{aligned}\|z_{j+1} - z_j\| &\leq \|z_{j+1} - x_{i_j+1}\| + \|x_{i_j+1} - z_j\| \\ &\leq 2^{-(j+1)} + 2c_{n_{i_j}} + 2^{-j} < 2(2^{-j} + c_{n_{i_j}}),\end{aligned}$$

which shows that $\{z_j\}$ is Cauchy and therefore converges strongly to a point z in F since F is closed. Now it is readly seen that $\{x_{i_j}\}$ converges strongly to z. Since the limit $\lim_{i\to\infty} \|x_i - z\|$ exists as in the proof of Lemma 3.2, $\{x_i\}$ itself converges strongly to $z \in F$. □

Remark 3.5. If $S = T$, Theorem 3.6 reduces to Theorem 3.6 due to Kim-Jung [12].

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