Carathéodory Functions and Associated Families of Analytic Functions

Carathéodory 함수들과 관련된 해석함수들의 족들

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by

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A Dissertation

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요 약

단엽 함수들 및 그 부분 족들에 속하는 함수들의 growth 및 왜곡정도, 여기에 수반되는 극치문제 등에 관한 연구는 기하함수이론에 있어서 유명한 계수문제인 Bieberbach conjecture가 해결되었음에도 불구하고 지금까지 많은 학자들에 의하여 활발하게 연구되고 있다.

본 논문에서는 Jack에 의하여 소개된 해석함수들의 최대치에 대한 정보를 이용하여, 위수 α의 Carathéodory 함수들의 여러 충분조건과 적분보존성을 조사하였고, 많은 학자들의 결과들을 확장·발전시켰다.

또한, Miller와 Mocanu에 의한 결과들을 응용한 새로운 연구방법을 도입하여, Carathéodory 함수들에 대한 다양한 기하학적 성질들과 성형함수들의 여러 충분조건들을 조사하였다.

더욱이, Carathéodory 함수들에 대한 적분보존성을 연구하여 위수 α의 성형, 볼록 및 close-to-convex 함수들이 확장된 Libera 및 Bernardi 적분연산자들에 대하여 닫혀있음을 밝혔다. 또한, 여러 가지 해석함수들의 기하학적 성질들을 조사함과 동시에 기존에 알려진 여러 결과들을 확장·발전시켰다.

마지막으로, Jack의 결과를 일반화시킨 최근의 Nunokawa의 결과를 이용하여 위수 α의 강 성형함수들의 여러 충분조건을 조사하였고, 일양 볼록함수들과 강 성형함수들의 관계에 대하여 연구하였다.

Preface

The theory of univalent functions is an old subject, which was born around the turn of the century and remains as an active field of current research. The progress is especially rapid in the recent years. The important study of univalent functions began in 1907 when Koebe [13] proved for the class $\mathcal{S}$ of univalent functions the existence of a positive constant c such that

$$\{\, w \,:\, |w| \,\le c \,\} \;\subset\; \bigcap_{f\in\mathcal{S}} f(\mathcal{U}),$$

where $\mathcal{U} = \{z \in \mathcal{C} : |z| < 1\}$ and $\mathcal{C}$ is the complex plane. In 1916, Bieberbach [3] proved that $c = 1/4$, which was an interesting result. This says that the open disk $|w| < 1/4$ is always covered by the map of $\mathcal{U}$ of any function $f \in \mathcal{S}$. Furthermore, he observed that the equality holds for "Koebe function "

$$k(z) = z(1-z)^{-2} = z + 2z^2 + 3z^3 + \cdots \qquad (z \in \mathcal{U})$$

or its rotations.

The Bieberbach conjecture has been considered so difficult to prove or disprove that some mathematicians believed it to be false. Many researchers have devoted their life to resolve it. This conjecture has inspired several development in geometric function theory by generating a lot of number of related problems.

In 1984, Branges [4] surprised the mathematics world by making a claim that he had resolved the Biberbach conjecture. Thus he ended the efforts of many mathematicians of almost seventy years. But there is no doubt that the study of this conjecture led to the development in geometric function theory by generating a lot of number of related problems.

In this thesis, we investigate various conditions for Carathéodory functions in the open unit disk. Also we give some applications of univalent functions as special cases. Furthermore, we derive some argument properties of analytic functions in the open unit disk.

The present thesis consists of five chapters and we give the outlines.

In chapter 1, we first introduce the class of univalent functions and its subclasses under geometric conditions. We also introduced the concept of Carathéodory functions and give some properties of it.

In chapter 2, we show some sufficient conditions for Carathéodory functions of order α by using well-known Jack's lemma [10] and extended some results obtained by many authors(Chichra [6], Marx [18], Miller [19], Miller, Mocanu and Reade [21], Nunokawa [25], Nunokawa, Ikeda, Koike, Ota and Saitoh [28], Ponnusamy and Karunakaran [29] and Strohhäcker [36]). Furthermore, we obtained another various conditions for Carathéodory functions.

In chapter 3, we give some conditions for Carathéodory functions by using Miller and Mocanu's lemma [20] and extend some results obtained by many authors(Miller [19], Miller, Mocanu and Reade [21], Nunokawa, Kwon and Cho [27] and R. Singh and S. Singh [35]). Further, we obtain sufficient conditions for univalent functions with some special cases.

In chapter 4, we obtain some properties of certain analytic functions by using the result of Miller and Mocanu [20] and extend some result obtained by Libera [15]. Furthermore, we obtain some integral preserving properties for certain analytic functions and extend the known results as special cases.

Finally, in chapter 5, we investigate some criteria for strongly starlike functions and the relationship between uniformly convex and strongly starlike functions.

Chapter 1

Univalent Functions

1.1 Basic results

In this chapter, we are mainly connected with functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.1.1)$$

that are analytic in the open unit disk $\mathcal{U} = \{z \in \mathcal{C} : |z| < 1\}$. The class of all functions of the form (1.1.1) that are analytic in $\mathcal{U}$, we'll be denoted by $\mathcal{A}$. Denote $\mathcal{S}$ by the subclass of $\mathcal{A}$ consisting of functions univalent (or schlicht) in $\mathcal{U}$; that is, $f \in \mathcal{S}$ if and only if $f \in \mathcal{A}$ and $f(z_1) \neq f(z_2)$ for all points z_1 and z_2 in $\mathcal{U}$ with $z_1 \neq z_2$. An analytic univalent function is called a conformal mapping, because of its angle-preserving property.

A typical example of a function of the class $\mathcal{S}$ is the Koebe function

$$k(z) = \frac{z}{(1-z)^{-2}} = z + 2z^2 + 3z^3 + \cdots \quad (z \in \mathcal{U}). \qquad (1.1.2)$$

The Koebe function is extremal for many problems relating $\mathcal{S}$. The function k maps $\mathcal{U}$ onto the entire plane minus the part of the negative real axis from

$-1/4$ to infinity.

Theorem 1.1.1 [3]. *If $f \in \mathcal{S}$, then $|a_2| \leq 2$, with equality if and only if f is a rotation of the Koebe function k, that is,*

$$k_\theta(z) = \frac{z}{(1 - e^{i\theta} z)^2} \qquad (\ \theta \in [0,\ 2\pi]\ ). \tag{1.1.3}$$

This theorem was given in 1916 and was the main basis for the famous Bieberbach conjecture [3] that the Taylor coefficients of each function of class $\mathcal{S}$ satisfy the inequality $|a_n| \leq n$. The Bieberbach conjecture was proved by Louis de Branges [4] after many partial results. The study of this conjecture led to the development of a greate number of different and deep methods that have solved many other problems.

As a first application of Biberbach's theorem, we will now introduce a covering theorem due to Koebe. Each function $f \in \mathcal{S}$ is an open mapping with $f(0) = 0$, so its range contains some disk centered at the origin. As early as 1907, Koebe [13] discovered that the ranges of all functions in $\mathcal{S}$ contain a common disk $|w| < \rho$ where ρ is an absolute constant. The Koebe function shows that $\rho \leq 1/4$ and Bieberbach [3] later established Koebe's conjecture that ρ may be taken to be $1/4$.

Theorem 1.1.2 [3]. *The range of every function of class $\mathcal{S}$ contains the disk $\{w : |w| < \frac{1}{4}\}$.*

Furthermore Biberbach's inequality $|a_2| \leq 2$ has implications in the geometric theory of conformal mapping. In particular, this inequality gives a basis estimate which leads to the distortion theorem, which provides sharp upper and lower bounds for $|f'(z)|$ as f ranges over the class $\mathcal{S}$, and related results.

Theorem 1.1.3 [7]. *For each* $f \in \mathcal{S}$,

$$\frac{1-r}{(1+r)^3} \leq \ |f'(z)| \ \leq \frac{1+r}{(1-r)^3}$$
$$\frac{r}{(1+r)^2} \leq \ |f(z)| \ \leq \frac{r}{(1-r)^2}. \tag{1.1.4}$$

For each $z \in \mathcal{U}$, $z \neq 0$, *equality occurs if and only if* f *is a rotation of the Koebe function of* (1.1.3).

1.2 Some special classes

We introduce some subclasses of $\mathcal{S}$ which are defined under natural geometric conditions.

A set $\mathcal{E}$ in the plane is said to be *starlike with respect to* w_0 an interior point of $\mathcal{E}$ if each ray with initial point w_0 intersects the interior of $\mathcal{E}$ in a set that is either a line segment or a ray. If a function f maps $\mathcal{E}$ onto a domain that is starlike with respect to w_0, then we say that f is *starlike with respect to* w_0. In the special case that $w_0 = 0$, we say that f is a *starlike function.*

A set $\mathcal{E}$ in the plane is called *convex* if for every pair of points w_1 and w_2 in the interior of $\mathcal{E}$, the line segment joining w_1 and w_2 is also in the interior of $\mathcal{E}$. If a function f maps $\mathcal{E}$ onto a convex domain, then f is called a *convex function.*

The subclasses of $\mathcal{S}$ consisting of starlike and convex functions are denoted by $\mathcal{S}^*$ and $\mathcal{K}$, respectively. Note that the Koebe function is starlike but not convex.

The following theorems give an analytic description of starlike and convex function [7].

Theorem 1.2.1. *Let $f \in \mathcal{A}$, then $f \in \mathcal{S}^*$ if and only if* $\text{Re}\ zf'/f > 0$.

Theorem 1.2.2. *Let $f \in \mathcal{A}$, then $f \in \mathcal{K}$ if and only if* $\text{Re}\ \{1 + zf''/f'\} > 0$.

The two preceding theorems reveal analytic connection between starlike and convex functions [1].

Theorem 1.2.3. *Let $f \in \mathcal{A}$, then $f \in \mathcal{K}$ if and only if $zf' \in \mathcal{S}^*$.*

A function $f(z)$ analytic in $\mathcal{U}$ is said to be *close-to-convex* if there exists a starlike function $g(z)$ such that

$$\text{Re}\ \frac{zf'(z)}{g(z)} > 0 \qquad (z \in \mathcal{U}).$$

We denote by $\mathcal{C}$ the class of close-to-convex functions f with the usual normalization $f(0) = f'(0) - 1 = 0$. Close-to-convex functions were introduced by Kaplan [12]. Every close-to-convex function is univalent. This can be inferred from the following simple but important criterion for univalence. The criterion is due to Noshiro [24] and Warschawski [37].

Theorem 1.2.4. *If f is analytic in a complex domain $\mathcal{D}$ and* $\text{Re}\ f'(z) > 0$ *there, then f is univalent in $\mathcal{D}$.*

We remark that the following chain of proper inclusions :

$$\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{C} \subset \mathcal{S} \subset \mathcal{A}.$$

1.3 Carathéodory functions

An analytic function p in $\mathcal{U}$ with $p(0) = 1$ is said to be a *Carathéodory function of order α* if it satisfies

$$\text{Re}\ p(z) > \alpha \quad (0 \leq \alpha < 1,\ z \in \mathcal{U}).$$

We denote by $\mathcal{P}(\alpha)$ the class of all Carathéodory functions of order α in $\mathcal{U}$ and $\mathcal{P} \equiv \mathcal{P}(0)$ [19].

Miller [19] and Miller, Mocanu and Reade [21] proved the following results, respectively. If p is analytic in $\mathcal{U}$, then

$$\text{Re}\ \{\ p(z)\ +\ \beta z p'(z)\ \}\ >\ 0\ \ (\beta \geq 0)\ \ \text{implies}\ \ p \in \mathcal{P} \tag{1.3.1}$$

and

$$\text{Re}\left\{\ p(z)\ +\ \beta\frac{zp'(z)}{p(z)}\ \right\}\ >\ 0\ \ \ (\beta \in \mathbf{R},\ \ p(z) \neq 0)\ \ \text{implies}\ \ p \in \mathcal{P}. \tag{1.3.2}$$

Recently, Nunokawa [26] gave many sufficient conditions for Carathéodory functions with an improvement of the result given by (1.3.1) above. Furthermore, the other interesting conditions for Carathédory functions may be found in various articles(for example, see [20, 22, 25]).

Chapter 2

Carathéodory Functions I

2.1 Introduction

In 1936, Robertson [30] introduced the concept of functions starlike and convex of order α as follows.

A function $f \in \mathcal{A}$ is said to be *starlike of order* α if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (0 \leq \alpha < 1,\ z \in \mathcal{U}).$$

The set of all such functions is denoted by $\mathcal{S}^*(\alpha)$ and especially, $\mathcal{S}^*(0) \equiv \mathcal{S}^*$. A function $f \in \mathcal{A}$ is said to be *convex of order* α if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (0 \leq \alpha < 1,\ z \in \mathcal{U}).$$

The set of all such functions is denoted by $\mathcal{K}(\alpha)$ and especially, $\mathcal{K}(0) \equiv \mathcal{K}$.

It is well-known [32] that the class $\mathcal{S}^*$ is a subclass of $\mathcal{S}$, whereas the functions in $\mathcal{S}^*$ need not be univalent if $\alpha < 0$. Furthermore, we note that any functions in $\mathcal{K}(\alpha)$ is univalent in $\mathcal{U}$ if $-1/2 \leq \alpha < 1$, and for $\alpha < -1/2$, there exists a function f in the class $\mathcal{K}(\alpha)$ which is not univalent.

A function $f \in \mathcal{A}$ is said to be a member of the class $\mathcal{R}(\alpha)$ if it satisfies

$$\operatorname{Re} f'(z) > \alpha \quad (\alpha < 1, \ z \in \mathcal{U}).$$

In particular, $\mathcal{R}(0) \equiv \mathcal{R}$ was introduced by MacGregor [17]. It is well known that $\mathcal{R}$ is a subclass of $\mathcal{C}$ and $\mathcal{R}(\alpha) \subset \mathcal{R}$ for $0 \leq \alpha < 1$. In 1952, Zomorovič [38] put the question whether $\mathcal{R}$ was a subclass of $\mathcal{S}^*$. Later on, Krzyz [14] showed by a counterexample that $\mathcal{R}$ is not a subclass of $\mathcal{S}^*$.

Let $\mathcal{T}(\alpha)$ be the subclass of $\mathcal{A}$ consisting of function f which satisfy

$$\operatorname{Re}\left\{\frac{f(z)}{z}\right\} > \alpha \ (0 \leq \alpha < 1, \quad z \in \mathcal{U}).$$

Furthermore, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_s(\alpha)$ if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > \alpha \quad (0 \leq \alpha < \frac{1}{2}, \quad z \in \mathcal{U}).$$

We note that $\mathcal{S}_s(0) \equiv \mathcal{S}_s$ is the class of starlike functions with respect to symmetrical points introduced by Sakaguchi [33].

The integral operator F_c for a function $f \in \mathcal{A}$ is defined by

$$F_c(z) = \frac{c+1}{z^c}\int_0^z f(t)t^{c-1}dt \quad (c > -1). \tag{2.1.1}$$

In particular, the operator F_c ($c \in \{1, \ 2, \ 3, \ \cdots\}$) was introduced by Bernardi [2]. In [1], Alexander showed that if $f \in \mathcal{S}^*$ then $F_0 \in \mathcal{K}$ where

$$F_0(z) = \int_0^z \frac{f(t)}{t}dt. \tag{2.1.2}$$

In [34], R. Singh and S. Singh showed that if $f \in \mathcal{R}$ then F_0 as given by (2.1.2) satisfies $F_0 \in \mathcal{S}^*$. Mocanu [22] proved that the same result holds if F_0 is replaced by

$$F_1(z) = \frac{2}{z}\int_0^z f(t)dt. \tag{2.1.3}$$

In this chapter, we prove some sufficient conditions for Carathéodory functions of order α which cover the form of assumptions given by (1.3.1) and (1.3.2) by using well-known Jack's Lemma [10] and extend some results obtained by many authors [6, 18, 19, 21, 25, 28, 29, 36]. Furthermore, we obtain another conditions for Carathéodory functions(of order α).

In order to prove our results, we need the following lemma given by Jack [10].

Lemma 2.1.1. *Suppose that the function w is analytic for $|z| \leq r$, $w(0) = 0$ and*

$$|w(z_0)| = \max_{|z|=r} |w(z)|.$$

Then $z_0 w'(z_0) = k w(z_0)$, where k is a real number with $k \geq 1$.

2.2 Conditions for Carathéodory functions of order α

In this section, we shall investigate some conditions for $\mathcal{P}(\alpha)$ by using Lemma 2.1.1 and extend some results by Miller [19], Nunokawa [25], Nunokawa, Ikeda, Koike, Ota and Saitoh [28], Ponnusamy and Karunakaran [29] and Miller, Mocanu and Reade [21].

Theorem 2.2.1. *Let p be analytic in $\mathcal{U}$ with $p(0) = 1$, $0 \leq \alpha < 1$ and $\beta \geq 0$. If*

$$\text{Re}\ \{p(z) + \beta z p'(z)\} > \alpha - \frac{\beta}{2(1-\alpha)}(1 - 2\alpha + |p(z)|^2), \qquad (2.2.1)$$

then $p \in \mathcal{P}(\alpha)$.

Proof. Define the function w by

$$p(z) = \frac{1+(1-2\alpha)w(z)}{1-w(z)}.$$

Then we see that w is analytic in $\mathcal{U}$ with $w(0)=0$. Suppose that there exists a point z_0 in $\mathcal{U}$ such that

$$\operatorname{Re} p(z) > \alpha \quad \text{for} \quad |z| < |z_0| \tag{2.2.2}$$

and

$$\operatorname{Re} p(z_0) = \alpha. \tag{2.2.3}$$

Then we have

$$|w(z)| < 1 \quad \text{for} \quad |z| < |z_0| \tag{2.2.4}$$

and

$$|w(z_0)| = 1. \tag{2.2.5}$$

By using Lemma 2.1.1, we get

$$z_0 w'(z_0) = k w(z_0), \tag{2.2.6}$$

where k is a real number with $k \geq 1$. Putting

$$p(z_0) = \alpha + iy \quad (y \in \mathbf{R}),$$

we obtain

$$w(z_0) = 1 - \frac{2(1-\alpha)^2}{(1-\alpha)^2+y^2} + i\frac{2(1-\alpha)y}{(1-\alpha)^2+y^2}. \tag{2.2.7}$$

Then, from (2.2.6) and (2.2.7), we have

$$\operatorname{Re}\{\, p(z_0) + \beta z_0 p'(z_0) \,\} = \alpha + \beta \operatorname{Re}\left\{ \frac{2(1-\alpha)z_0 w'(z_0)}{(1-w(z_0))^2} \right\}$$

$$
\begin{aligned}
&= \alpha \ + \ 2\beta(1-\alpha)k \ \mathrm{Re} \left\{ \frac{w(z_0)}{(1-w(z_0))^2} \right\} \\
&= \alpha \ - \ \beta k \ \frac{(1-\alpha)^2 + y^2}{2(1-\alpha)} \\
&\leq \alpha \ - \ \frac{\beta}{2(1-\alpha)} \left(1 - 2\alpha + |p(z_0)|^2 \right).
\end{aligned}
$$

This contradicts the assumption (2.2.1). Therefore we complete the proof of Theorem 2.2.1.

Taking $\alpha = 0$ and $\beta = 1$ in Theorem 2.2.1, we have the following result by Nunokawa et al. [28].

Corollary 2.2.1. *Let p be analytic in $\mathcal{U}$ with $p(0) = 1$. If*

$$
\mathrm{Re} \{ \ p(z) \ + \ zp'(z) \ \} \ > \ -\frac{1 + |p(z)|^2}{2},
$$

then $p \in \mathcal{P}$.

Remark 2.2.1. Corollary 2.2.1 is an improvement of the result by Miller [19].

The right hand side of the assumption (2.2.1) in Theorem 2.2.1 depend on $|p(z)|$. But applying the same method as the proof of Theorem 2.2.1, we can derive a similar result (Theorem 2.2.1$'$ below) without depending on $|p(z)|$ in the assumption (2.2.1) of Theorem 2.2.1.

Theorem 2.2.1$'$. *Let p be analytic in $\mathcal{U}$ with $p(0) = 1$. If*

$$
\mathrm{Re} \{ \ p(z) \ + \ \beta zp'(z) \ \} \ > \ \alpha - \frac{\beta(1-\alpha)}{2} \quad (0 \leq \alpha < 1, \ 0 \leq \beta),
$$

then $p \in \mathcal{P}(\alpha)$.

Letting $\alpha = 0$ and $\beta = 1$ in Theorem 2.2.1$'$, we have

Corollary 2.2.1′. *Let p be analytic in $\mathcal{U}$ with $p(0)=1$. If*

$$\mathrm{Re}\ \{\ p(z)\ +\ zp'(z)\ \}\ >\ \frac{3\alpha-1}{2}\quad (0\le\alpha<1),$$

then $p\in\mathcal{P}(\alpha)$.

Remark 2.2.2. Corollary 2.2.1′ is an improvement of the result by Nunokawa [25].

Example 2.2.1. Taking p in above results by various types of analytic functions we have the following results : If $f\in\mathcal{A}$, then

(i) $\mathrm{Re}\ \{f'(z)+\beta zf''(z)\}>\frac{\beta}{2}(1+|f'(z)|^2)\ (\beta>0)$ implies $f\in\mathcal{R}$ [6],

(ii) $\mathrm{Re}\ \{f'(z)+zf''(z)\}>-\frac{1}{2}$ implies $f\in\mathcal{R}$,

(iii) $\mathrm{Re}\ f'(z)\ >\ -\frac{1}{2}\left(1+\left|\frac{f(z)}{z}\right|^2\right)$ implies $\mathrm{Re}\ \frac{f(z)}{z}\ >\ 0$,

(iv) $\mathrm{Re}\left\{\frac{zf'(z)f^{\gamma-1}(z)}{z^\gamma}\right\}\ >\ \alpha\ -\ \frac{1-\alpha}{2(c+\gamma)}$ implies

$\mathrm{Re}\left\{\frac{zF'(z)F^{\gamma-1}(z)}{z^\gamma}\right\}\ >\ \alpha,$ where $F(z)=\left(\frac{c+\gamma}{z^c}\int_0^z t^{c-1}f^\gamma(t)dt\right)^{\frac{1}{\gamma}}$,

$0\le\alpha<1,\quad c>-\gamma$ and $\gamma>0$ [29].

Theorem 2.2.2. *Let p be analytic in $\mathcal{U}$ with $p(0)=1$. If*

$$\mathrm{Re}\ \left\{p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\right\}\ >\ \delta\left(\alpha,\beta,\gamma,|p(z)|\right), \tag{2.2.8}$$

where

$$\delta\left(\alpha,\beta,\gamma,|p(z)|\right)\ =\ \alpha\ -\ \frac{(\alpha\beta+\gamma)(1-2\alpha+|p(z)|^2)}{2(1-\alpha)(\gamma^2+2\alpha\beta\gamma+\beta^2|p(z)|^2)} \tag{2.2.9}$$
$$(0\le\alpha<1,\ \beta\ne0,\ \alpha\beta+\gamma>0),$$

then $p\in\mathcal{P}(\alpha)$.

Proof. At first, we note that $p(z) \neq -\gamma/\beta$ for $z \in \mathcal{U}$. In fact, if $\beta p(z) + \gamma$ has a zero of order m at $z = z_1 \in \mathcal{U}$, then we can write

$$\beta p(z) \ + \ \gamma \ = \ (z - z_1)^m p_1(z) \quad (m \in \mathbf{N}),$$

where p_1 is analytic in $\mathcal{U}$ and $p_1(z_1) \neq 0$. Then we have

$$p(z) \ + \ \frac{zp'(z)}{\beta p(z) + \gamma} \ = \ \frac{1}{\beta}\left\{(z - z_1)^m p_1(z) - \gamma + \frac{mz}{z - z_1} + \frac{zp_1'(z)}{p_1(z)}\right\}. \quad (2.2.10)$$

Thus choosing $z \to z_1$ suitably, the real part of the right hand side of (2.2.10) can take any negative infinite values, which contradicts the hypothesis (2.2.8).

Defining w by

$$p(z) \ = \ \frac{1 + (1 - 2\alpha)w(z)}{1 - w(z)},$$

we see that the function w is analytic in $\mathcal{U}$ with $w(0) = 0$. If there exists a point $z_0 \in \mathcal{U}$ such that the conditions (2.2.2) and (2.2.3) are satisfied, then (by Lemma 2.1.1) we obtain (2.2.6) under the restrictions (2.2.4) and (2.2.5). Setting

$$p(z_0) \ = \ \alpha \ + \ iy \quad (y \in \mathbf{R}),$$

we have

$$w(z_0) \ = \ \frac{-(1 - \alpha)^2 + y^2}{(1 - \alpha)^2 + y^2} \ + \ i\frac{2(1 - \alpha)y}{(1 - \alpha)^2 + y^2}.$$

Then, by a simple calculation, we obtain

$$\begin{aligned}
&\mathrm{Re}\left\{p(z_0) + \frac{z_0 p'(z_0)}{\beta p(z_0) + \gamma}\right\} \\
&= \alpha + \mathrm{Re}\left\{\frac{2(1 - \alpha)z_0 w'(z_0)}{(1 - w(z_0))(\beta + \gamma + (\beta - 2\alpha\beta - \gamma)w(z_0))}\right\} \\
&= \alpha \ + \ 2(1 - \alpha)k\mathrm{Re}\left\{\frac{w(z_0)}{(1 - w(z_0))(\beta + \gamma + (\beta - 2\alpha\beta - \gamma)w(z_0))}\right\} \\
&\leq \alpha \ - \ \frac{1 - 2\alpha + \alpha^2 + y^2}{2(1 - \alpha)(\gamma^2 + 2\alpha\beta\gamma + \beta^2(\alpha^2 + y^2))}
\end{aligned}$$

$$
\begin{aligned}
&= \alpha - \frac{(\alpha\beta+\gamma)(1-2\alpha+|p(z_0)|^2)}{2(1-\alpha)(\gamma^2+2\alpha\beta\gamma+\beta^2|p(z_0)|^2)} \\
&= \delta(\alpha, \beta, \gamma, |p(z_0|),
\end{aligned}
$$

where $\delta(\alpha, \beta, \gamma, |p(z_0)|)$ is given by (2.2.9), which contradicts the assumption (2.2.8). Therefore we complete the proof of Theorem 2.2.2.

Remark 2.2.3. For $\gamma = 0$, Theorem 2.2.2 is the improvement of result by Miller et al. [21].

Taking $\beta = 1$ and $\gamma = 0$ in Theorem 2.2.2, we have

Corollary 2.2.2. *Let p be analytic in $\mathcal{U}$ with $p(0) = 1$ and $0 \le \alpha < 1$. If*

$$
\operatorname{Re}\left\{ p(z) + \frac{zp'(z)}{p(z)} \right\} > \frac{\alpha(1-2\alpha)(|p(z)|^2-1)}{2(1-\alpha)|p(z)|^2},
$$

then $p \in \mathcal{P}(\alpha)$.

By using the same method as the proof of Theorem 2.2.2, we have

Theorem 2.2.2′. *Let p be analytic in $\mathcal{U}$ with $p(0) = 1$ and $\beta > 0$, $0 \le \alpha < 1$. If one of the following conditions*

$$
\text{(i)} \qquad \operatorname{Re}\left\{p(z) + \frac{zp'(z)}{\beta p(z)+\gamma}\right\} > \alpha - \frac{\alpha\beta+\gamma}{2\beta^2(1-\alpha)} \quad (-\alpha\beta < \gamma < \beta(1-2\alpha)),
$$

$$
\text{(ii)} \qquad \operatorname{Re}\left\{p(z) + \frac{zp'(z)}{\beta+\gamma}\right\} > \alpha - \frac{1}{2\beta} \quad (\gamma = \beta(1-2\alpha)),
$$

$$
\text{(iii)} \qquad \operatorname{Re}\left\{p(z) + \frac{zp'(z)}{\beta p(z)+\gamma}\right\} > \alpha - \frac{1-\alpha}{2(\alpha\beta+\gamma)} \quad (\gamma > \beta(1-2\alpha))
$$

is satisfied, then $p \in \mathcal{P}(\alpha)$.

Letting $\beta = 1$ and $\gamma = 0$ in Theorem 2.2.2′, we have

Corollary 2.2.2′. *Let p be analytic in $\mathcal{U}$ with $p(0) = 1$. If one of the following conditions*

$$\text{(i)} \qquad \mathrm{Re}\left\{p(z) + \frac{zp'(z)}{p(z)}\right\} > \frac{\alpha - 2\alpha^2}{2(1-\alpha)} \quad \left(0 \leq \alpha < \frac{1}{2}\right),$$

$$\text{(ii)} \qquad \mathrm{Re}\left\{p(z) + \frac{zp'(z)}{p(z)}\right\} > 0 \quad \left(\alpha = \frac{1}{2}\right),$$

$$\text{(iii)} \qquad \mathrm{Re}\left\{p(z) + \frac{zp'(z)}{p(z)}\right\} > \frac{(1+\alpha)(2\alpha - 1)}{2\alpha} \quad \left(\frac{1}{2} < \alpha < 1\right)$$

is satisfied, then $p \in \mathcal{P}(\alpha)$.

Remark 2.2.4. Taking $p(z) = zf'(z)/f(z)$ in Corollary 2.2.2′, we have the classical result by Marx [18] and Strohhäcker [36] : that is, $\mathcal{K} \subset \mathcal{S}^*(1/2)$.

Applying the same method of the proof in Theorem 2.2.1 and Theorem 2.2.2, we obtain the following result. We only will introduce the statements without the proof.

Theorem 2.2.3. *Let p be analytic in $\mathcal{U}$ with $p(0) = 1$. If*

$$|\, p(z) + zp'(z) - \beta \,| < \delta(\alpha, \beta, |p(z)|),$$

where $0 \leq \alpha < 1$, $\{1 - 3\alpha\}/2 \leq \beta$ and

$$\delta(\alpha, \beta, |p(z)|) = \left\{\left(1 - \alpha + \beta + \frac{|p(z)|^2 - 1}{2(1-\alpha)}\right)^2 + |p(z)|^2 - \alpha^2\right\}^{\frac{1}{2}},$$

then $p \in \mathcal{P}(\alpha)$.

Taking $\alpha = 0$ and $\beta = 1$ in Theorem 2.2.3, we have

Corollary 2.2.3. *Let p be analytic in $\mathcal{U}$ with $p(0) = 1$. If*

$$|\, p(z) + zp'(z) - 1 \,| < \sqrt{\left(\frac{3 + |p(z)|^2}{2}\right)^2 + |p(z)|^2},$$

then $p \in \mathcal{P}$.

Next, we derive another condition for Carathéodory functions of order α in Theorem 2.2.4 below.

Theorem 2.2.4. *Let p be analytic in $\mathcal{U}$ with $p(0) = 1$ and $0 \leq \alpha < 1$. If*

$$\frac{zp'(z)}{p(z) - \alpha} \neq iA \quad (|A| \geq 1), \tag{2.2.11}$$

then $p \in \mathcal{P}(\alpha)$.

Proof. Let

$$q(z) = \frac{1}{1-\alpha}(p(z) - \alpha).$$

Then q is analytic in $\mathcal{U}$ with $q(0) = 1$. Here, we note that $p(z) \neq \alpha$ for $z \in \mathcal{U}$. In fact, if there exists a point $z_0 \in \mathcal{U}$ such that $p(z_0) = \alpha$ and hence $q(z_0) = 0$ then q can written by

$$q(z) = (z - z_1)^m q_1(z) \quad (m \in \mathbf{N}),$$

where q_1 is analytic in $\mathcal{U}$ and $q_1(z_1) \neq 0$. Hence we have

$$\frac{zp'(z)}{p(z) - \alpha} = \frac{zq'(z)}{q(z)} = \frac{mz}{z - z_1} + \frac{zq_1'(z)}{q_1(z)}. \tag{2.2.12}$$

But, the imaginary part of the right-hand side of (2.2.12) can take any value when z approaches z_0. This contradicts our assumption (2.2.11).

Suppose that there exists a point $z_0 \in \mathcal{U}$ such that

$$\text{Re}\, q(z) > 0 \quad \text{for} \quad |z| < |z_0|$$

and

$$\text{Re}\, q(z_0) = 0 \quad (q(z_0) \neq 0).$$

Setting

$$\phi(z) = \frac{1 - q(z)}{1 + q(z)},$$

we have

$$|\phi(z)| < 1 \quad \text{for} \quad |z| < |z_0|$$

and

$$|\phi(z_0)| = 1 \quad (\phi(0) = 0).$$

Let $q(z_0) = ia$ ($a \in \mathbf{R}$). Then, by Lemma 2.1.1, we obtain

$$\frac{z_0\phi'(z_0)}{\phi(z_0)} = \frac{-2z_0q'(z_0)}{1-q^2(z_0)} = \frac{-2z_0q'(z_0)}{1+a^2} = k,$$

where k is a real number with $k \geq 1$ and so

$$-z_0q'(z_0) \geq \frac{1+a^2}{2}.$$

Therefore, $z_0q'(z_0)$ is a negative real number.

At first, suppose that $a > 0$. Then we have

$$\frac{z_0p'(z_0)}{p(z_0)-\alpha} = \frac{z_0q'(z_0)}{q(z_0)} = \frac{-iz_0q'(z_0)}{a} \equiv iA.$$

Hence we obtain

$$A = \frac{-z_0q'(z_0)}{a} \geq \frac{1+a^2}{2a} \geq 1,$$

which contradicts the assumption (2.2.11).

Next, for $a < 0$, we have

$$\frac{z_0p'(z_0)}{p(z_0)-\alpha} = \frac{z_0q'(z_0)}{q(z_0)} = \frac{iz_0q'(z_0)}{|q(z_0)|} = \frac{iz_0q'(z_0)}{|a|} \equiv -iA$$

and A is a real number with $A \geq 1$. This also contradicts the assumption (2.2.11). Hence we complete the proof of Theorem 2.2.4.

Remark 2.2.5. Taking p by appropriate analytic functions in Theorem 2.2.4, we can find the conditions for univalence, starlikeness, convexity and so on.

2.3 Another properties

In this section, we consider the second-order differential inequality for Carathéodory functions and investigate some conditions for the class $\mathcal{P}$.

Theorem 2.3.1. *Let* B *and* C *be analytic in* $\mathcal{U}$ *with* $\text{Re } B(z) > A$ $(A \in \mathbf{R})$. *If* p *is analytic in* $\mathcal{U}$ *with* $p(0) = 1$ *and*

$$\text{Re } \left\{ Az^2p''(z) + B(z)zp'(z) + C(z)p(z) \right\} > \delta\left(A, B(z), C(z)\right),$$

where

$$\delta\left(A, B(z), C(z)\right) = \frac{(\text{Im } C(z))^2 - (\text{Re } B(z) - A)^2}{2\left(\text{Re } B(z) - A\right)},$$

then $p \in \mathcal{P}$.

Proof. Define the function w by

$$p(z) = \frac{1 + w(z)}{1 - w(z)}.$$

Then we see that w is analytic in $\mathcal{U}$ with $w(0) = 0$. Suppose that there exists a point z_0 in $\mathcal{U}$ such that

$$\text{Re } p(z) > 0 \quad \text{for} \quad |z| < |z_0| \tag{2.3.1}$$

and

$$\text{Re } p(z_0) = 0. \tag{2.3.2}$$

Then (by Lemma 2.1.1) we obtain (2.2.6) under the restrictions (2.2.4) and (2.2.5). Putting

$$p(z_0) = iy \quad (y \in \mathbf{R}),$$

we obtain

$$w(z_0) = \frac{-1 + y^2}{1 + y^2} + i\frac{2y}{1 + y^2}.$$

Then we have

$$\begin{aligned}
& \text{Re } \left\{ A z_0^2 p''(z_0) + B(z_0) z_0 p'(z_0) + C(z_0) p(z_0) \right\} \\
= & \text{ Re } \left\{ z_0 p'(z_0) \left(kAp(z_0) - A + B(z_0) \right) + C(z_0) p(z_0) \right\} \\
= & \text{ Re } \left\{ \frac{2kw(z_0)}{(1-w(z_0))^2} \left(kAp(z_0) - A + B(z_0) \right) + C(z_0) p(z_0) \right\} \\
= & \text{ Re } \left\{ -\frac{(1+y^2)k}{2} \left(iykA - A + B(z_0) \right) + iyC(z_0) \right\} \\
= & - \left\{ \frac{k}{2}(1+y^2) \left(\text{Re } B(z_0) - A \right) + y \text{Im } C(z_0) \right\} \\
\leq & - \left\{ \frac{k^2 \left(\text{Re } B(z_0) - A \right)^2 - \left(\text{Im } C(z_0) \right)^2}{2k \left(\text{Re } B(z_0) - A \right)} \right\} \\
\leq & \frac{\left(\text{Im } C(z_0) \right)^2 - \left(\text{Re } B(z_0) - A \right)^2}{2 \left(\text{Re } B(z_0) - A \right)}.
\end{aligned}$$

But this contradicts our assumption. Hence the proof is completed.

Taking $A = 0$ and $B(z) = C(z) = 1$ in Theorem 2.3.1, then we have

Corollary 2.3.1. *Let p be analytic in $\mathcal{U}$ with $p(0) = 1$. Then*

$$\text{Re } \{ \, p(z) \; + \; zp'(z) \, \} \; > \; -\frac{1}{2} \quad \textit{implies} \quad \text{Re } p(z) \; > \; 0.$$

Remark 2.3.1. Corollary 2.3.1 is the result obtained by Miller [19]. And this is also shown by Corollary 2.2.1′.

Letting $p(z) = \{F(z)/z\}^{\beta}$, $A = 0$, $B(z) = 1$ and $C(z) = \gamma + \beta$ in Theorem 2.3.1, we have

Corollary 2.3.2. *Let $f \in \mathcal{A}$ and let β and γ be complex numbers with*

$$\text{Re } (\beta + \gamma) \; > \; 0 \quad (\beta \neq 0).$$

If

$$\text{Re } \left\{ (\gamma + \beta) \left(\frac{f(z)}{z} \right)^{\beta} \right\} \; > \; \frac{1}{2} \left\{ (\text{Im } \; (\gamma + \beta))^2 - 1 \right\},$$

then

$$\operatorname{Re}\left(\frac{F(z)}{z}\right)^{\beta} > 0,$$

where

$$F(z) = \left(\frac{\gamma+\beta}{z^{\beta}}\int_0^z f(t)^{\beta}t^{\gamma-1}dt\right)^{\frac{1}{\beta}}.$$

Theorem 2.3.2. *Let $0 < \alpha \leq 1$. If p is analytic in $\mathcal{U}$ with $p(0) = 1$ and*

$$\operatorname{Re}\left\{ p^{\alpha}(z)\left(1+\frac{zp'(z)}{p(z)}\right)\right\} > h(\delta(\alpha),\alpha),$$

where

$$h(x,\alpha) = -\frac{1}{2}\left(x^{\alpha+1}\sin\frac{\alpha\pi}{2} - 2x^{\alpha}\cos\frac{\alpha\pi}{2} + x^{\alpha-1}\sin\frac{\alpha\pi}{2}\right)$$

and

$$\delta(\alpha) = \frac{1}{(1+\alpha)\sin\frac{\alpha\pi}{2}}\left(\alpha\cos\frac{\alpha\pi}{2} + \sqrt{(1-2\alpha^2)\sin^2\frac{\alpha\pi}{2}+\alpha^2}\right),$$

then $p \in \mathcal{P}$.

Proof. Firstly, we note that $p(z) \neq 0$ for $0 < \alpha < 1$. Define the function w by

$$p(z) = \frac{1+w(z)}{1-w(z)}.$$

Then we see that w is analytic in $\mathcal{U}$ with $w(0) = 0$. If there exists a point $z_0 \in \mathcal{U}$ such that the conditions (2.3.1) and (2.3.2) are satisfied, then (by lemma 2.1.1) we obtain (2.2.6) under the restrictions (2.2.4) and (2.2.5). Putting

$$p(z_0) = iy \begin{cases} y \in \mathbf{R} - \{0\} & \text{for } 0 < \alpha < 1, \\ y \in \mathbf{R} & \text{for } \alpha = 1. \end{cases}$$

We obtain

$$w(z_0) = \frac{-1+y^2}{1+y^2} + i\frac{2y}{1+y^2}.$$

Then we have

$$
\begin{aligned}
\operatorname{Re}\left\{p^{\alpha}(z_0)\left(1+\frac{z_0p'(z_0)}{p(z_0)}\right)\right\} &= \operatorname{Re}\left\{\left(p(z_0)+z_0p'(z_0)\right)p^{\alpha-1}(z_0)\right\}\\
&= \operatorname{Re}\left\{\left(iy-\frac{(1+y^2)k}{2}\right)(iy)^{\alpha-1}\right\}\\
&= \operatorname{Re}\left\{\left(iy-\frac{(1+y^2)k}{2}\right)|y|^{\alpha-1}\left(\cos\left(\pm\frac{(\alpha-1)\pi}{2}\right)+i\sin\left(\pm\frac{(\alpha-1)\pi}{2}\right)\right)\right\}.
\end{aligned}
$$

At first, we consider the case $0<\alpha<1$.

(i) For the case $y>0$, we have

$$
\begin{aligned}
&\operatorname{Re}\left\{p^{\alpha}(z_0)\left(1+\frac{z_0p'(z_0)}{p(z_0)}\right)\right\}\\
&= \operatorname{Re}\left\{\left(-\frac{k}{2}\left(y^{\alpha-1}+y^{\alpha+1}\right)+iy^{\alpha}\right)\left(\sin\frac{\alpha\pi}{2}-i\cos\frac{\alpha\pi}{2}\right)\right\}\\
&= -\frac{k}{2}\left(y^{\alpha-1}+y^{\alpha+1}\right)\sin\frac{\alpha\pi}{2}+y^{\alpha}\cos\frac{\alpha\pi}{2}\\
&\le \frac{1}{2}\left(-y^{\alpha+1}\sin\frac{\alpha\pi}{2}+2y^{\alpha}\cos\frac{\alpha\pi}{2}-y^{\alpha-1}\sin\frac{\alpha\pi}{2}\right)\\
&= h(y,\alpha).
\end{aligned}
$$

Then, by a simple calculation, we obtain

$$
\begin{aligned}
h(y,\alpha) &\le h\left(\frac{1}{(1+\alpha)\sin\frac{\alpha\pi}{2}}\left(\alpha\cos\frac{\alpha\pi}{2}+\sqrt{(1-2\alpha^2)\sin^2\frac{\alpha\pi}{2}+\alpha^2}\right),\alpha\right)\\
&= h(\delta(\alpha),\alpha).
\end{aligned}
$$

(ii) For the case $y<0$, we have

$$
\begin{aligned}
&\operatorname{Re}\left\{p^{\alpha}(z_0)\left(1+\frac{z_0p'(z_0)}{p(z_0)}\right)\right\}\\
&= \operatorname{Re}\left\{\left(-\frac{k}{2}\left(|y|^{\alpha-1}+|y|^{\alpha+1}\right)+i|y|^{\alpha}\right)\left(\sin\frac{\alpha\pi}{2}+i\cos\frac{\alpha\pi}{2}\right)\right\}\\
&= -\frac{k}{2}\left(|y|^{\alpha-1}+|y|^{\alpha+1}\right)\sin\frac{\alpha\pi}{2}+|y|^{\alpha}\cos\frac{\alpha\pi}{2}\\
&= h(|y|,\alpha)\\
&\le h(\delta(\alpha),\alpha).
\end{aligned}
$$

These contradict our assumption.

Next, we consider the case $\alpha = 1$.

$$\mathrm{Re}\ \{\ p(z_0)\ +\ z_0 p'(z_0)\ \}\ =\ -\frac{k}{2}(1+y^2)\ \leq\ -\frac{1}{2}\ =\ \delta(h(1),1).$$

This contradicts our assumption. So, the proof is completed.

Remark 2.3.2. Taking $\alpha = 1$ in Theorem 2.3.2, we obtain the same result of Corollary 2.3.1.

Taking $p(z) = f(z)/z$ in Theorem 2.3.2, we have

Example 2.3.1. Let $f \in \mathcal{A}$ and $0 < \alpha \leq 1$. If

$$\mathrm{Re}\left\{ f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1}\right\} > h(\delta(\alpha),\alpha) \quad (z \in \mathcal{U}),$$

where h and $\delta(\alpha)$ are given in Theorem 2.3.2, respectively, then

$$\mathrm{Re}\,\frac{f(z)}{z} > 0.$$

Taking $\alpha = 1/2$ in Example 2.3.1, we have $h(\delta(\alpha),\alpha) = 0$. So we obtain the following example.

Example 2.3.2. Let $f \in \mathcal{A}$. Then

$$\mathrm{Re}\left\{ f'(z)\left(\frac{z}{f(z)}\right)^{\frac{1}{2}}\right\} > 0 \text{ implies } \mathrm{Re}\,\frac{f(z)}{z} > 0.$$

Theorem 2.3.3. *Let p be analytic in $\mathcal{U}$ with $p(0) = 1$. If*

$$\frac{zp'(z)}{p(z)}\frac{p^2(z)-1}{p^2(z)+1} \neq iA \quad (|A| \geq 1), \tag{2.3.3}$$

then $p \in \mathcal{P}$.

Proof. From the assumption (2.3.3), we note that $p(z) \neq 0$ and $p^2(z) \neq -1$ for $z \in \mathcal{U}$. Let

$$p(z) = \frac{1+w(z)}{1-w(z)}.$$

Then we see that w is analytic in $\mathcal{U}$ with $w(0) = 0$. We claim that $|w(z)| < 1$ in $\mathcal{U}$. Suppose that there exists a point $z_0 \in \mathcal{U}$ such that

$$\max_{|z|<|z_0|} |w(z)| = |w(z_0)| = 1 \quad (w(z_0) \neq -1).$$

Then, by Lemma 2.1.1, we have

$$z_0 w'(z_0) = k w(z_0) \quad (k \geq 1).$$

Writing $w(z_0) = e^{i\theta}$ $(0 < |\theta| < \pi)$, we note that

$$p(z_0) = \frac{1+e^{i\theta}}{1-e^{i\theta}} = i \cot \frac{\theta}{2}.$$

Hence we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{2kw(z_0)}{1-w^2(z_0)} = i \frac{k}{\sin \theta}.$$

For the case $0 < \theta < \pi$, we have

$$\mathrm{Im} \frac{z_0 p'(z_0)}{p(z_0)} \frac{p^2(z_0)-1}{p^2(z_0)+1} = \mathrm{Im} \left\{ i \frac{k}{\sin \theta} \frac{(i \cot \frac{\theta}{2})^2 - 1}{(i \cot \frac{\theta}{2})^2 + 1} \right\} = -\frac{k}{\sin \theta} g(t),$$

where

$$g(t) = \frac{1+t^2}{1-t^2} \quad \left(t = \cot \frac{\theta}{2}\right). \tag{2.3.4}$$

Since $g(t)$ is an increasing function for $t > 0$, we obtain

$$\begin{aligned} \mathrm{Im} \frac{z_0 p'(z_0)}{p(z_0)} \frac{p^2(z_0)-1}{p^2(z_0)+1} &\leq -\frac{k}{\sin \theta} g(0^+) \\ &= -\frac{k}{\sin \theta} \\ &\leq -1, \end{aligned}$$

which contradicts the assumption (2.3.3). Similarly, for the case $-\pi < \theta < 0$, we have

$$\begin{aligned}\operatorname{Im} \frac{z_0 p'(z_0)}{p(z_0)} \frac{p^2(z_0) - 1}{p^2(z_0) + 1} &= -\frac{k}{\sin \theta} g(t) \\ &\geq -\frac{k}{\sin \theta} g(0^-) \\ &= -\frac{k}{\sin \theta} \\ &\geq 1,\end{aligned}$$

where g is given by (2.3.4). This contradicts the assumption (2.3.3). Therefore we complete the proof of Theorem 2.3.3.

Remark 2.3.3. Taking p by appropriate analytic functions in Theorem 2.3.3, we can find the conditions for univalence, starlikeness, convexity and so on.

Theorem 2.3.4. *Let* $f \in \mathcal{A}$ *and* $h(z) = \sqrt{f(z^2)}$. *If* $h \in \mathcal{S}_s(\beta)$, *then* $h \in \mathcal{T}(\alpha)$, *where*

$$\beta = \frac{(2-3\alpha)}{4(1-\alpha)} \quad \text{and} \quad 0 \leq \alpha \leq \frac{1}{2}.$$

Proof. Let

$$p(z) = \frac{h(z)}{z} = \frac{1 + (1-2\alpha)w(z)}{1 - w(z)}.$$

Then we see that w is analytic in $\mathcal{U}$ with $w(0) = 0$. We claim that $|w(z)| < 1$ in $\mathcal{U}$. Suppose that there exists a point $z_0 \in \mathcal{U}$ such that $\max_{|z|<|z_0|} |w(z)| = |w(z_0)| = 1$. Then, by Lemma 2.1.1, we get (2.2.6). Here, we note that

$$\begin{aligned}h(z) &= \sqrt{f(z^2)} \\ &= \sqrt{z^2 + \sum_{n=2}^{\infty} a_n z^{2n}} \\ &= z\left\{1 + a_2 z^2 + a_3 z^4 + \cdots\right\}^{\frac{1}{2}} \\ &= z + c_3 z^3 + c_5 z^5 + \cdots\end{aligned}$$

So, h is an odd function with $h(0) = 0$. Therefore we have

$$\frac{zh'(z)}{h(z) - h(-z)} = \frac{zh'(z)}{2h(z)}.$$

Putting $p(z_0) = \alpha + iy \quad (y \in \mathcal{R})$, we have

$$w(z_0) = \frac{-(1-\alpha)^2 + y^2}{(1-\alpha)^2 + y^2} + i\frac{2(1-\alpha)y}{(1-\alpha)^2 + y^2}.$$

Then

$$\begin{aligned}
\text{Re}\left\{ \frac{z_0 h'(z_0)}{h(z_0) - h(-z_0)} \right\} &= \frac{1}{2}\text{Re}\left\{ 1 + \frac{z_0 p'(z_0)}{p(z_0)} \right\} \\
&= \frac{1}{2}\left\{ 1 + \text{Re}\,\frac{(1-2\alpha)kw(z_0)}{1 + (1-2\alpha)w(z_0)} + \text{Re}\,\frac{kw(z_0)}{1 - w(z_0)} \right\} \\
&= \frac{1}{2}\left\{ 1 - k\frac{\alpha\left((1-\alpha)^2 + y^2\right)}{2(1-\alpha)(\alpha^2 + y^2)} \right\} \\
&\leq \frac{1}{4(1-\alpha)} g(t),
\end{aligned}$$

where

$$g(t) = \frac{\alpha(1-\alpha)(3\alpha - 1) + (2 - 3\alpha)t}{\alpha^2 + t} \qquad (\, t = y^2 \geq 0 \,).$$

Since g is an increasing function on $[0, \infty)$, and so

$$g(t) \leq \lim_{t \to \infty} g(t) = 2 - 3\alpha.$$

Hence we have

$$\text{Re}\left\{ \frac{z_0 h'(z_0)}{h(z_0) - h(-z_0)} \right\} \leq \frac{2 - 3\alpha}{4(1-\alpha)} = \beta,$$

which contradicts the our assumption. Therefore we have Theorem 2.3.4.

Taking $\alpha = 1/2$ in Theorem 2.3.4, then we have the following result.

Corollary 2.3.3. *If $h \in \mathcal{S}_s(1/4)$, then $h \in \mathcal{T}(1/2)$.*

Chapter 3

Carathéodory Functions II

3.1 Introduction

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{P}(m, M)$ if

$$| f(z) - m | < M \qquad (z \in \mathcal{U},\ |m-1| < M \le m).$$

The class $\mathcal{P}(m, M)$ was introduced by Jakubowski [11]. It is clear that $m > 1/2$ and $\mathcal{P}(m, M) \subset \mathcal{S}^*(m - M) \subset \mathcal{S}^*$.

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{SP}(\alpha, \beta)$ if

$$\left| \frac{zf'(z)}{f(z)} - (\alpha + \beta) \right| < \mathrm{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} - \alpha + \beta,$$

where $0 \le \alpha < 1$ and β is a positive real number. This means that for $f \in \mathcal{SP}(\alpha, \beta)$ and $z \in \mathcal{U}$, zf'/f lies in

$$\Omega = \{ w : | w - (\alpha + \beta) | < \mathrm{Re}\, w - \alpha + \beta \},$$

that is, the portion of the plane which contains $w = 1$ and is bounded by the parabola $y^2 = 4\beta(x - \alpha)$ whose vertex is the point $w = \alpha$. Under the choice of α and β, $\Omega \subset \{ w : \mathrm{Re}\, w > \alpha \}$ and hence $\mathcal{SP}(\alpha, \beta) \subset \mathcal{S}^*(\alpha)$.

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}(A, B)$ if

$$\left| \frac{f'(z) - 1}{A - Bf'(z)} \right| < 1,$$

where $-1 \leq B < A \leq 1$. In particular, the class $\mathcal{R}(1, -1) \equiv \mathcal{R}$ coincides with the class of functions studied by MacGregor [17].

In this chapter, we investigate some conditions for Carathéodory functions. Also, we obtain sufficient conditions for univalent functions with some special cases.

In proving our results, we need the following lemma due to Miller and Mocanu [20].

Lemma 3.1.1. *Let $q(z) = a + q_1 z + q_2 z^2 + \cdots$ be regular in $\mathcal{U}$ with $q(z) \not\equiv a$. If $z_0 = r_0 e^{i\theta}$, $0 < r_0 < 1$ and*

$$| q(z_0) | = \max_{|z| \leq r_0} | q(z) |.$$

Then

$$\frac{z_0 q'(z_0)}{q(z_0)} = k,$$

where

$$k \geq \frac{| q(z_0) | - | a |}{| q(z_0) | + | a |}. \tag{3.1.1}$$

3.2 Conditions for the class $\mathcal{P}(m, M)$

In this section, we obtain some conditions for the class $\mathcal{P}(m, M)$ with another interesting geometric properties. Further, we extend some results by Nunokawa, Kwon and Cho [27], Miller [19], Miller, Mocanu and Reade [21] and R. Singh and S. Singh [35].

Theorem 3.2.1. *Let p be analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \not\equiv 1$. If*

$$| \, p(z) \; + \; zp'(z) \; - \; m \, | \; < \; \gamma(m, M),$$

where

$$\gamma(m, M) \; = \; \frac{2M^2}{M \; + \; |1 - m|} \quad \text{and} \quad |m - 1| \; < \; M \; \leq \; m,$$

then $p \in \mathcal{P}(m, M)$.

Proof. Letting $q(z) \; = \; p(z) \; - \; m$. Then we can see that q is analytic in $\mathcal{U}$ with $q(0) \; = \; 1 \; - \; m$ and $q(z) \not\equiv 1 \; - \; m$. Suppose that there exists a point z_0 in $\mathcal{U}$ such that

$$| \, p(z) \; - \; m \, | \; < \; M \qquad \text{for} \qquad |z| \; < \; |z_0|$$

and

$$| \, p(z_0) \; - \; m \, | \; = \; M.$$

From Lemma 3.1.1, we have

$$\frac{z_0 p'(z_0)}{p(z_0) - m} \; = \; \frac{z_0 q'(z_0)}{q(z_0)} \; = \; k, \tag{3.2.1}$$

where

$$k \; \geq \; \frac{M \; - \; |1 - m|}{M \; + \; |1 - m|}. \tag{3.2.2}$$

For the simplicity, we now put

$$x \; = \operatorname{Re} p(z_0) \;\; \text{and} \;\; y \; = \operatorname{Im} p(z_0) \;\; (x, \; y \; \in \mathbf{R}). \tag{3.2.3}$$

By using Lemma 3.1.1, (3.2.1), (3.2.2) and (3.2.3), we obtain,

$$| \, p(z_0) \; + \; z_0 p'(z_0) \; - \; m \, | \; = \; \left| x \; + \; iy \; + \; \frac{z_0 q'(z_0)}{q(z_0)}(x \; - \; m \; + \; iy) \; - \; m \right|$$

$$
\begin{aligned}
&= \mid x - m + iy + k(x - m + iy) \mid \\
&= (1+k)\sqrt{(x-m)^2 + y^2} \\
&= (1+k)M \\
&\geq \gamma(m, M).
\end{aligned}
$$

This is the contradiction to the hypothesis. Therefore we complete the proof of Theorem 3.2.1.

Taking $m = M = 1$ in Theorem 3.2.1, we have the following.

Corollary 3.2.1. *Let p be analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \not\equiv 1$. If*

$$
\mid p(z) + zp'(z) - 1 \mid < 2,
$$

then $p \in \mathcal{P}(1, 1)$.

Remark 3.2.1. Corollary 3.2.1 is an improvement of the result by Nunokawa et al. [27].

Example 3.2.1. Let $f \in \mathcal{A}$ with $f'(z) \not\equiv 1$. Then

$$
\mid f'(z) + zf''(z) - 1 \mid < 2 \quad \text{implies} \quad f' \in \mathcal{P}(1, 1).
$$

Example 3.2.2. Let $f \in \mathcal{A}$ with $f(z)/z \not\equiv 1$. Then

$$
\mid f'(z) - 1 \mid < 2 \quad \text{implies} \quad f/z \in \mathcal{P}(1, 1).
$$

In Theorem 3.2.1, if M approaches to ∞, then we obtain the following result by Miller [19].

Corollary 3.2.2. *Let p be analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \not\equiv 1$. If*

$$
\text{Re} \{ p(z) + zp'(z) \} > 0,
$$

then $p \in \mathcal{P}$.

Theorem 3.2.2. *Let p be analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \not\equiv 1$. If*

$$\left| p(z) + \frac{zp'(z)}{p(z)} - m \right| < \gamma(m, M),$$

where

$$\gamma(m, M) = \frac{M}{m+M}\left\{ m + M + \frac{M - |1-m|}{M + |1-m|} \right\} \tag{3.2.4}$$

and $|m-1| < M \leq m$, then $p \in \mathcal{P}(m, M)$.

Proof. Applying Lemma 3.1.1 with $x = \mathrm{Re}\, p(z_0)$ and $y = \mathrm{Im}\, p(z_0)$ at the point z_0 and using the similar method as in the proof of Theorem 3.2.1, we have,

$$\begin{aligned} \left| p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)} - m \right| &= \left| p(z_0) + \frac{z_0 q'(z_0)}{q(z_0)} \frac{p(z_0) - m}{p(z_0)} - m \right| \\ &= M \left\{ \frac{(x+k)^2 + y^2}{x^2 + y^2} \right\}^{\frac{1}{2}} \\ &= M\sqrt{h(x)}, \end{aligned}$$

where

$$h(x) = \frac{2(m+k)x + M^2 - m^2 + k^2}{2mx + M^2 - m^2}$$

and k is given by (3.1.1).

Here, by a simple calculation, we see that h is a decreasing function for $x \in \mathbf{R}$. Hence we have

$$\left| p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)} - m \right| \geq M\sqrt{h(m+M)} = \gamma(m, M),$$

where $\gamma(m, M)$ is defined by (3.2.4). This is the contradiction to the hypothesis. Therefore we completes the proof of Theorem 3.2.2.

Taking $m = M = 1$ in Theorem 3.2.2, we obtain the following.

Corollary 3.2.3. *Let p be analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \not\equiv 1$. If*

$$\left| p(z) + \frac{zp'(z)}{p(z)} - 1 \right| < \frac{3}{2},$$

then $p \in \mathcal{P}(1,1)$.

Example 3.2.3. Let $f \in \mathcal{A}$ with $f(z)/z \not\equiv 1$. Then

$$\left| \frac{zf''(z)}{f'(z)} \right| < \frac{3}{2} \quad \text{implies} \quad \frac{zf'}{f} \in \mathcal{P}(1,1).$$

Example 3.2.4. Let $f \in \mathcal{A}$ with $zf'(z)/f(z) \not\equiv 1$. Then

$$\left| \frac{f(z)}{z} + \frac{zf'(z)}{f(z)} - 2 \right| < \frac{3}{2} \quad \text{implies} \quad \frac{f}{z} \in \mathcal{P}(1,1).$$

In Theorem 3.2.2, if M approaches to ∞, then we obtain the result by Miller, Mocanu and Reade [21].

Corollary 3.2.4. *Let p be analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \not\equiv 1$. If*

$$\operatorname{Re} \left\{ p(z) + \frac{zp'(z)}{p(z)} \right\} > 0,$$

then $p \in \mathcal{P}$.

Theorem 3.2.3. *Let p be analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \not\equiv 1$. If*

$$| p(z) - m |^{1-\alpha} \left| p(z) - m + \frac{zp'(z)}{p(z)} \right|^{\alpha} < \gamma(m, M),$$

where

$$\gamma(m, M) = M \left(1 + \frac{1}{m+M} \, \frac{M - |1-m|}{M + |1-m|} \right)^{\alpha}, \qquad (3.2.5)$$

$0 \leq \alpha \leq 1$ *and* $|m - 1| < M \leq m$, *then* $p \in \mathcal{P}(m, M)$.

Proof. Applying Lemma 3.1.1 at the point z_0 and using the similar method as in the proof of Theorem 3.2.1 with $x = \text{Re p}(z_0)$ and $y = \text{Im p}(z-0)$, we obtain

$$\begin{aligned} &| p(z_0) - m |^{1-\alpha} \left| p(z_0) - m + \frac{z_0 p'(z_0)}{p(z_0)} \right|^{\alpha} \\ &= | q(z_0) | \left| 1 + \frac{z_0 q'(z_0)}{q(z_0)} \frac{1}{p(z_0)} \right|^{\alpha} \\ &= M \left\{ 1 + \frac{2kx + k^2}{x^2 + y^2} \right\}^{\frac{\alpha}{2}} \\ &= M \ \{h(x)\}^{\frac{\alpha}{2}} \end{aligned}$$

where

$$h(x) = 1 + \frac{2kx + k^2}{2mx + M^2 - m^2}$$

and k is given by (3.1.1). Here, by a simple calculation, we see that h is a decreasing function for $x \in \mathbf{R}$. Hence we have

$$\begin{aligned} &| p(z_0) - m |^{1-\alpha} \left| p(z_0) - m + \frac{z_0 p'(z_0)}{p(z_0)} \right|^{\alpha} \\ &\geq M \ \{h(m+M)\}^{\frac{\alpha}{2}} = \gamma(m, M), \end{aligned}$$

where $\gamma(m, M)$ is given by (3.2.5). This is the contradiction to the hypothesis. Therefore we complete the proof of Theorem 3.2.3.

Taking $m = 1$ in Theorem 3.2.3, we have the following.

Corollary 3.2.5. *Let p be analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \not\equiv 1$. If*

$$| p(z) - 1 |^{1-\alpha} \left| p(z) - 1 + \frac{zp'(z)}{p(z)} \right|^{\alpha} < M \left(\frac{M+2}{M+1} \right)^{\alpha},$$

then $p \in \mathcal{P}(1, M)$, where $0 \leq \alpha \leq 1$ and $0 < M \leq 1$.

Example 3.2.5. Let $f \in \mathcal{A}$ with $zf'(z)/f(z) \not\equiv 1$. Then

$$\left| \frac{zf'(z)}{f(z)} - 1 \right|^{1-\alpha} \left| \frac{zf''(z)}{f'(z)} \right|^{\alpha} < M \left(\frac{M+2}{M+1}\right)^{\alpha} \text{ implies } \frac{zf'}{f} \in \mathcal{P}(1, M),$$

where $0 \leq \alpha \leq 1$ and $0 < M \leq 1$.

Remark 3.2.2. This example 3.2.5 is the improvement of result obtained by Singh and Singh [35].

Taking $\alpha = 1$ and $\alpha = m = 1$ in Example 3.2.5 respectively, we obtain the following.

Example 3.2.6. Let $f \in \mathcal{A}$ with $zf'(z)/f(z) \not\equiv 1$. Then

$$(i) \quad \left| \frac{zf''(z)}{f'(z)} \right| < \frac{M(M+2)}{M+1} \quad \text{implies} \quad zf'/f \in \mathcal{P}(1, M),$$

$$(ii) \quad \left| \frac{zf''(z)}{f'(z)} \right| < 2 \quad \text{implies} \quad zf'/f \in \mathcal{P}(1, 1).$$

3.3 Conditions for the classes $\mathcal{SP}(\alpha, \beta)$ and $\mathcal{R}(A, B)$

To derive the results for the classes $\mathcal{SP}(\alpha, \beta)$ and $\mathcal{R}(A, B)$, we recall Lemma 2.1.1 and Lemma 3.1.1. In this section, we give some interesting conditions for the classes $\mathcal{SP}(\alpha, \beta)$ and $\mathcal{R}(A, B)$ with applications in the univalent function theory.

Theorem 3.3.1. *Let p is analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \not\equiv 1$, where $0 \leq \alpha < 1$, $\max\{\alpha/2,\ 1-\alpha\} \leq \beta$. If*

$$\left| \frac{zp'(z)}{p(z)} \right| < \frac{1-\alpha}{2(2\beta+\alpha-1)} \sqrt{3 - \frac{\alpha}{\beta}},$$

then

$$|\, p(z) - (\alpha + \beta) \,| < \operatorname{Re} p(z) - \alpha + \beta.$$

Proof. Let $q(z) = p(z) - (\alpha + \beta)$. Then q is analytic in $\mathcal{U}$ with $q(0) = 1 - (\alpha + \beta)$ and $q(z) \not\equiv 1 - (\alpha + \beta)$. Suppose that there exists a point z_0 in $\mathcal{U}$ such that

$$| p(z) - (\alpha + \beta) | < \operatorname{Re} p(z) - \alpha + \beta \quad \text{for} \quad |z| < |z_0| \qquad (3.3.1)$$

and

$$| p(z_0) - (\alpha + \beta) | = \operatorname{Re} p(z_0) - \alpha + \beta. \qquad (3.3.2)$$

From (3.3.1) and (3.3.2), we have

$$| q(z_0) | = \max_{|z| \leq r_0} | q(z) | \quad (z_0 = r_0 e^{i\theta} \text{ and } r_0 < 1). \qquad (3.3.3)$$

For the simplicity, we put,

$$x = \operatorname{Re} p(z_0) \qquad \text{and} \qquad y = \operatorname{Im} p(z_0) \qquad (x, \ y \in \mathbf{R}).$$

Then, from (3.3.2), we have

$$x = \frac{1}{4\beta} y^2 + \alpha.$$

Also, from lemma 3.1.1, (3.3.2) and (3.3.3), we obtain,

$$\frac{z_0 q'(z_0)}{q(z_0)} = k \qquad \text{and} \qquad k \geq \frac{1-\alpha}{2\beta + \alpha - 1}.$$

Then, by a simple calculation, we have

$$\begin{aligned} \left| \frac{z_0 p'(z_0)}{p(z_0)} \right| &= \left| \frac{z_0 p'(z_0)}{p(z_0) - (\alpha + \beta)} \right| \left| \frac{p(z_0) - (\alpha + \beta)}{p(z_0)} \right| \\ &= |k| \frac{| p(z_0) - (\alpha + \beta) |}{|p(z_0)|} \\ &= k\sqrt{h(t)}, \end{aligned}$$

where

$$h(t) = \frac{\left(\frac{1}{4\beta}t - \beta\right)^2 + t}{\left(\frac{1}{4\beta}t + \alpha\right)^2 + t}$$

and

$$t = y^2 = \{\text{Im}\ p(z_0)\}^2 \in \mathbf{R}.$$

Here, we see easily that h has the minimum value at the point

$$t = 4\beta(2\beta - \alpha).$$

Hence we have the following inequality

$$\left| \frac{zp'(z)}{p(z)} \right| \geq \frac{1-\alpha}{2(2\beta + \alpha - 1)} \sqrt{3 - \frac{\alpha}{\beta}} ,$$

which is the contradiction to the hypothesis. Therefore we complete the proof of Theorem 3.3.1.

Taking $\alpha = \beta = 1/2$ in Theorem 3.3.1, we obtain the following.

Corollary 3.3.1. *Let p is analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \not\equiv 1$. If*

$$\left| \frac{zp'(z)}{p(z)} \right| < \frac{1}{\sqrt{2}},$$

then $p \in \mathcal{SP}\left(\frac{1}{2}, \frac{1}{2}\right)$.

Example 3.3.1. Let $f \in \mathcal{A}$ with $zf'(z)/f(z) \not\equiv 1$. Then

$$\left| 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right| < \frac{1}{\sqrt{2}} \text{ implies } \frac{zf'}{f} \in \mathcal{SP}\left(\frac{1}{2}, \frac{1}{2}\right).$$

Example 3.3.2. Let $f \in \mathcal{A}$ and $f(z)/z \not\equiv 1$. Then

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < \frac{1}{\sqrt{2}} \text{ implies } \frac{f}{z} \in \mathcal{SP}\left(\frac{1}{2}, \frac{1}{2}\right).$$

Theorem 3.3.2. *Let p be an analytic function in $\mathcal{U}$ with $p(0) = 1$. If*

$$\mathrm{Re}\left\{ p^2(z) + zp'(z) \right\} < \frac{1}{\pi},$$

then $p \in \mathcal{SP}\left(\frac{1}{2}, \frac{1}{2}\right)$.

Proof. Let

$$p(z) = 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{w(z)}}{1-\sqrt{w(z)}}\right)^2.$$

Then we have w is analytic in $\mathcal{U}$ with $w(0) = 0$. Now, suppose that there exists a point $z_0 \in \mathcal{U}$ such that

$$\max_{|z|\le|z_0|} |w(z)| = |w(z_0)| = 1.$$

By using Lemma 2.1.1, we get that $z_0 w'(z_0) = kw(z_0),\ k \ge 1$. Taking $w(z_0) = e^{i\theta}$ $(0 < \theta < 2\pi)$, we have

$$\begin{aligned}
&\mathrm{Re}\left\{ p^2(z_0) + z_0 p'(z_0) \right\} \\
=\ &\mathrm{Re}\left\{\left(1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{w(z_0)}}{1-\sqrt{w(z_0)}}\right)^2\right)^2 + \frac{4}{\pi^2}\left(\log\frac{1+\sqrt{w(z_0)}}{1-\sqrt{w(z_0)}}\right)\frac{k\sqrt{w(z_0)}}{1-\sqrt{w(z_0)}}\right\} \\
=\ &\mathrm{Re}\left\{\left(1+\frac{2}{\pi^2}\left(\log\left(\cot\frac{\theta}{4}\right)+i\frac{\pi}{2}\right)^2\right)^2 + \frac{4}{\pi^2}\left(\log\left(\cot\frac{\theta}{4}\right)+i\frac{\pi}{2}\right)\left(\frac{-ik}{2\sin\frac{\theta}{2}}\right)\right\} \\
=\ &\frac{1}{4} + \frac{4}{\pi^4}\left\{\log\left(\cot\frac{\theta}{4}\right)\right\}^4 - \frac{2}{\pi^2}\left\{\log\left(\cot\frac{\theta}{4}\right)\right\}^2 + \frac{k}{\pi\sin\frac{\theta}{2}} \\
\ge\ &g(t),
\end{aligned}$$

where

$$g(t) = \frac{1}{4} + \frac{1}{\pi} + \frac{4}{\pi^4}t^4 - \frac{2}{\pi^2}t^2 \quad \text{and} \quad t = \left\{\log\left(\cot\frac{\theta}{4}\right)\right\} \in \mathbf{R}.$$

Then, by the simple calculation, we obtain

$$g(t) \geq g\left(\frac{\pi}{2}\right) = \frac{1}{\pi}.$$

But this contradicts our assumption. Therefore we complete the proof of Theorem 3.3.2.

Taking $p(z) = zf'(z)/f(z)$ in Theorem 3.3.2, we obtain the following.

Corollary 3.3.2. *Let $f \in \mathcal{A}$. Then*

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} < \frac{1}{\pi} \ \ \textit{implies} \ \ \frac{zf'}{f} \in \mathcal{SP}\left(\frac{1}{2},\frac{1}{2}\right).$$

Theorem 3.3.3. *Let p is analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \not\equiv 1$. If*

$$\left| \frac{p(z) + zp'(z) - 1}{A - B(p(z) + zp'(z))} \right| < 2 - |B|, \tag{3.3.4}$$

where $-1 \leq B < A \leq 1$, then

$$\left| \frac{p(z) - 1}{A - Bp(z)} \right| < 1.$$

Proof. At first, we note that $p(z) \neq -A/B$ for $z \in \mathcal{U}$. In fact, if $A - Bp(z)$ has a zero of order m at $z = z_1 \in \mathcal{U}$, then we can write

$$A - Bp(z) = (z - z_1)^m p_1(z) \quad (m \in \mathbf{N}),$$

where p_1 is analytic in $\mathcal{U}$ and $p_1(z_1) \neq 0$. Then we have

$$\begin{aligned} &\left| \frac{p(z) + zp'(z) - 1}{A - B(p(z) + zp'(z))} \right| \qquad (3.3.5) \\ &= \left| -\frac{1}{B} + \frac{A}{(z-z_1)^{m-1}\{(2z - z_1)p_1(z) + (z - z_1)zp_1'(z)\}} \right|. \end{aligned}$$

Choosing $z \to z_1$ suitably, the real part of the right hand side of (3.3.5) can take any infinite values, which contradicts the hypothesis (3.3.4). Now letting

$$q(z) = \frac{p(z) - 1}{A - Bp(z)},$$

then q is analytic in $\mathcal{U}$ with $q(0) = 0$ and $q(z) \not\equiv 0$. Suppose that there exists a point z_0 in $\mathcal{U}$ such that

$$\left| \frac{p(z) - 1}{A - Bp(z)} \right| < 1 \quad \text{for} \quad |z| < |z_0| \tag{3.3.6}$$

and

$$\left| \frac{p(z_0) - 1}{A - Bp(z_0)} \right| = 1. \tag{3.3.7}$$

From (3.3.6) and (3.3.7), we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = k \geq 1. \tag{3.3.8}$$

For the simplicity, we put,

$$x = \operatorname{Re} p(z_0) \quad \text{and} \quad y = \operatorname{Im} p(z_0) \quad (x, \ y \in \mathbf{R}).$$

Then, from (3.3.7), we have

$$y^2 = \left(\frac{A-B}{1-B^2} \right)^2 - \left(x - \frac{1-AB}{1-B^2} \right)^2. \tag{3.3.9}$$

Also, from Lemma 3.1.1, (3.3.7), (3.3.8) and (3.3.9), we obtain

$$\begin{aligned} \left| \frac{p(z_0) + z_0 p'(z_0) - 1}{A - B(p(z_0) + z_0 p'(z_0))} \right| &= |q(z_0)| \left| \frac{A - B + k(A - Bp(z_0))}{A - B - Bk(p(z_0) - 1)} \right| \\ = \left| \frac{A - B + k(A - Bp(z_0))}{A - B - Bk(p(z_0) - 1)} \right| &= \sqrt{h(x)}. \end{aligned}$$

Here, we let $h(x) = n(x)/d(x)$, where

$$\begin{cases} n(x) &= -2Bk(k+1-B^2)x + (A+B)k^2 + 2A(1-B^2)k + (1-B^2)(A-B) \\ d(x) &= -2Bk(B^2k+1-B^2)x + B^2(A+B)k^2 + 2B(1-B^2)k + (1-B^2)(A-B). \end{cases}$$

And then, by a simple calculation, we see that h is an increasing function for $B \geq 0$ and a decreasing function for $B \leq 0$. Hence we have

$$h(x) \geq h\left(\frac{1-A}{1-B}\right) = \left(\frac{k+1-B}{Bk+1-B}\right)^2 \geq (2-B)^2 \quad \text{for} \quad B \geq 0$$

and

$$h(x) \geq h\left(\frac{1+A}{1+B}\right) = \left(\frac{k+1+B}{-Bk+1+B}\right)^2 \geq (2+B)^2 \quad \text{for} \quad B \leq 0.$$

That is,

$$\left| \frac{p(z_0) + z_0 p'(z_0) - 1}{A - B(p(z_0) + z_0 p'(z_0))} \right| = \sqrt{h(x)} \geq (2-|B|)^2.$$

This is a contradiction to the assumption (3.3.4). Therefore we complete the proof of Theorem 3.3.3.

Remark 3.3.1. Taking $A = 1$ and $B = 0$ in Theorem 3.3.3, we have the same result Corollary 3.2.1, which is obtained by Nunokawa et al. [27].

Taking $A = \beta$, $B = -\beta$ and $0 < \beta \leq 1$ in Theorem 3.3.3, we obtain the following.

Corollary 3.3.3. *Let p is analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \not\equiv 1$. If*

$$\left| \frac{p(z) + zp'(z) - 1}{1 + p(z) + zp'(z)} \right| < \beta(2-\beta),$$

then

$$\left| \frac{p(z) - 1}{p(z) + 1} \right| < \beta.$$

Example 3.3.3. Let $f \in \mathcal{A}$ with $f'(z) \not\equiv 1$. Then

$$\left| \frac{f'(z) + zf''(z) - 1}{1 + f'(z) + zf''(z)} \right| < \beta(2-\beta) \quad \text{implies} \quad \left| \frac{f'(z) - 1}{f'(z) + 1} \right| < \beta.$$

Taking $\beta = 1$ in Example 3.3.3, then we obtain the following.

Example 3.3.4. Let $f \in \mathcal{A}$ with $f'(z) \not\equiv 1$. Then

$$\left| \frac{f'(z) + zf''(z) - 1}{1 + f'(z) + zf''(z)} \right| < 1 \quad \text{implies} \quad f \in \mathcal{R}(1, -1)$$

Chapter 4

Geometric Properties of Analytic Functions

4.1 Introduction

A function $f \in \mathcal{A}$ is said to be *close-to-convex of order α and type β* if there is a function $g \in \mathcal{S}^*(\beta)$ such that

$$\operatorname{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > \alpha \quad (0 \leq \alpha,\ \beta < 1,\ z \in \mathcal{U}),$$

which was studied by Goodman [8]. We denote by $\mathcal{C}(\alpha, \beta)$ the class of all close-to-convex functions of order α and type β.

In this chapter, we prove some integral preserving properties for certain analytic functions which contained the result of Libera [15]. Further, we extend the known results as special cases.

In order to prove our results, we need the following lemma given by Miller and Mocanu [20].

Lemma 4.1.1. *Let $p(z) = a + p_n z^n + p_{n+1} z^{n+1} + \dots$ be analytic in $\mathcal{U}$ with $p(z) \not\equiv a$ and $n \geq 1$. If $z_0 = r_0 e^{i\theta_0}$ $(0 < r_0 < 1)$ and*

$$\text{Re } p(z_0) = \min_{|z| \leq r_0} \text{Re } p(z)$$

then

$$z_0 p'(z_0) \leq -\frac{n|a - p(z_0)|^2}{2 \text{ Re } (a - p(z_0))}. \tag{4.1.1}$$

We note that, for the case $a = n = 1$ and $\text{Re } p(z_0) = \alpha$ $(0 \leq \alpha < 1)$ in Lemma 4.1.1, the condition (4.1.1) is replaced by the simpler condition

$$z_0 p'(z_0) \leq -\frac{(1-\alpha)^2 + (\text{Im } p(z_0))^2}{2(1-\alpha)}. \tag{4.1.1$'$}$$

4.2 Integral preserving properties

In this section, we investigate the integral preserving properties where the integral operators are defined by (2.1.1), (2.1.2) and (2.1.3) and extend the result by Libera [15].

Theorem 4.2.1. *Let $f \in \mathcal{A}$ and $0 \leq \alpha < 1$. If $f \in \mathcal{S}^*(\beta(\alpha))$ then $F_c \in \mathcal{S}^*(\alpha)$, where F_c is the integral operator defined by* (2.1.1) *and*

$$\beta(\alpha) = \begin{cases} \alpha - \frac{c+\alpha}{2(1-\alpha)}, & -\alpha \leq c \leq 1 - 2\alpha \\ \alpha - \frac{1-\alpha}{2(c+\alpha)}, & c \geq 1 - 2\alpha \end{cases} \tag{4.2.1}$$

Proof. Define the function

$$p(z) = \frac{zF_c'(z)}{F_c(z)}.$$

Then we see that p is analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \not\equiv 1$. If there exists a point $z_0 \in \mathcal{U}$ such that the conditions (2.2.2) and (2.2.3) are satisfied, then, from Lemma 4.1.1, we have $z_0 p'(z_0)$ is a negative real number with the condition $(4.1.1)'$. From the condition (2.2.3), we have

$$p(z_0) = \alpha + i \operatorname{Im} p(z_0) = \alpha + iy \quad (y \in \mathbf{R}).$$

Then, using the condition $(4.1.1)'$, we obtain

$$\begin{aligned}
\operatorname{Re} \frac{z_0 f'(z_0)}{f(z_0)} &= \operatorname{Re} \left(p(z_0) + \frac{z_0 p'(z_0)}{c + p(z_0)} \right) \\
&= \operatorname{Re} \left(\alpha + iy + \frac{z_0 p'(z_0)}{\alpha + c + iy} \right) \\
&= \alpha + \frac{(c+\alpha) z_0 p'(z_0)}{(c+\alpha)^2 + y^2} \\
&\leq \alpha - \frac{c+\alpha}{2(1-\alpha)} \left\{ \frac{(1-\alpha)^2 + y^2}{(c+\alpha)^2 + y^2} \right\} \\
&= \alpha - \frac{c+\alpha}{2(1-\alpha)} h(t),
\end{aligned}$$

where

$$h(t) = \frac{(1-\alpha)^2 + t}{(c+\alpha)^2 + t}, \qquad t = y^2 \geq 0.$$

Here, we know that h is a decreasing function for $-\alpha \leq c \leq 1 - 2\alpha$ and an increasing function for $c \geq 1 - 2\alpha$, respectively. At first, for $-\alpha \leq c \leq 1 - 2\alpha$, we have

$$\begin{aligned}
\operatorname{Re} \frac{z_0 f'(z_0)}{f(z_0)} &\leq \alpha - \frac{c+\alpha}{2(1-\alpha)} \lim_{t \to \infty} h(t) \\
&= \alpha - \frac{c+\alpha}{2(1-\alpha)}
\end{aligned}$$

and for $c \geq 1 - 2\alpha$, we obtain

$$\begin{aligned}
\operatorname{Re} \frac{z_0 f'(z_0)}{f(z_0)} &\leq \alpha - \frac{c+\alpha}{2(1-\alpha)} h(0) \\
&= \alpha - \frac{1-\alpha}{2(c+\alpha)}.
\end{aligned}$$

This is a contradiction to the hypothesis. Therefore we complete the proof of Theorem 4.2.1.

Putting $c = 0$ and $c = 1$ in Theorem 4.2.1, we obtain the following corollaries, respectively.

Corollary 4.2.1. *Let $f \in \mathcal{A}$ and $0 \leq \alpha < 1$. If $f \in \mathcal{S}^*(\beta(\alpha))$, then $F_0 \in \mathcal{S}^*(\alpha)$, where F_0 is the integral operator defined by* (2.1.2) *and*

$$\beta(\alpha) = \begin{cases} \alpha - \frac{\alpha}{2(1-\alpha)}, & 0 \leq \alpha \leq \frac{1}{2} \\ \alpha - \frac{1-\alpha}{2\alpha}, & \frac{1}{2} \leq \alpha < 1 \end{cases} \tag{4.2.2}$$

Corollary 4.2.2. *Let $f \in \mathcal{A}$ and $0 \leq \alpha < 1$. If $f \in \mathcal{S}^*(\beta(\alpha))$, then $F_1 \in \mathcal{S}^*(\alpha)$, where F_1 is the integral operator defined by* (2.1.3) *and*

$$\beta(\alpha) = \alpha - \frac{1-\alpha}{2(1+\alpha)}. \tag{4.2.3}$$

Remark 4.2.1. If we let $\alpha = 0$ in Corollary 4.2.2, then we have an improvement of the result by Libera [15].

Theorem 4.2.2. *Let $f \in \mathcal{A}$ and $0 \leq \alpha < 1$. If $f \in \mathcal{K}(\beta(\alpha))$, then $F_c \in \mathcal{K}(\alpha)$, where F_c is the integral operator defined by* (2.1.1) *and $\beta(\alpha)$ is given by* (4.2.1).

Proof. Define a function

$$p(z) = 1 + \frac{zF_c''(z)}{F_c'(z)}. \tag{4.2.4}$$

Then p is analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \not\equiv 1$. Differentiating (4.2.4) logarithmically, we have

$$1 + \frac{z_0 f''(z_0)}{f'(z_0)} = p(z_0) + \frac{z_0 p'(z_0)}{c + p(z_0)}.$$

Applying Lemma 4.2.1 to p at the point $z_0 \in \mathcal{U}$ and using the same method as in the proof of Theorem 4.2.1, we have the result.

Putting $c = 0$ and $c = 1$ in Theorem 4.2.2, we obtain the following corollaries, respectively.

Corollary 4.2.3. *Let $f \in \mathcal{A}$ and $0 \leq \alpha < 1$. If $f \in \mathcal{K}(\beta(\alpha))$, then $F_0 \in \mathcal{K}(\alpha)$, where F_0 is the integral operator defined by* (2.1.2) *and $\beta(\alpha)$ is given by* (4.2.2).

Corollary 4.2.4. *Let $f \in \mathcal{A}$ and $0 \leq \alpha < 1$. If $f \in \mathcal{K}(\beta(\alpha))$, then $F_1 \in \mathcal{K}(\alpha)$, where F_1 is the integral operator defined by* (2.1.3) *and $\beta(\alpha)$ is given by* (4.2.3).

4.3 Conditions for close-to-convex functions

In this section, we give a condition for close-to-convex functions by using Lemma 4.1.1 with some special cases.

Theorem 4.3.1. *Let $f,\ g \in \mathcal{A}$ with $zf'(z)/g(z) \not\equiv 1$ in $\mathcal{U}\backslash\{0\}$ and $0 \leq \alpha < 1$. If*

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)}\right\} > \beta(\alpha),$$

then

$$\operatorname{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > \alpha,$$

where

$$\beta(\alpha) = \begin{cases} -\frac{\alpha}{2(1-\alpha)}, & 0 \leq \alpha \leq \frac{1}{2} \\ -\frac{1-\alpha}{2\alpha}, & \frac{1}{2} \leq \alpha < 1 \end{cases} \tag{4.3.1}$$

Proof. Let us put

$$p(z) = \frac{zf'(z)}{g(z)}.$$

Then p is analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \not\equiv 1$. If there exists a point $z_0 \in \mathcal{U}$ such that the conditions (2.2.2) and (2.2.3) are satisfied, then by Lemma 4.1.1, we obtain (4.1.1)$'$. Suppose that $p(z_0) = \alpha + iy$ and $y = \text{Im } p(z_0) \in \mathbf{R}$. Then, from , (2.2.2), (2.2.3) and (4.1.1)$'$, we have

$$\begin{aligned}
\text{Re}\left\{1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \frac{z_0 g'(z_0)}{g(z_0)}\right\} &= \text{Re}\,\frac{z_0 p'(z_0)}{p(z_0)} \\
&= \frac{\alpha}{\alpha^2 + y^2} z_0 p'(z_0) \\
&\leq -\frac{\alpha}{2(1-\alpha)}\left\{\frac{(1-\alpha)^2 + y^2}{\alpha^2 + y^2}\right\} \\
&= -\frac{\alpha}{2(1-\alpha)} h(t),
\end{aligned}$$

where

$$h(t) = \frac{(1-\alpha)^2 + t}{\alpha^2 + t}, \qquad t = y^2 \geq 0.$$

Here, we know that h is a decreasing function for $0 \leq \alpha \leq \frac{1}{2}$ and an increasing function for $\frac{1}{2} \leq \alpha < 1$, respectively. At first, for $0 \leq \alpha \leq \frac{1}{2}$, we have

$$\begin{aligned}
\text{Re}\left\{1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \frac{z_0 g'(z_0)}{g(z_0)}\right\} &\leq -\frac{\alpha}{2(1-\alpha)} \lim_{t \to \infty} h(t) \\
&= -\frac{\alpha}{2(1-\alpha)}
\end{aligned}$$

and for $\frac{1}{2} \leq \alpha < 1$, we obtain

$$\begin{aligned}
\text{Re}\left\{1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \frac{z_0 g'(z_0)}{g(z_0)}\right\} &\leq -\frac{\alpha}{2(1-\alpha)} \, h(0) \\
&= -\frac{1-\alpha}{2\alpha}.
\end{aligned}$$

This is a contradiction to the hypothesis. Therefore we complete the proof of Theorem 4.3.1.

Remark 4.3.1. In particular, if we take $g \in \mathcal{S}^*(\gamma)$ in the above theorem, then $f \in \mathcal{C}(\alpha, \gamma)$.

Putting $g(z) = f(z)$ and $g(z) = z$ in Theorem 4.3.1 respectively, we obtain the following corollaries.

Corollary 4.3.1. *Let $f \in \mathcal{A}$ with $zf'(z)/f(z) \not\equiv 1$ in $\mathcal{U}\backslash\{0\}$ and $0 \leq \alpha < 1$. If*

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right\} > \beta(\alpha),$$

then $f \in \mathcal{S}^*(\alpha)$, where $\beta(\alpha)$ is given by (4.3.1).

Corollary 4.3.2. *Let $f \in \mathcal{A}$ with $f'(z) \not\equiv 1$ in $\mathcal{U}\backslash\{0\}$ and $0 \leq \alpha < 1$. If $f \in \mathcal{K}(1 + \beta(\alpha))$, then $f \in \mathcal{R}(\alpha)$, where $\beta(\alpha)$ is given by* (4.3.1).

Chapter 5

Strongly Starlike Functions

5.1 Introduction

Let $\mathcal{UCV}$ be the class of uniformly convex functions as introduced by Goodman [9]. Geometrically, the property of uniform convexity of a function means that the image of every circular arc contained in $\mathcal{U}$, with center $\zeta \in \mathcal{U}$, is convex. Further, Ronning [31](also, see [16]) gave a more applicable one variable analytic characterization for $\mathcal{UCV}$. That is, a function $f \in \mathcal{A}$ is in $\mathcal{UCV}$ if and only if

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} \geq \left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in \mathcal{U}).$$

A function $f \in \mathcal{A}$ is called strongly starlike of order δ if it satisfies

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi}{2}\delta \quad (0 < \delta \leq 1;\ z \in \mathcal{U}).$$

We denote by $\mathcal{S}(\delta)$ the class of all strongly starlike functions of order δ. For $\delta = 1$, $\mathcal{S}(1) \equiv \mathcal{S}^*$ is the well known class of normalized starlike functions with respect to origin. Furthermore, if $0 < \delta < 1$, then the class $\mathcal{S}(\delta)$ consists only of bounded starlike functions [5] and therefore the inclusion, $\mathcal{S}(\delta) \subset \mathcal{S}^*$, is proper. The class $\mathcal{S}(\delta)$ and the related classes have been extensively studied by Mocanu [23] and Nunokawa [25].

In this chapter , we investigate some criteria for the class $\mathcal{S}(\delta)$, applying the result of Nunokawa [25], and give the relationship between the classes $\mathcal{UCV}$ and $\mathcal{S}(\delta)$ as a special case.

In order to prove our results, we need the following lemma given by Nunokawa [25].

Lemma 5.1.1. *Let p be analytic in $\mathcal{U}$, $p(0) = 1$, and $p(z) \neq 0$ in $\mathcal{U}$. Suppose that there exists a point $z_0 \in \mathcal{U}$ such that*

$$| \arg p(z) | \;<\; \frac{\pi}{2}\delta \quad \textit{for} \quad |z| < |z_0| \tag{5.1.1}$$

and

$$| \arg p(z_0) | \;=\; \frac{\pi}{2}\delta \qquad (\delta > 0). \tag{5.1.2}$$

Then

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\delta, \tag{5.1.3}$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \textit{when} \quad \arg p(z_0) = \frac{\pi}{2}\delta, \tag{5.1.4}$$

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \quad \textit{when} \quad \arg p(z_0) = -\frac{\pi}{2}\delta, \tag{5.1.5}$$

and

$$\{p(z_0)\}^{\frac{1}{\delta}} = \pm ia \qquad (a > 0). \tag{5.1.6}$$

5.2 Criteria for strongly starlike functions of order δ

With the help of Lemma 5.1.1, we now derive

Theorem 5.2.1. *Let $f \in \mathcal{A}$. If*

$$\text{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} \geq |\lambda|\left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in \mathcal{U}),$$

where

$$|\lambda| > \left(\frac{(1+\delta^2)\cos^2\frac{\pi}{2}\delta}{\delta(\delta+\sin\pi\delta)+(1-\delta^2)\sin^2\frac{\pi}{2}\delta}\right)^{1/2} \qquad (0<\delta<\delta_0)$$

and δ_0 is the solution of the equation $\delta\tan(\pi\delta/2)=1$, then $f \in \mathcal{S}(\delta)$.

Proof. Let

$$p(z) = \frac{zf'(z)}{f(z)} \quad (z \in \mathcal{U}).$$

Then p is analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \neq 0$ in $\mathcal{U}$. If there exist a point $z_0 \in \mathcal{U}$ such that the condition (5.1.1) and (5.1.2) are satisfied, then (by Lemma 5.1.1) we obtain (5.1.3) under the restrictions (5.1.4), (5.1.5) and (5.1.6).

At first, suppose that

$$p(z_0) = (ia)^{\delta} \quad (a>0).$$

Then we have

$$\frac{\text{Re}\left\{1+\frac{z_0f''(z_0)}{f'(z_0)}\right\}}{\left|\frac{z_0f''(z_0)}{f'(z_0)}\right|} = \frac{\text{Re}\left\{p(z_0)+\frac{z_0p'(z_0)}{p(z_0)}\right\}}{\left|p(z_0)+\frac{z_0p'(z_0)}{p(z_0)}-1\right|}$$
$$= \left(\frac{a^{2\delta}\cos^2\frac{\pi}{2}\delta}{a^{2\delta}+2a^{\delta}\left(k\delta\sin\frac{\pi}{2}\delta-\cos\frac{\pi}{2}\delta\right)+k^2\delta^2+1}\right)^{1/2}$$

$$
\begin{aligned}
&\le \left(\frac{a^{2\delta}\cos^2\frac{\pi}{2}\delta}{a^{2\delta}+2a^{\delta}\left(\delta\sin\frac{\pi}{2}\delta-\cos\frac{\pi}{2}\delta\right)+\delta^2+1}\right)^{1/2}\\
&\equiv \sqrt{g(t)} \quad \left(t=a^{\delta}>0\right).
\end{aligned}
$$

Then g has a maximum value at

$$
t_0 = \frac{\delta^2+1}{\cos\frac{\pi}{2}\delta-\delta\sin\frac{\pi}{2}\delta} \quad (0<\delta<\delta_0).
$$

Hence we obtain

$$
\begin{aligned}
\sqrt{g(t)} &\le \sqrt{g(t_0)} \\
&= \left(\frac{(1+\delta^2)\cos^2\frac{\pi}{2}\delta}{\delta(\delta+\sin\pi\delta)+(1-\delta^2)\sin^2\frac{\pi}{2}\delta}\right)^{1/2}\\
&< |\lambda|.
\end{aligned} \tag{5.2.1}
$$

This contradicts the assumption of Theorem 5.2.1.

For the case of $p(z_0)=(-ia)^{\delta}$ $(a>0)$, applying the same method as the above, we also have (5.2.1). Therefore we complete the proof of Theorem 5.2.1.

If we take $\delta=1/2$, we have the following

Corollary 5.2.1. *If $f\in\mathcal{UCV}$, then $f\in\mathcal{S}(1/2)$.*

Theorem 5.2.2. *Let $f\in\mathcal{A}_0$ and $\alpha\ge 0$. If*

$$
\left|\frac{zf''(z)}{f'(z)}-\alpha\right| < |\lambda|\left|\frac{zf'(z)}{f(z)}\right| \quad (z\in\mathcal{U}),
$$

where

$$
|\lambda| \le \frac{(\alpha+1)\sin\frac{\pi}{2}\delta+\delta\cos\frac{\pi}{2}\delta}{\sqrt{(\alpha+1)^2+\delta^2}} \quad (0<\delta<\delta_0)
$$

and δ_0 is the solution of the equation $\delta\tan(\pi\delta/2)=\alpha+1$, then $f\in\mathcal{S}(\delta)$.

Proof. Let

$$p(z) = \frac{zf'(z)}{f(z)} \quad (z \in \mathcal{U}).$$

Then p is analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \neq 0$ in $\mathcal{U}$.

Suppose that there exist a point $z_0 \in \mathcal{U}$ such that the condition (5.1.1) and (5.1.2) are satisfied, then (by Lemma 5.1.1) we obtain (5.1.3) under the restrictions (5.1.4), (5.1.5) and (5.1.6).

If we assume that

$$p(z_0) = (ia)^\delta \quad (a > 0),$$

then we have

$$\begin{aligned}
\frac{|\lambda| \left| \frac{z_0 f'(z_0)}{f(z_0)} \right|}{\left| \frac{z_0 f''(z_0)}{f'(z_0)} - \alpha \right|} &= \frac{|\lambda| \mid p(z_0) \mid}{\left| p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)} - (\alpha+1) \right|} \\
&= |\lambda| \left(\frac{a^{2\delta}}{a^{2\delta} + 2a^\delta \left(k\delta \sin \frac{\pi}{2}\delta - (\alpha+1)\cos \frac{\pi}{2}\delta \right) + k^2\delta^2 + (\alpha+1)^2} \right)^{1/2} \\
&\leq |\lambda| \sqrt{g(t)},
\end{aligned}$$

where

$$g(t) = \frac{t^2}{t^2 + 2t\left(\delta \sin \frac{\pi}{2}\delta - (\alpha+1)\cos \frac{\pi}{2}\delta\right) + \delta^2 + (\alpha+1)^2} \quad (t = a^\delta > 0).$$

Then g has a maximum value at

$$t_0 = \frac{(\alpha+1)^2 + \delta^2}{(\alpha+1)\cos \frac{\pi}{2}\delta - \delta \sin \frac{\pi}{2}\delta} \quad (0 < \delta < \delta_0).$$

Hence we have

$$\begin{aligned}
\sqrt{g(t)} &\leq |\lambda| \sqrt{g(t_0)} \\
&= |\lambda| \frac{\sqrt{(\alpha+1)^2 + \delta^2}}{(\alpha+1)\sin \frac{\pi}{2}\delta + \delta \cos \frac{\pi}{2}\delta} \\
&\leq 1,
\end{aligned}$$

which is a contradiction to the assumption of Theorem 5.2.2.

For the case of $p(z_0) = (-ia)^\delta$ $(a > 0)$, by using the same method of the proof as the above, we have the contradiction to the assumption. Therefore we have the theorem.

Theorem 5.2.3. *Let $f \in \mathcal{A}_0$ and $\alpha > 0$. If*

$$\left| (1-\alpha)\left(\frac{zf'(z)}{f(z)} - 1\right) + \alpha\frac{zf''(z)}{f'(z)} \right| < |\lambda| \left| \frac{zf'(z)}{f(z)} \right| \quad (z \in \mathcal{U}),$$

where

$$|\lambda| \le \frac{\sin\frac{\pi}{2}\delta + \alpha\delta\cos\frac{\pi}{2}\delta}{\sqrt{1+\alpha^2\delta^2}} \quad (0 < \delta < \delta_0)$$

and δ_0 is the solution of the equation $\delta\tan(\pi\delta/2) = 1/\alpha$, then $f \in \mathcal{S}(\delta)$.

Proof. Let

$$p(z) = \frac{zf'(z)}{f(z)} \quad (z \in \mathcal{U}).$$

Then p is analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \neq 0$ in $\mathcal{U}$. If there exist a point $z_0 \in \mathcal{U}$ such that the condition (5.1.1) and (5.1.2) are satisfied, then (by Lemma 5.1.1) we obtain (5.1.3) under the restrictions (5.1.4), (5.1.5) and (5.1.6).

For the case of $p(z_0) = (ia)^\delta$ $(a > 0)$, we have

$$\begin{aligned}
\frac{|\lambda| \left| \frac{z_0 f'(z_0)}{f(z_0)} \right|}{\left| (1-\alpha)\left(\frac{z_0f'(z_0)}{f(z_0)} - 1\right) + \alpha\frac{z_0f''(z_0)}{f'(z_0)} \right|} &= \frac{|\lambda| |\, p(z_0)\,|}{\left| p(z_0) + \alpha\frac{z_0p'(z)}{p(z_0)} - 1 \right|} \\
&= |\lambda| \left(\frac{a^{2\delta}}{a^{2\delta} + 2a^\delta\left(\alpha k\delta\sin\frac{\pi}{2}\delta - \cos\frac{\pi}{2}\delta\right) + 1 + \alpha^2k^2\delta^2} \right)^{1/2} \\
&\le |\lambda|\sqrt{g(t)},
\end{aligned}$$

where

$$g(t) = \frac{t^2}{t^2 + 2t\left(\alpha\delta\sin\frac{\pi}{2}\delta - \cos\frac{}{\pi}2\delta\right) + 1 + \alpha^2\delta^2} \qquad (t = a^\delta > 0).$$

Then g has a maximum value at

$$t_0 = \frac{1 + \alpha^2\delta^2}{\cos\frac{\pi}{2}\delta - \alpha\delta\sin\frac{\pi}{2}\delta} \quad (0 < \delta < \delta_0).$$

The remaining part of the proof is a similar to that of Theorem 5.2.3 and so we omit it.

Taking $\alpha = 0$ in Theorem 5.2.2 or $\alpha = 1$ in Theorem 5.2.3, we have the following results.

Corollary 5.2.2. *Let $f \in \mathcal{A}_0$. If*

$$\left|\frac{zf''(z)}{f'(z)}\right| < |\lambda|\left|\frac{zf'(z)}{f(z)}\right| \qquad (z \in \mathcal{U}),$$

where

$$|\lambda| \leq \frac{\sin\frac{\pi}{2}\delta + \delta\cos\frac{\pi}{2}\delta}{\sqrt{1+\delta^2}} \qquad (0 < \delta < \delta_0) \tag{5.2.2}$$

and δ_0 is given by Theorem 5.2.1, then $f \in \mathcal{S}(\delta)$.

Corollary 5.2.3. *Let $f \in \mathcal{A}_0$. If*

$$\left|\frac{zf''(z)}{f'(z)}\right| < |\lambda|\left|\frac{zf'(z)}{f(z)}\right| \qquad (|\lambda| \leq 3/\sqrt{10}),$$

then $f \in \mathcal{S}(1/2)$.

Theorem 5.2.4. *Let $f \in \mathcal{A}_0$. If*

$$\left|-\frac{zf'(z)}{f(z)} + 2 + \frac{zf''(z)}{f'(z)} - \frac{f(z)}{zf'(z)}\right| < |\lambda|\left|\frac{f(z)}{zf'(z)}\right|, \tag{5.2.3}$$

where λ and δ_0 are given by (5.2.2) and Theorem 5.2.1, respectively, then $f \in \mathcal{S}(\delta)$.

Proof. If we set

$$p(z) = \frac{zf'(z)}{f(z)} \quad (z \in \mathcal{U}), \tag{5.2.4}$$

then p is analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \neq 0$ in $\mathcal{U}$. Taking differentiation in both sides of (5.2.4) and simplifying, (5.2.3) can be written as

$$| \, zp'(z) + p(z) - 1 \, | < |\lambda| \quad (z \in \mathcal{U}).$$

If there exist a point $z_0 \in \mathcal{U}$ such that the condition (5.1.1) and (5.1.2) are satisfied, then (by Lemma 5.1.1) we obtain (5.1.3) under the restrictions (5.1.4), (5.1.5) and (5.1.6).

At first, we suppose that $p(z_0) = (ia)^\delta$ $(a > 0)$. Then we have

$$\begin{aligned}
& | \, z_0 p'(z_0) + p(z_0) - 1 \, | \\
&= \sqrt{(1 + k^2\delta^2)a^{2\delta} + 2\left(\delta k \sin\frac{\pi}{2}\delta - \cos\frac{\pi}{2}\delta\right)a^\delta + 1} \\
&\geq \sqrt{(1 + \delta^2)a^2 + 2\left(\delta \sin\frac{\pi}{2}\delta - \cos\frac{\pi}{2}\delta\right)a + 1} \\
&\equiv \sqrt{g(t)} \quad (t = a^\delta > 0).
\end{aligned}$$

Then the function g has a minimum value at

$$t_0 = \frac{\cos\frac{\pi}{2}\delta - \delta \sin\frac{\pi}{2}\delta}{1 + \delta^2}.$$

Hence we obtain

$$\begin{aligned}
\sqrt{g(t)} &\geq \sqrt{g(t_0)} \\
&= \frac{\sin\frac{\pi}{2}\delta + \delta \cos\frac{\pi}{2}\delta}{\sqrt{1 + \delta^2}} \\
&\geq |\lambda|,
\end{aligned}$$

where we have used the inequality (5.2.2). This is a contradiction to the assumption (5.2.2). Similarly, for the case of $p(z_0) = (-ia)^\delta$ $(a > 0)$, we also have the contradiction. Therefore we complete the proof of Theorem 5.2.4.

Theorem 5.2.5. *Let $f \in \mathcal{A}_0$. If*

$$\left| 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} - \frac{f(z)}{zf'(z)} \right| < |\lambda| \left| \frac{f(z)}{zf'(z)} \right|, \tag{5.2.5}$$

where $|\lambda| \leq 1$, then $f \in \mathcal{S}^*$.

Proof. Letting

$$p(z) = \frac{zf'(z)}{f(z)} \quad (z \in \mathcal{U}),$$

we see that p is analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \neq 0$ in $\mathcal{U}$. Further, the condition (5.2.5) can be written as

$$| zp'(z) - 1 | \leq |\lambda| \quad (z \in \mathcal{U}).$$

The remaining part of the proof is a similar to that of Theorem 5.2.4 and so we omit it.

References

[1] J. W. Alexander, *Functions which map the interior of the unit circle upon simple regions*, Ann. Math., **17**(1915), 12-22.

[2] S. D. Bernardi, *Convex and starlike univalent functions*, Trans. Amer. Math. Soc., **135**(1969), 429-446.

[3] L. Bieberbach, *Über die koeffizienten derjenigen potenzveihen,* welche eire schlichte Abbidurg des Einheitskreises vermitteln, S.-B. Press. Akad. Wiss., (1916), 940-955.

[4] L. de Branges, *A proof of the Biberbach conjecture*, Acta Math., **154**(1985), 137-152.

[5] D. Brannan and W. Kirwan, *On some classes of bounded univalent functions*, J. London Math. Soc., **1(2)**(1969), 431-443.

[6] P. N. Chichra, *New subclasses of the class of close-to-convex functions*, Proc. Amer. math. Soc., **62**(1977), 37-43.

[7] P. L. Duren, *Univalent functions*, Springer-Verlag, New York Inc., 1983.

[8] A. W. Goodman, *Univalent functions*, Mariner Publ. Comp., Tampa Florida, (1983).

[9] A. W. Goodman, *On uniformly convex functions* , Ann. Polon. Math., **56**(1991), 87-92.

[10] I. S. Jack, *Functions starlike and convex of order* α, J. London Math. Soc., (2)**3**(1971), 469-474.

[11] Z. J. Jakubowski, *On the coefficients of starlike functions of some classes*, Ann. Polon. Math., **26**(1972), 305-313.

[12] W. Kaplan, *Close-to convex schlicht functions*, Michigan Math. J., **1**(1952), 169-185.

[13] P. Koebe, *Über die Uniformisierung beliebiger analytischer Kurven*, Nachr. Akad. Wiss. Göttingen, Math.-Phys. K1., (1907), 191-210.

[14] J. Krzyz, *A counterexample concerning univalent functions*, Folias Soc. Scient. Lubliniensis, Mat. Fiz. Chem., **2**(1962), 57-58.

[15] R. J. Libera, *Some classes of regular univalent functions*, Proc. Amer. Math. Soc., **16**(1965), 755-758.

[16] W. Ma and D. Minda, *Uniformly convex functions* , Ann. Polon. Math., **57**(1992), 166-175.

[17] T. H. MacGregor, *Functions whose derivative has a positive real part*, Trans. Amer. Math. Soc., **104**(1962), 532-537.

[18] A. Marx, *Untersuchungen über schlichte Abbildungen*, Math. Ann., **107**(1932/1933), 40-67.

[19] S. S. Miller, *Differential inequalities and Carathéodory functions*, Bull. Amer. Math. Soc., **81**(1975), 79-81.

[20] S. S. Miller and P. T. Mocanu, *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl., **65**(1978), 289-305.

[21] S. S. Miller, P. T. Mocanu and M. O. Reade, *All alpha-convex functions are starlike*, Rev. Roumaine Math. Purse Appl., **17**(1972), 1395-1397.

[22] P. T. Mocanu, *On starlikeness of Libera transform*, Mathematica, **28**(51)(1986), 153-155.

[23] P. T. Mocanu, *Alpha-convex integral operators and strongly starlike functions*, Studia Univ. Babes-Bolyai Mathematica, **34**(1989), 18-24.

[24] K. Noshiro, *On the theory of schlicht functions*, J. Fac. Sci. Hokkaido Univ., **2**(1934-35), 129-155.

[25] M. Nunokawa, *Differential inequalities and Carathéodory functions*, Proc.Japan Acad. Appl., **65**, Ser. A(1989), 326-328.

[26] M. Nunokawa, *On the order of strongly starlikeness of strongly convex functions*, Proc. Japan Acad. Ser. A Math. Sci., **69**(1993), 234-237.

[27] M. Nunokawa, O. S. Kwon and N. E. Cho, *On the multivalent functions*, Tsukuba J. Math., **15**(1991), 141-143.

[28] M. Nunokawa, A. Ikeda, N. Koike, Y. Ota and H. Saitoh, *Differential inequalities and Carathéodory function*, J. Math. Anal. Appl., **212**(1997), 324-332.

[29] S. Ponnusamy and V. Karunakaran, *Differential subordination and conformal mappings*, Complex Variables, **11**(1989), 79-86.

[30] M. S. Robertson, *On the theory of univalent functions*, Ann. Math., **32**(1936), 374-408.

[31] F. Ronning, *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc., **118**(1993), 190-196.

[32] St. Ruscheweyh, *Convolution in geometric function theory*, Press de l'Université de Montréal, Montréal (1982).

[33] K. Sakaguchi, *On a certain univalent mapping*, J. Math. Soc. Japan, **11**(1959), 72-75.

[34] R. Singh and S. Singh, *Starlikeness of and convexity of certain integrals*, Ann. Univ. Mariae Curie-Sklodowska Sect. A., **16**(1981), 145-148.

[35] R. Singh and S. Singh, *Some sufficient conditions for univalence and starlikeness*, Coll. Math., **47**(1982), 309-314.

[36] E. Strohhäcker, *Beiträge zur Theorie der schlichten funktionen*, Math. Z., **37**(1933), 356-380.

[37] S. E. Warschawski, *On the higher derivatives at the boundary in the conformal mapping*, Trans. Amer. Soc., **38**(1935), 310-340.

[38] V. A. Zomorovič, *On some problems in the theory of univalent functions*, Nauk. Zapiski, Kiev. Derjavni Univ., **11**(1952), 83-94.

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