Argument Estimates of Meromorphic Functions Defined by Certain Differential Operators

(어떤 미분연산자들에 의하여 정의된 유리형함수들의 편각추정)

Advisor : Prof. Nak Eun Cho

by

Hye Young Lee

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A dissertation

by

Hye Young Lee

Approved by :

(Chairman) Jin Mun Jeong, Ph. D.

(Member) Jun Yong Shin, Ph. D.

(Member) Nak Eun Cho, Ph. D.

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어떤 미분연산자들에 의하여 정의된 유리형 함수들의 편각추정

이 혜 영

부경대학교 교육대학원 수학교육전공

요 약

기하함수이론은 지금까지 많은 학자들에 의하여 다양하게 연구되어 왔다. 특히, Miller와 Mocanu[3]은 미분종속이론을 소개하여 해석함수들의 여러 기하학적 성질들을 조사하였다.

본 논문에서는 Uralegaddi와 Somanatha[7,8]에 의하여 소개된 유리형 함수들의 미분연산자들과 미분 종속이론을 이용하여 유리형 함수들의 새로운 부분족들을 소개하고 그들의 포함관계를 조사하였다.

또한, Nunokawa[4]의 결과를 응용하여 유리형 close-to-convex 함수들의 편각추정을 하였으며, sector상에서 적분보존성질들을 조사하였다.

1. Introduction

Let Σ denote the class of functions of the form

$$f(z) = \frac{a_{-1}}{z} + \sum_{n=0}^{\infty} a_n z^n \quad (a_{-1} \neq 0), \tag{1.1}$$

which are analytic in the punctured open unit disk $\mathcal{D} = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\}$. We denote by $\Sigma^*(\beta)$ the subclass of Σ consisting of all functions which is meromorphic starlike of order β in $\mathcal{U} = \mathcal{D} \cup \{0\}(0 \leq \beta < 1)$. Analytically, a function f of the form (1.1) belongs to the class $\Sigma^*(\beta)$ if and only if

$$-\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta \quad (0 \leq \beta < 1 \; ; \; z \in \mathcal{U}).$$

For analytic functions g and h with $g(0) = h(0)$, g is said to be subordinate to h if there exists an analytic function w such that $w(0) = 0, |w(z)| < 1 \;\; (z \in \mathcal{U})$, and $g(z) = h(w(z))$. We denote this subordination by $g \prec h$ or $g(z) \prec h(z)$.

Following Uralegaddi and Somanatha [7,8], we define

$$D^0 f(z) = f(z),$$
$$D^1 f(z) = \frac{a_{-1}}{z} + 2a_0 + 3a_1 z + 4a_2 z^2 + \cdots,$$
$$D^2 f(z) = D^1(D^1 f(z)),$$

and

$$\begin{aligned} D^n f(z) &= D^1(D^{n-1} f(z)), \\ &= \frac{a_{-1}}{z} + \sum_{m=2}^{\infty} m^n a_{m-2} z^{m-2} \;\; (n \in \mathbb{N}). \end{aligned} \tag{1.2}$$

Let

$$\Sigma[n; A, B] = \left\{ f \in \Sigma : -\frac{z(D^n f(z))'}{D^n f(z)} \prec \frac{1 + Az}{1 + Bz}, \; z \in \mathcal{U} \right\}, \tag{1.3}$$

where $-1 < B < A \leq 1$. In particular, we note that $\Sigma[0; 1, -1]$ is the well known class of meromorphic starlike functions. From (1.3), we observe [5] that a function f is in $\Sigma[n; A, B]$ if and only if

$$\left|\frac{z(D^n f(z))'(z)}{D^n f(z)}+\frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2} \quad (-1<B<A\leq 1;\ z\in\mathcal{U}). \tag{1.4}$$

A function $f\in\Sigma$ is said to be in the class $\Sigma_c(\beta,\gamma)$ if there is a meromorphic starlike function g of order β such that

$$-\mathrm{Re}\left\{\frac{zf'(z)}{g(z)}\right\}>\gamma \quad (0\leq\gamma<1\ ;\ z\in\mathcal{U}).$$

Libera and Robertson [2] showed that $\Sigma_c(0,0)$, the class of meromorphic close-to-convex functions, is not univalent. Also, $\Sigma_c(\beta,\gamma)$ provides an interesting generalization of the class of meromorphic close-to-convex functions [6].

The object of the present paper is to give some argument estimates of meromorphic functions belonging to Σ and the integral preserving properties for meromorphic close-to convex functions in connection with the differential operators D^n defined by (1.2).

2. Main results

In proving our results below, we need the following lemmas.

Lemma 2.1 [1]. *Let h be convex univalent in $\mathcal{U}$ with $h(0)=1$ and $Re(\beta h(z)+\gamma)>0(\beta,\gamma\in\mathbb{C})$. If p is analytic in $\mathcal{U}$ with $p(0)=1$, then*

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\prec h(z) \quad (z\in\mathcal{U})$$

implies

$$p(z)\prec h(z) \quad (z\in\mathcal{U}).$$

Lemma 2.2 [3]. *Let h be convex univalent in $\mathcal{U}$ and λ be analytic in $\mathcal{U}$ with* $\mathrm{Re}\,\lambda(z)\geq 0$. *If p is analytic in $\mathcal{U}$ and $p(0)=h(0)$, then*

$$p(z)+\lambda(z)zp'(z)\prec h(z) \quad (z\in\mathcal{U})$$

implies

$$p(z)\prec h(z) \quad (z\in\mathcal{U}).$$

Lemma 2.3 [4]. *Let p be analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \neq 0$ in $\mathcal{U}$. Suppose that there exists a point $z_0 \in \mathcal{U}$ such that*

$$\left| \arg\ p(z) \right| \quad < \quad \frac{\pi}{2}\alpha \ \textit{for}\ |z| < |z_0| \tag{2.1}$$

and

$$\left| \arg\ p(z_0) \right| \quad = \quad \frac{\pi}{2}\alpha \quad (0 < \alpha \leq 1). \tag{2.2}$$

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} \quad = \quad ik\alpha, \tag{2.3}$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \textit{when}\ \ \arg\ p(z_0) = \frac{\pi}{2}\alpha \tag{2.4}$$

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \quad \textit{when}\ \ \arg\ p(z_0) = -\frac{\pi}{2}\alpha \tag{2.5}$$

and

$$p(z_0)^{\frac{1}{\alpha}} = \pm ia \ \ (a > 0). \tag{2.6}$$

At first, with the help of Lemma 2.1, we obtain the following

Proposition 2.1. *Let h be convex univalent in $\mathcal{U}$ with $h(0) = 1$ and* Re h *be bounded in $\mathcal{U}$. If $f \in \Sigma$ satisfies the condition*

$$-\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} \prec h(z) \quad (z \in \mathcal{U}),$$

then

$$-\frac{z(D^n f(z))'}{D^n f(z)} \prec h(z) \quad (z \in \mathcal{U})$$

for $\max_{z\in\mathcal{U}} \operatorname{Re} h(z) < 2$ *(provided $D^n f(z) \neq 0$ in $\mathcal{U}$).*

Proof. Let

$$p(z) = -\frac{z(D^n f(z))'}{D^n f(z)}.$$

By using the equation

$$z(D^n f(z))' = D^{n+1} f(z) - 2\ D^n f(z), \tag{2.7}$$

we get

$$p(z) - 2 = -\frac{D^{n+1} f(z)}{D^n f(z)}. \tag{2.8}$$

Taking logarithemic derivatives in both sides of (2.8) and multiplying by z, we have

$$\frac{zp'(z)}{-p(z)+2} + p(z) = -\frac{z(D^{n+1} f(z))'}{D^{n+1} f(z)} \prec h(z) \quad (z \in \mathcal{U}).$$

From Lemma 2.1, it follows that $p(z) \prec h(z)$ for Re $(-h(z)+2) > 0$ $(z \in \mathcal{U})$, which means

$$-\frac{z(D^n f(z))'}{D^n f(z)} \prec h(z) \quad (z \in \mathcal{U})$$

for $\max_{z \in \mathcal{U}} \mathrm{Re}\ h(z) < 2$.

Taking

$$h(z) = \frac{1+Az}{1+Bz} \quad (-1 < B < A \leq 1;\ z \in \mathcal{U})$$

in Proposition 2.1, we have

Corollary 2.1. *Let $f \in \Sigma$ and $1 + A \leq 2(1+B)$ $(-1 < B < A \leq 1)$. Then for any non-negative integer n, we have*

$$\Sigma[n+1; A, B] \subset \Sigma[n; A, B],$$

where $D^n f(z) \neq 0$ in $\mathcal{U}$.

Proposition 2.2. *Let h be convex univalent in $\mathcal{U}$ with $h(0) = 1$ and* Re *h be bounded in $\mathcal{U}$. Let F be the integral operator defined by*

$$F(z) = \frac{c}{z^{c+1}} \int_0^z t^c f(t)dt \quad (c > 0). \tag{2.9}$$

If $f \in \Sigma$ satisfies the condition

$$-\frac{z(D^n f(z))'}{D^n f(z)} \prec h(z) \quad (z \in \mathcal{U}),$$

then
$$-\frac{z(D^nF(z))'}{D^nF(z)} \prec h(z) \quad (z\in\mathcal{U})$$
for $\max_{z\in\mathcal{U}} \operatorname{Re} h(z) < c+1$ *(provided* $D^nF(z)\neq 0$ *in* $\mathcal{U}$*).*

Proof. From (2.9), we have
$$z(D^nF(z))' = c\ D^nf(z) - (c+1)\ D^nF(z). \tag{2.10}$$
Let
$$p(z) = -\frac{z(D^nF(z))'}{D^nF(z)}.$$
Then, by using (2.10), we get
$$p(z) - (c+1) = -c\ \frac{D^nf(z)}{D^nF(z)}. \tag{2.11}$$
Taking logarithemic derivatives in both sides of (2.11) and multiplying by z, we have
$$\frac{zp'(z)}{-p(z)+(c+1)} + p(z) = -\frac{z(D^nf(z))'}{D^nf(z)} \prec h(z) \quad (z\in\mathcal{U}).$$
Therefore, by Lemma 2.1, we have
$$-\frac{z(D^nF(z))'}{D^nF(z)} \prec h(z) \quad (z\in\mathcal{U})$$
for $\max_{z\in\mathcal{U}}\operatorname{Re} h(z) < c+1$ (provided $D^nF(z)\neq 0$ in $\mathcal{U}$).

Taking
$$h(z) = \frac{1+Az}{1+Bz}\ (-1<B<A\leq 1;\ z\in\mathcal{U})$$
in Proposition 2.2, we have

Corollary 2.2. *Let* $f\in\Sigma$ *and choose a positive number* c *such that*
$$c\geq\frac{1+A}{1+B}-1\ (-1<B<A\leq 1).$$
If $f\in\Sigma[n;A,B]$, *then* $F\in\Sigma[n;A,B]$, *where the integral operator* F *is defined by* (2.9) *and* $D^nF(z)\neq 0$ *in* $\mathcal{U}$.

Applying Proposition 2.1, we now derive

Theorem 2.1. *Let* $f \in \Sigma$ *and* $1 + A \leq 2(1 + B)$ $(-1 < B < A \leq 1)$. *If*

$$\left|\arg\left(-\frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta \ \ (0 \leq \gamma < 1; 0 < \delta \leq 1)$$

for some $g \in \Sigma[n+1; A, B]$, *then*

$$\left|\arg\left(-\frac{z(D^{n}f(z))'}{D^{n}g(z)} - \gamma\right)\right| < \frac{\pi}{2}\alpha,$$

where α $(0 < \alpha \leq 1)$ *is the solution of the equation*

$$\delta = \alpha + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha \sin\frac{\pi}{2}(1 - t(A, B))}{\frac{2(1-B)+A-1}{1-B} + \alpha\cos\frac{\pi}{2}(1 - t(A, B))}\right) \tag{2.12}$$

when

$$t(A, B) = \frac{2}{\pi}\sin^{-1}\left(\frac{A - B}{2(1 - B^2) - (1 - AB)}\right). \tag{2.13}$$

Proof. Let

$$p(z) = -\frac{1}{1-\gamma}\left(\frac{z(D^n f(z))'}{D^n g(z)} + \gamma\right).$$

By (2.7), we have

$$\begin{aligned}(1-\gamma)zp'(z)D^n g(z) + (1-\gamma)p(z)z(D^n g(z))' - 2z(D^n f(z))' \\ = -z(D^{n+1}f(z))' - \gamma z(D^n g(z))'(z).\end{aligned} \tag{2.14}$$

Dividing (2.14) by $D^n g(z)$ and simplifying, we get

$$p(z) + \frac{zp'(z)}{-q(z) + 2} = -\frac{1}{1-\gamma}\left(\frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)} + \gamma\right), \tag{2.15}$$

where

$$q(z) = -\frac{z(D^n g(z))'}{D^n g(z)}.$$

Since $g \in \Sigma[n+1; A, B]$, from Corollary 2.1, we have

$$q(z) \prec \frac{1+Az}{1+Bz}.$$

From (1.4), we have

$$-q(z) + 2 = \rho e^{i\frac{\pi}{2}\phi},$$

where

$$\begin{cases} \frac{2(1+B)-(1+A)}{1+B} < \rho < \frac{2(1-B)+A-1}{1-B} \\ -t(A,B) < \phi < t(A,B) \end{cases}$$

when $t(A,B)$ is given by (2.13). Let h be a function which maps $\mathcal{U}$ onto the angular domain $\{w : |\arg w| < \frac{\pi}{2}\delta\}$ with $h(0) = 1$. Applying Lemma 2.2 for this h with $\lambda(z) = \frac{1}{-q(z)+2}$, we see that $\mathrm{Re}\, p(z) > 0$ in $\mathcal{U}$ and hence $p(z) \neq 0$ in $\mathcal{U}$.

If there exists a point $z_0 \in \mathcal{U}$ such that the conditions (2.1) and (2.2) are satisfied, then(by Lemma 2.3) we obtain (2.3) under the restrictions (2.4), (2.5) and (2.6).

At first, suppose that $p(z_0)^{\frac{1}{\alpha}} = ia$ $(a > 0)$. Then we obtain

$$\begin{aligned} &\arg\left[-\frac{1}{1-\gamma}\left(\frac{z_0(D^{n+1}f(z_0))'}{D^{n+1}g(z_0)} + \gamma\right)\right] = \arg\left(p(z_0) + \frac{z_0p'(z_0)}{-q(z_0)+2}\right) \\ &= \frac{\pi}{2}\alpha + \arg\left(1 + i\alpha k(\rho e^{i\frac{\pi}{2}\phi})^{-1}\right) \\ &= \frac{\pi}{2}\alpha + \tan^{-1}\left(\frac{\eta k \sin\frac{\pi}{2}(1-\phi)}{\rho + \alpha k \cos\frac{\pi}{2}(1-\phi)}\right) \\ &\geq \frac{\pi}{2}\alpha + \tan^{-1}\left(\frac{\alpha \sin\frac{\pi}{2}(1-t(A,B))}{\frac{2(1-B)+A-1}{1-B} + \alpha\cos\frac{\pi}{2}(1-t(A,B))}\right) \\ &= \frac{\pi}{2}\delta, \end{aligned}$$

where δ and $t(A,B)$ are given by (2.12) and (2.13), respectively. This is a contradiction to the assumption of our theorem.

Next, suppose that $p(z_0)^{\frac{1}{\alpha}} = -ia$ $(a > 0)$. Applying the same method as the above, we have

$$\arg\left[-\frac{1}{1-\gamma}\left(\frac{z_0(D^{n+1}f(z_0))'}{D^{n+1}g(z_0)}+\gamma\right)\right]$$
$$\leq -\frac{\pi}{2}\alpha-\tan^{-1}\left(\frac{\alpha\sin\frac{\pi}{2}(1-t(A,B))}{\frac{2(1-B)+A-1}{1-B}+\alpha\cos\frac{\pi}{2}(1-t(A,B))}\right)$$
$$=-\frac{\pi}{2}\delta,$$

where δ and $t(A,B)$ are given by (2.12) and (2.13), respectively, which contradicts the assumption. Therefore we complete the proof of our theorem.

Letting $A=1,\ B=0$ and $\delta=1$ in Theorem 2.1, we have

Corollary 2.3. *Let $f\in\Sigma$. If*

$$-\text{Re}\left\{\frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)}\right\}>\gamma\ \ (0\leq\gamma<1)$$

for some $g\in\Sigma$ satisfying the condition

$$\left|\frac{z(D^{n+1}g(z))'}{D^{n+1}g(z)}+1\right|<1,$$

then

$$-\text{Re}\left\{\frac{z(D^{n}f(z))'}{D^{n}g(z)}\right\}>\gamma.$$

Taking $A=1,\ B=0$ and $g(z)=\frac{1}{z}$ in Theorem 2.1, we have

Corollary 2.4. *Let $f\in\Sigma$. If*

$$\left|\arg\left[-z^2(D^{n+1}f(z))'-\gamma\right]\right|<\frac{\pi}{2}\delta\ \ (0\leq\gamma<1;\ 0<\delta\leq 1),$$

then

$$\left|\arg\left[-z^2(D^{n}f(z))'-\gamma\right]\right|<\frac{\pi}{2}\delta.$$

Making $n=0$ and $\delta=1$ in Corollary 2.4, we have

Corollary 2.5. *Let $f \in \Sigma$. If*

$$-\text{Re}\ \{z^2(zf''(z) + 3f'(z))\} > \gamma\ \ (0 \le \gamma < 1),$$

then

$$-\text{Re}\ \{z^2 f'(z)\} > \gamma.$$

By the same techniques as in the proof of Theorem 2.1, we obtain

Theorem 2.2. *Let $f \in \Sigma$ and $1 + A \le 2(1 + B)$ $(-1 < B < A \le 1)$. If*

$$\left|\arg\left(\frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)} + \gamma\right)\right| < \frac{\pi}{2}\delta\ \ (\gamma > 1; 0 < \delta \le 1)$$

for some $g \in \Sigma[n+1; A, B]$, then

$$\left|\arg\left(\frac{z(D^n f(z))'}{(D^n g)(z)} + \gamma\right)\right| < \frac{\pi}{2}\alpha,$$

where α $(0 < \alpha \le 1)$ is the solution of the equation given by (2.12).

Next, we prove

Theorem 2.3. *Let $f \in \Sigma$ and choose a positive number c such that*

$$c \ge \frac{1+A}{1+B} - 1\ \ (-1 < B < A \le 1).$$

If

$$\left|\arg\left(-\frac{z(D^n f(z))'}{D^n g(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta\ \ (0 \le \gamma < 1,\ 0 < \delta \le 1)$$

for some $g \in \Sigma[n; A, B]$, then

$$\left|\arg\left(-\frac{z(D^n F(z))'}{D^n G(z)} - \gamma\right)\right| < \frac{\pi}{2}\alpha,$$

where F is the integral operator given by (2.9),

$$G(z) = \frac{c}{z^{c+1}} \int_0^z t^c g(t)dt\ \ (c > 0) \tag{2.16}$$

and $\alpha(0 < \alpha \leq 1)$ *is the solution of the equation*

$$\delta = \alpha + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha\sin\frac{\pi}{2}(1-t(A,B,c))}{\frac{(c+1)(1-B)+A-1}{1-B} + \alpha\cos\frac{\pi}{2}(1-t(A,B,c))}\right) \tag{2.17}$$

when

$$t(A,B,c) = \frac{2}{\pi}\sin^{-1}\left(\frac{A-B}{(c+1)(1-B^2)-(1-AB)}\right).$$

Proof. Let

$$p(z) = -\frac{1}{1-\gamma}\left(\frac{z(D^nF(z))'}{D^nG(z)} + \gamma\right).$$

Since $g \in \Sigma[n; A, B]$, from Corollary 2.2, $G \in \Sigma[n; A, B]$.
Using (2.10), we have

$$(1-\gamma)p(z)D^nG(z) - (c+1)D^nF(z) = -cD^nf(z) - \gamma D^nG(z).$$

Then, by a simple calculation, we get

$$(1-\gamma)(zp'(z) + p(z)(-q(z)+c+1)) + \gamma(-q(z)+c+1) = -\frac{cz(D^nf(z))'}{D^nG(z)},$$

where

$$q(z) = -\frac{z(D^nG(z))'}{D^nG(z)}.$$

Hence we have

$$p(z) + \frac{zp'(z)}{-q(z)+c+1} = -\frac{1}{1-\gamma}\left(\frac{z(D^nf(z))'}{D^ng(z)} + \gamma\right).$$

The remaining part of the proof is similar to that of Theorem 2.1 and so we omit it.

Letting $n = 0$, $A = 1$, $B = 0$ and $\delta = 1$ in Theorem 2.3, we have

Corollary 2.6. *Let* $f \in \Sigma$. *If*

$$-\text{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > \gamma \ \ (0 \le \gamma < 1)$$

for some $g \in \Sigma$ satisfying the condition

$$\left|\frac{zg'(z)}{g(z)} + 1\right| < 1,$$

then

$$-\text{Re}\left\{\frac{zF'(z)}{G(z)}\right\} > \gamma,$$

where F and G are the integral operators given by (2.9) *and* (2.16), *respectively.*

Taking $n = 0$, $B \to A$ and $g(z) = \frac{1}{z}$ in Theorem 2.3, we have

Corollary 2.7. *Let $f \in \Sigma$ and $c > 0$. If*

$$|\arg\,(-z^2 f'(z) - \gamma)| < \frac{\pi}{2}\delta \ \ (0 \le \gamma < 1; 0 < \delta \le 1),$$

then

$$|\arg\,(-z^2 F'(z) - \gamma)| < \frac{\pi}{2}\alpha,$$

where F is the integral operator given by (2.9) *and α* $(0 < \alpha \le 1)$ *is the solution of the equation*

$$\delta = \alpha + \frac{2}{\pi}\tan^{-1}\frac{\alpha}{c}.$$

By using the same methods as in proving Theorem 2.3, we have

Theorem 2.4. *Let $f \in \Sigma$ and choose a positive number c such that*

$$c \ge \frac{1+A}{1+B} - 1 \ \ (-1 < B < A \le 1).$$

If

$$\left|\arg\left(\frac{z(D^n f(z))'}{D^n g(z)} + \gamma\right)\right| < \frac{\pi}{2}\delta \ \ (\gamma > 1;\ 0 < \delta \le 1)$$

for some $g \in \Sigma[n; A, B]$, then

$$\left|\arg\left(\frac{z(D^nF(z))'}{D^nG(z)}+\gamma\right)\right|<\frac{\pi}{2}\alpha,$$

where F and G are the integral operators given by (2.9) *and* (2.16), *respectively, and $\alpha(0<\alpha\leq 1)$ is the solution of the equation given by* (2.17).

Finally, we derive

Theorem 2.5. *Let $f\in\Sigma$ and $1+A\leq 2(1+B)(-1<B<A\leq 1)$. If*

$$\left|\arg\left(-\frac{z(D^nf(z))'}{D^ng(z)}-\gamma\right)\right|<\frac{\pi}{2}\delta\ \ (0\leq\gamma<1;\ 0<\delta\leq 1)$$

for some $g\in\Sigma[n+1;A,B]$, then

$$\left|\arg\left(-\frac{z(D^{n+1}F(z))'}{D^{n+1}G(z)}-\gamma\right)\right|<\frac{\pi}{2}\delta,$$

where F and G are the integral operators given by (2.9) *and* (2.16) *with $c=1$, respectively.*

Proof. From (2.7) and (2.10) with $c=1$, we have $D^nf(z)=D^{n+1}F(z)$ Therefore

$$\frac{z(D^nf(z))'}{D^ng(z)}=\frac{z(D^{n+1}F(z))'}{D^{n+1}G(z)}$$

and the result follows.

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