Duality for a Class of Nondifferentiable Multiobjective Variational Problems

(미분불가능한 다목적 변분문제에 대한 쌍대성)

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A thesis submitted in partial fulfillment of the requirement for the degree of

Master of Education

Graduate School of Education
Pukyong National University

August 2005

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A dissertation

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August 31, 2005

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미분불가능한 다목적 변분문제에 대한 쌍대성

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요 약

본 논문의 목적은 부등식 및 등식제약조건을 갖는 미분불가능한 다목적 최적화 문제에 대한 Wolfe형 쌍대정리와 Mond-Weir형 쌍대정리를 정립하는 것이다.

본 논문에서는 다목적 변분문제에 관해서 벡터함수에 대한 볼록함수를 정의하고 미분불가능한 다목적 변분문제에 대한 효율해를 정의하였다. 또한, Chandra, Craven과 Husain의 미분불가능한 변분문제에 관한 필요 최적정리를 이용하여 부등식 및 등식제약조건을 갖는 미분불가능한 다목적 변분문제에 대하여 Wolfe, Mond-Weir 형태의 쌍대문제를 만들어 원문제와 쌍대문제 사이에서 약쌍대성과 강쌍대성이 성립함을 보였다. 마지막으로 본 논문에서 다룬 변분문제들과 연관되어 있는 몇가지 변분문제들을 소개하였다.

1 Introduction

Duality for multiobjective variational problems has been of much interest in recent years, and contributions have been made to its development. Their optimum, i.e., proper efficiency, efficiency and weak efficiency etc., are the concept of solutions that appears to be the natural extension of the optimization to a single objective to one of multiobjectives. Hanson ([5]) extended the duality results of mathematical programming to a class of functions subsequently called invex. Since that time, it has been shown [12, 14] that many results in mathematical programming previously established for convex functions actually hold for the wider class of invex functions. Mond et al. ([12]) extended the concept of invexity to the continuous case and used it to generalize earlier duality results for a class of variational problems.

Mond and Smart ([13]) extended the duality theorems for a class of static non-differentiable problems with Wolfe type and Mond-Weir type duals, and further extended these for the continuous analogies. Mishra and Mukherjee ([10]) extended the work of Mond et al. ([12]) for multiobjective variational problems which in particular extended and earlier work of Bector and Husain ([5]) for invex functions. Jeyakumar and Mond ([6]) introduced a wider class than that of invex functions subsequently known as V-invex functions, which preserves the sufficient optimality and duality results in the scalar case, and avoids the major difficulty of verifying that the inequality holds for the same function. Mukherjee and Mishra ([10]) extended the work of [6] to variational problems with the concept of weak minima.

Recently, Kim and Kim ([7]) established duality relations for nondifferentiable multiobjective variational problems with inequality constraints assumptions by employ a characterization of efficient solution due to Chankong and Haimes ([3]). The purpose of this thesis, Wolfe and Mond-Weir type duals for a class of nondifferentiable multiobjective variational problems with equality constraints are formulated. Under convexity assumptions on the objective and constraint functions involved, weak and strong duality theorems are proved to related properly efficient solutions for primal and dual problems.

This thesis is organized as follows: Section 2 gives notations, definitions of convexity and necessary optimality conditions for single objective variational problems. In Section 3, we establish duality theorem of Wolfe type under the conditions of convex functions. Using the concept of proper efficiency. In Section 4, in the same way of Section 3, we establish duality theorem of Mond-Weir type for a nondifferentiable multiobjective variational problem. In Section 5, some problems concerning to these variational problems are introduced.

2 Preliminaries and Notations

The following convention for vectors x and y in R^n will be used:

$$x > y \Longleftrightarrow x_i > y_i \text{ for all } i = 1, \cdots, n,$$

$$x \geqq y \Longleftrightarrow x_i \geqq y_i \text{ for all } i = 1, \cdots, n,$$

$$x \geq y \Longleftrightarrow x_i \geqq y_i \text{ for all } i = 1, \cdots, n, \quad \text{but } x \neq y,$$

$$x \not\geq y \text{ is the negation of } x \geq y.$$

Throughout this paper, we will use the following notations.

Let $I = [a,b]$ be a real interval, let $f := (f^1, \cdots, f^p) : I \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^p$, $g := (g^1, \cdots, g^m) : I \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $h := (h^1, \cdots, h^q) : I \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^q$ be continuously differentiable functions. In order to consider $f(t, x(t), \dot{x}(t))$, where $x : I \to \mathbb{R}^n$ is differentiable with derivative $\dot{x}$, denote the partial derivatives of f by

$$f_x^i = \left[\frac{\partial f^i}{\partial x_1}, \cdots, \frac{\partial f^i}{\partial x_n}\right], \quad f_{\dot{x}}^i = \left[\frac{\partial f^i}{\partial \dot{x}_1}, \cdots, \frac{\partial f^i}{\partial \dot{x}_n}\right], \quad i = 1, \cdots, p.$$

Let $C(I, \mathbb{R}^m)$ denote the space of continuous functions $\phi : I \longrightarrow \mathbb{R}^m$, with the uniform norm; $C_+(I, \mathbb{R}^m)$ is the cone of nonnegative functions in $C(I, \mathbb{R}^m)$. Denote by X the space of piecewise smooth functions $x : I \longrightarrow \mathbb{R}^n$, with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differentiation operator D is given by

$$u = Dx \iff x(t) = \alpha + \int_a^t u(s)ds,$$

where α is a given boundary value: thus $D = d/dt$ except at discontinuities. For each $t \in I$, let $B_i(t)$ be a positive semidefinite $n \times n$ matrix with $B_i(\cdot)$ continuous on I, $i = 1, 2, \cdots, p$ and the symbol T denotes the transposition.

Our problem is the multiobjective variational problem (VP) defined as follows;

$$\text{(VP)} \quad \text{Minimize} \quad \left(\int_{t_0}^{t_1} \left[f^1(t, x(t), \dot{x}(t)) + (x(t)^T B_1(t) x(t))^{\frac{1}{2}}\right] dt, \cdots, \int_{t_0}^{t_1} \left[f^p(t, x(t), \dot{x}(t)) + (x(t)^T B_p(t) x(t))^{\frac{1}{2}}\right] dt\right)$$

$$\text{subject to} \quad x(t_0) = x_0, \ x(t_1) = x_1, \tag{1}$$

$$g(t, x(t), \dot{x}(t)) \geqq 0, \ h(t, x(t), \dot{x}(t)) = 0, t \in I. \tag{2}$$

Let X_0 be the set of feasible solutions for (VP), that is,

$$X_0 := \{x \in X \mid x(t_0) = x_0, \ x(t_1) = x_1, \ g(t, x(t), \dot{x}(t)) \geqq 0, \ h(t, x(t), \dot{x}(t)) = 0, \ \forall t \in I\}.$$

By using the generalized Schwarz inequality, we derive the following lemma in order to prove the weak duality theorems for multiobjective variational problem (VP).

Lemma 2.1 *Let $A(t)$ be an $n \times n$ positive semidefinite (symmetric) matrix, with $A(\cdot)$ continuous on I, and $w(t)^T A(t) w(t) \leqq 1$. Then*

$$\int_{t_0}^{t_1} (x(t)^T A(t) x(t))^{\frac{1}{2}} dt \geqq \int_{t_0}^{t_1} x(t)^T A(t) w(t) dt.$$

Proof. With the generalized Schwarz inequality, we obtain

$$\int_{t_0}^{t_1} (x(t)^T A(t) x(t))^{\frac{1}{2}} (w(t)^T A(t) w(t))^{\frac{1}{2}} dt \geqq \int_{t_0}^{t_1} x(t)^T A(t) w(t) dt.$$

Since $w(t)^T A(t) w(t) \leqq 1$,

$$\int_{t_0}^{t_1} (x(t)^T A(t) x(t))^{\frac{1}{2}} dt \geqq \int_{t_0}^{t_1} x(t)^T A(t) w(t) dt.$$

□

Definition 2.1 *(1) A point $x^* \in X_0$ is said to be an efficient solution of (VP) if there exists no other feasible point $x \in X_0$ such that*

$$\int_{t_0}^{t_1} \left[f^i(t, x(t), \dot{x}(t)) + (x(t)^T B_i(t) x(t))^{\frac{1}{2}} \right] dt$$
$$\leqq \int_{t_0}^{t_1} \left[f^i(t, x^*(t), \dot{x}^*(t)) + x^*(t)^T B_i(t) x^*(t))^{\frac{1}{2}} dt \right] dt \quad \text{for all} \quad i \in \{1, \cdots, p\}$$

and

$$\int_{t_0}^{t_1} \left[f^j(t, x(t), \dot{x}(t)) + (x(t)^T B_j(t) x(t))^{\frac{1}{2}} \right] dt$$
$$< \int_{t_0}^{t_1} \left[f^j(t, x^*(t), \dot{x}^*(t)) + x^*(t)^T B_j(t) x^*(t))^{\frac{1}{2}} dt \right] dt \quad \text{for some} \quad j \in \{1, \cdots, p\}.$$

(2) A point $x^* \in X_0$ is said to be a properly efficient solution of (VP) if it is an efficient for (VP) and if there exists a scalar $M > 0$ such that for all $i \in \{1, \cdots, p\}$,

$$\begin{aligned}
&\textstyle\int_{t_0}^{t_1} \left[f^i(t, x^*(t), \dot{x}^*(t)) + (x^*(t)^T B_i(t) x^*(t))^{\frac{1}{2}}\right] dt \\
&\quad - \textstyle\int_{t_0}^{t_1} \left[f^i(t, x(t), \dot{x}(t)) + (x(t)^T B_i(t) x(t))^{\frac{1}{2}}\right] dt \\
&\quad \leqq M \left\{ \textstyle\int_{t_0}^{t_1} \left[f^j(t, x(t), \dot{x}(t)) + (x(t)^T B_j(t) x(t))^{\frac{1}{2}}\right] dt \right. \\
&\qquad \left. - \textstyle\int_{t_0}^{t_1} \left[f^j(t, x^*(t), \dot{x}^*(t)) + (x^*(t)^T B_j(t) x^*(t))^{\frac{1}{2}}\right] dt \right\},
\end{aligned}$$

for some j such that

$$\begin{aligned}
&\textstyle\int_{t_0}^{t_1} \left[f^j(t, x(t), \dot{x}(t)) + (x(t)^T B_j(t) x(t))^{\frac{1}{2}}\right] dt \\
&\quad > \textstyle\int_{t_0}^{t_1} \left[f^j(t, x^*(t), \dot{x}^*(t)) + (x^*(t)^T B_j(t) x^*(t))^{\frac{1}{2}}\right] dt
\end{aligned}$$

whenever $x \in X_0$ and

$$\begin{aligned}
&\textstyle\int_{t_0}^{t_1} \left[f^i(t, x(t), \dot{x}(t)) + (x(t)^T B_i(t) x(t))^{\frac{1}{2}}\right] dt \\
&\quad < \textstyle\int_{t_0}^{t_1} \left[f^i(t, x^*(t), \dot{x}^*(t)) + (x^*(t)^T B_i(t) x^*(t))^{\frac{1}{2}}\right] dt.
\end{aligned}$$

Let Λ^+ be

$$\Lambda^+ = \{\tau \in \mathbb{R}^p \mid \tau > 0,\ \tau^T e = 1,\ e = (1, 1, \cdots, 1)^T \in \mathbb{R}^p\}$$

then for $\tau \in \Lambda^+$, the related single objective problem

$$\begin{aligned}
(\mathrm{VP}_\tau) \quad \text{Minimize} \quad & \sum_{i=1}^{p} \tau_i \int_{t_0}^{t_1} \left[f^i(t, x(t), \dot{x}(t)) + (x(t)^T B_i(t) x(t))^{\frac{1}{2}}\right] dt \\
\text{subject to} \quad & x(t_0) = x_0,\ x(t_1) = x_1, \\
& g(t, x(t), \dot{x}(t)) \geqq 0,\ h(t, x(t), \dot{x}(t)) = 0, t \in I.
\end{aligned}$$

Problem (VP) and (VP_τ) are equivalent in the sense of Geoffrion's ([4]) Theorem 1 and 2, which are valid when $\mathbb{R}^n$ is replaced by some normed space of functions as the proofs of these theorems do not depend on the dimensionality of the space in which the feasible set of (VP) lies, where for each $t \in I,\ B(t) = 0$. For our variational problems the feasible set X_0 lies in the normed space $C(I, \mathbb{R}^n)$. For completeness we shall merely state these theorems characterizing proper vector minima of (VP) in terms of solutions of (VP$_\tau$).

Theorem 2.1 *Let $\tau \in \Lambda^+$ be any fixed. If x^* is optimal for (VP$_\tau$), then x^* is properly efficient for (VP).*

Proof. We can easily prove this theorem. □

Theorem 2.2 *Let $\tau \in \Lambda^+$ be any fixed. If x^* is a properly efficient for (VP), then x^* is optimal for (VP$_\tau$) for some $\tau \in \Lambda^+$.*

Proof. Assume that x^* is not optimal for (VP$_\tau$). Then there exists $\bar{x} \in X_0$ such that

$$\begin{aligned} &\sum_{i=1}^{p} \tau_i \int_{t_0}^{t_1} \left[f^i(t, \bar{x}(t), \dot{\bar{x}}(t)) + (\bar{x}(t)^T B_i(t) \bar{x}(t))^{\frac{1}{2}} \right] dt \\ &\quad < \sum_{i=1}^{p} \tau_i \int_{t_0}^{t_1} \left[f^i(t, x^*(t), \dot{x}^*(t)) + (x^*(t)^T B_i(t) x^*(t))^{\frac{1}{2}} \right] dt \end{aligned} \tag{3}$$

This implies that there exists a subset P' of $\{1, 2, \cdots, p\}$ such that $P' \neq \emptyset$ and for $i \in P'$

$$\begin{aligned} &\int_{t_0}^{t_1} \left[f^i(t, \bar{x}(t), \dot{\bar{x}}(t)) + (\bar{x}(t)^T B_i(t) \bar{x}(t))^{\frac{1}{2}} \right] dt \\ &\quad < \int_{t_0}^{t_1} \left[f^i(t, x^*(t), \dot{x}^*(t)) + (x^*(t)^T B_i(t) x^*(t))^{\frac{1}{2}} \right] dt \end{aligned}$$

and for $j \in \{1, 2, \cdots, p\} \setminus P'$

$$\int_{t_0}^{t_1} \left[f^j(t, \bar{x}(t), \dot{\bar{x}}(t)) + (\bar{x}(t)^T B_j(t) \bar{x}(t))^{\frac{1}{2}} \right] dt$$
$$> \int_{t_0}^{t_1} \left[f^j(t, x^*(t), \dot{x}^*(t)) + (x^*(t)^T B_j(t) x^*(t))^{\frac{1}{2}} \right] dt.$$

From (3), there exists $i \in \{1, 2, \cdots, p\}$ such that for all $j \in \{1, 2, \cdots, p\} \setminus P'$

$$\tau_j \left\{ \int_{t_0}^{t_1} \left[f^j(t, \bar{x}(t), \dot{\bar{x}}(t)) + (\bar{x}(t)^T B_j(t) \bar{x}(t))^{\frac{1}{2}} \right] dt - \int_{t_0}^{t_1} \left[f^j(t, x^*(t), \dot{x}^*(t)) + (x^*(t)^T B_j(t) x^*(t))^{\frac{1}{2}} \right] dt \right\}$$
$$< |P'| \tau_i \left\{ \int_{t_0}^{t_1} \left[f^i(t, x^*(t), \dot{x}^*(t)) + (x^*(t)^T B_i(t) x^*(t))^{\frac{1}{2}} \right] dt - \int_{t_0}^{t_1} \left[f^i(t, \bar{x}(t), \dot{\bar{x}}(t)) + (\bar{x}(t)^T B_i(t) \bar{x}(t))^{\frac{1}{2}} \right] dt \right\},$$

where $|P'|$ denotes the element number of P'.

Let $M = \tau_j / (|P'| \tau_i)$. For all $\tau \in \Lambda^+$, M may be sufficiently large, and

$$\int_{t_0}^{t_1} \left[f^i(t, x^*(t), \dot{x}^*(t)) + (x^*(t)^T B_i(t) x^*(t))^{\frac{1}{2}} \right] dt - \int_{t_0}^{t_1} \left[f^i(t, \bar{x}(t), \dot{\bar{x}}(t)) + (\bar{x}(t)^T B_i(t) \bar{x}(t))^{\frac{1}{2}} \right] dt$$
$$> M \left\{ \int_{t_0}^{t_1} \left[f^j(t, \bar{x}(t), \dot{\bar{x}}(t)) + (\bar{x}(t)^T B_j(t) \bar{x}(t))^{\frac{1}{2}} \right] dt - \int_{t_0}^{t_1} \left[f^j(t, x^*(t), \dot{x}^*(t)) + (x^*(t)^T B_j(t) x^*(t))^{\frac{1}{2}} \right] dt \right\},$$

which contradicts that x^* is properly efficient for (VP). □

Before presenting two distinct duals to (VP) we start the following Kuhn-Tucker type necessary optimality conditions for (VP_τ) and point out that they can be easily derived by invoking the results of [2].

Proposition 2.1 *If x is optimal for (VP_τ) and is normal, there exists a piecewise smooth $y : I \longrightarrow \mathbb{R}^m_+$, $\mu : I \rightarrow \mathbb{R}^q_+$ and $w : I \rightarrow \mathbb{R}^n$ satistying for all $t \in I$,*

$$\sum_{i=1}^{p} \tau_i \Big[f^i_x(t, x, \dot{x}) + w(t)^T B_i(t)\Big] - y(t)^T g_x(t, x(t), \dot{x}(t)) - \mu(t)^T h_x(t, x(t), \dot{x}(t))$$

$$= \frac{d}{dt}\Big[\sum_{i=1}^{p} \tau_i f^i_{\dot{x}}(t, x, \dot{x}) - y(t)^T g_{\dot{x}}(t, x(t), \dot{x}(t)) - \mu(t)^T h_{\dot{x}}(t, x(t), \dot{x}(t))\Big]$$

$$y(t)^T g(t, x(t), \dot{x}(t)) = 0,$$

$$w(t)^T B_i(t) w(t) \leqq 1,\ i = 1, \cdots, p$$

$$x(t)^T B_i(t) w(t) = (x(t)^T B_i(t) x(t))^{\frac{1}{2}},\ i = 1, \cdots, p.$$

3 Wolfe Type Duality

Now we formulate the following Wolfe type dual to (VP) is

(WD) Maximize

$$\Big(\int_{t_0}^{t_1} \Big[f^1(t, u(t), \dot{u}(t)) + u(t)^T B_1(t) w(t) - y(t)^T g(t, u(t), \dot{u}(t)) - \mu(t)^T h(t, u(t), \dot{u}(t))\Big] dt$$

$$, \cdots ,$$

$$\int_{t_0}^{t_1} \Big[f^p(t, u(t), \dot{u}(t)) + u(t)^T B_p(t) w(t) - y(t)^T g(t, u(t), \dot{u}(t)) - \mu(t)^T h(t, u(t), \dot{u}(t))\Big] dt\Big)$$

subject to

$$u(t_0) = x_0,\ u(t_1) = x_1, \quad ($$

$$\sum_{i=1}^{p} \tau_i \Big[f^i_x(t, u(t), \dot{u}(t)) + w(t)^T B_i(t)\Big] - y(t)^T g_x(t, u(t), \dot{u}(t)) - \mu(t)^T h_x(t, u(t), \dot{u}(t))$$

$$= \frac{d}{dt}\left[\sum_{i=1}^{p} \tau_i f^i_{\dot{x}}(t, u(t), \dot{u}(t)) - y(t)^T g_{\dot{x}}(t, u(t), \dot{u}(t)) - \mu(t)^T h_{\dot{x}}(t, u(t), \dot{u}(t))\right], \quad ($$

$$w(t)^T B(t) w(t) \leqq 1, \quad ($$

$$y(t) \geqq 0,\ y(t) \in \mathbb{R}^m_+,\ \mu(t) \in \mathbb{R}^n,\ w(t) \in \mathbb{R}^n,\ t \in I,\ \tau \in \Lambda^+.$$

Theorem 3.1 (Weak Duality) *Assume that x is feasible for problem (VP), and (u, y, μ, w) is feasible for problem (WD). Let $f(t,\cdot,\cdot)$, $-g(t,\cdot,\cdot)$ be convex and $h(t,\cdot,\cdot)$ be affine, for each $t \in I$. Then the following cannot hold:*

$$\int_{t_0}^{t_1} \left[f^i(t, x(t), \dot{x}(t)) + (x(t)^T B_i(t) x(t))^{\frac{1}{2}} \right] dt$$
$$\leq \int_{t_0}^{t_1} \left[f^i(t, u(t), \dot{u}(t)) + u(t)^T B_i(t) w(t) - y(t)^T g(t, u(t), \dot{u}(t)) - \mu(t)^T h(t, u(t), \dot{u}(t)) \right] dt \tag{7}$$

for all $i \in \{1, 2, \cdots, p\}$ and

$$\int_{t_0}^{t_1} \left[f^j(t, x(t), \dot{x}(t)) + (x(t)^T B_j(t) x(t))^{\frac{1}{2}} \right] dt$$
$$< \int_{t_0}^{t_1} \left[f^j(t, u(t), \dot{u}(t)) + u(t)^T B_j(t) w(t) - y(t)^T g(t, u(t), \dot{u}(t)) - \mu(t)^T h(t, u(t), \dot{u}(t)) \right] dt \tag{8}$$

for some $j \in \{1, 2, \cdots, p\}$.

Proof. Suppose that (7) and (8) hold. Then (7) and (8) imply that

$$\begin{aligned} &\sum_{i=1}^{p} \tau_i \int_{t_0}^{t_1} \left[f^i(t, x(t), \dot{x}(t)) + (x(t)^T B_i(t) x(t))^{\frac{1}{2}} \right] dt \\ &< \sum_{i=1}^{p} \tau_i \int_{t_0}^{t_1} \Big[f^i(t, u(t), \dot{u}(t)) + u(t)^T B_i(t) w(t) - y(t)^T g(t, u(t), \dot{u}(t)) \\ &\qquad - \mu(t)^T h(t, u(t), \dot{u}(t)) \Big] dt. \end{aligned} \tag{9}$$

Using Lemma 2.1, we have

$$\sum_{i=1}^{p}\tau_i\int_{t_0}^{t_1}\Big[f^i(t,x(t),\dot{x}(t))+(x(t)^TB_i(t)x(t))^{\frac{1}{2}}\Big]dt$$

$$-\sum_{i=1}^{p}\tau_i\int_{t_0}^{t_1}\Big[f^i(t,u(t),\dot{u}(t))+u(t)^TB_i(t)w(t)-y(t)^Tg(t,u(t),\dot{u}(t))$$

$$-\mu(t)^Th(t,u(t),\dot{u}(t))\Big]dt$$

$$\geqq\sum_{i=1}^{p}\tau_i\int_{t_0}^{t_1}\Big[f^i(t,x(t),\dot{x}(t))+x(t)^TB_i(t)w(t)\Big]dt$$

$$-\sum_{i=1}^{p}\tau_i\int_{t_0}^{t_1}\Big[f^i(t,u(t),\dot{u}(t))+u(t)^TB_i(t)w(t)-y(t)^Tg(t,u(t),\dot{u}(t))$$

$$-\mu(t)^Th(t,u(t),\dot{u}(t))\Big]dt.$$

Since f, $-g$ is convex and h is affine,

$$\sum_{i=1}^{p}\tau_i\int_{t_0}^{t_1}\Big[f^i(t,x(t),\dot{x}(t))+x(t)^TB_i(t)w(t)\Big]dt$$

$$-\sum_{i=1}^{p}\tau_i\int_{t_0}^{t_1}\Big[f^i(t,u(t),\dot{u}(t))+u(t)^TB_i(t)w(t)-y(t)^Tg(t,u(t),\dot{u}(t))$$

$$-\mu(t)^Th(t,u(t),\dot{u}(t))\Big]dt$$

$$\geqq\sum_{i=1}^{p}\tau_i\int_{t_0}^{t_1}\Big\{(x(t)-u(t))^T\Big[f_x^i(t,u(t),\dot{u}(t))+w(t)^TB_i(t)-y(t)^Tg_x(t,u(t),\dot{u}(t))$$

$$-\mu(t)^Th_x(t,u(t),\dot{u}(t))\Big]-(\dot{x}(t)-\dot{u}(t))^T\Big[f_{\dot{x}}^i(t,u(t),\dot{u}(t))$$

$$-y(t)^Tg_{\dot{x}}(t,u(t),\dot{u}(t))-\mu(t)^Th_{\dot{x}}(t,u(t),\dot{u}(t))\Big]\Big\}dt$$

$$+\int_{t_0}^{t_1}y(t)^Tg(t,x(t),\dot{x}(t))dt+\int_{t_0}^{t_1}\mu(t)^Th(t,x(t),\dot{x}(t))dt$$

$$= \int_{t_0}^{t_1} \Bigg\{ (x(t)-u(t))^T \frac{d}{dt}\Big[\sum_{i=1}^{p} \tau_i f^i_{\dot{x}}(t,u(t),\dot{u}(t)) - y(t)^T g_{\dot{x}}(t,u(t),\dot{u}(t))$$

$$-\mu(t)^T h_{\dot{x}}(t,u(t),\dot{u}(t))\Big] - (\dot{x}(t)-\dot{u}(t))^T \Big[\sum_{i=1}^{p} \tau_i f^i_{\dot{x}}(t,u(t),\dot{u}(t))$$

$$-y(t)^T g_{\dot{x}}(t,u(t),\dot{u}(t)) - \mu(t)^T h_{\dot{x}}(t,u(t),\dot{u}(t))\Big]\Bigg\} dt$$

$$+ \int_{t_0}^{t_1} y(t)^T g(t,x(t),\dot{x}(t))dt + \int_{t_0}^{t_1} \mu(t)^T h(t,x(t),\dot{x}(t))dt.$$

Using integration by parts and boundary conditions (1),(5), we obtain

$$\sum_{i=1}^{p} \tau_i \int_{t_0}^{t_1} \Big[f^i(t,x(t),\dot{x}(t)) + x(t)^T B_i(t) w(t)\Big] dt$$

$$-\sum_{i=1}^{p} \tau_i \int_{t_0}^{t_1} \Big[f^i(t,u(t),\dot{u}(t)) + u(t)^T B_i(t) w(t) - y(t)^T g(t,u(t),\dot{u}(t))$$

$$-\mu(t)^T h(t,u(t),\dot{u}(t))\Big] dt$$

$$\geqq 0.$$

Hence

$$\sum_{i=1}^{p} \tau_i \int_{t_0}^{t_1} \Big[f^i(t,x(t),\dot{x}(t)) + (x(t)^T B_i(t) x(t))^{\frac{1}{2}}\Big] dt$$

$$\geqq \sum_{i=1}^{p} \tau_i \int_{t_0}^{t_1} \Big[f^i(t,u(t),\dot{u}(t)) + u(t)^T B_i(t) w(t) - y(t)^T g(t,u(t),\dot{u}(t)) \tag{10}$$

$$-\mu(t)^T h(t,u(t),\dot{u}(t))\Big] dt,$$

which contradicts (9). Hence the result follows. □

Theorem 3.2 (Strong Duality) *Let $f(t,\cdot,\cdot)$, $-g(t,\cdot,\cdot)$ be convex and $h(t,\cdot,\cdot)$ be affine, for each $t \in I$. Let x^* be a normal and properly efficient solution for (VP). Then there exist $\tau \in \Lambda^+$ and a piecewise smooth $y^* : I \to \mathbb{R}^m$ such that*

(x^*, y^*, μ^*, w^*) *is a properly efficient solution of (WD) and*

$$\int_{t_0}^{t_1} \Big[f^i(t, x^*(t), \dot{x}^*(t)) + (x^*(t)^T B_i(t) x^*(t))^{\frac{1}{2}}\Big] dt$$

$$= \int_{t_0}^{t_1} \Big[f^i(t, x^*(t), \dot{x}^*(t)) + x^*(t)^T B_i(t) w^*(t) - y^*(t)^T g(t, x^*(t),\ \dot{x}^*(t))$$
$$- \mu^*(t)^T h(t, x^*(t), \dot{x}^*(t))\Big] dt, \quad i = 1, 2, \cdots, p.$$

Proof. Since x^* is a properly efficient solution of (VP), by Theorem 2.2, there exists $\tau \in \Lambda^+$ such that x^* is optimal for (P_τ). Now we will use this τ. Therefore, by Proposition 2.1, there exists a piecewise smooth $y^* : I \longrightarrow \mathbb{R}^m$ such that for $t \in I$

$$\sum_{i=1}^{p} \tau_i \Big[f_x(t, x^*(t), \dot{x}^*(t)) + w^*(t)^T B(t)\Big] - y^*(t)^T g_x(t, x^*(t), \dot{x}^*(t))$$
$$-\mu^*(t)^T h_x(t, x^*(t), \dot{x}^*(t))$$
$$= \frac{d}{dt}\Big[\sum_{i=1}^{p} \tau_i f^i_{\dot{x}}(t, x^*(t), \dot{x}^*(t)) + w^*(t)^T B(t) - y^*(t)^T g_{\dot{x}}(t, x^*(t), \dot{x}^*(t))$$
$$-\mu^*(t)^T h_{\dot{x}}(t, x^*(t), \dot{x}^*(t))\Big] \tag{11}$$

$$y^*(t)^T g(t, x^*(t), \dot{x}^*(t)) = 0, \tag{12}$$

$$w^*(t)^T B_i(t) w^*(t) \leqq 1,\ i = 1, \cdots, p, \tag{13}$$

$$x^*(t)^T B_i(t) w^*(t) = (x^*(t)^T B_i(t) x^*(t))^{\frac{1}{2}},\ i = 1, \cdots, p. \tag{14}$$

This implies that (x^*, y^*, μ^*, w^*) is a feasible solution of (WD). Suppose that

(x^*, y^*, μ^*, w^*) is not an efficient solution of (WD). Then there exists a feasible solution $(\bar{x}, \bar{y}, \bar{\mu}, \bar{w})$ of (WD) such that for all $i \in \{1, 2, \cdots, p\}$

$$\int_{t_0}^{t_1} \Big[f^i(t, \bar{x}(t), \dot{\bar{x}}(t)) + \bar{x}(t)^T B_i(t) \bar{w}(t) - \bar{y}(t)^T g(t, \bar{x}(t), \dot{\bar{x}}(t)) - \bar{\mu}(t)^T h(t, \bar{x}(t), \dot{\bar{x}}(t))\Big] dt$$

$$\geq \int_{t_0}^{t_1} \Big[f^i(t, x^*(t), \dot{x}^*(t)) + x^*(t)^T B_i(t) w^*(t) - y^*(t)^T g(t, x^*(t), \dot{x}^*(t))$$
$$- \mu^*(t)^T h(t, x^*(t), \dot{x}^*(t))\Big] dt$$

and for some $j \in \{1, 2, \cdots, p\}$

$$\int_{t_0}^{t_1} \Big[f^j(t, \bar{x}(t), \dot{\bar{x}}(t)) + \bar{x}(t)^T B_j(t)\bar{w}(t) - \bar{y}(t)^T g(t, \bar{x}(t), \dot{\bar{x}}(t)) - \bar{\mu}(t)^T h(t, \bar{x}(t), \dot{\bar{x}}(t))\Big] dt$$
$$> \int_{t_0}^{t_1} \Big[f^j(t, x^*(t), \dot{x}^*(t)) + x^*(t)^T B_j(t) w^*(t) - y^*(t)^T g(t, x^*(t), \dot{x}^*(t))$$
$$- \mu^*(t)^T h(t, x^*(t), \dot{x}^*(t))\Big] dt.$$

Using (12) and (13), for all $i \in \{1, 2, \cdots, p\}$

$$\int_{t_0}^{t_1} \Big[f^i(t, \bar{x}(t), \dot{\bar{x}}(t)) + \bar{x}(t)^T B_i(t)\bar{w}(t) - \bar{y}(t)^T g(t, \bar{x}(t), \dot{\bar{x}}(t)) - \bar{\mu}(t)^T h(t, \bar{x}(t), \dot{\bar{x}}(t))\Big] dt$$
$$\geq \int_{t_0}^{t_1} \Big[f^i(t, x^*(t), \dot{x}^*(t)) + (x^*(t)^T B_i(t) x^*(t))^{\frac{1}{2}}\Big] dt$$

and for some $j \in \{1, 2, \cdots, p\}$

$$\int_{t_0}^{t_1} \Big[f^j(t, \bar{x}(t), \dot{\bar{x}}(t)) + \bar{x}(t)^T B_j(t)\bar{w}(t) - \bar{y}(t)^T g(t, \bar{x}(t), \dot{\bar{x}}(t)) - \bar{\mu}(t)^T h(t, \bar{x}(t), \dot{\bar{x}}(t))\Big] dt$$
$$> \int_{t_0}^{t_1} \Big[f^j(t, x^*(t), \dot{x}^*(t)) + (x^*(t)^T B_j(t) x^*(t))^{\frac{1}{2}}\Big] dt,$$

which contradicts Theorem 3.1. Hence (x^*, y^*, μ^*, w^*) is efficient.

Now we assume that (x^*, y^*, μ^*, w^*) is not properly efficient for (WD); i.e., there exists a feasible solution $(\widetilde{x}, \widetilde{y}, \widetilde{\mu}, \widetilde{w})$ such that for some i and some $M > 0$

$$\int_{t_0}^{t_1} \Big[f^i(t, \widetilde{x}(t), \dot{\widetilde{x}}(t)) + \widetilde{x}(t)^T B_i(t)\widetilde{w}(t) - \widetilde{y}(t)^T g(t, \widetilde{x}(t), \dot{\widetilde{x}}(t)) - \widetilde{\mu}(t)^T h(t, \widetilde{x}(t), \dot{\widetilde{x}}(t))\Big] dt$$
$$> \int_{t_0}^{t_1} \Big[f^i(t, x^*(t), \dot{x}^*(t)) + x^*(t)^T B_i(t) w^*(t) - y^*(t)^T g(t, x^*(t), \dot{x}^*(t))$$
$$- \mu^*(t)^T h(t, x^*(t), \dot{x}^*(t)\Big] dt \tag{15}$$

and

$$\int_{t_0}^{t_1}\Big[f^i(t,\widetilde{x}(t),\dot{\widetilde{x}}(t))+\widetilde{x}(t)^TB_i(t)\widetilde{w}(t)-\widetilde{y}(t)^Tg(t,\widetilde{x}(t),\dot{\widetilde{x}}(t))-\widetilde{\mu}(t)^Th(t,\widetilde{x}(t),\dot{\widetilde{x}}(t))\Big]dt$$

$$-\int_{t_0}^{t_1}\Big[f^i(t,x^*(t),\dot{x}^*(t))+x^*(t)^TB_i(t)w^*(t)-y^*(t)^Tg(t,x^*(t),\dot{x}^*(t))$$

$$-\mu^*(t)^Th(t,x^*(t),\dot{x}^*(t))\Big]dt$$

$$>M\Bigg\{\int_{t_0}^{t_1}\Big[f^j(t,x^*(t),\dot{x}^*(t))+x^*(t)^TB_j(t)w^*(t)-y^*(t)^Tg(t,x^*(t),\dot{x}^*(t))$$

$$-\mu^*(t)^Th(t,x^*(t),\dot{x}^*(t)\Big]dt$$

$$-\int_{t_0}^{t_1}\Big[f^j(t,\widetilde{x}(t),\dot{\widetilde{x}}(t))+\widetilde{x}(t)^TB_j(t)\widetilde{w}(t)-\widetilde{y}(t)^Tg(t,\widetilde{x}(t),\dot{\widetilde{x}}(t))$$

$$-\widetilde{\mu}(t)^Th(t,\widetilde{x}(t),\dot{\widetilde{x}}(t))\Big]dt\Bigg\}$$

and $\forall j\in\{1,2,\cdots,p\}$ such that

$$\int_{t_0}^{t_1}\Big[f^j(t,\widetilde{x}(t),\dot{\widetilde{x}}(t))+\widetilde{x}(t)^TB_j(t)\widetilde{w}(t)-\widetilde{y}(t)^Tg(t,\widetilde{x}(t),\dot{\widetilde{x}}(t))-\widetilde{\mu}(t)^Th(t,\widetilde{x}(t),\dot{\widetilde{x}}(t))\Big]dt$$

$$<\int_{t_0}^{t_1}\Big[f^j(t,x^*(t),\dot{x}^*(t))+x^*(t)^TB_j(t)w^*(t)-y^*(t)^Tg(t,x^*(t),\dot{x}^*(t))$$

$$-\mu^*(t)^Th(t,x^*(t),\dot{x}^*(t))\Big]dt. \tag{16}$$

Since $y^*(t)^Tg(t,x^*(t),\dot{x}^*(t))=0$, $h(t,x^*(t),\dot{x}^*(t))=0$ and $x^*(t)^TB_i(t)w^*(t)=(x^*(t)^TB_i(t)x^*(t))^{\frac{1}{2}}$, (15), (16) become

$$\int_{t_0}^{t_1}\Big[f^i(t,\widetilde{x}(t),\dot{\widetilde{x}}(t))+\widetilde{x}(t)^TB_i(t)\widetilde{w}(t)-\widetilde{y}(t)^Tg(t,\widetilde{x}(t),\dot{\widetilde{x}}(t))-\widetilde{\mu}(t)^Th(t,\widetilde{x}(t),\dot{\widetilde{x}}(t))\Big]dt$$

$$>\int_{t_0}^{t_1}\Big[f^i(t,x^*(t),\dot{x}^*(t))+(x^*(t)^TB_i(t)x^*(t))^{\frac{1}{2}}\Big]dt \tag{17}$$

and

$$\int_{t_0}^{t_1}\Big[f^j(t,\widetilde{x}(t),\dot{\widetilde{x}}(t))+\widetilde{x}(t)^TB_j(t)\widetilde{w}(t)-\widetilde{y}(t)^Tg(t,\widetilde{x}(t),\dot{\widetilde{x}}(t))-\widetilde{\mu}(t)^Th(t,\widetilde{x}(t),\dot{\widetilde{x}}(t))\Big]dt$$
$$<\int_{t_0}^{t_1}\Big[f^j(t,x^*(t),\dot{x}^*(t))+(x^*(t)^TB_i(t)x^*(t))^{\frac{1}{2}}\Big]dt. \tag{18}$$

Note that $\tau\in\Lambda^+$, (17), (18) contradict (10). Thus $(x^*(t),y^*(t),\mu^*(t),w^*(t))$ is a properly efficient solution for (WD). Furthermore, from (12) and (14),

$$\int_{t_0}^{t_1}\Big[f^i(t,x^*(t),\dot{x}^*(t))+(x^*(t)^TB_i(t)x^*(t))^{\frac{1}{2}}\Big]dt$$
$$=\int_{t_0}^{t_1}\Big[f^i(t,x^*(t),\dot{x}^*(t))+x^*(t)^TB_i(t)w^*(t)-y^*(t)^Tg(t,x^*(t),\ \dot{x}^*(t))$$
$$-\mu^*(t)^Th(t,x^*(t),\dot{x}^*(t))\Big]dt.$$

□

4 Mond-Weir Type Duality

Now we formulate the following Mond-Weir type dual to (VP) is

(MD) Maximize

$$\left(\int_{t_0}^{t_1}\left[f^1(t,u(t),\dot{u}(t))+u(t)^TB_1(t)w(t)\right]dt,\cdots,\right.$$
$$\left.\int_{t_0}^{t_1}\left[f^p(t,u(t),\dot{u}(t))+u(t)^TB_p(t)w(t)\right]dt\right)$$

subject to

$$u(t_0)=x_0,\ u(t_1)=x_1, \tag{19}$$

$$\sum_{i=1}^{p}\tau_i\left[f_x^i(t,u(t),\dot{u}(t))+w(t)^TB_i(t)\right]-y(t)^Tg_x(t,u(t),\dot{u}(t))-\mu(t)^Th_x(t,u(t),\dot{u}(t))$$
$$=\frac{d}{dt}\left[\sum_{i=1}^{p}\tau_if_{\dot{x}}^i(t,u(t),\dot{u}(t))-y(t)^Tg_{\dot{x}}(t,u(t),\dot{u}(t))-\mu(t)^Th_{\dot{x}}(t,u(t),\dot{u}(t))\right], \tag{20}$$

$$y(t)^Tg(t,u(t),\dot{u}(t))\leqq 0,\ \mu(t)^Th(t,u(t),\dot{u}(t))=0, \tag{21}$$

$$w(t)^TB(t)w(t)\leqq 1, \tag{22}$$

$$y(t)\geqq 0,\ y(t)\in\mathbb{R}_+^m,\ \mu(t)\in\mathbb{R}^n,\ w(t)\in\mathbb{R}^n,\ t\in I,\ \tau\in\Lambda^+.$$

Weak and strong duality results between (VP) and (MD) are similar to those contained in the above section. Here, we state the following duality results.

Theorem 4.1 (Weak Duality) *Assume that x is feasible for problem (VP), and (u,y,μ,w) is feasible for problem (MD). Let $f(t,\cdot,\cdot)$, $-g(t,\cdot,\cdot)$ be convex and $h(t,\cdot,\cdot)$ be affine, for each $t\in I$. Then the following cannot hold:*

$$\int_{t_0}^{t_1}\left[f^i(t,x(t),\dot{x}(t))+(x(t)^TB_i(t)x(t))^{\frac{1}{2}}\right]dt$$
$$\leq\int_{t_0}^{t_1}\left[f^i(t,u(t),\dot{u}(t))+u(t)^TB_i(t)w(t)\right]dt \tag{23}$$

for all $i \in \{1, 2, \cdots, p\}$ *and*

$$\begin{aligned} &\int_{t_0}^{t_1} \left[f^j(t, x(t), \dot{x}(t)) + (x(t)^T B_j(t) x(t))^{\frac{1}{2}} \right] dt \\ &\qquad < \int_{t_0}^{t_1} \left[f^j(t, u(t), \dot{u}(t)) + u(t)^T B_j(t) w(t) \right] dt \end{aligned} \tag{24}$$

for some $j \in \{1, 2, \cdots, p\}$.

Proof. Suppose that (23) and (24) hold. Then (23) and (24) imply that

$$\begin{aligned} &\sum_{i=1}^{p} \tau_i \int_{t_0}^{t_1} \left[f^i(t, x(t), \dot{x}(t)) + (x(t)^T B_i(t) x(t))^{\frac{1}{2}} \right] dt \\ &\qquad < \sum_{i=1}^{p} \tau_i \int_{t_0}^{t_1} \left[f^i(t, u(t), \dot{u}(t)) + u(t)^T B_i(t) w(t) \right] dt. \end{aligned} \tag{25}$$

Using Lemma 2.1, we have

$$\begin{aligned} &\sum_{i=1}^{p} \tau_i \int_{t_0}^{t_1} \left[f^i(t, x(t), \dot{x}(t)) + (x(t)^T B_i(t) x(t))^{\frac{1}{2}} \right] dt \\ &\qquad\qquad - \sum_{i=1}^{p} \tau_i \int_{t_0}^{t_1} \left[f^i(t, u(t), \dot{u}(t)) + u(t)^T B_i(t) w(t) \right] dt \\ &\quad \geqq \sum_{i=1}^{p} \tau_i \int_{t_0}^{t_1} \left[f^i(t, x(t), \dot{x}(t)) + x(t)^T B_i(t) w(t) \right] dt \\ &\qquad\qquad - \sum_{i=1}^{p} \tau_i \int_{t_0}^{t_1} \left[f^i(t, u(t), \dot{u}(t)) + u(t)^T B_i(t) w(t) \right] dt \\ &\quad \geqq \sum_{i=1}^{p} \tau_i \int_{t_0}^{t_1} \left[f^i(t, x(t), \dot{x}(t)) + x(t)^T B_i(t) w(t) \right] dt \\ &\qquad\qquad - \sum_{i=1}^{p} \tau_i \int_{t_0}^{t_1} \Big[f^i(t, u(t), \dot{u}(t)) + u(t)^T B_i(t) w(t) - y(t)^T g(t, u(t), \dot{u}(t)) \\ &\qquad\qquad\qquad - \mu(t)^T h(t, u(t), \dot{u}(t)) \Big] dt. \end{aligned}$$

Since f, $-g$ is convex and h is affine,

$$\sum_{i=1}^{p}\tau_i\int_{t_0}^{t_1}\Big[f^i(t,x(t),\dot{x}(t))+x(t)^TB_i(t)w(t)\Big]dt$$
$$-\sum_{i=1}^{p}\tau_i\int_{t_0}^{t_1}\Big[f^i(t,u(t),\dot{u}(t))+u(t)^TB_i(t)w(t)$$
$$-y(t)^Tg(t,u(t),\dot{u}(t))-\mu(t)^Th(t,u(t),\dot{u}(t))\Big]dt$$
$$\geqq\sum_{i=1}^{p}\tau_i\int_{t_0}^{t_1}\Big\{(x(t)-u(t))^T\Big[f_x^i(t,u(t),\dot{u}(t))+w(t)^TB_i(t)-y(t)^Tg_x(t,u(t),\dot{u}(t))$$
$$-\mu(t)^Th_x(t,u(t),\dot{u}(t))\Big]\ dt-(\dot{x}(t)-\dot{u}(t))^T\Big[f_{\dot{x}}^i(t,u(t),\dot{u}(t))$$
$$-y(t)^Tg_{\dot{x}}(t,u(t),\dot{u}(t))-\mu(t)^Th_{\dot{x}}(t,u(t),\dot{u}(t))\Big]\Big\}\ dt$$
$$+\int_{t_0}^{t_1}y(t)^Tg(t,x(t),\dot{x}(t)dt+\int_{t_0}^{t_1}\mu(t)^Th(t,x(t),\dot{x}(t)dt$$
$$=\int_{t_0}^{t_1}\Big\{(x(t)-u(t))^T\frac{d}{dt}\Big[\sum_{i=1}^{p}\tau_if_{\dot{x}}^i(t,u(t),\dot{u}(t))-y(t)^Tg_{\dot{x}}(t,u(t),\dot{u}(t))$$
$$-\mu(t)^Th_{\dot{x}}(t,u(t),\dot{u}(t))\Big]-(\dot{x}(t)-\dot{u}(t))^T\Big[\sum_{i=1}^{p}\tau_if_{\dot{x}}^i(t,u(t),\dot{u}(t))$$
$$-y(t)^Tg_{\dot{x}}(t,u(t),\dot{u}(t))-\mu(t)^Th_{\dot{x}}(t,u(t),\dot{u}(t))\Big]\Big\}\ dt$$
$$+\int_{t_0}^{t_1}y(t)^Tg(t,x(t),\dot{x}(t))dt+\int_{t_0}^{t_1}\mu(t)^Th(t,x(t),\dot{x}(t))dt.$$

Using integration by parts and boundary conditions (1) and (19), we obtain

$$\sum_{i=1}^{p}\tau_i\int_{t_0}^{t_1}\Big[f^i(t,x(t),\dot{x}(t))+x(t)^TB_i(t)w(t)\Big]dt-\sum_{i=1}^{p}\tau_i\int_{t_0}^{t_1}\Big[f^i(t,u(t),\dot{u}(t))$$
$$+u(t)^TB_i(t)w(t)-y(t)^Tg(t,u(t),\dot{u}(t))-\mu(t)^Th(t,u(t),\dot{u}(t))\Big]dt$$
$$\geqq 0.$$

Hence

$$\sum_{i=1}^{p} \tau_i \int_{t_0}^{t_1} \left[f^i(t, x(t), \dot{x}(t)) + (x(t)^T B_i(t) x(t))^{\frac{1}{2}} \right] dt$$
$$- \sum_{i=1}^{p} \tau_i \int_{t_0}^{t_1} \left[f^i(t, u(t), \dot{u}(t)) + u(t)^T B_i(t) w(t) \right] dt \geqq 0, \tag{26}$$

which contradiction (25). Hence the result follows. □

Theorem 4.2 (Strong Duality) *Let $f(t, \cdot, \cdot)$, $-g(t, \cdot, \cdot)$ be convex and $h(t, \cdot, \cdot)$ be affine, for each $t \in I$. Let x^* be a normal and a properly efficient solution for (VP). Then for some $\tau \in \Lambda^+$, there exits a piecewise smooth $y^* : I \to \mathbb{R}^m$ such that (x^*, y^*, μ^*, w^*) is a properly efficient solution of (WD) and*

$$\int_{t_0}^{t_1} \left[f^i(t, x^*(t), \dot{x}^*(t)) + (x^*(t)^T B_i(t) x^*(t))^{\frac{1}{2}} \right] dt$$
$$= \int_{t_0}^{t_1} \left[f^i(t, x^*(t), \dot{x}^*(t)) + x^*(t)^T B_i(t) w^*(t)) \right] dt, \ i = 1, 2, \cdots, p.$$

Proof. Since x^* is a properly efficient solution of (VP), by Theorem 2.2, x^* is optimal for (P_τ) for some $\tau \in \Lambda^+$. Therefore, by Proposition 2.1, there exists a piecewise smooth $y^* : I \to \mathbb{R}^m$ such that for $t \in I$

$$\sum_{i=1}^{p} \tau_i \left[f_x(t, x^*(t), \dot{x}^*(t)) + w^*(t)^T B(t) \right] - y^*(t)^T g_x(t, x^*(t), \dot{x}^*(t))$$
$$-\mu^*(t)^T h_x(t, x^*(t), \dot{x}^*(t))$$
$$= \frac{d}{dt} \Big[\sum_{i=1}^{p} \tau_i f^i_{\dot{x}}(t, x^*(t), \dot{x}^*(t)) + w^*(t)^T B(t) - y^*(t)^T g_{\dot{x}}(t, x^*(t), \dot{x}^*(t))$$
$$-\mu^*(t)^T h_{\dot{x}}(t, x^*(t), \dot{x}^*(t)) \Big] \tag{27}$$

$$y^*(t)^T g(t, x^*(t), \dot{x}^*(t)) = 0, \tag{28}$$

$$w^*(t)^T B_i(t) w^*(t) \leqq 1, \ i = 1, \cdots, p, \tag{29}$$

$$x^*(t)^T B_i(t) w^*(t) = (x^*(t)^T B_i(t) x^*(t))^{\frac{1}{2}}, \ i = 1, \cdots, p. \tag{30}$$

This implies that (x^*, y^*, μ^*, w^*) is a feasible solution of (MD). Suppose that (x^*, y^*, μ^*, w^*) is not an efficient solution of (MD). Then there exists a feasible solution $(\bar{x}, \bar{y}, \bar{\mu}, \bar{w})$ of (MD) such that for all $i \in \{1, 2, \cdots, p\}$

$$\int_{t_0}^{t_1} \Big[f^i(t, \bar{x}(t), \dot{\bar{x}}(t)) + \bar{x}(t)^T B_i(t)\bar{w}(t))\Big] dt \geq \int_{t_0}^{t_1} \Big[f^i(t, x^*(t), \dot{x}^*(t)) + x^*(t)^T B_i(t) w^*(t))\Big] dt$$

and for some $j \in \{1, 2, \cdots, p\}$

$$\int_{t_0}^{t_1} \Big[f^j(t, \bar{x}(t), \dot{\bar{x}}(t)) + \bar{x}(t)^T B_j(t)\bar{w}(t)\Big] dt > \int_{t_0}^{t_1} \Big[f^j(t, x^*(t), \dot{x}^*(t)) + x^*(t)^T B_j(t) w^*(t)\Big] dt.$$

Using (30),

$$\int_{t_0}^{t_1} \Big[f^i(t, \bar{x}(t), \dot{\bar{x}}(t)) + \bar{x}(t)^T B_i(t)\bar{w}(t)\Big] dt \geq \int_{t_0}^{t_1} \Big[f^i(t, x^*(t), \dot{x}^*(t)) + (x^*(t)^T B_i(t) x^*(t))^{\frac{1}{2}}\Big] dt$$

and for some $j \in \{1, 2, \cdots, p\}$

$$\int_{t_0}^{t_1} \Big[f^j(t, \bar{x}(t), \dot{\bar{x}}(t)) + \bar{x}(t)^T B_j(t)\bar{w}(t)\Big] dt > \int_{t_0}^{t_1} \Big[f^j(t, x^*(t), \dot{x}^*(t)) + (x^*(t)^T B_j(t) x^*(t))^{\frac{1}{2}}\Big] dt.$$

which contradicts Theorem 4.1. Hence (x^*, y^*, μ^*, w^*) is efficient.

Now we assume that (x^*, y^*, μ^*, w^*) is not properly efficient for (MD); i.e., there exists a feasible solution $(\widetilde{x}, \widetilde{y}, \widetilde{\mu}, \widetilde{w})$ such that for some i and some $M > 0$

$$\int_{t_0}^{t_1} \Big[f^i(t, \widetilde{x}(t), \dot{\widetilde{x}}(t)) + \widetilde{x}(t)^T B_i(t)\widetilde{w}(t)\Big] dt > \int_{t_0}^{t_1} \Big[f^i(t, x^*(t), \dot{x}^*(t)) + x^*(t)^T B_i(t) w^*(t)\Big] dt$$

and

$$\begin{aligned}
&\int_{t_0}^{t_1} \Big[f^i(t, \widetilde{x}(t), \dot{\widetilde{x}}(t)) + \widetilde{x}(t)^T B_i(t)\widetilde{w}(t)\Big] dt \\
&\qquad - \int_{t_0}^{t_1} \Big[f^i(t, x^*(t), \dot{x}^*(t)) + x^*(t)^T B_i(t) w^*(t)\Big] dt \\
&> M \Big\{ \int_{t_0}^{t_1} \Big[f^j(t, x^*(t), \dot{x}^*(t)) + x^*(t)^T B_j(t) w^*(t)\Big] dt \\
&\qquad - \int_{t_0}^{t_1} \Big[f^j(t, \widetilde{x}(t), \dot{\widetilde{x}}(t)) + \widetilde{x}(t)^T B_j(t)\widetilde{w}(t)\Big] dt \Big\}
\end{aligned}$$

and $\forall j \in \{1, 2, \cdots, p\}$ such that

$$\int_{t_0}^{t_1} \left[f^j(t, \tilde{x}(t), \dot{\tilde{x}}(t)) + \tilde{x}(t)^T B_j(t) \tilde{w}(t) \right] dt < \int_{t_0}^{t_1} \left[f^j(t, x^*(t), \dot{x}^*(t)) + x^*(t)^T B_j(t) w^*(t) \right] dt.$$

Since $x^*(t)^T B_i(t) w^*(t) = (x^*(t)^T B_i(t) x^*(t))^{\frac{1}{2}}$,

$$\int_{t_0}^{t_1} \left[f^i(t, \tilde{x}(t), \dot{\tilde{x}}(t)) + \tilde{x}(t)^T B_i(t) \tilde{w}(t) \right] dt$$
$$> \int_{t_0}^{t_1} \left[f^i(t, x^*(t), \dot{x}^*(t)) + (x^*(t)^T B_i(t) x^*(t))^{\frac{1}{2}} \right] dt \tag{31}$$

and

$$\int_{t_0}^{t_1} \left[f^j(t, \tilde{x}(t), \dot{\tilde{x}}(t)) + \tilde{x}(t)^T B_j(t) \tilde{w}(t) \right] dt$$
$$< \int_{t_0}^{t_1} \left[f^j(t, x^*(t), \dot{x}^*(t)) + (x^*(t)^T B_i(t) x^*(t))^{\frac{1}{2}} \right] dt. \tag{32}$$

Note that $\tau \in \Lambda^+$, (31), (32) contradict (26). Thus (x^*, y^*, μ^*, w^*) is a properly efficient solution for (WD). Furthermore, from (30),

$$\int_{t_0}^{t_1} \left[f^i(t, x^*(t), \dot{x}^*(t)) + (x^*(t)^T B_i(t) x^*(t))^{\frac{1}{2}} \right] dt$$
$$= \int_{t_0}^{t_1} \left[f^i(t, x^*(t), \dot{x}^*(t)) + x^*(t)^T B_i(t) w^*(t) \right] dt.$$

□

5 Related Problems

As in [1], the above duality results can be reduced to the corresponding problems given below by omitting h and the boundary conditions (1) for (VP) and (WD) and

(MD) with "natural boundary conditions" (and omitting h and μ). That is,

(VP1) Minimize

$$\left(\int_{t_0}^{t_1}\left[f^1(t,x(t),\dot{x}(t))+(x(t)^TB_1(t)x(t))^{\frac{1}{2}}\right]dt,\cdots,\right.$$
$$\left.\int_{t_0}^{t_1}\left[f^p(t,x(t),\dot{x}(t))+(x(t)^TB_p(t)x(t))^{\frac{1}{2}}\right]dt\right)$$

subject to $\quad g(t,x(t),\dot{x}(t)) \geqq 0,\ t\in I.$

(WD1) Maximize

$$\left(\int_{t_0}^{t_1}\left[f^1(t,u(t),\dot{u}(t))+u(t)^TB_1(t)w(t)-y(t)g(t,u(t),\dot{u}(t))\right]dt,\cdots,\right.$$
$$\left.\int_{t_0}^{t_1}\left[f^p(t,u(t),\dot{u}(t))+u(t)^TB_p(t)w(t)-y(t)g(t,u(t),\dot{u}(t))\right]dt\right)$$

subject to

$$\sum_{i=1}^{p}\tau_i\left[f_x^i(t,u(t),\dot{u}(t))+w(t)^TB_i(t)\right]-y(t)^Tg_x(t,u(t),\dot{u}(t))$$
$$=\frac{d}{dt}\left[\sum_{i=1}^{p}\tau_i f_{\dot{x}}^i(t,u(t),\dot{u}(t))-y(t)^Tg_{\dot{x}}(t,u(t),\dot{u}(t))\right],$$

$f_{\dot{x}}^i(t,u(t),\dot{u}(t))-y(t)^Tg_{\dot{x}}(t,u(t),\dot{u}(t))=0$ when $t=t_0,\ t=t_1,$

$w(t)^TB(t)w(t)\leqq 1,$

$y(t)\geqq 0,\ y(t)\in\mathbb{R}_+^m,\ w(t)\in\mathbb{R}^n,\ t\in I,\ \tau\in\Lambda^+.$

and

(MD1) Maximize

$$\left(\int_{t_0}^{t_1}\Big[f^1(t,u(t),\dot{u}(t)) + u(t)^T B_1(t)w(t)\Big]dt, \cdots, \int_{t_0}^{t_1}\Big[f^p(t,u(t),\dot{u}(t)) + u(t)^T B_p(t)w(t)\Big]dt\right)$$

subject to

$$\sum_{i=1}^{p}\tau_i\Big[f_x^i(t,u(t),\dot{u}(t)) + w(t)^T B_i(t)\Big] - y(t)^T g_x(t,u(t),\dot{u}(t)) = \frac{d}{dt}\left[\sum_{i=1}^{p}\tau_i f_{\dot{x}}^i(t,u(t),\dot{u}(t)) - y(t)^T g_{\dot{x}}(t,u(t),\dot{u}(t))\right],$$

$$f_{\dot{x}}^i(t,u(t),\dot{u}(t)) - y(t)^T g_{\dot{x}}(t,u(t),\dot{u}(t)) = 0 \text{ when } t = t_0,\ t = t_1,$$

$$y(t)^T g(t,u(t),\dot{u}(t)) \leqq 0,$$

$$w(t)^T B(t)w(t) \leqq 1,$$

$$y(t) \geqq 0,\ y(t) \in \mathbb{R}_+^m,\ \mu(t) \in \mathbb{R}^n,\ w(t) \in \mathbb{R}^n,\ t \in I,\ \tau \in \Lambda^+.$$

In particular if (VP1), (WD1) and (MD1) are independent of t, thus if f_i, $B_i(t)$(for $i = 1, 2, \cdots, p$), and g do not depend explicitly on t, then these problems essentially reduce to the static cases of nondifferentiable programming studied by Mond([11]) , namely

(VP2) Minimize $\left(f^1(x) + (x^T B_1 x)^{\frac{1}{2}}, \cdots, f^p(x) + (x^T B_p x)^{\frac{1}{2}}\right)$

subject to $g(x) \geqq 0.$

The Wolfe type dual to (P) is

$$\text{(WD2)}\quad \text{Maximize}\quad \left(f^1(u) + u^T B_1 w - y^T g(u), \cdots, f^p(u) + u^T B_p w - y^T g(u)\right)$$

$$\text{subject to}\quad \sum_{i=1}^{p} \tau_i \left[f_x^i(u) + w^T B_i \right] = y^T g_x(u)$$

$$w^T B_i w \leqq 1, \; i = 1, 2, \cdots, p,$$

$$y \geqq 0, \; w \in \mathbb{R}.$$

and

$$\text{(MD2)}\quad \text{Maximize}\quad \left(f^1(u) + u^T B_1 w, \cdots, f^p(u) + u^T B_p w\right)$$

$$\text{subject to}\quad \sum_{i=1}^{p} \tau_i \left[f_x^i(u) + w^T B_i \right] = y^T g_x(u)$$

$$y^T g(u) \leqq 0,$$

$$w^T B_i w \leqq 1, \; i = 1, 2, \cdots, p,$$

$$y \geqq 0, \; w \in \mathbb{R}.$$

Lal, Nath and Kumar ([8]) have given the weak dual theorems for (VP2) under invexity type of assumptions. Some results in [2, 7, 8, 12, 13] are included in our conclusions.

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