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Error Estimates for Mixed Finite Element Approximations to a Linear Stefan Problem

by

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Graduate School of Education
Pukyong National University

August 2002

Error Estimates for Mixed Finite Element Approximations to a Linear Stefan Problem

선형 스테판 문제에 대한 혼합 유한요소
근사해에 관한 오차 추정

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by
Min-Hee Ann

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A Dissertation

by

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요 약

본 논문에서는 한쪽 경계에서는 동질의 Neumann 경계조건을 가지고, 자유 경계에서는 동질의 Dirichlet 경계조건을 가지는 선형 스테판 문제에 대한 혼합 유한요소 근사해의 오차 추정에 관해 연구하였다. 주어진 스테판 문제를 Landau 변환을 이용하여 고정 영역상의 문제로 변환시킨 후, 혼합 약한 형식과 혼합 유한요소 근사해를 도입하고, 혼합 유한 요소 근사해의 일의적 존재를 보였다. 또한, 보조 사영을 소개하고 여러 가지 추정 결과를 이용하여 혼합 유한요소 근사해의 오차를 추정하였다. 아울러, 혼합 유한요소 근사해를 이용하여 주어진 선형 스테판 문제의 근사해를 구성하고 H^1 및 L_2 노름에 대한 오차를 추정하였다.

1. Introduction

In this thesis, we consider a mixed finite element Galerkin method for the following single-phase linear Stefan problem in one space dimension.

Problem (P1): Find $U(y,\tau)$ and $S(\tau)$ such that U satisfies

$$U_\tau - U_{yy} = 0, \quad (y,\tau) \in \Omega(\tau) \times (0, T_0] \tag{1.1}$$

with the initial and boundary conditions

$$U(y,0) = g(y), \quad 0 < y < 1 \tag{1.2}$$

$$U_y(0,\tau) = 0, \;\; U(S(\tau),\tau) = 0, \;\; 0 < \tau \le T_0 \tag{1.3}$$

and further, on the free boundary, S satisfies

$$\frac{dS}{d\tau} + U_y(S(\tau),\tau) = 0, \quad 0 < \tau \le T_0 \tag{1.4}$$

with $S(0) = 1$, where $\Omega(\tau) = \{(y,\tau) |\; 0 < y \le S(\tau),\; 0 < \tau \le T_0\}$.

Regarding the existence, uniqueness, and regularity results of the solution $\{U, S\}$, the reader may refer to Fasano and Primicerio [2].

Earlier Nitsche [7, 8] initiated the study of error analysis for semidiscrete finite element approximations to single-phase linear Stefan problems by using the fixing domain method. In [1, 3, 5, 9, 11, 12], the authors constructed semidiscrete finite element approximations to linear or quasilinear Stefan problems in one space dimension with various types of boundary conditions. They proved the local existence or global existence of semidiscrete approximations and proved the optimal order of convergence of semidiscrete approximations with respect to L_2, H^1, and H^2 norms, provided that finite element space consists of piecewise polynomials of degree r with $r \geq 3$. And also they assume that the finite element space S^h is a subspace of $H^2 \cap H_0^1$.

Under the similar conditions on a finite element space, the authors in [3, 4, 12, 13] constructed fully discrete approximations and proved the optimal order of convergence of fully discrete approximations with respect to L_2, H^1 and H^2 norms.

Recently, for a linear Stefan problem $(1.1)-(1.4)$ with a homogeneous Diriclet boundary condition at $x = 0$ instead of a homogeneous Neumann boundary condition at $x = 0$, the authors in [6, 10] adopted a mixed finite element method to improve the restrictions on the order of finite elements and to relax the requirement of the smoothness of finite elements. They constructed semidiscrete finite element approximations and analyzed the error estimates of semidiscrete finite element approximations with respect to L_2 and H^1 norms.

In this thesis, we apply the ideas in [6, 10] to construct mixed finite element approximations to the linear Stefan problem $(1.1) - (1.4)$ and to analyize the error estimates for the mixed finite element approximations with respect to L_2 and H^1 norms.

This thesis is organized as follows. In Section 2, a mixed formulation is derived and the local existence of semidiscrete approximations is proved. Auxiliary projections and related estimates are given in Section 3. Section 4 is related to error estimates for semidiscrete approximations with respect to L_2 and H^1 norms. The global existence of semidiscrete approximations is proved in Section 5 using the Schauder's fixed point theorem.

2. Mixed formulation and local existence

With the help of the Landau transformations,

$$x = \frac{y}{S(\tau)} \tag{2.1}$$

and

$$t = t(\tau) = \int_0^{\tau} \frac{1}{S^2(\tau')} d\tau', \tag{2.2}$$

the problem (P1) can be transformed into a problem (P2).

Problem (P2): Find $u(x,t) = U(y,\tau)$, $s(t) = S(\tau)$ such that u satisfies

$$u_t - u_{xx} = -x u_x(1) u_x \qquad (x,t) \in I \times (0,T] \tag{2.3}$$

with the initial and boundary conditions

$$u(x,0) = g(x), \qquad x \in I \tag{2.4}$$

$$u_x(0,t) = 0, u(1,t) = 0, \quad 0 < t \leq T \tag{2.5}$$

and s satisfies

$$\frac{ds}{dt} = -u_x(1)s, \quad 0 < t \leq T \tag{2.6}$$

with $s(0) = 1$, where $t = T$ corresponds to $\tau = T_0, I = (0,1)$, and $u_x(1) = u_x(1,t)$. Further the integal in (2.2) can be rewritten as

$$\frac{d\tau}{dt} = s^2(t) \quad 0 < t \leq T \tag{2.7}$$

with $\tau(0) = 0$.

Assume that all the uniqueness and regularity properties for U and S can be carried over to the solution u and s, and that

Condition R:

$$u \in W^{1,\infty}(0,T; H^{r+1}(I)), \;\; s \in W^{1,\infty}(0,T) \quad \text{for some } r \geq 2, \tag{2.8}$$

where $H^{r+1}(I)$ is the usual Sobolev space with the usual Sobolev norm $\|\cdot\|_{r+1}$, simply deonted by H^{r+1}, $W^{1,\infty}(0,T; H^{r+1}(I))$ is the Banach space of functions v whose essential supremums of v and v_t with respect to the norm $\|\cdot\|_{r+1}$ over $(0,T)$ are bounded and $W^{1,\infty}(0,T)$ is the Banach space of functions s whose essential supremums of s and s_t with respect to the absolute value over $(0,T)$ are bounded.

Let K_1 denote the bound of the functions in all the norms of the spaces in Condition R and let $\bar{K}_1$ denote the bound for $\{U, S\}$ in the normed space appearing in Condition R.

Denote $H^1_+ = \{v \in H^1; \;\; v(1) = 0\}$ and $H^1_- = \{v \in H^1; \;\; v(0) = 0\}$. Let $v = u_x$. Then the equation (2.3) can be rewritten as a system:

$$u_x = v \tag{2.9}$$

$$u_t - v_x = -xv(1)v. \tag{2.10}$$

Multiplying (2.9) by w_x with $w \in H^1_+$ and (2.10) by z_x with $z \in H^1_-$ and integrating by parts the first term in the left hand side of (2.10), we obtain

$$(u_x, w_x) = (v, w_x) \quad \text{for} \quad w \in H^1_+$$

and

$$(u_{tx}, z) + (v_x, z_x) = v(1)(xv, z_x) \quad \text{for} \quad z \in H^1_-.$$

The mixed weak formulation of (2.3)-(2.5) is to find a pair $(u(t), v(t)) \in H_+^1 \times H_-^1$, $t \in (0, T]$ such that

$$(u_x, w_x) = (v, w_x) \ \text{ for } \ w \in H_+^1 \tag{2.11}$$

and

$$(v_t, z) + (v_x, z_x) = v(1)(xv, z_x) \ \text{ for } \ z \in H_-^1. \tag{2.12}$$

And for $t = 0$

$$u(x, 0) = g(x) \quad \text{ and } \quad v(x, 0) = g'(x), \quad x \in I.$$

Let $S_+^h \in H_+^1$ and $S_-^h \in H_-^1$ be two finite dimensional subspaces satisfying the following properties:

(i) Approximation property: for $1 \leq k \leq r$, there exists a constant K_0 such that for $j = 0, 1$ and $w \in H^k \cap H_+^1$,

$$\inf_{\chi \in S_+^h} \|w - \chi\|_j \leq K_0 h^{k-j} \|w\|_k, \tag{2.13}$$

where r is a positive constant.

Note that for $w \in H^k \cap H_-^1$ the infimum is taken over all $\chi \in S_-^h$.

(ii) Inverse property: for $\chi \in S_+^h$ (or S_-^h),

$$\|\chi\|_{L^\infty} \leq K_0 h^{-\frac{1}{2}} \|\chi\|. \tag{2.14}$$

Now mixed semidiscrete finite element approximations for the problem (P2) are defined as follows: Find $u^h \in S_+^h$, $v^h \in S_-^h$, s^h such that for $t \in (0, T]$

$$(u_x^h, w_x) = (v^h, w_x) \quad \text{for} \quad w \in S_+^h \tag{2.15}$$

$$(u_t^h, s) + (v_x^h, s_x) = v^h(1)(xv^h, s_x) \quad \text{for} \quad s \in S_-^h \tag{2.16}$$

$$\frac{ds^h}{dt} = -v^h(1)s^h \tag{2.17}$$

and for $t = 0$

$$u^h(x,0) = Q_h g(x), \quad v^h(x,0) = P_h g'(x), \quad s^h(0) = 1,$$

where Q_h and P_h are appropriate projections to be defined later. Moreover, the Galerkin approximation τ^h of τ is defined as follows:

$$\frac{d\tau^h}{dt} = [s^h(t)]^2, \quad t > 0 \tag{2.18}$$

with $\tau^h(0) = 0$.

In the next theorem, we prove the local existences of u^h and v^h.

Theorem 2.1. *There exist u^h and v^h satisfying (2.15) and (2.16) respectively in a certain interval $(0, t^*)$ where t^* depends on g and g' but not on S_+^h or S_-^h.*

Proof. Taking $s = v^h$ in (2.16), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v^h\|^2 + \|v_x^h\|^2 &= v^h(1)(xv^h, v_x^h) \\ &\le |v^h(1)|\|v^h\|\|v_x^h\| \\ &\le \frac{1}{\epsilon^4}\|v^h\|^6 + \frac{3}{4}\epsilon^{\frac{4}{3}}\|v_x^h\|^2.\end{aligned} \tag{2.19}$$

Therefore we have

$$\frac{d}{dt}\|v^h\|^2 + (2 - \frac{3}{2}\epsilon^{\frac{4}{3}})\|v_x^h\|^2 \le \frac{2}{\epsilon^4}\|v^h\|^6.$$

For a sufficiently small ϵ, there exists $\beta > 0$ such that

$$\frac{d}{dt}\|v^h\|^2 + \beta\|v_x^h\|^2 \le \frac{2}{\epsilon^4}\|v^h\|^6.$$

Thus we have

$$\frac{d}{dt}\|v^h\|^2 \le \frac{2}{\epsilon^4}\|v^h\|^6. \tag{2.20}$$

Now, we put $\bar{\lambda}(t) = \|v^h(t)\|^2 + 1$. By (2.20) we have

$$\frac{d}{dt}\bar{\lambda} \le c\bar{\lambda}^3(t) \tag{2.21}$$

for some constant $c > 0$. By integrating both sides of (2.21) one can show that

$$\bar{\lambda}(t) \le \frac{\bar{\lambda}^2(0)}{1 - 2ct\bar{\lambda}^2(0)} \tag{2.22}$$

with $\bar{\lambda}(0) = \|v^h(0)\|^2 + 1 = \|P_n g'\|^2 + 1$. Therefore there exists v^h on the time interval $[0, \frac{1}{2c(\|P_n g'\|^2+1)^2}]$ and so there exists u^h satisfying (2.15) on the time interval $[0, \frac{1}{2c(\|P_n g'\|^2+1)^2}]$. This completes the proof.

3. Auxiliary projections and related estimates

For $u, w, z \in H^1$, set

$$A(u, w) = (u_x, w_x) \tag{3.1}$$

$$B(v; w, z) = (w_x, z_x) - v(1)(xw, z_x) + \lambda(w, z) \tag{3.2}$$

where $\lambda > 0$ is a sufficiently large constant so that there is a constant $\alpha > 0$ such that

$$B(v; z, z) \geq \alpha \|z\|_1^2, \quad z \in H_-^1. \tag{3.3}$$

It is easy to show that

$$|A(u, w)| \leq \|u\|_1 \|w\|_1, \quad u, w \in H_+^1 \tag{3.4}$$

$$A(u, u) \geq c\|u\|_1^2, \quad u \in H_+^1 \tag{3.5}$$

$$|B(v; w, z)| \leq K_2 \|w\|_1 \|z\|_1, \quad w \in H_+^1, \quad z \in H_-^1, \tag{3.6}$$

for some positive constant K_2 and c. Here K_2 depends only on $\|v\|_\infty$. Now define auxiliary projections $\bar{u} \in S_+^h$ and $\bar{v} \in S_-^h$ of $u \in H_+^1$ and $v \in H_-^1$ with respect to A and B, respectively as follows:

$$A(u - \bar{u}, w) = 0, \quad \forall \quad w \in S_+^h \tag{3.7}$$

$$B(v; v - \bar{v}, z) = 0, \quad \forall \quad z \in S_-^h. \tag{3.8}$$

Existence and uniqueness for $(\bar{u}, \bar{v})$ follow from the Lax-Milgram theorem. Let $\eta = u - \bar{u}$ and $\xi = v - \bar{v}$. Then the following estimates can be derived by the similar methods in Das and Pani[1, 11].

Lemme 3.1. *For $j = 0, 1$ and $1 \leq m \leq r$, There exists a constant $K_3 = K_3(K_0, K_1, K_2)$, independent of h, such that*

$$\|\eta\|_j \leq K_3 h^{m-j} \|u\|_m$$

$$\|\xi\|_j \leq K_3 h^{m-j} \|v\|_m$$

$$\|\xi_t\|_j \leq K_3 h^{m-j} [\|v\|_m + \|v_t\|_m]$$

$$|\xi(1, t)| \leq K_3 h^{2(m-1)} \|v\|_m$$

hold.

4. Error estimates for semidiscrete approximations

To analyze the error estimates for semidiscrete finite element approximations, we temporarily assume that there exists a positive constant K^* such that

$$\|v^h\|_{L^\infty(H^1)} \le K^*. \tag{4.1}$$

Let $\theta = u^h - \bar{u}, \psi = v^h - \bar{v}, \zeta = u - u^h = \eta - \theta$, and $\sigma = v - v^h = \xi - \psi$. In other to maintain a uniform degree of approximation, we define $P_h g' = \bar{v}(x, 0)$ where $\bar{v}$ is the projection of v onto S^h_- defined by (3.8). Using Lemma 3.1 we will estimates θ and ψ in the following theorem.

Theorem 4.1. *Assume that (4.1) and the regularity condition (2.8) hold. Then there exists positive constants β and $K_4 = K_4(K_1, K_3, K^*, \lambda)$ such that*

$$\|\theta\|_{L^\infty(H^1)} + \|\psi\|_{L^\infty(L^2)} + \beta\|\psi\|_{L^2(H^1)} \le K_4 h^m \tag{4.2}$$

holds for $2 \le m \le r$.

Proof. From (2.15), (3.7), and (2.11), it follows that

$$(\theta_x, w_x) = (v^h - v, w_x) \qquad \forall \quad w \in S^h_+. \tag{4.3}$$

Setting $w = \theta$ in (4.3), we have

$$\|\theta_x\| \le \|\xi\| + \|\psi\|$$

and

$$\|\theta_x\|^2 \ge \frac{1}{2}\|\theta_x\|^2 + \frac{1}{2}\|\theta\|^2 = \frac{1}{2}\|\theta\|_1^2.$$

Therefore we have

$$\frac{1}{2}\|\theta\|_1^2 \leq [\|\xi\| + \|\psi\|]^2,$$

that is,

$$\|\theta\|_1 \leq \sqrt{2}[\|\xi\| + \|\psi\|]. \tag{4.4}$$

From (2.12), (2.16), and (3.8), we have

$$\begin{aligned}
&(\psi_t, z) + (\psi_x, z_x) \\
&\quad = v^h(1)(xv^h, z_x) - v(1)(x\bar{v}, z_x) + (\xi_t, z) - \lambda(\xi, z) \\
&\quad = (v^h(1) - v(1))(xv^h, z_x) + v(1)(x\psi, z_x) + (\xi_t, z) - \lambda(\xi, z) \quad \forall z \in S_-^h. \qquad (4.5)
\end{aligned}$$

Choosing $z = \psi$ in (4.5), we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\psi\|^2 + \|\psi_x\|^2 &\leq K^*(v^h(1) - \bar{v}(1) - v(1) + \bar{v}(1))(x, \psi_x) \\
&\quad + K_1(x\psi, \psi_x) + (\xi_t, \psi) - \lambda(\xi, \psi).
\end{aligned}$$

Therfore we have

$$\begin{aligned}
&\frac{d}{dt}\|\psi\|^2 + 2\|\psi_x\|^2 \\
&\leq 2K^*|\psi(1) - \xi(1)|\|\psi_x\| + 2K_1\|\psi\|\|\psi_x\| + 2\|\xi_t\|\|\psi\| + 2\lambda\|\xi\|\|\psi\| \\
&\leq 2\sqrt{2}K^*\|\psi\|^{\frac{1}{2}}\|\psi_x\|^{\frac{3}{2}} + 2K^*|\xi(1)|\|\psi_x\| + 2K_1\|\psi\|\|\psi_x\| + 2\|\xi_t\|\|\psi\| + 2\lambda\|\xi\|\|\psi\| \\
&\leq \|\xi_t\|^2 + \|\xi\|^2 + K(K^*, \epsilon)|\xi(1)|^2 + (\frac{3}{4}\epsilon^{\frac{4}{3}} + 2\epsilon)\|\psi_x\|^2 \\
&\qquad + K(K_1, K^*, \lambda, \epsilon)\|\psi\|^2. \qquad (4.6)
\end{aligned}$$

Choosing ϵ sufficiently small so that $2-(\frac{3}{4}\epsilon^{\frac{4}{3}}+2\epsilon)=\bar{\beta}>0$ and integrating both sides of (4.6) with respect to t, we obtain

$$\|\psi\|^2+\bar{\beta}\int_0^t\|\psi\|_1^2dt'\leq K(K^*)\int_0^t[\|\xi_t\|^2+\|\xi\|^2+|\xi(1,t')|^2]dt'$$
$$+K(K_1,K^*,\lambda)\int_0^t\|\psi\|^2dt'.$$

An application of Gronwall's inequality to the above inequality yields

$$\|\psi\|^2+\bar{\beta}\int_0^t\|\psi\|_1^2dt'\leq K(K_1,K_3,K^*,\lambda)h^{2m}.$$

And therefore taking the supremum over all $t\in(0,t)$ gives us

$$\|\psi\|^2_{L^\infty(L^2)}+\bar{\beta}\|\psi\|^2_{L^2(H^1)}\leq K(K_1,K_3,K^*,\lambda)h^{2m},$$

that is,

$$\|\psi\|_{L^\infty(L^2)}+\bar{\beta}\|\psi\|_{L^2(H^1)}\leq K(K_1,K_3,K^*,\lambda)h^m.$$

Thus, by (4.4) and Lemma 3.1, we obtain

$$\|\theta\|_{L^\infty(H^1)}\leq\sqrt{2}(\|\xi\|_{L^\infty(L^2)}+\|\psi\|_{L^\infty(L^2)})\leq K(K_1,K_3,K^*,\lambda)h^m$$

and

$$\|\theta\|_{L^\infty(H^1)}+\|\psi\|_{L^\infty(L^2)}+\bar{\beta}\|\psi\|_{L^2(H^1)}\leq K_4h^m,$$

which completes the proof.

Theorem 4.2. *There exists a constant* $K_5 = K_5(K_0, K_1, K_3, K_4, K^*)$ *such that*

$$\|\psi_t\|_{L^2(L^2)} + \|\psi\|_{L^\infty(H^1)} \le K_5 h^{m-1} \tag{4.7}$$

holds for $2 \le m \le r$.

Proof. Setting $s = \psi_t$ in (4.5), we have

$$\begin{aligned}
&\|\psi_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\psi_x\|^2 \\
&\quad = v^h(1)(xv^h, \psi_{tx}) - v(1)(x\bar{v}, \psi_{tx}) + (\xi_t, \psi_t) - \lambda(\xi, \psi_t) \\
&\quad = (v^h(1) - v(1))(xv^h, \psi_{tx}) + v(1)(x\psi, \psi_{tx}) + (\xi_t, \psi_t) - \lambda(\xi, \psi_t).
\end{aligned}$$

$$\begin{aligned}
2\|\psi_t\|^2 + \frac{d}{dt}\|\psi_x\|^2 &\le 2|v^h(1) - v(1)|\{|v^h(1)\psi_t(1)| + \|v^h\|\|\psi_t\| + \|v^h_x\|\|\psi_t\|\} \\
&\quad + 2v(1)\{|\psi(1)\psi_t(1)| + \|\psi\|\|\psi_t\| + \|\psi_t\|\|\psi_x\|\} + 2\|\xi_t\|\|\psi_t\| \\
&\quad + 2\lambda\|\xi\|\|\psi_t\| \\
&\le 2(K_0 h^{-\frac{1}{2}}\|\psi\| + |\xi(1)|)\{K^* K_0 h^{-\frac{1}{2}}\|\psi_t\| + 2K^*\|\psi_t\|\} \\
&\quad + 2K_1\{K_0 h^{-\frac{1}{2}}\|\psi\| K_0 h^{-\frac{1}{2}}\|\psi_t\| + \|\psi\|\|\psi_t\| + \|\psi_x\|\|\psi_t\|\} \\
&\quad + 2\|\xi_t\|\|\psi_t\| + 2\lambda\|\xi\|\|\psi_t\| \\
&\le K(K_0, K_1, K^*, \epsilon)h^{-2}\|\psi\|^2 + K(K_0, K^*, \epsilon)h^{-1}|\xi(1)|^2 \\
&\quad + K(K_1, \epsilon)\|\psi_x\|^2 + K(\epsilon)\|\xi_t\|^2 + K(\lambda, \epsilon)\|\xi\|^2 + 8\epsilon\|\psi_t\|^2.
\end{aligned}$$

Chossing ϵ sufficiently small, integrating the previous inequality with respect to

t, and applying Lemma 3.1 and Theorem 4.1, we obtain

$$\int_0^t \|\psi_t\|^2 dt' + \|\psi_x\|^2 \le K(K_1)\int_0^t \|\psi_x\|^2 dt' + K(K_0, K_1)h^{-2}\int_0^t \|\psi\|^2 dt'$$

$$+ K(K_0, K^*, \lambda)\int_0^t [\|\xi_t\|^2 + \|\xi\|^2 + |\xi(1)|^2] dt'.$$

Therefore we get

$$\int_0^t \|\psi_t\|^2 dt' + \|\psi_x\|^2 \le K(K_0, K^*, \lambda)\int_0^t [\|\xi_t\|^2 + \|\xi\|^2 + |\xi(1)|^2] dt'$$

$$+ K(K_0, K_1)h^{-2}\int_0^t \|\psi\|^2 dt'$$

$$\le K(K_0, K_3, K^*, \lambda)h^{2m} + K(K_0, K_1, K_4)h^{2m-2}$$

$$\le K(K_0, K_1, K_3, K_4, \lambda)h^{2(m-1)}.$$

. This implies that

$$\|\psi_t\|_{L^2(L^2)} + \|\psi\|_{L^\infty(H^1)} \le K_5 h^{m-1},$$

which completes the proof.

From Theorem 4.1, 4.2 and Lemma 3.1, we get the following theorem.

Theorem 4.3. *Let u be the solution of (2.3)-(2.5) satisfying the regularity conditions (2.8). Further assume that there are positive constants h_0 and K^* with $K^* \ge 2K_1$, such that the approximate solution $(u^h, v^h) \in S_+^h \times S_-^h$ of (2.15) and (2.16) satisfying (4.1) exists in $I \times [0, T]$ for $0 < h \le h_0$. Then the following estimates hold: for $2 \le m \le r$,*

$$\|\zeta\|_{L^\infty(L^2)} + \|\sigma\|_{L^\infty(L^2)} + \beta\|\sigma\|_{L^2(L^2)} \le K_6 h^m \tag{4.8}$$

where $K_6 = K_6(K_0, K_1, K_3, K_4, K^*)$ *and* $\beta > 0$ *and*

$$\|\zeta\|_{L^\infty(H^1)} + \|\sigma\|_{L^\infty(H^1)} + \beta\|\sigma_t\|_{L^2(L^2)} \le K_7 h^{m-1} \tag{4.9}$$

where $K_7 = K_7(K_0, K_3, K_5, K^*)$. *Besides, for a sufficiently small* h *and* $2 \le m \le r$,

$$\|v^h\|_{L^\infty(H^1)} \le 2K_1 \le K^* \tag{4.10}$$

and therefore K_6 *as well as* K_7 *can be chosen independent of* K^*.

Proof. From Lemma 3.1 and Theorem 4.1, we obtain

$$\begin{aligned}
&\|\zeta\|_{L^\infty(L^2)} + \|\sigma\|_{L^\infty(L^2)} + \beta\|\sigma\|_{L^2(L^2)} \\
&\quad \le \|\eta\|_{L^\infty(L^2)} + \|\theta\|_{L^\infty(L^2)} + \|\xi\|_{L^\infty(L^2)} + \|\psi\|_{L^\infty(L^2)} \\
&\qquad + \beta\|\xi\|_{L^2(L^2)} + \beta\|\psi\|_{L^2(L^2)} \\
&\quad \le K_4(K_1, K_3, K^*, \lambda)h^m + K_0K_3h^m + K_0K_3h^m + \beta K_0K_3h^m \\
&\quad \le K_6 h^m.
\end{aligned}$$

By Lemma 3.1, Theorem 4.1 and 4.2, we get

$$\begin{aligned}
&\|\zeta\|_{L^\infty(H^1)} + \|\sigma\|_{L^\infty(H^1)} + \|\sigma_t\|_{L^2(L^2)} \\
&\qquad = \|\eta - \theta\|_{L^\infty(H^1)} + \|\xi - \psi\|_{L^\infty(L^2)} + \|\xi_t - \psi_t\|_{L^2(L^2)} \\
&\qquad \le \|\eta\|_{L^\infty(H^1)} + \|\xi\|_{L^\infty(L^2)} + \|\xi_t\|_{L^2(L^2)} + \|\theta\|_{L^\infty(H^1)} \\
&\qquad\quad + \|\psi\|_{L^\infty(L^2)} + \|\psi_t\|_{L^2(L^2)} \\
&\qquad \le K_7 h^{m-1}.
\end{aligned}$$

Furthermore from (2.8) and the above inequality, we get

$$\begin{aligned}\|v^h\|_{L^\infty(H^1)} &\le \|v\|_{L^\infty(H^1)} + \|v - v^h\|_{L^\infty(H^1)}\\ &\le K_1 + \|\sigma\|_{L^\infty(H^1)}\\ &\le K_1 + K_7 h^{m-1}\\ &\le 2K_1 \le K^*,\end{aligned}$$

for a sufficiently small h. Now the proof is completed.

Finally, the Galerkin approximation of the solution $U(y,\tau)$, $S(\tau)$ of the problem (P1) can be define as

$$U^h(y,\tau) = u^h(x,t) \tag{4.11}$$

$$S^h(\tau) = s^h(t) \tag{4.12}$$

$$U_y(y^h,\tau^h) = v^h(x,t)\frac{1}{s^h(t)} \tag{4.13}$$

where y and τ are given as follows :

$$y = s^h(t)x \tag{4.14}$$

$$\tau = \tau^h(t). \tag{4.15}$$

Theorem 4.4. *Assume that the assumption of Theorem 4.3, and the regularity conditions for U and S hold. Then the following estimates hold for $2 \le m \le r$,*

$$\|S - S^h\|_{L^\infty(0,T_0)} = O(h^m) \tag{4.16}$$

$$\|\tau - \tau^h\|_{L^\infty(0,T_0)} = O(h^m) \tag{4.17}$$

$$\|U - U^h\|_{L^\infty(0,T_0;H^j(\bar\Omega(\tau)))} = O(h^{m-j}) \quad \text{for} \quad j = 0, 1, \tag{4.18}$$

where $\bar{\Omega}(\tau) = (0, min(S(\tau), S^h(\tau)))$ *for* $\tau \in (0, T_0)$.

Proof. Subtracting (2.17) from (2.6) and integrating with respect to t yield

$$
\begin{aligned}
|s - s^h| &\le \int_0^t |v(1)s - v^h(1)s^h| dt' \\
&\le \int_0^t |v(1)s - \bar{v}(1)s| + |v^h(1)s - \bar{v}(1)s| dt' \\
&\quad + \int_0^t |v^h(1)||s - s^h| dt' \\
&\le \int_0^t [|\xi(1)| + |\psi(1)|]|s| dt' + \int_0^t |v^h(1)||s - s^h| dt'
\end{aligned}
$$

Applying Gronwall's inequality, we obtain

$$|s - s^h| \le K(K_1, K^*) \int_0^t [|\xi(1)| + |\psi(1)|] dt'$$

and so, we get

$$\|s - s^h\|_{L^\infty} \le K(K_1, K_3, K_4, K^*) h^m \quad \text{for } 2 \le m \le r.$$

The estimate (4.15) follows immediately from the fact that

$$\|S - S^h\|_{L^\infty(0,T_0)} = \|s - s^h\|_{L^\infty(0,T)}$$

Moreover, by using (4.16), the estimate (4.17) can be proved. Finally, we will show that

$$\|U - U^h\|_{L^\infty(0,T_0; H^j(\bar{\Omega}(\tau)))} = O(h^m) \quad \text{for} \quad j = 0, 1.$$

For $j = 0$, we have

$$
\begin{aligned}
\|U(y,\tau) - U^h(y^h,\tau^h)\| &\le \|U(y,\tau) - U(y^h,\tau)\| + \|U(y^h,\tau) - U(y^h,\tau^h)\| \\
&\quad + \|U(y^h,\tau^h) - U^h(y^h,\tau^h)\| \\
&\le \|U_y(y',\tau)(y-y^h)\| + \|U_\tau(y^h,\tau')(\tau-\tau^h)\| \\
&\quad + \|u(x,t) - u^h(x,t)\| \\
&\le K_1|s-s^h| + K_1|\tau-\tau^h| + \|\zeta\|_{L^2} \\
&\le K(K_1,K_3,K_4,K_6)h^m.
\end{aligned}
$$

Similarly, we have

$$
\|U_y(y,\tau) - U_y^h(y^h,\tau^h)\| \le K(K_1,K_3,K_4,K_6)h^m.
$$

Hence by Lemma 3.1 and Theorem 4.1 the estimate (4.18) is obtained. This completes the proof.

5. Global existence for semidiscrete approximations

In this section we consider the global existence of semidiscrete approximations $\{u^h, v^h\}$. Let us recall (4.5) and (3.2) to obtain

$$
\begin{aligned}
&(\psi_t, z) + B(v;\psi,z) \\
&\qquad = (\xi_t, z) + \psi(1)(xv^h, z_x) - \xi(1)(xv^h, z_x) + \lambda(\psi,z) - \lambda(\xi,z). \qquad (5.1)
\end{aligned}
$$

Replacing v^h by $v - E$ in (5.1), for any $E = E(x,t) \in H^1_-$, we obtain

$$
\begin{aligned}
(\psi_t, z) + B(v;\psi,z) &= (\xi_t, z) + \psi(1)(xv - xE, z_x) \\
&\quad - \xi(1)(xv - xE, z_x) + \lambda(\psi,z) - \lambda(\xi,z) \quad \forall z \in S^h_-. \qquad (5.2)
\end{aligned}
$$

This is a linear ordinary differential equation in ψ as a function of t. Therefore there exists a unique solution ψ in the interval $[0, T]$ of (5.2) with $\psi(x, 0) = 0$. This equation defines as operator $\Im$ such that $\psi = \Im(E)$ for each $E \in H^1_-$.

Since $\sigma = \xi - \psi$, then

$$\sigma = \xi - \Im(E) \quad \text{for each } E \in H^1_- \tag{5.3}$$

To claim the existence of a solution v^h in (2.17), we need to show that the operator $\Im$ in equation (5.3) has a fixed point.

Theorem 5.1. *Assume that the finite element space S^h_+(or S^h_-) satisfies the inverse property (2.15) and u is the unique solution of (2.3) - (2.5) satisfying the regularity condition (2.8). For any $\rho > 0$, there exists a solution $v^h \in S^h_-$ of (2.16) satisfying $\|v - v^h\|_{L^\infty(H^1)} \le \rho$ for a sufficiently small h.*

Proof. Setting $s = \psi$ in (5.2), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\psi\|^2 + \alpha\|\psi\|_1^2 \le \|\xi_t\|\|\psi\| + |\psi(1)|(\|v\| + \|E\|)\|\psi_x\|$$
$$+ |\xi(1)|(\|v\| + \|E\|)\|\psi_x\| + \lambda\|\psi\|^2 + \lambda\|\xi\|\|\psi\|$$

And so we obtain

$$\frac{d}{dt}\|\psi\|^2 + 2\alpha\|\psi\|_1^2 \le \|\xi_t\|^2 + \|\psi\|^2 + 2\sqrt{2}(K_1 + \|E\|)\|\psi\|^{\frac{1}{2}}\|\psi_x\|^{\frac{3}{2}}$$
$$+ 2(K_1 + \|E\|)|\xi(1)|\|\psi_x\| + 2\lambda\|\psi\|^2 + 2\lambda\|\xi\|\|\psi\|$$
$$\le \|\xi_t\|^2 + \|\xi\|^2 + (K_1 + \|E\|)^2|\xi(1)|^2$$
$$+ \{1 + \frac{16}{3}(K_1 + \|E\|)^4 + 2\lambda + \lambda^2\}\|\psi\|^2 + (\frac{3}{4}\epsilon^{\frac{4}{3}} + 1)\|\psi_x\|^2.$$

This implies that for sufficiently small $\epsilon > 0$

$$\frac{d}{dt}\|\psi\|^2 + \beta\|\psi\|_1^2 \leq \|\xi_t\|^2 + \|\xi\|^2 + (K_1 + \|E\|)^2|\xi(1)|^2$$
$$+ \{1 + \frac{16}{3}(K_1 + \|E\|)^4 + 2\lambda + \lambda^2\}\|\psi\|^2$$

and so

$$\|\psi\|^2 + \beta\int_0^t \|\psi\|_1^2 dt' \leq \int_0^t [\|\xi_t\|^2 + \|\xi\|^2 + (K_1 + \|E\|)^2|\xi(1)|^2]dt'$$
$$+ K(K_1, E, \lambda)\int_0^t \|\psi\|^2 dt'.$$

By Gronwall's inequality

$$\|\psi\|^2 + \beta\int_0^t \|\psi\|_1^2 dt' \leq K(K_1, K_3, E)h^{2m} \quad \text{for } 2 \leq m \leq r.$$

Therefore we have

$$\|\psi\|_{L^\infty(L^2)} + \beta\|\psi\|_{L^2(H^1)} \leq K(K_1, K_3, E)h^m \quad \text{for } 2 \leq m \leq r.$$

and by using Theorem 4.2, we obtain

$$\|\sigma\|_{L^\infty(H^1)} = \|\xi - \psi\|_{L^\infty(H^1)} \leq K(K_3, K_5)h^{m-1} \quad \text{for } 2 \leq m \leq r.$$

Thus for a sufficiently small h, we have

$$\|\sigma\|_{L^\infty(H^1)} \leq \rho$$

Consequently the mapping $\Im$ defined by (5.3) maps a ball

$$B_\rho = \{w \in L^\infty(H^1) \mid \|w\|_{L^\infty(H^1)} \le \rho\}$$

into itself for a sufficiently small h. Thus Schauder's fixed point theorem guarantees the existence of an E with $\sigma = E$. This completes the proof.

Remark. The global existence of u^h follows from the global existence of v^h and (2.15).

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