Comparison of Confidence Intervals on Variance Component in a Simple Linear Regression Model with Unbalanced Nested Error Structure

불균형 중첩 오차구조를 갖는 단순선형 회귀모형의 분산의 신뢰구간의 비교

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TABLE OF CONTENTS

LIST OF TABLES

불균형 중첩 오차구조를 갖는 단순선형 회귀모형의 분산의 신뢰구간의 비교

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요약

불균형 중첩 오차구조를 갖는 단순선형 회귀모형에 나타나는 주 샘플링 단위의 분산 σ_A^2에 대한 세 가지 신뢰구간을 구하였다. 그 중에서 두 가지 신뢰구간은 ANOVA 방법으로부터 구한 기대평균제곱을 σ_A^2에 관한 식을 구하여 Ting et al.(1990) 방법을 적용하여 구하였다. 나머지 한 방법은 Khuri et al.(1998)이 제안한 일반화 p-값을 활용하여 σ_A^2에 관한 신뢰구간을 구하였다. 세 가지 신뢰구간을 비교하기 위하여 시뮬레이션을 시행하고 실제 자료에 적용하여 시뮬레이션의 결과와 일관성 있는 결과를 보이는 것을 확인하였다.

1. INTRODUCTION

1.1 Experimental Design

Statisticians often use experimental design to compare the differences of treatments in their experimental research. Experimental design is to design and conduct experiments to obtain best information at a minimum cost. The relationships of the effects of different levels in an experimental design are expressed as a linear model. The linear model of an experimental design is divided into three kinds: fixed effects model, random effects model, and mixed effects model. The fixed effects model with I populations and a random sample of size J from each population is written as

$$Y_{ij} = \mu + \alpha_i + E_{ij} \tag{1.1}$$

$$i = 1, ..., I; \;\; j = 1, ..., J$$

$$\sum_{i=1}^{I} \alpha_i = 0$$

where Y_{ij} is the jth observed sample value from the ith population, the quantities μ and α_i are unobservable fixed constants called parameters, E_{ij} is a random error term with mean zero and variance σ_E^2. The objective of a fixed effects model is to make inferences about all treatment levels included in the experiment.

Suppose that four specific types of training methods are used by a company to train operators to fill bottles with vegetable oil. The purpose of the experiment is to compare four training methods. Model (1.1) can be used for this experiment where Y_{ij} is the weight of the jth bottle filled with oil by the ith method, μ is a constant representing the average bottle weight filled with oil,

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α_i is the ith training method effect, and E_{ij} is an independent normal random variable with mean zero and variance σ_E^2. Specifically, $I = 4$ represents the number of training methods, J is the number of bottles filled with oil by each training method, and σ_E^2 is a measure of variability of bottle weight filled with oil for a particular method. The investigator is interested in making inferences about four training methods in this experiment.

Consider a simple experiment in which treatment levels for a factor are randomly selected from a large population. A random effects model is written as

$$Y_{ij} = \mu + A_i + E_{ij} \tag{1.2}$$

$$i = 1, ..., I; \;\; j = 1, ..., J$$

where A_i and E_{ij} are random variables with means of zero and variances σ_A^2 and σ_E^2, respectively. In this model we are interested in the variability of the A_i as measured by σ_A^2. The objective of a random effects model is inferences concerning functions of variances. The random effects model is also referred to as a variance component model.

As an example, consider machines in a large plant that are used to fill bottles with vegetable oil. Five machines are selected at random from which a sample of bottles are filled and weighted. The purpose of the experiment is to determine how much the weight variability in the bottles is attributed to variability among machines. The experimental model is represented as (1.2.) where Y_{ij} represents the weight of the jth bottle filled with oil by the ith machine, μ is a constant representing the average bottle weight filled with oil, and A_i and E_{ij} are mutually independent normal random variables with means of zero and variances σ_A^2 and σ_E^2, respectively. In the context example, $I = 5$ represents the number of sampled machines, and J is the number of bottles filled with oil by each machine. The variance component σ_A^2 is a measure of variability of bottle weight filled with oil across machines and σ_E^2 is a measure of weight variability for any particular machine. The investigator is primarily interested in determining the amount of variability among machines in the population.

Suppose that a second factor with specific treatment levels of interest is add to model (1.2). If each level of factor A is crossed with each level of factor B and each combination is replicated k times, the model is expressed as

$$Y_{ijk} = \mu + A_i + \beta_j + E_{ijk} \tag{1.3}$$

$$i = 1, ..., I; \;\; j = 1, ..., J; \;\; k = 1, 2, ...K$$

$$\sum_{j=1}^{J} \beta_j = 0$$

where Y_{ijk} is the kth observed sample value in the ith level of factor A and the jth level of factor B, A_i is independently distributed as $N(0, \sigma_A^2)$, β_j is the effect of the jth level of factor B, E_{ijk} is independently distributed as $N(0, \sigma_E^2)$ and A_i and E_{ijk} are independent. Factor B in (1.3) is a fixed effect where the selected treatment levels in the experiment are of interest. That is, an investigator is interested in estimating functions of β_j. Model (1.3) includes both a random effect (factor A) and a fixed effect (factor B). This type of model is called a mixed effects model. The objective of a mixed effects model is to make inferences concerning functions of variance σ_A^2 and inferences of β_j.

Suppose that two types of training courses are used by the company to train operators to use the filling machines. After five machines are randomly selected from the population of machines, three bottles are filled by operators using each training course. The problem of interest is to not only determine weight variability among the machines but to also determine the difference between the two training courses. Model (1.3) can be used for this experiment where Y_{ijk} is the weight of the kth bottle filled on the ith machine by the jth training course, μ is a constant representing the average bottle weight filled with oil, A_i and E_{ij} are mutually independent normal random variables with zero means and variances σ_A^2 and σ_E^2, respectively and β_j is the jth training course effect. Additionally, $I = 5$ represents the number of machines for each training course, $J = 2$ represents the number of training courses, and $K = 3$ represents the number of bottles filled with oil by each combination of machine and training course.

1.2 Statistical Inferences

Statistical inferences are largely divided into estimation and test of hypotheses. Estimation includes point estimation and interval estimation. The selection of a function of the sample values that will best represent the parameter of interest is concerned with point estimation.

Interval estimation is generally more informative than point estimation because it is not enough to obtain a single value for the parameter under investigation and a point estimate has no information about confidence and bound of error. An interval estimation provides this information. Let θ represent a parameter of interest. A confidence interval is a random interval whose endpoints L and U, where $L \leq U$ are functions of the sample values such that $\mathrm{P}[L \leq \theta \leq U] = 1 - \alpha$. The term $1 - \alpha$ is the confidence coefficient and is selected prior to data collection. A confidence interval [L, U] that satisfies $\mathrm{P}[L \leq \theta \leq U] = 1 - \alpha$ is called an exact two-sided $1 - \alpha$ confidence interval. Often exact $1 - \alpha$ confidence intervals do not exist and $\mathrm{P}[L \leq \theta \leq U]$ is only approximately equal to $1 - \alpha$. These intervals are referred to as approximate intervals. An approximate interval is conservative if $\mathrm{P}[L \leq \theta \leq U] > 1 - \alpha$ and liberal if $\mathrm{P}[L \leq \theta \leq U] < 1 - \alpha$.

Hypothesis testing refers to the process of trying to decide the truth or falsity of hypotheses on the basis of experimental evidence. Confidence intervals and tests of hypotheses are procedures for making statistical inferences that attach measures of uncertainty to the inferences. It is almost always the case, however, that confidence intervals are "uniformly more informative" than tests of hypotheses for making decisions based on parametric values. Thus, tests of hypotheses are seldom needed if confidence intervals are available.

1.3 Literature Review

If the number of observations in cells is not equal, then experimental designs are unbalanced. The unbalanced one-fold nested design model is written as

$$Y_{ij} = \mu + A_i + E_{ij} \qquad (1.4)$$
$$i = 1, ..., I; \;\; j = 1, ..., J_i$$

where μ is an unknown constant, A_i and E_{ij} are mutually independent normal random variables with means of zero and variances σ_A^2 and σ_E^2, respectively, $I \geq 2$, $J_i \geq 1$, and $J_i > 1$ for at least one value of i. The analysis of variance table for the one-fold nested design is given in Table 1.1.

TABLE 1.1 ANOVA for One-fold Nested Design

SV	DF	MS	EMS
Among Groups	$n_1 = I - 1$	S_1^2	$\theta_1 = \sigma_E^2 + c_1\sigma_A^2$
Within Groups	$n_2 = N - I$	S_2^2	$\theta_2 = \sigma_E^2$
Total	$N - 1$		

$$N = \sum_{i=1}^{I} J_i, \qquad (1.5)$$

$$S_1^2 = \sum_{i=1}^{I}\sum_{j=1}^{J_i}(\bar{Y}_{i.} - \bar{Y}_{..})^2,$$

$$S_2^2 = \sum_{i=1}^{I}\sum_{j=1}^{J_i}(Y_{ij} - \bar{Y}_{i.})^2, \qquad \text{and}$$

$$c_1 = \frac{N - \sum_{i=1}^{I} J_i^2/N}{I - 1}.$$

In the balanced design where all $J_i = J$, $N = IJ$, and $c_1 = J$, $n_1 S_1^2/\theta_1$ and $n_2 S_2^2/\theta_2$ are independent chi-squared random variables with n_1 and n_2 degrees of freedom, respectively. In the unbalanced design, S_1^2 and S_2^2 are still independent and $n_2 S_2^2/\theta_2$ has a chi-squared distribution with n_2 degrees of freedom. However, unless $\sigma_A^2 = 0$, $n_1 S_1^2/\theta_1$ no longer has a chi-squared distribution.

Thomas and Hultquist (1978) proposed a statistic that can be used for constructing a confidence interval on σ_A^2 in the unbalanced model. The proposed statistic is

$$\frac{n_1 S_{1U}^2}{\theta_{1U}} \tag{1.6}$$

where

$$n_1 S_{1U}^2 = \sum_{i=1}^{I} \bar{Y}_{i.}^2 - (\frac{1}{I})(\sum_{i=1}^{I} \bar{Y}_{i.})^2,$$

$$\bar{Y}_{i.} = \sum_{j=1}^{J_i} \frac{Y_{ij}}{J_i},$$

$$\theta_{1U} = E(S_{1U}^2) = \sigma_A^2 + (\frac{1}{h})\sigma_E^2, \qquad \text{and}$$

$$h = \frac{I}{\sum_{i=1}^{I} \frac{1}{J_i}}.$$

The term $n_1 S_{1U}^2$ means the unweighted sums of squares of the treatment means and h represents the harmonic mean of the J_i values. They showed that the moment generation function on $n_1 S_{1U}^2/\theta_{1U}$ approaches that of a chi-squared random variable with n_1 degrees of freedom as all J_i approach a constant or if either $\lambda_A = \sigma_A^2/\sigma_E^2$ or all J_i approach infinity. They showed that $n_1 S_1^2/\theta_{1U}$ is well approximated by a chi-squared random variables when $\lambda_A > 0.25$. In situations where the Thomas-Hultquist approximation works well, an interval on σ_A^2 can be formed by replacing S_1^2 with hS_{1U}^2 and J with h in the balanced design equations.

In extremely unbalanced designs where $\lambda_A < 0.25$, the chi-squared approximation for $n_1 S_{1U}^2/\theta_{1U}$ is not good and this substitution can yield a liberal confidence interval. A method that works well over the entire range of λ_A was developed by Burdick and Eickman (1986). The Burdick-Eickman approximate $100(1-\alpha)\%$ confidence interval is

$$\left[\frac{hS_{1U}^2 L^*}{F_{\alpha_{11}:n_1,\infty}(1+hL^*)}; \frac{hS_{1U}^2 U^*}{F_{1-\alpha_{21}:n_1,\infty}(1+hU^*)}\right] \tag{1.7}$$

where

$$L^* = \frac{S_{1U}^2}{(F_{\alpha_{12}:n_1,n_2} S_2^2)} - \frac{1}{m},$$

$$U^* = \frac{S_{1U}^2}{(F_{1-\alpha_{22}:n_1,n_2} S_2^2)} - \frac{1}{M},$$

$$m = \min(J_1, J_2, ..., J_I),$$

$$M = \max(J_1, J_2, ..., J_I), \qquad \text{and}$$

$$\alpha_{11} + \alpha_{21} = \alpha_{12} + \alpha_{22} = \alpha.$$

Burdick and Eickman conducted a simulation study to show (1.7) is generally conservative. They showed that their method can always be recommended over the Thomas-Hultquist approximation. The average interval lengths of these two methods never differed by more than 5% and the Burdick and Eickman method maintains its confidence coefficient over a wider range of unbalanced designs than does the Thomas-Hultquist method.

The variance component model with one explanatory variable is

$$Y_{ij} = \mu + \beta X_{ij} + A_i + E_{ij} \tag{1.8}$$

$$i = 1, ..., I; \;\; j = 1, ..., J_i$$

where A_i is the cluster effect and assumed to be a random sample from $N(0, \sigma_A^2)$, and E_{ij} is an observational error within a cluster and assumed to be a random sample from $N(0, \sigma_E^2)$. The random variables A_i and E_{ij} are independent. This model is also referred to as a simple regression model with nested error structure. In the balanced case where $J_i = J$, $\hat{\beta}$ is the ordinary least squares estimator of β. Several methods for point estimation of the regression coefficients have been proposed for model (1.8) and its various extensions.

Researches have also been done in confidence intervals and tests of hypothesis in the variance component model with one explanatory variable in the balanced case where $J_i = J$. In this case the model is also called simple regression model with balanced nested error structure. Tong and Cornelius (1989) compared four estimators of regression coefficient β in the model with respect to their mean squared error in a Monte Carlo simulation study. Tong and Cornelius (1991) investigated properties of tests of hypothesis for regression coefficient β in the model and compared with respect to type I error rate and power of test in a Monte Carlo simulation study. Guven (1995) derived explicit maximum likelihood estimators of regression coefficient β in the model.

Park and Burdick (1993) derived three approximate confidence intervals on σ_A^2 using distributional results for sums of squares associated with the model.

Park and Burdick (1994) proposed several confidence intervals on the regression coefficient β in the model and the intervals were compared using computer simulation. Park and Hwang (2002) derived exact and approximate confidence intervals for the mean response for a given level of the independent variable in the simple linear regression model with nested error structure. Yu and Burdick (1995) extended the model and considered confidence intervals on the variance components in regression models with balanced (Q-1)-fold nested error structure. They used a method proposed by Ting, Burdick, Graybill, Jeyaratnam, and Lu (1990). That is, the regression model with two-fold nested error structure was first considered and then results were generalized to the (Q-1)-fold nested error structure.

2. A SIMPLE REGRESSION MODEL WITH AN UNBALANCED ONE-FOLD NESTED ERROR STRUCTURE

The simple regression model with an unbalanced one-fold nested error structure is written as

$$Y_{ij} = \mu + \beta X_{ij} + A_i + E_{ij} \tag{2.1}$$
$$i = 1, ..., I; \;\; j = 1, ..., J_i$$

where Y_{ij} is the jth observation in the ith primary level, μ and β are unknown constants, X_{ij} is a fixed predictor variable, and A_i and E_{ij} are jointly independent normal random variables with zero means and variances σ_A^2 and σ_E^2, respectively, $I \geq 2$, $J_i \geq 1$, and $J_i > 1$ for at least one value of i. A_i is an error term associated with the first-stage sampling unit and E_{ij} is an error term associated with the second-stage sampling unit. Model (2.1) is unbalanced since the number of observations in cells are not all equal. This model is referred to as either a single-factor covariance model with one covariate or a variance component model with one explanatory variable. Since the X_{ij} and β are fixed, model (2.1) is a mixed model. This error structure yields response variables that are correlated. That is,

$$Cov(Y_{ij}, Y_{i'j'}) = \begin{cases} \sigma_A^2 + \sigma_E^2 & \text{if} \quad i = i', j = j'; \\ \sigma_A^2 & \text{if} \quad i = i', j \neq j'; \\ 0 & \text{if} \quad i \neq i'. \end{cases} \tag{2.2}$$

In order to form confidence intervals on linear functions of the variance components, an appropriate set of sums of squares is needed. One possible partitioning of model (2.1) is shown in Table 2.1.

TABLE 2.1 ANOVA for Model (2.1)

SV	DF	SS
Mean	1	$J_{.}\bar{Y}_{..}^2$
Covariate after mean	1	$\hat{\beta}_L^2(S_{wxxa} + S_{wxxe})$
Primary units adjusted for regression	$I - 1$	R_{WB}+R$_L$
Residual	$J_{.} - I - 1$	R_T
Total	$J_{.}$	$\sum_{i=1}^{I} \sum_{j=1}^{J_i} Y_{ij}^2$

The notation in Table 2.1 is defined as

$$J_{.} = \sum_{i=1}^{I} J_i,$$

$$\bar{X}_{i.} = \frac{\sum_{j=1}^{J_i} X_{ij}}{J_i},$$

$$\bar{Y}_{i.} = \frac{\sum_{j=1}^{J_i} Y_{ij}}{J_i},$$

$$\bar{X}_{..} = \frac{\sum_{i=1}^{I} \sum_{j=1}^{J_i} X_{ij}}{J_{.}} = \frac{\sum_{i=1}^{I} \bar{X}_{i.} J_i}{J_{.}},$$

$$\bar{Y}_{..} = \frac{\sum\limits_{i=1}^{I} \sum\limits_{j=1}^{J_i} Y_{ij}}{J_.} = \frac{\sum\limits_{i=1}^{I} \bar{Y}_{i.} J_i}{J_.},$$

$$S_{wxxa} = \sum_{i=1}^{I} (\bar{X}_{i.} - \bar{X}_{..})^2 J_i,$$

$$S_{wxxe} = \sum_{i=1}^{I} \sum_{j=1}^{J_i} (X_{ij} - \bar{X}_{i.})^2,$$

$$S_{wxya} = \sum_{i=1}^{I} (\bar{X}_{i.} - \bar{X}_{..})(\bar{Y}_{i.} - \bar{Y}_{..}) J_i,$$

$$S_{wxye} = \sum_{i=1}^{I} \sum_{j=1}^{J_i} (X_{ij} - \bar{X}_{i.})(Y_{ij} - \bar{Y}_{i.}),$$

$$S_{wyya} = \sum_{i=1}^{I} (\bar{Y}_{i.} - \bar{Y}_{..})^2 J_i,$$

$$S_{wyye} = \sum_{i=1}^{I} \sum_{j=1}^{J_i} (Y_{ij} - \bar{Y}_{i.})^2,$$

$$\hat{\beta}_{WB} = \frac{S_{wxya}}{S_{wxxa}},$$

$$\hat{\beta}_L = \frac{(S_{wxya} + S_{wxye})}{(S_{wxxa} + S_{wxxe})},$$

$$\hat{\beta}_T = \frac{S_{wxye}}{S_{wxxe}},$$

$$R_{WB} = S_{wyya} - \hat{\beta}_{WB}^2 S_{wxxa},$$

$$R_L = \hat{\beta}_{WB}^2 S_{wxxa} + \hat{\beta}_T^2 S_{wxxe}$$
$$- \hat{\beta}_L^2 (S_{wxxa} + S_{wxxe}), \quad \text{and}$$

$$R_T = S_{wyye} - \hat{\beta}_T^2 S_{wxxe}.$$

Model (2.1) is written in matrix notation,

$$\underline{Y} = \mathbf{X}\underline{\alpha} + \mathbf{Z}\underline{U} + \underline{E} \tag{2.3}$$

where $\underline{Y}$ is a $J_. \times 1$ vector of observations, $\mathbf{X}$ is a $J_. \times 2$ matrix of known values with a column of 1's in the first column and a column of X_{ij}'s in the second column, $\underline{\alpha}$ is a 2×1 vector of parameters with μ and β as elements, $\mathbf{Z}$ is a $J_. \times I$ design matrix with 0's and 1's, i.e. $\mathbf{Z} = \overset{I}{\underset{i=1}{\oplus}} \underline{1}_{J_i \times 1}$, $\underline{U}$ is an $I \times 1$ vector of random effects, and $\underline{E}$ is a $J_. \times 1$ vector of random error terms. In particular,

$$\underline{Y} = \begin{pmatrix} Y_{11} \\ \vdots \\ Y_{1J_1} \\ Y_{21} \\ \vdots \\ Y_{2J_2} \\ \vdots \\ Y_{I1} \\ \vdots \\ Y_{IJ_I} \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} 1 & X_{11} \\ \vdots & \vdots \\ 1 & X_{1J_1} \\ 1 & X_{21} \\ \vdots & \vdots \\ 1 & X_{2J_2} \\ \vdots & \vdots \\ 1 & X_{I1} \\ \vdots & \vdots \\ 1 & X_{IJ_I} \end{pmatrix}, \quad \underline{\alpha} = \begin{pmatrix} \mu \\ \beta \end{pmatrix}, \quad \underline{U} = \begin{pmatrix} A_1 \\ \vdots \\ A_I \end{pmatrix},$$

and

$$\mathbf{Z} = \overset{I}{\underset{i=1}{\oplus}} \underline{1}_{J_i} = \begin{pmatrix} \underline{1}_{J_1} & \underline{0}_{J_1} & \cdots & \underline{0}_{J_1} \\ \underline{0}_{J_2} & \underline{1}_{J_2} & \cdots & \underline{0}_{J_2} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{0}_{J_I} & \underline{0}_{J_I} & \cdots & \underline{1}_{J_I} \end{pmatrix}$$

where $\underline{1}_{J_i}$ is a $J_i \times 1$ column vector of 1's and $\oplus$ is the direct sum operator. The direct sum of two matrices **P** and **Q** is defined as

$$\mathbf{P} \oplus \mathbf{Q} = \begin{pmatrix} \mathbf{P} & 0 \\ 0 & \mathbf{Q} \end{pmatrix}.$$

By the assumptions in (2.1) the response variables have a multivariate normal distribution

$$\underline{Y} \sim N(\mathbf{X}\underline{\alpha}, \sigma_A^2 \mathbf{Z}\mathbf{Z}' + \sigma_E^2 \mathbf{D}_{J.}) \tag{2.4}$$

where $\mathbf{D}_{J.}$ is a $J. \times J.$ identity matrix.

$$\begin{aligned} E(\underline{Y}) &= E(\mathbf{X}\underline{\alpha} + \mathbf{Z}\underline{U} + \underline{E}) \\ &= \mathbf{X}\underline{\alpha} + E(\mathbf{Z}\underline{U}) + E(\underline{E}) \\ &= \mathbf{X}\underline{\alpha}, \qquad \text{and} \\ V(\underline{Y}) &= V(\mathbf{X}\underline{\alpha} + \mathbf{Z}\underline{U} + \underline{E}) \\ &= V(\mathbf{Z}\underline{U}) + V(\underline{E}) \\ &= \mathbf{Z}V(\underline{U})\mathbf{Z}' + \sigma_E^2 \mathbf{D}_{J.} \\ &= \sigma_A^2 \mathbf{Z}\mathbf{Z}' + \sigma_E^2 \mathbf{D}_{J.} \end{aligned}$$

In order to define unweighted sums of squares, the vector of means of response variables of primary level and associated variance component matrix are needed. These are defined in matrix notation as

$$\mathbf{M}\underline{Y} = [\bar{Y}_{1.}, \bar{Y}_{2.}, ..., \bar{Y}_{I.}]' = \underline{Y}_M \tag{2.5}$$

where

$$\mathbf{M} = \overset{I}{\underset{i=1}{\oplus}} [J_i^{-1} \underline{1}'_{J_i \times 1}] = \begin{pmatrix} \frac{1}{J_1}\underline{1}_{J_1} & \underline{0} & \cdots & \underline{0} \\ \underline{0} & \frac{1}{J_2}\underline{1}_{J_2} & \cdots & \underline{0} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{0} & \underline{0} & \cdots & \frac{1}{J_I}\underline{1}_{J_I} \end{pmatrix}$$

and $\bar{Y}_{i.}$ is the mean of response variables of the ith primary level. The expectation and variance of vector of means of response variables of the primary level are

$$\begin{aligned} E(\underline{Y}_M) &= E(\mathbf{M}\underline{Y}) \\ &= \mathbf{MX}\underline{\alpha} \\ &= \mathbf{X}_M\underline{\alpha}, \qquad \text{and} \end{aligned}$$

where $\mathbf{X}_M = \mathbf{MX}$

$$\begin{aligned} V(\underline{Y}_M) &= V(\mathbf{M}\underline{Y}) \\ &= \mathbf{M}(\sigma_A^2\mathbf{ZZ}' + \sigma_E^2\mathbf{D}_{J.})\mathbf{M}' \\ &= \sigma_A^2\mathbf{MZZ}'\mathbf{M}' + \sigma_E^2\mathbf{MD}_{J.}\mathbf{M}' \\ &= \sigma_A^2\mathbf{D}_I + \sigma_E^2\mathbf{MM}' \end{aligned} \tag{2.6}$$

since $\mathbf{MM}' = diag[J_i^{-1}]$ and $\mathbf{MZ} = \mathbf{D}_I$ where $\mathbf{D}_I$ is an $I \times I$ identity matrix. Thus, the vector of means of response variables of primary level has a multivariate normal distribution

$$\underline{Y}_M \sim N(\mathbf{X}_M\underline{\alpha},\ \mathbf{V}_M) \tag{2.7}$$

where $\mathrm{V}_M = \sigma_A^2\mathbf{D}_I + \sigma_E^2\mathbf{MM}'$.

3. DISTRIBUTIONAL PROPERTY OF ERROR SUMS OF SQUARES

In this section we report distributional results used to derive confidence intervals. Four regression coefficient estimators are considered. Consider weighted between regression coeficient estimator $\hat{\beta}_{WB}$ that is obtained from least squares regression of $\bar{Y}_{i.}$ on $\bar{X}_{i.}$ with weigh J_i for each primary level i and $\hat{\beta}_{WB}$ is written as

$$\hat{\beta}_{WB} = \frac{S_{wxya}}{S_{wxxa}}.$$

The weighted between regression coefficient estimator is the second element of the vector

$$(\mathbf{X}'_M\mathbf{W}\mathbf{X}_M)^{-1}\mathbf{X}'_M\mathbf{W}\underline{Y}_M \tag{3.1}$$

where $\mathbf{W} = diag[J_i]$. The error sum of squares

$$\begin{aligned} R_{WB} &= S_{wyya} - \hat{\beta}^2_{WB}S_{wxxa} \\ &= \underline{Y}'_M\mathbf{A}_W\underline{Y}_M \end{aligned} \tag{3.2}$$

where $\mathbf{A}_W = \mathbf{W} - \mathbf{W}\mathbf{X}_M(\mathbf{X}'_M\mathbf{W}\mathbf{X}_M)^{-1}\mathbf{X}'_M\mathbf{W}$

Unweighted between regression coeffcient estimator considering primary level's means and their unwighted mean is used as an alternative of between regression coefficient estimator. Unweighted between regression coeficient estimator $\hat{\beta}_{UB}$ is obtained from the least squares regression of $\bar{Y}_{i.}$ on $\bar{X}_{i.}$ and $\hat{\beta}_{UB}$ is written as

$$\hat{\beta}_{UB} = \frac{S_{uxya}}{S_{uxxa}}$$

where,

$$S_{uxya} = \sum_{i=1}^{I} (\bar{X}_{i.} - \bar{\bar{X}}_{..})(\bar{Y}_{i.} - \bar{\bar{Y}}_{..})$$

$$S_{uxxa} = \sum_{i=1}^{I} (\bar{X}_{i.} - \bar{\bar{X}}_{..})^2,$$

$$\bar{\bar{X}}_{..} = \frac{\sum_{i=1}^{I} \bar{X}_{i.}}{I}, \qquad \text{and}$$

$$\bar{\bar{Y}}_{..} = \frac{\sum_{i=1}^{I} \bar{Y}_{i.}}{I}.$$

The unweighted between regression coefficient estimator is the second element of the vector

$$(\mathbf{X}'_M \mathbf{X}_M)^{-1} \mathbf{X}'_M \underline{Y}_M \tag{3.3}$$

The error sum of squares R_A associated with this regression model is

$$\begin{aligned} R_{UB} &= S_{uyya} - \hat{\beta}^2_{UB} S_{uxxa} \\ &= \underline{Y}'_M \mathbf{A}_U \underline{Y}_M \end{aligned} \tag{3.4}$$

where $\mathbf{A}_U = \mathbf{D}_I - \mathbf{X}_M (\mathbf{X}'_M \mathbf{X}_M)^{-1} \mathbf{X}'_M$.

The within regression coefficient estimator $\hat{\beta}_T = S_{wxye}/S_{wxxe}$ is obtained from the least squares regression of Y_{ij} on X_{ij} and the grouping variables. The point estimator $\hat{\beta}_T$ is the second element of the vector

$$(\mathbf{X}^{*'} \mathbf{X}^*)^{-} \mathbf{X}^{*'} \underline{Y} \tag{3.5}$$

where $\mathbf{X}^* = [\mathbf{X}\ \ \mathbf{Z}]$, and $(\mathbf{X}^{*'}\mathbf{X}^*)^-$ is a generalized inverse of $\mathbf{X}^{*'}\mathbf{X}^*$. The error sum of squares R_T associated with this regression model is

$$\begin{aligned} R_T &= S_{wyye} - \hat{\beta}_T^2 S_{wxxe} \\ &= \underline{Y}'\mathbf{T}\underline{Y} \end{aligned} \tag{3.6}$$

where $\mathbf{T} = \mathbf{D}_{J.} - \mathbf{X}^*(\mathbf{X}^{*'}\mathbf{X}^*)^-\mathbf{X}^{*'}$. and $\mathbf{D}_{J.}$ is an identity matrix of order $J_.$.

Finally, the total regression coefficient estimator

$$\hat{\beta}_L = \frac{(S_{wxya} + S_{wxye})}{(S_{wxxa} + S_{wxxe})}.$$

is obtained from the least squares regression of Y_{ij} on X_{ij}. The point estimator $\hat{\beta}_L$ is the second element of the vector

$$(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\underline{Y}. \tag{3.7}$$

The error sum of squares R_L associated with this regression model is

$$\begin{aligned} R_L &= (S_{wyya} + S_{wyye}) - \hat{\beta}_L^2(S_{wxxa} + S_{wxxe}) - R_{WB} - R_T \\ &= \underline{Y}'(\mathbf{L} - \mathbf{M}'\mathbf{A}_W\mathbf{M} - \mathbf{T})\underline{Y} \end{aligned} \tag{3.8}$$

where $\mathbf{L} = \mathbf{D}_{J.} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$.

Theorem 1.

R_T/σ_E^2 a chi-squared random variable with $J_. - I - 1$ degree of freedom.

Proof. Notice that $\mathbf{T}$ is idempotent. It can be shown that $\mathbf{X}^*(\mathbf{X}^{*'}\mathbf{X}^*)^-\mathbf{X}^{*'}\mathbf{X} =$

$\mathbf{X}$ and $\mathbf{X}^*(\mathbf{X}^{*'}\mathbf{X}^*)^-\mathbf{X}^{*'}\mathbf{Z} = \mathbf{Z}$ by Theorem 7.1 in Searle (1987, p. 218). Therefore, as may be easily verified, $\mathbf{TX} = \mathbf{0}$ and $\mathbf{TZ} = \mathbf{0}$. It follows that

$$\begin{aligned}
E(R_T) &= E(\underline{Y}'\mathbf{T}\underline{Y}) \\
&= tr(\mathbf{TV}) + \underline{\alpha}'\mathbf{X}'\mathbf{TX}\underline{\alpha} \\
&= tr((\mathbf{D}_{J.} - \mathbf{X}^*(\mathbf{X}^{*'}\mathbf{X}^*)^-\mathbf{X}^{*'})(\sigma_A^2\mathbf{ZZ}' + \sigma_E^2\mathbf{D}_{J.})) \\
&= tr(\sigma_E^2(\mathbf{D}_{J.} - \mathbf{X}^*(\mathbf{X}^{*'}\mathbf{X}^*)^-\mathbf{X}^{*'})) \\
&= tr(\sigma_E^2\mathbf{T}) \\
&= \sigma_E^2 r(\mathbf{T}) \\
&= (J_. - I - 1)\sigma_E^2.
\end{aligned}$$

The distribution of R_T is determined by writing $R_T/\sigma_E^2 = \underline{Y}'(\mathbf{T}/\sigma_E^2)\underline{Y}$ and noting $(\mathbf{T}/\sigma_E^2)\mathbf{V} = \mathbf{T}(\sigma_A^2\mathbf{ZZ}' + \sigma_E^2\mathbf{D}_{J.})/\sigma_E^2 = \mathbf{T}$. By Theorem 2 in Searle (1971, p. 57) R_T/σ_E^2 is a che-squared random variable with $J_. - I - 1$ degree of freedom.

Theorem 2.

If $\sigma_A^2 = 0$, then R_{WB}/σ_E^2 is a chi-squared random variable with $I - 2$ degree of freedom.

Proof.

Notice that

$$\begin{aligned}
\mathbf{A}_W\mathbf{V}_M =&(\mathbf{W} - \mathbf{WX}_M(\mathbf{X}_M'\mathbf{WX}_M)^{-1}\mathbf{X}_M'\mathbf{W})(\sigma_A^2\mathbf{D}_I + \sigma_E^2\mathbf{M}') \\
=&\sigma_A^2\mathbf{W} - \sigma_A^2\mathbf{WX}_M(\mathbf{X}_M'\mathbf{WX}_M)^{-1}\mathbf{X}_M'\mathbf{W} \\
&+ \sigma_E^2\mathbf{D}_I - \sigma_E^2\mathbf{WX}_M(\mathbf{X}_M'\mathbf{WX}_M)^{-1}\mathbf{X}_M'
\end{aligned}$$

since $\mathbf{WMM}' = diag[J_i] \cdot diag[J_i^{-1}] = \mathbf{D}_I$ and

$$tr(\mathbf{WX}_M(\mathbf{X}_M'\mathbf{WX}_M)^{-1}\mathbf{X}_M'\mathbf{W}) = k_1$$

where

$$k_1 = (\sum_{i=1}^{I} J_i\bar{X}_{i.} \sum_{i=1}^{I} J_i^2 - 2\sum_{i=1}^{I} J_i\bar{X}_{i.} \sum_{i=1}^{I} J_i^2\bar{X}_{i.} + \sum_{i=1}^{I} J_i \sum_{i=1}^{I} J_i^2\bar{X}_{i.}^2)/(J_. S_{xxa}).$$

It follows that

$$\begin{aligned} E(R_{WB}) &= E(\underline{Y}_M'\mathbf{A}_W\underline{Y}_M) \\ &= tr(\mathbf{A}_W\mathbf{V}_M) + \underline{\alpha}'\mathbf{X}_M'\mathbf{A}_W\mathbf{X}_M\underline{\alpha} \\ &= \sigma_A^2(tr(\mathbf{W}) - k_1) + \sigma_E^2 tr(\mathbf{D}_I - \mathbf{WX}_M(\mathbf{X}_M'\mathbf{WX}_M)^{-1}\mathbf{X}_M') \\ &= (J_. - k_1)\sigma_A^2 + (I-2)\sigma_E^2 \end{aligned}$$

since $\mathbf{A}_W\mathbf{X}_M = \mathbf{0}$. The distribution of R_{WB} is determined by writing

$$R_{WB}/\sigma_E^2 = \underline{Y}_M'(\mathbf{A}_W/\sigma_E^2)\underline{Y}_M$$

and noting

$$\begin{aligned} (\mathbf{A}_W/\sigma_E^2)\mathbf{V}_M &= (\sigma_A^2/\sigma_E^2)\mathbf{A}_W + \mathbf{A}_W\mathbf{MM}' \\ &= (\sigma_A^2/\sigma_E^2)\mathbf{A}_W + \mathbf{D}_I - \mathbf{WX}_M(\mathbf{X}_M'\mathbf{WX}_M)^{-1}\mathbf{X}_M' \end{aligned}$$

since $\mathbf{WMM}' = \mathbf{D}_I$. Note that $\mathbf{D}_I - \mathbf{WX}_M(\mathbf{X}_M'\mathbf{WX}_M)^{-1}\mathbf{X}_M'$ is idempotent. It follows that R_{WB}/σ_E^2 is a chi-squared random variable with $I-2$ degrees of freedom if $\sigma_A^2 = 0$.

Theorem 3.

If $\sigma_E^2 = 0$, then R_{UB}/σ_A^2 is a chi-squared random variable with $I-2$ degrees of freedom.

Proof.

Notice that

$$\begin{aligned}\mathbf{A}_U\mathbf{V}_M =&(\mathbf{D}_I - \mathbf{X}_M(\mathbf{X}_M'\mathbf{X}_M)^{-1}\mathbf{X}_M')(\sigma_A^2\mathbf{D}_I + \sigma_E^2\mathbf{MM}')\\ =&\sigma_A^2(\mathbf{D}_I - \mathbf{X}_M(\mathbf{X}_M'\mathbf{X}_M)^{-1}\mathbf{X}_M')\\ &+\sigma_E^2(\mathbf{MM}' - \mathbf{X}_M(\mathbf{X}_M'\mathbf{X}_M)^{-1}\mathbf{X}_M'\mathbf{MM}'),\end{aligned}$$

$$tr(\mathbf{MM}') = \sum_{i=1}^{I}(1/J_i), \qquad \text{and}$$

$$tr(\mathbf{X}_M(\mathbf{X}_M'\mathbf{X}_M)^{-1}\mathbf{X}_M'\mathbf{MM}') = k_2$$

where

$$k_2 = \frac{\sum_{i=1}^{I}\sum_{k=1}^{I}(\bar{X}_{i.} - \bar{X}_{k.})^2/J_k}{I\cdot S_{uxxa}}$$

and $\mathbf{A}_U\mathbf{X}_M = \mathbf{0}$. It follows that

$$\begin{aligned}E(R_{UB}) &= E(\underline{Y}_M'\mathbf{A}_U\underline{Y}_M)\\ &= tr(\mathbf{A}_U\mathbf{V}_M) + \underline{\alpha}'\mathbf{X}_M'\mathbf{A}_U\mathbf{X}_M\underline{\alpha}\\ &= \sigma_A^2 tr(\mathbf{D}_I - \mathbf{X}_M) + \sigma_E^2(\sum_{i=1}^{I}(1/J_i) - k_2)\\ &= (I-2)\sigma_A^2 + (\sum_{i=1}^{I}(1/J_i) - k_2)\sigma_E^2.\end{aligned}$$

The distribution of R_{UB} is determined by writing

$$R_{UB}/\sigma_A^2 = \underline{Y}_M'(\mathbf{A}_U/\sigma_A^2)\underline{Y}_M$$

and noting $(\mathbf{A}_U/\sigma_A^2)\mathbf{V}_M = \mathbf{A}_U + (\sigma_E^2/\sigma_A^2)\mathbf{A}_U\mathbf{MM}'$. Note that $\mathbf{A}_U$ is idempotent. Thus R_{UB}/σ_A^2 is a chi-squared random variable with $I-2$ degrees of freedom if $\sigma_E^2 = 0$.

Theorem 4.

R_{WB}/σ_E^2 and R_T/σ_E^2 are independent and R_{UB}/σ_A^2 and R_T/σ_E^2 are independent.

Proof.

Notice that

$$\begin{aligned}\mathbf{M}'\mathbf{A}_W\mathbf{M}(\sigma_A^2\mathbf{ZZ}' + \sigma_E^2\mathbf{D}_{J.})\mathbf{T} &= \sigma_A^2\mathbf{M}'\mathbf{A}_W\mathbf{MZZ}'\mathbf{T} + \sigma_E^2\mathbf{M}'\mathbf{A}_W\mathbf{MT}\\ &= \mathbf{0}\end{aligned}$$

using $\mathbf{MT} = \mathbf{MM}'\mathbf{Z}'\mathbf{T} = \mathbf{0}$ since $\mathbf{M} = \mathbf{MM}'\mathbf{Z}'$ and $\mathbf{Z}'\mathbf{T} = \mathbf{0}$. Accordinly R_{WB}/σ_E^2 and R_T/σ_E^2 are independent. Note that

$$\begin{aligned}\mathbf{M}'\mathbf{A}_U\mathbf{M}(\sigma_A^2\mathbf{ZZ}' + \sigma_E^2\mathbf{D}_{J.})\mathbf{T} &= \sigma_A^2\mathbf{M}'\mathbf{A}_U\mathbf{MZZ}'\mathbf{T} + \sigma_E^2\mathbf{M}'\mathbf{A}_U\mathbf{MT}\\ &= \mathbf{0}\end{aligned}$$

Thus R_{UB}/σ_A^2 and R_T/σ_E^2 are independent.

Olsen et al.(1976), Thomas and Hultquist(1978), and El-Bassiouni(1994) used spectral decomposition method to obtain following statistics. They proposed a statistic $SSM = \mathbf{U}'\mathbf{U}$ which is asymptotically chi-squared distributed. In particular,

$$\frac{\mathbf{U}'\mathbf{U}}{(\sigma_A^2 + \sigma_E^2/\lambda_H)} \to \chi^2_{(I-1)} \quad \text{as} \quad \sigma_E^2 \to 0$$

where $\mathbf{U} = \mathbf{C}^{+}\mathbf{Z}'(\mathbf{D}_{J.} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')\mathbf{Z}$, λ_H is the harmonic mean of positive eigenvalues, λ_i, of C,

$$\lambda_H = \frac{\sum_{i=1}^{I} r_i}{(\sum_{i=1}^{I} r_i/\lambda_i)},$$

and r_i is the multiplicity of positive eigenvalue λ_i. Thus,

$$E(\mathbf{U}'\mathbf{U}) \cong (I-1)(\sigma_A^2 + \frac{\sigma_E^2}{\lambda_H}).$$

It was also shown that $\mathbf{U}'\mathbf{U}/(\sigma_A^2 + \sigma_E^2/\lambda_H)$ and R_T/σ_E^2 are independent.

If the covariate values within each group are same, this proposed statistics becomes the error sum of squares associated with unweighted between regression coefficient and the total regresion coefficient estimator reduces to the weighted between regression coefficient estimator. That is, if $X_{ij} = X_i$ for all j, then $SSM = R_{UB}$ and $\hat{\beta}_L = \hat{\beta}_{WB}$. If group means of the covariate values are all same, i.e., $\bar{X}_{i.} = \bar{X}_{..} = \bar{\bar{X}}_{..}$ for all i, then $\mathbf{X}_M$ is linearly dependent and $\hat{\beta}_{WB}$ and $\hat{\beta}_{UB}$ are not defined.

4. CONFIDENCE INTERVAL ON σ_A^2

The expected mean squares are sumarized using the distributional property of error sums of squares.

$$E(S_{WB}^2) = c_1\sigma_A^2 + \quad \sigma_E^2 = \theta_{WB}, \qquad (4.1a)$$

$$E(S_{UB}^2) = \quad \sigma_A^2 + c_2\sigma_E^2 = \theta_{UB}, \quad \text{and} \qquad (4.1b)$$

$$E(S_T^2) = \qquad \sigma_E^2 = \theta_T \qquad (4.1c)$$

where

$$S_{WB}^2 = \frac{R_{WB}}{(I-2)},$$

$$S_{UB}^2 = \frac{R_{UB}}{(I-2)},$$

$$S_T^2 = \frac{R_T}{(J_. - I - 1)},$$

$$c_1 = \frac{(J_. - k_1)}{(I-2)}, \qquad \text{and}$$

$$c_2 = \frac{(\sum_i(1/J_.) - k_2)}{(I-2)}.$$

The mean square errors, S_{WB}^2 and S_{UB}^2, are independent of S_T^2 and they are exactly chi-squared distributed depending on cases where $\sigma_A^2 = 0$ and $\sigma_E^2 = 0$, respectively.

In the case where $\sigma_A^2 \to 0$, S_{WB}^2 and S_T^2 should be used to construct confidence intervals on σ_A^2. The variance component σ_A^2 can be represented by functions of expected mean squares in (4.1a) and (4.1c),

$$\sigma_A^2 = (\theta_{WB} - \theta_T)/c_1$$

An approximate confidence interval on σ_A^2 can be constructed using the method of Ting et al.(1990). In particular, the $1-2\alpha$ two-sided confidence interval for this form of σ_A^2 is

$$\frac{1}{c_1}[(S_{WB}^2 - S_T^2) - (G_1^2 S_{WB}^4 + G_2^2 S_T^4 + G_{12} S_{WB}^2 S_T^2)^{\frac{1}{2}};$$

$$(S_{WB}^2 - S_T^2) + (H_1^2 S_{WB}^4 + H_2^2 S_T^4 + H_{12} S_{WB}^2 S_T^2)^{\frac{1}{2}}] \qquad (4.2)$$

where

$$F_1 = F_{(\alpha:I-2,J.-I-1)},$$

$$F_2 = F_{(1-\alpha:I-2,J.-I-1)},$$

$$G_1 = 1 - \frac{1}{F_{(\alpha:I-2,\infty)}},$$

$$G_2 = \frac{1}{F_{(1-\alpha:J.-I-1,\infty)}} - 1,$$

$$G_{12} = \frac{[(F_1-1)^2 - G_1^2 F_1^2 - G_2^2]}{F_1},$$

$$H_1 = \frac{1}{F_{(1-\alpha:I-2,\infty)}} - 1,$$

$$H_2 = 1 - \frac{1}{F_{(\alpha:J.-I-1,\infty)}},$$

$$H_{12} = \frac{[(1-F_2)^2 - H_1^2 F_2^2 - H_2^2]}{F_2},$$

and $F_{(\delta:n_1,n_2)}$ is the F-value for n_1 and n_2 degrees of freedom with δ area to the right. Since $\sigma_A^2 > 0$, any negative bound is defined to be zero. Interval (4.2) is referred to as TINGW method.

Another approach is adapting generalized p-values method proposed by Khuri et al.(1998) to construct an approximate confidence interval on σ_A^2. It was shown in Chapter 3 that $(I-1)S_M^2/(\sigma_A^2+\sigma_E^2/\lambda_H)$ is chi-squared distributed with $(I-1)$ degrees of freedom as σ_E^2 approaches zero,

$$\frac{(J_. - I - 1)S_T^2}{\sigma_E^2} \sim \chi^2_{(J_.-I-1)},$$

and they are independent where $S_M^2 = SSM/(I-1)$. Thus, using this property, the estimators of σ_E^2 are obtained by $(J_. - I - 1)s_T^2/U_1$ where s_T^2 is an observed value of S_T^2 and U_1 has a chi-squared distribution with $(J_. - I - 1)$ degrees of freedom. The estimators of σ_{AE}^2 are obtained by $(I-1)s_M^2/U_2$ where $\sigma_{AE}^2 = \sigma_A^2 + \sigma_E^2/\lambda_H$, s_M^2 is an observed value of S_M^2, and U_2 has a chi-squared distribution with $(I-1)$ degrees of freedom. Thus, a generalized pivotal quantity σ_A^2 can by represented as

$$\sigma_A^2 = \frac{(I-1)s_M^2}{U_2} - \frac{1}{\lambda_H} \cdot \frac{(J_. - I - 1)s_T^2}{U_1}.$$

Accordingly, an approximate $1-2\alpha$ two-sided confidence interval for this form of σ_A^2 is

$$[C_\alpha \quad ; \quad C_{1-\alpha}] \tag{4.3}$$

where C_α is the αth percentile of the distribution constructed by the generalized pivotal quantity. Interval (4.3) is referred to as GPQ method.

When σ_E^2 approaches zero, S_{UB}^2 and S_T^2 can be used and σ_A^2 is represented

$$\sigma_A^2 = \theta_{UB} - c_2\theta_T$$

from (4.1b) and (4.1c). The Ting et al. $1-2\alpha$ two-sided confidence interval for this form of σ_A^2 is

$$\begin{aligned}&[S_{UB}^2 - c_2S_T^2 - (G_1^2S_{UB}^4 + c_2^2G_2^2S_T^4 + c_2G_{12}S_{UB}^2S_T^2)^{\frac{1}{2}};\\ &S_{UB}^2 - c_2S_T^2 + (H_1^2S_{UB}^4 + c_2^2H_2^2S_T^4 + c_2H_{12}S_{UB}^2S_T^2)^{\frac{1}{2}}]\end{aligned} \tag{4.4}$$

Interval (4.4) is referred to as TINGU method.

If $I = 3$, then $c_2 = 1/c_1$ and $c_2A_W = A_U$. Thus $S_{WB}^2/c_1 = c_2S_{WB}^2 = S_{UB}^2$ and TINGW and TINGU methods are same.

5. SIMULATION AND EXAMPLE

5.1 Simulation Study

The methods proposed in Chapter 4 are now compared using simulation study. The criteria for analyzing the performance of the methods are ; 1) their ability to maintain stated confidence coefficient, and 2) the average length of two-sided confidence intervals. Although shorter average interval lengths are preferable, it is necessary that the methods first maintain the stated confidence coefficient. Four unbalanced patterns were selected for simulation study and are shown in Table 5.1

TABLE 5.1 Unbalanced Patterns Used in Simulation

Pattern	I	J$_i$
1	3	3 5 10
2	5	1 3 5 7 10
3	7	1 2 4 6 8 10
4	10	1 1 1 5 5 5 5 10 10 10

Let $\rho = \sigma_A^2/(\sigma_A^2 + \sigma_E^2)$. Without loss of generality $\sigma_A^2 = 1 - \sigma_E^2$ so that $\rho = \sigma_A^2$ and $1 - \rho = \sigma_E^2$. A_i and E_{ij} are independently generated from normal populations with zero means and variance ρ and $1-\rho$, respectively, using RANNOR routines of SAS. Values of μ and β are respectively varied from -3 to 3 in increments of 1 so that 49 different combinations of μ and β are used. Any fixed values of X_{ij}'s are given. Then Y_{ij}'s are calculated according to model (2.1) and R_{WB}, R_T, SSM, and R_{UB} are computed as shown in Chapter 3. Simulated values for S_{WB}^2, S_T^2, S_M^2, and S_{UB}^2 are substituted into appropriate formula and the intervals are computed. Values of ρ are varied from 0.001 to 0.999 in increments of 0.1. Each value of ρ is simulated 2000 times for each pattern. Two-sided intervals are computed based on equal tailed F-values. Confidence coefficients are determined by counting the number of the intervals that contain σ_A^2. Using the normal approximation to the binomial, if the true coefficient is 0.90, there is less than a 2.5% chance that an estimated confidence coefficient based on 2000 replications will be less than 0.8866. The average lengths of the two-sided confidence intervals are also calculated.

Table 5.2 and 5.3 present the results of the simulation for stated 90% confidence intervals on σ_A^2. The numbers in the body of Table 5.2 and 5.3 respectively report range of simulated confidence coefficients and average interval lengths and minimum and maximum values for the range as ρ ranges from 0.001 to 0.999. Different combinations of μ and β do not change the trend of simultion results and the change of minimum values of stated confidence coefficients is at most 0.012.

TABLE 5.2 90% Range of Simulated Confidence Coefficients

Pattern	1			2		
ρ	TINGW	GPQ	TINGU	TINGW	GPQ	TINGU
0.001	0.9005	0.9035	0.9005	0.8905	0.894	0.87
0.1	0.908	0.9045	0.908	0.8925	0.894	0.883
0.2	0.898	0.9085	0.898	0.8875	0.8955	0.884
0.3	0.893	0.904	0.893	0.89	0.896	0.898
0.4	0.9095	0.905	0.9095	0.8845	0.8985	0.907
0.5	0.9015	0.905	0.9015	0.898	0.8985	0.905
0.6	0.897	0.905	0.897	0.88	0.897	0.9045
0.7	0.899	0.9065	0.899	0.8655	0.896	0.8905
0.8	0.898	0.907	0.898	0.87	0.897	0.893
0.9	0.8975	0.905	0.8975	0.8635	0.896	0.8935
0.999	0.902	0.905	0.902	0.872	0.896	0.8925
MAX	0.9095	0.9085	0.9095	0.898	0.8985	0.907
MIN	0.893	0.9035	0.893	0.8635	0.894	0.87
Pattern	3			4		
0.001	0.9	0.908	0.854	0.897	0.899	0.8135
0.1	0.8965	0.9095	0.866	0.901	0.8995	0.853
0.2	0.8955	0.907	0.888	0.8845	0.8985	0.868
0.3	0.8865	0.906	0.8955	0.8885	0.9015	0.8765
0.4	0.863	0.9065	0.883	0.869	0.905	0.8915
0.5	0.871	0.904	0.884	0.862	0.902	0.884
0.6	0.865	0.905	0.882	0.8715	0.903	0.891
0.7	0.8645	0.9055	0.895	0.862	0.9025	0.913
0.8	0.8735	0.901	0.907	0.857	0.902	0.898
0.9	0.858	0.9005	0.895	0.841	0.902	0.899
0.999	0.8685	0.8995	0.9045	0.856	0.9	0.904
MAX	0.9	0.9095	0.907	0.901	0.905	0.913
MIN	0.858	0.8995	0.854	0.841	0.8985	0.8135

TABLE 5.3 90% Range of Average Interval Lengths

Pattern	1			2		
ρ	TINGW	GPQ	TINGU	TINGW	GPQ	TINGU
0.001	44.670385	4.7037761	44.670385	1.6963909	1.7565652	2.4211901
0.1	59.306305	6.203999	59.306305	2.3232264	2.1452094	2.9775824
0.2	88.336569	7.7152441	88.336569	2.9723129	2.5340456	3.5968303
0.3	107.57395	9.2208541	107.57395	3.7117547	2.9165519	4.2278956
0.4	120.93906	10.721291	120.93906	4.1990992	3.2901384	4.6446833
0.5	142.92995	12.216601	142.92995	4.8447865	3.6554658	5.2186535
0.6	168.20474	13.706938	168.20474	5.6089767	4.0127826	5.7815406
0.7	185.53583	15.192167	185.53583	6.0088214	4.3633191	6.2783312
0.8	216.90426	16.673825	216.90426	7.0531375	4.7080909	7.0924004
0.9	245.96681	18.151819	245.96681	7.5738493	5.0491171	7.6091444
0.999	246.10563	19.612088	246.10563	8.4534132	5.3857702	8.5034603
MAX	246.10563	19.612088	246.10563	8.4534132	5.3857702	8.5034603
MIN	44.670385	4.7037761	44.670385	1.6963909	1.7565652	2.4211901
Pattern	3			4		
0.001	0.8841862	1.2104748	1.4681376	0.3847124	0.7396307	0.7652631
0.1	1.3228924	1.5056313	1.8549788	0.6214983	0.9254198	0.9536777
0.2	1.848266	1.8000139	2.327806	0.8308489	1.1062396	1.1387063
0.3	2.2514865	2.0867147	2.6258684	1.0543185	1.2748048	1.3411049
0.4	2.7162211	2.3635822	3.127743	1.2549925	1.4296433	1.5096349
0.5	3.1438896	2.6303032	3.4974074	1.4559445	1.5735124	1.6897315
0.6	3.568139	2.8878889	3.7790143	1.5924883	1.7080275	1.7819612
0.7	4.0057759	3.1385151	4.1079702	1.806329	1.8377487	1.9402982
0.8	4.5275753	3.3852496	4.5825573	1.9948288	1.9655106	2.0982791
0.9	4.7874478	3.6297314	4.8973924	2.2045574	2.0937172	2.2359168
0.999	5.1810588	3.8715458	5.1928903	2.4126914	2.2214677	2.3882471
MAX	5.1810588	3.8715458	5.1928903	2.4126914	2.2214677	2.3882471
MIN	0.8841862	1.2104748	1.4681374	0.3847124	0.7396307	0.7652631

Simulation results are consistent with our study since TINGW method improves as ρ approaches zero while TINGU method performs well as ρ is closed to one across all values of ρ for patterns 1. However, only GPQ method keeps the stated confidence coefficients for all ρ values of four patterns. The average interval lengths of three methods generate wider intervals as ρ increases for all four patterns. For smaller ρ value, say $\rho \leq 0.1$, in pattern 3 and 4, TINGW method has shortest interval lengths. For other values of ρ in four patterns, GPQ method has shortest interval length.

5.2 Numerical Example

The results of the simulation study are applied to a data set. Scheffe (1959, p216) wrote a data set of 94 observations for seven types of starch film and the data set was reproduced with permission of the author and publisher from Industrial Statistics by Freeman (1942). The dependent variable in the data set is the breaking strength in grams and the independent variable is the thickness in 10^{-4} inch from tests of starch film. The data set was constructed by selecting three types of starch, Potato, Canna, and Wheat. Three observations are selected from Potato, five from Canna, and ten from Wheat. This data set has the form of pattern 1 in Table 5.1 and is used to fit the simple linear regression model of the breaking strength on the thickness of starch film assuming an unbalanced nested error structure.

The selected data set is listed in Table 5.4 In order to apply the methods proposed in Chapter 4 to the data set a SAS code was programmed and 90% confidence intervals on σ_A^2 were calculated. The resulting intervals were given in Table 5.5 From SAS output the estimators $\hat{\sigma}_A^2$ and $\hat{\sigma}_E^2$ are computed as 8479.97 and 3063.89, respectively. Therefore, the estimate of the ratio of variance in primary unit to total variance $\hat{\rho}$ is 0.7345. GPQ should be used because it keeps the stated confidence level and generates the shortest interval length among three methods in patern 1 of Tables 5.2 and 5.3. The calculated interval lengths in Table 5.5 are consistent with the results in Table 5.3.

TABLE 5.4 The Data Set Used for The Example

Type	Potato		Canna		Wheat	
Obs.	X	Y	X	Y	X	Y
1	13.0	983.3	7.7	791.7	5.0	263.7
2	13.3	958.8	6.3	610.0	3.5	130.8
3	10.7	747.8	8.6	710.0	4.7	382.9
4			11.8	940.7	4.3	302.5
5			12.4	990.0	3.8	213.3
6					3.0	132.1
7					4.2	292.0
8					4.5	315.5
9					4.3	262.4
10					4.1	314.4

TABLE 5.5 90% Confidence Intervals on σ_A^2

Methods	Lower bound	Upper bound	Length
TINGW(TINGU)	2702.9	3359262.4	3356559.5
GPQ	915.6	216887.0	215971.4

6. CONCLUSIONS

Three approximate confidence intervals on the variance component of the primary level in a simple linear regression model with unbalanced nested error structure were proposed. The simulation study was conducted to compare the proposed intervals on the selected unbalanced patterns in Table 5.1 From Tables 5.2 and 5.3 if $\rho < 0.1$ in pattern 2 and ρ □ 0.1 in patterns 3 and 4, TINGW method is recommended because it keeps the stated confidence coefficients as well as shortest average interval lengths. For other values of ρ in four patterns GPQ method is recommended.

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