Fixed Point Theorems for Involution Maps in 2-Banach Spaces

2-바나흐공간내에서 대합사상에 대한 부동점 정리

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요 약

본 논문은 바나흐공간내에서 Khamsi의 1-local retract을 이용하여 Kim-Kirk에 의하여 연구된 어떤 부동점 정리를 2-바나흐공간으로 일반화시킨 것이다. 2절에서는 Gähler에 의하여 소개된 2-노름의 개념, 정의 및 예제를 소개한다. 3절에서는 먼저 Khamsi에 의하여 소개된 1-local retract의 개념을 소개한다. 유사한 방법으로 2-노름공간에서 1-local retract의 개념을 정의한 후 2-노름과 관련된 몇몇 성질들을 밝힌다. 마지막 4절에서는Assad-Sessa에 의하여 소개된 어떤 축약조건을 적용하여 2-바나흐공간내에서 대합사상(involution maps)에 대한 부동점정리를 증명한다. 이것은 Khan-Khan에 의하여 얻어진 결과를 곧바로 확장하는 것이다. 또한 본 연구의 주 정리에 대한 여러 가지 응용을 다루며, 우리의 정리를 뒷받침하는 유클리드 2-바나흐공간에서 한 예를 소개한다.

1. Introduction

Let K be a nonempty subset of a Banach space X. A mapping $T : K \to X$ is called *k-lipschitzian* if $||Tx - Ty|| \leq k \; ||x - y||$ for all $x, y \in K$. It is called *nonexpansive* if the same condition with $k = 1$ holds. A mapping $T : K \to K$ is called an *involution* if $T^2 = I$, where I denotes the identity map. Recall that the *modulus of convexity* of X is the function $\delta_X : [0, 2] \to [0, 1]$ defined by

$$\delta_X(\varepsilon) = \inf\{1 - \frac{||x + y||}{2} : ||x||, \; ||y|| \leq 1, ||x - y|| \geq \varepsilon\}.$$

Goebel [4] and Geobel-Złotkiewicz [6] investigated that if K is a closed convex subset of a Banach space and if a mapping $T : K \to K$ is k-lipschitzian involution where k satisfies

$$\frac{k}{2}\left(1 - \delta\left(\frac{2}{k}\right)\right) < 1,$$

(called as *Goebel's Lipschitz condition* in [12]), then T have a fixed point in K. The proofs of these facts are straightforward verifications that starting from any $x \in K$, the sequence of iterates $\{G^n(x)\}$ for $G = \frac{1}{2}(I + T)$ always converges to a fixed point of T. Later on, Assad-Sessa [1] extended a fixed point theorem of Goebel-Złotkiewicz [6] to an involution mapping satisfying the contractive condition introduced by Delbosco [2]. Also, Górniki [7] revisited the theorem due to [6] to establish some fixed point theorems of k-lipschitzian involutions. Gähler[3] introduced the concept of 2-metric spaces and studied some examples and topological properties for such spaces. Khan-Khan [11] established an analogue of a fixed point theorem due to Assad-Sessa [1] on such 2-Banach spaces. In section 2 of this paper, we will give some definitions and an example relating to 2-normed spaces. In section 3, we introduce the well-known concept originally Khamsi [10], called a 1-*local retract*, and shall construct the 2-norms' version and present some properties related with 1-local retract and 2-norms. Finally, in section 4, we shall prove a fixed point theorem (see Theorem 4.2) for an involution map in 2-Banach spaces which is an extension of one proved earlier by Khan-Khan [11]. Next we shall give some applications of this theorem in 2-Banach spaces and a sharper example to support our main theorem 4.2.

2. Preliminaries and examples

In this section, we introduce some concepts and properties of 2-normed spaces. The following notions are essentially due to Gähler [3].

Definition 2.1. Let X be a linear space, and $\|\cdot,\cdot\|$ be a real-valued function defined on X. Then the pair $(X, \|\cdot,\cdot\|)$ is called a *2-normed space* if, for $a, b, c \in X$,

(i) $\|a, b\| = 0$ if and only if a and b are linearly dependent,

(ii) $\|a, b\| = \|b, a\|$,

(iii) $\|a, \beta b\| = |\beta| \|a, b\| \qquad (\beta \in \mathbb{R})$,

(iv) $\|a, b + c\| \leq \|a, b\| + \|a, c\|$.

Here $\|\cdot,\cdot\|$ is called a *2-norm* and is a non-negative function.

Let X be a 2-normed space. For all real $r > 0$, the set

$$U_r(a, b) = \{x \in X : \|x - a, b - a\| < r\}$$

will be called a *r-neighborhood* of two points $a, b \in X$. Obviously, $U_0(a, b) = \emptyset$ and $U_r(a, b)$ always contains the line joining a and b. Note also that $U_r(a, b) = X$ for $r > 0$ in a case that $a = b$. For more detailed topological properties, see [3].

First, consider an example for 2-norms in the n-dimensional Euclidean space.

EXAMPLE 2.1. Let $X := \mathbb{R}^n$ be a $n(\geq 2)$-dimensional Euclidean space. Let $k > 0$ be fixed. For $a = (a_1, a_2, \dots, a_n)$ and $b = (b_1, b_2, \dots, b_n)$ in $\mathbb{R}^n$, define

$$\|a, b\|_k = k\left\{\sum_{i<j} \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}^2\right\}^{1/2}.$$

Then the function $\|\cdot, \cdot\|_k$ is just a 2-norm on X.

REMARK 2.1. Since the usual norm on $\mathbb{R}^n$ is given by

$$\|a - b\| = \left(\sum_{i=1}^{n}(a_i - b_i)^2\right)^{1/2}$$

the following relation between a 2-norm $\|\cdot, \cdot\|_1$ $(k = 1)$ and the usual norm on $\mathbb{R}^n$ is easily observed:

$$\|a - b\| = \left(\frac{1}{n-1}\sum_{i=1}^{n}\|e_i, a - b\|_1^2\right)^{1/2}$$

where the unit vectors $e_i = (0, 0, \dots, \overset{i}{1}, 0, \cdots, 0)$ $(i = 1, 2, \dots, n)$ form a normalized basis of $\mathbb{R}^n$. Also, we note that for $r > 0$ and $k > 0$, r-neighborhood of a and b in $\mathbb{R}^n$

$$U_r(a, b) = \{x \in X : \|x - a, b - a\|_k < r\}$$

denotes an (infinitely long) open cylinder, formed with the axis going through a and b, and radius $\rho = \frac{r}{k\|a-b\|}$. This shows the relation between the usual distance $\|a - b\|$ and the cylinder's size, which means the cylinder approaches to the whole space as $\|a - b\| \to 0$, while the cylinder contracts near the axis going through a and b as $\|a - b\| \to \infty$.

DEFINITION 2.2. A sequence $\{x_n\}$ in a 2-normed space X is called a *convergent sequence* if there is an element $x \in X$ such that the $\lim_{n\to\infty} \|x_n - x, a\| = 0$ for all $a \in X$. If $\{x_n\}$ converges to x, we write $x_n \to x$ and call x the *limit* of $\{x_n\}$. Of course, here $dimX \geq 2$ otherwise every sequence of points in such a space would converge to every point of the space.

DEFINITION 2.3. A sequence $\{x_n\}$ in a 2-normed space X is called a *Cauchy sequence* if $\lim_{n,m\to\infty} \|x_m - x_n, a\| = 0$ for every $a \in X$. A 2-normed space in which every Cauchy sequence is a convergent sequence is called a 2-*Banach space.*

We also need the following notion from Assad-Sessa [1]. Let $\mathbb{R}_+$ be the set of nonnegative real numbers, i.e., $\mathbb{R}_+ = [0, \infty)$ and let Φ be the family of continuous functions $\phi : \mathbb{R}_+^3 \to \mathbb{R}_+$ satisfying the following properties:

(i) $\phi(1, 1, 1) := c < 2$.

(ii) for $s \geq 0,\ t \geq 0$, the inequality $s \leq \phi(t, 2t, s)$ implies that $s \leq ht$ for some $h \in [c, 2)$.

Here we give some examples of functions belonging in Φ.

EXAMPLE 2.2. For $p, q, r \in \mathbb{R}_+$ define

$$\phi_1(p, q, r) = \alpha \max\{2p, q, r\} \tag{2.1}$$

with $0 \leq \alpha < 1$, or

$$\phi_2(p,q,r) = \alpha\, p + \beta\, q + \gamma\, r \tag{2.2}$$

where $\alpha, \beta \geq 0, 0 \leq \gamma < 1$, $1 \leq \alpha + \beta + \gamma$ and $\alpha + 2(\beta + \gamma) < 2$. Then $\phi_i \in \Phi$ $(i = 1, 2)$ is obvious. Indeed, take $c = h = 2\alpha < 2$ in (2.1), while we note $c = \alpha + \beta + \gamma \leq \alpha + 2(\beta + \gamma) < 2$ and $h = \frac{\alpha+2\beta}{1-\gamma} \in [c, 2)$ in (2.2).

Assad-Sessa [1] developed the above notion to prove the following fixed theorem for involution self-maps in Banach spaces: Let C be a closed convex subset of a Banach space X and $T : C \to C$ be an involution and assume that there exists a $\phi \in \Phi$ such that

$$\|Tx - Ty\| \leq \phi(\|x - y\|, \|x - Tx\|, \|y - Ty\|)$$

for all $x, y \in C$. Then T has a fixed point in C. In particular, taking $\phi(p,q,r) = (\frac{\alpha}{2}+2\beta)\max\{2p, q, r\}$ for all $(p,q,r) \in \mathbb{R}^3_+$ and applying the above theorem due to Assad-Sessa [1] with this $\phi \in \Phi$ and $h = k = \alpha + 4\beta < 2$, we easily obtain the fixed point theorem for involution self-maps originally due to Iseki [8]: Let C be a closed convex subset of a Banach space X and $T : C \to C$ be an involution such that

$$d(Tx, Ty) \leq \alpha d(x,y) + \beta[d(x, Tx) + d(y, Ty)]$$

for all $x, y \in C$, where $\alpha \geq 0$, $\beta \geq 0$ and $\alpha + 4\beta < 2$, then T has a fixed point in C.

3. Some properties of 1-local retract in 2-normed spaces

In this section we introduce the well-known concept originally due to Khamsi [10], called a 1-*local retract*, and shall construct the 2-norm version in 2-normed spaces. Next, we shall give some properties of such a concept in 2-normed spaces.

Let X be a normed space and $F \subset K \subset X$. Then recall that F is said to be a 1-*local retract* of K if every family $\{B_i; i \in I\}$ of closed balls centered at point of F has the property: $(\bigcap_{i\in I} B_i) \cap K \neq \emptyset \Rightarrow (\bigcap_{i\in I} B_i) \cap F \neq \emptyset$. This concept is due to Khamsi ([9],[10]), who used it to prove the existence of common fixed points for commuting families of nonexpansive mappings in a more general context. It is easy to see that a 1-local retract of a convex set is metrically convex, and a 1-local retract of a closed set must itself be closed. It is easy to check [10] that nonexpanisve retracts are always 1-local retracts (but not conversely).

Now let us consider the 2-norm version of the concept and properties introduced above. Let X be a 2-normed space. Recall from [3] that the family

$$\{W_\Sigma(a) := \bigcap_{i=1}^{n} U_{r_i}(a, b_i) \,|\, \Sigma = \{(b_i, r_i) : i = 1, 2, \ldots, n\}, n \in \mathbb{N}\}$$

forms the neighborhood system of $a \in X$, where $U_{r_i}(a, b_i) = \{x \in X : \|x - a, b_i - a\| < r_i\}$ $(i = 1, 2, \ldots, n)$ for a finitely many points $b_1, b_2, \ldots, b_n \in X$.

Denote by K' the set of all *accumulation points* of $K(\neq \emptyset) \subset X$. Then note that

$$(3.2) \quad a \in K' \quad \text{iff} \quad \forall\, \Sigma = \{(b_i, r_i) : i = 1, 2, \ldots, n\},\ W_\Sigma(a) \cap K \setminus \{a\} \neq \emptyset.$$

This is immediately equivalent to the following fact: For every finitely many points $b_1, \ldots, b_n \in X$, there exists a sequence $\{x_j\}$, $x_j(\neq a) \in K$, such that

$$\lim_{j \to \infty} \|x_j - a, b_i - a\| = 0$$

for all $i = 1, 2, \ldots, n$. Now consider the *closure* of K, that is, $\overline{K} = K' \cup K$. Also, we similarly note that

$$(3.3) \qquad a \in \overline{K} \quad \text{iff} \quad \forall\, \Sigma = \{(b_i, r_i) : i = 1, 2, \ldots, n\},\ W_\Sigma(a) \cap K \neq \emptyset.$$

Now we introduce the following characterization of $\overline{K}$, the closure of K, which is originally due to Gähler [3]. Here, we shall give a modified proof in 2-normed spaces for the sake of convenience.

PROPOSITION 3.1 ([3]). *Let K be a nonempty subset of a 2-normed space X. Then*

$$\overline{K} = \bigcap_{\Sigma} W_\Sigma(K),$$

where $W_\Sigma(K) = \cup_{a \in K} W_\Sigma(a)$.

Proof. Let $x \in K$. From (3.3), for every $\Sigma = \{(b_i, r_i) : i = 1, 2, \ldots, n\}$ there exists a $x_\Sigma \in K$ such that

$$x_\Sigma \in W_\Sigma(x) = \bigcap_{i=1}^{n} U_{r_i}(x, b_i).$$

For all $i = 1, 2, \ldots, n$, since

$$x_\Sigma \in U_{r_i}(x, b_i) \quad \text{iff} \quad x \in U_{r_i}(x_\Sigma, b_i) \tag{3.4}$$

from (iii) of Definition 2.1, this immediately yields $x \in W_\Sigma(x_\Sigma)$ and hence $x \in W_\Sigma(K)$ for the Σ. This gives $\overline{K} \subset \cap_\Sigma W_\Sigma(K)$. On the other hand, for the converse inclusion, let $x \in \cap_\Sigma W_\Sigma(K)$. Then for every Σ there exists $x_\Sigma \in K$ such that $x \in W_\Sigma(x_\Sigma)$. Immediately, it follows from (3.4) that $x_\Sigma \in W_\Sigma(x)$. Hence $x \in \overline{K}$ and the proof is complete. □

We say that a subset K of X is *closed* if $\overline{K} = K$. As the usual notation, different from the r-neighborhood $U_r(a, b)$ of $a, b \in X$, we set

$$B_r(a, b) = \{x \in X : \|x - a, b - a\| \le r\}.$$

Is the set $B_r(a, b)$ closed? Here we present a positive answer for this question.

PROPOSITION 3.2. *Let X be a 2-normed space. For $r \ge 0$ and $a_0, b_0 \in X$, $B_r(a_0, b_0)$ is a closed set in X.*

Proof. If $r = 0$, it is obvious. Now let $r > 0$ and $K := B_r(a_0, b_0)$. We claim that

$$x \in \overline{K} = \bigcap_{\Sigma} W_{\Sigma}(K) \quad \Rightarrow \quad x \in K.$$

Let $\Sigma_n := \{(a_0, 1/n), (b_0, 1/n)\}$ for each $n \in \mathbb{N}$ and $x \in \overline{K}$. By above property, it must be $x \in W_{\Sigma}(K) = \cup_{a \in K} W_{\Sigma}(a)$. That is, there exists $a \in K$ such that $x \in W_{\Sigma}(a) = U_{1/n}(a, a_0) \cap U_{1/n}(a, b_0)$. This implies

$$\|x - a, a_0 - a\| < 1/n \quad \text{and} \quad \|x - a, b_0 - a\| < 1/n. \tag{3.5}$$

Since $a \in K = B_r(a_0, b_0)$, we have

$$\|a - a_0, b_0 - a_0\| \leq r.$$

This combined with (3.5) and properties of Definition 2.1 yield

$$\begin{aligned}
\|x - a_0, b_0 - a_0\| &= \|(x - a) + (a - a_0), b_0 - a_0\| \\
&\leq \|x - a, b_0 - a_0\| + \|a - a_0, b_0 - a_0\| \\
&= \|x - a, (b_0 - a) + (a - a_0)\| + \|a - a_0, b_0 - a_0\| \\
&\leq \|x - a, b_0 - a\| + \|x - a, a - a_0\| + \|a - a_0, b_0 - a_0\| \\
&< 1/n + 1/n + r = r + 2/n
\end{aligned}$$

for all Σ_n. Since n is arbitrarily given, it follows that $\|x - a_0, b_0 - a_0\| \leq r$ and so $x \in K$, which completes the proof. $\square$

Now consider the analogous concept of 1-*local retract* in a 2-normed space. How can we explain the closed balls in such a 2-Banach space? Let X be a 2-normed space. For a fixed $a \in X$, set

$$B_{r_a}(a) = \bigcap_{\Sigma} B_{\Sigma}(a),$$

where

$$B_{\Sigma}(a) = \bigcap_{i=1}^{n} B_{r_{(\Sigma,a)}}(a, b_i)$$

for every finite set $\Sigma = \{b_1, b_2, \dots, b_n\}$ in X and $r_a = \inf_{\Sigma} r_{(\Sigma,a)} \geq 0$. From now on, the set $B_{r_a}(a)$ in X will called the closed *ball* centered at $a \in X$ with radius $r(a)$. Note especially that if $r_{(\Sigma,a)} = r$ for all Σ, then $r_a = r$ and so $B_r(a) = \cap_{b \in X} B_r(a, b)$.

As before, if $F \subset K \subset X$, then F is said to be a 1-*local retract* of K if every family $\{B_i; i \in I\}$ of closed balls centered at point of F has the property: $(\cap_{i \in I} B_i) \cap K \neq \emptyset \Rightarrow (\cap_{i \in I} B_i) \cap F \neq \emptyset$.

As an analogous version, we say that $T : K \to K$ is k-lipschtzian if $\|Tx - Ty, a\| \leq k\|x - y, a\|$ for all $x, y \in K$ and $a \in X$. It is called *nonexpansive* it the same condition with $k = 1$ holds. Also K is said to be a *retract* of X if there exists a *retraction* r of X onto K, that is, a continuous mapping $r : X \to K$ such that

$$r(x) = x \quad (x \in K).$$

If r is nonexpansive, then K is said to be a *nonexpansive retract* of X. Then we have the following easy result.

PROPOSITION 3.3. *Let X be a 2-normed space. Each nonexpansive retract A of X is a 1-local retract of X.*

Proof. Let $\{B_{r_\alpha}(x_\alpha)\}$ be a family of closed balls $B_{r_\alpha}(x_\alpha)$, each of radius $r_\alpha = \inf_\Sigma r_{(\Sigma,x_\alpha)}$, centered at $x_\alpha \in A$ for every α. Suppose that

$$\bigcap_\alpha B_{r_\alpha}(x_\alpha) \neq \emptyset$$

and let r be a nonexpansive retract of X onto A. Then it is obvious that for $z \in \cap_\alpha B_{r_\alpha}(x_\alpha)$, $r(z) \in A \cap [\cap_\alpha B_{r_\alpha}(x_\alpha)]$. Indeed, clearly $r(a) \in A$. For every α and $\Sigma = \{b_1, b_2, \ldots, b_n\}$,

$$\begin{aligned} \|r(z) - x_\alpha, b_i - a\| &= \|r(z) - r(x_\alpha), b_i - a\| \\ &\leq \|z - x_\alpha, b_i - a\| \leq r_{(\Sigma,x_\alpha)} \end{aligned}$$

for every $i = 1, 2, \ldots, n$ and so

$$r(z) \in \cap_{i=1}^n B_{r_{(\Sigma,x_\alpha)}}(x_\alpha, b_i) = B_\Sigma(x_\alpha)$$

for $\Sigma = \{b_1, b_2, \ldots, b_n\}$. Since Σ is arbitrarily given and $r_\alpha = \inf_\Sigma r_{(\Sigma,x_\alpha)}$,

$$r(z) \in \bigcap_\Sigma B_\Sigma(x_\alpha) = B_{r_\alpha}(x_\alpha)$$

for every α. Therefore, $r(z) \in \cap_\alpha B_{r_\alpha}(x_\alpha)$. $\square$

In normed space, every 1-local retract of a closed set must itself be closed. Is it possible in 2-normed spaces? First Gähler [3] introduced the following property,

PROPERTY (K). *Let* $\{x_n\}$ *be a sequence in a 2-normed space* X *and* $x \in X$. *If there exists two points* b *and* c *in* X *such that*

$$\|c-x, b-x\| \neq 0, \quad \lim_{i\to\infty} \|x_i - x, b-x\| = 0 \quad \textit{and} \quad \lim_{i\to\infty} \|x_i - x, c-x\| = 0,$$

then $\lim_{i\to\infty} \|x_i - x, a-x\| = 0$ *for all* $a \in X$.

PROPOSITION 3.4. *Let* K *be a closed subset of a 2-normed space* X, *and suppose* $A(\subset K)$ *is a 1-local retract of* K. *Then* A *is closed in* X.

Proof. We claim: $x \in \overline{A} \Rightarrow x \in A$. Let $x \in \overline{A}$. Using the easy equivalent form of (3.3), for every finitely many points $b_1, b_2, \ldots, b_n \in X$ there exists a sequence $\{x_n\}$ in A such that $\lim_{j\to\infty} \|x_j - x, b_i - x\| = 0$ for all $i = 1, \ldots, n$. Let $b \in X$. Taking $b_i = b + x$ for all i yields

$$\lim_{j\to\infty} \|x_j - x, b\| = 0$$

for every $b \in X$. That means, for any $k \in \mathbb{N}$, there is $N_k \in \mathbb{N}$ such that $\|x_j - x, b\| < 1/k$ for all $j \geq N_k$ and $b \in X$. In particular,

$$\|x_{N_k} - x, a - x_{N_k}\| < 1/k \leq 1/k \tag{3.6}$$

and so $x \in B_{1/k}(x_{N_k}, a)$ for every $a \in X$. For each $k \in \mathbb{N}$, setting $\Sigma = \{(a, 1/k)\}$, $r_{(\Sigma, x_{N_k})} = 1/k$ and $r_k = \inf_\Sigma r_{(\Sigma, x_{N_k})} = 1/k$, we have $x \in B_\Sigma(x_{N_k})$ for every $\Sigma = \{(a, 1/k)\}$. Therefore $x \in B_{r_k}(x_{N_k}) = B_{1/k}(x_{N_k})$ for $k \in \mathbb{N}$. Since K is closed, $x \in K$ naturally. This implies

$$x \in K \cap \bigcap_{k=1}^{\infty} B_{1/k}(x_{N_k}).$$

Since A is 1-local retract of K, it must be

$$A \cap \bigcap_{k=1}^{\infty} B_{1/k}(x_{N_k}) \neq \emptyset.$$

On choosing $z \in A \cap \bigcap_{k=1}^{\infty} B_{1/k}(x_{N_k})$, we can prove $x = z$. For this end, from $z \in B_{1/k}(x_{N_k}) = \cap_{a \in X} B_{1/k}(x_{N_k}, a)$, it follows that

$$\|z - x_{N_k}, a - x_{N_k}\| \leq 1/k$$

for every $a \in X$. This combined with (3.6) yields

$$\|z - x, a - x_{N_k}\| \leq \|z - x_{N_k}, a - x_{N_k}\| + \|x_{N_k} - x, a - x_{N_k}\| \leq 2/k$$

for all $a \in X$. In particular, we have $\|z - x, c\| \leq 2/k$ for every $c \in X$ and $k \in \mathbb{N}$. Letting $k \to \infty$ gives $\|z - x, c\| = 0$ for every $c \in X$. Since $dim X \geq 2$, the only way $z - x$ can be linearly dependent with all $a \in X$, that is, $z - x = 0$. Since $z \in A$, this yields $x \in A$ and completes the proof. □

By looking over the proof of Lemma 2.2 in [11] in Banach spaces we can similarly prove the following result.

PROPOSITION 3.5. *Let K be a nonempty subset of a 2-normed space X. Suppose K is a nonempty 1-local retract of $\overline{conv}(K)$, where $\overline{conv}(K)$ means the closed convex hull of K. Let $T : K \to K$ be a mapping. Then there exists a mapping $G : K \to K$ such that for each $x \in K$*

$$\|x - Gx, a\| = \|Gx - Tx, a\| = \frac{1}{2}\|x - Tx, a\|, \tag{3.7}$$

and

$$\|z - Gx, a\| \leq \max\{\|z - x, a\|, \|z - Tx, a\|\} \tag{3.8}$$

for all $z \in K$ and $a \in X$.

Proof. Let $x \in K$. If $Tx = x$, then take $Gx = x$. Now assume that $Tx \neq x$. For every $z \in K$ and $\Sigma = \{b_1, b_2, \dots, b_n\}$, we set

$$r_{(\Sigma,z)} = \max\{\|x - z, b_i - z\|, \|Tx - z, b_i - z\| : b_i \in \Sigma\}$$

$$r_{(\Sigma,x)} = \max\{\|x - Tx, b_i - x\|/2 : b_i \in \Sigma\}$$

$$r_{(\Sigma,Tx)} = \max\{\|x - Tx, b_i - Tx\|/2 : b_i \in \Sigma\}$$

and $r_a = \inf_\Sigma r_{(\Sigma,a)}$ for $a = z, x$ and Tx. First we can observe that for $a = z, x$ and Tx respectively,

$$\|(x + Tx)/2 - a, b_i - a\| \leq r_{(\Sigma,a)}$$

and so $(x + Tx)/2 \in B_{r_{(\Sigma,a)}}(a)$. Since Σ is arbitrarily given and $r_a = \inf_\Sigma r_{(\Sigma,a)}$,

$$(x + Tx)/2 \in \bigcap_\Sigma B_{r_{(\Sigma,a)}}(a) = B_{r_a}(a)$$

for $a = z, x$ and Tx respectively. This immediately implies

$$(x + Tx)/2 \in \bigcap_{z \in K} B_{r_z}(z) \cap B_{r_x}(x) \cap B_{r_{Tx}}(Tx).$$

Since K is a 1-local retract of $\overline{conv}(K)$, it must be

$$\bigcap_{z \in K} B_{r_z}(z) \cap B_{r_x}(x) \cap B_{r_{Tx}}(Tx) \cap K \neq \emptyset$$

Defining an element in the above nonempty set by Gx, we obtain the required mapping $G : K \to K$, which (3.7) and (3.8) are satisfied by taking $\Sigma = \{a\}$ specially for each $a \in X$. $\square$

Following the convention of [5], we shall use $\frac{1}{2}x \oplus \frac{1}{2}Tx$ to denote the point Gx for each $x \in K$.

4. Fixed Point Theorems for involutions

Let K be a nonempty subset of a 2-Banach space X and let $T : K \to K$ be a mapping. We say that a sequence $\{x_n\}$ satisfying $\|x_n - Tx_n, a\| \to 0$ for all $a \in X$ as $n \to \infty$ is called *approximate fixed point* (in short, a.f.p.) with respect to T. Further, it is called an *a.f.p. Cauchy* with respect to T if it is both Cauchy and a.f.p. with respect to T.

First we begin with the following easy lemma for our argument.

Lemma 4.1. *Let K be a nonempty closed subset of a 2-Banach space X with $dimX \geq 2$. Let $T : K \to K$ be a mapping. Assume that there exists a $\phi \in \Phi$ such that*

$$\|Tx - Ty, a\| \leq \phi(\|x - y, a\|, \|x - Tx, a\|, \|y - Ty, a\|) \tag{4.1}$$

for every $x, y \in K$ and $a \in X$. If there exists an a.f.p. Cauchy sequence in K with respect to T, then T has a fixed point in K.

Proof. Let $\{x_n\}$ be an a.f.p. Cauchy sequence in K with respect to T and let $x_n \to x^* \in K$. Since $\|Tx_n - x_n, a\| \to 0$ for every $a \in X$ as $n \to \infty$, we have

$$\begin{aligned}\|x_n - Tx^*, a\| &\leq \|x_n - Tx_n, a\| + \|Tx_n - Tx^*, a\| \\ &\leq \|x_n - Tx_n, a\| + \phi(\|x_n - x^*, a\|, \|x_n - Tx_n, a\|, \|x^* - Tx^*, a\|),\end{aligned}$$

for every $a \in X$ and each $n \in \mathbb{N}$. Since ϕ is continuous, it follows as $n \to \infty$ that for all $a \in X$

$$||x^* - Tx^*, a|| \leq \phi(0, 0, ||x^* - Tx^*, a||),$$

which in turn implies that $||x^* - Tx^*, a|| = 0$ for every $a \in X$ by the property (ii) above. Since $dimX \geq 2$, the only way $(x^* - Tx^*)$ can be linearly dependent with all $a \in X$, that is, $x^* - Tx^* = 0$. Hence $x^* = Tx^*$ as required. This completes the proof. □

THEOREM 4.1. *Let K be a nonempty closed subset of a 2-Banach space X with $dimX \geq 2$. Let $T : K \to K$ be a mapping. Assume that there exists a $\phi \in \Phi$ such that*

$$||Tx - Ty, a|| \leq \phi(||x - y, a||, ||x - Tx, a||, ||y - Ty, a||)$$

for all $x, y \in K$ and $a \in X$. If there exists a mapping $G : K \to K$ satisfying the following properties;

(a) $||TGx - Gx, a|| \leq \alpha \; ||Tx - x, a||$,

(b) $||Gx - x, a|| \leq \beta \; ||Tx - x, a||$

for all $x \in K$ and $a \in X$, where $0 \leq \alpha < 1$, $\beta > 0$, then T has a fixed point in K and $Fix(T) = Fix(G)$, where $Fix(T)$ denotes the set of all fixed points of T.

Proof. Let $x \in K$ be an arbitrary point and assume $x \neq Tx$. By using conditions (a) and (b), a simple calculation implies that $\|G^{n+1}x - G^n x, a\| \leq \beta \cdot \alpha^n \|Tx - x, a\|$ for every $a \in X$, and this immediately yields that $\{G^n x\}$ is a Cauchy sequence. By (a) it is obvious that $\{G^n x\}$ is an a.f.p. sequence in K with respect to T. Hence it is an a.f.p. Cauchy sequence in K with respect to T, as required in Lemma 4.1, and so T has a fixed point in K. By using (a) and (b) again we readily see that $Fix(G) = Fix(T)$. □

REMARK 4.1. Note that T is not continuous. By looking over the proofs of Lemma 4.1 and Theorem 4.1, we readily see that if $T : K \to K$ is a continuous mapping and if there exists a continuous mapping $G : K \to K$ satisfying the conditions (a) and (b) as in Theorem 4.1, then $Fix(T)$ is in fact a nonempty retract of K, that is, there exists a retraction $R : K \to Fix(T)$ such that $Rx = \lim_{n\to\infty} G^n x$ for each $x \in K$ and $TR = R = GR$.

Using Proposition 3.5, we can prove a fixed point theorem for involution maps in 2 -Banach spaces.

THEOREM 4.2. *Let X be a 2-Banach space, let K be a nonempty 1-local retract of $\overline{conv}(K)$. If $T : K \to K$ is an involution map satisfying the following property;*

$$\|Tx - Ty, a\| \leq \phi(\|x - y, a\|, \|x - Tx, a\|, \|y - Ty, a\|)$$

for all $x, y \in K$ and $a \in X$ and some $\phi \in \Phi$, then T has a fixed point in K.

Proof. Let $G : K \to K$ be defined by $Gx = \frac{1}{2}x \oplus \frac{1}{2}Tx$ for each $x \in K$ as in Proposition 3.5. To this end, it suffices to show that the mapping G satisfies two conditions (a) and (b) of Theorem 4.1 Let $x \in K$ and $a \in X$ be given. By (3.7) and $T^2 = I$, we have

$$\begin{aligned}\|x - TGx, a\| &= \|T^2x - TGx, a\| \\ &\leq \phi(\|Tx - Gx, a\|, \|Tx - x, a\|, \|Gx - TGx, a\|) \\ &= \phi(\|x - Gx, a\|, 2\|x - Gx, a\|, \|Gx - TGx, a\|)\end{aligned}$$

and similarly,

$$\|Tx - TGx, a\| \leq \phi(\|x - Gx, a\|, 2\|x - Gx, a\|, \|Gx - TGx, a\|).$$

By replacing z by TGx in (3.8), we get

$$\begin{aligned}\|TGx - Gx, a\| &\leq \max\{\|TGx - x, a\|, \|TGx - Tx, a\|\} \\ &\leq \phi(\|x - Gx, a\|, 2\|x - Gx, a\|, \|TGx - Gx, a\|).\end{aligned}$$

Combined with the property (ii), this immediately implies that

$$\begin{aligned}\|TGx - Gx, a\| &\leq h\|x - Gx, a\| \\ &= \frac{h}{2}\|x - Tx, a\|\end{aligned}$$

for all $x \in K$ and $a \in X$, where $h \in [k, 2)$. Since all assumptions of Theorem 4.1 with $a = h/2 < 1$ and $b = 1/2$ are fulfilled, T has a fixed point in K. $\square$

From now on, we will give some applications of Theorem 4.2. As the first direct consequence of Theorem 4.2, we shall give the following result, which reduces to a fixed point theorem of Khan and Khan [11] in only case $K = X$.

COROLLARY 4.1. *Let K be a nonempty closed convex subset of a linear 2-Banach space X. If $T : K \to K$ is an involution map satisfying the following property;*

$$\|Tx - Ty, a\| \leq \phi(\|x - y, a\|, \|x - Tx, a\|, \|y - Ty, a\|)$$

for all $x, y \in K$ and $a \in X$ and some $\phi \in \Phi$, then T has a fixed point in K.

Proof. Since K is closed and convex, $\overline{conv}(K) = K$. Obviously, K is itself 1-local retract of K, and so the consequence immediately follows from Theorem 4.2. □

Now we shall give an analogous 2-norm version of the result due to Assad and Sessa [1] which is a generalization of a fixed point theorem of Geobel and Złotkiewicz [6] in Banach spaces.

COROLLARY 4.2. *Let X be a 2-Banach space, let K be a nonempty 1-local retract of $\overline{conv}(K)$. If $T : K \to K$ is an involution map satisfying the following property;*

$$(4.2) \qquad \|Tx - Ty, a\| \leq \alpha \|x - y, a\| + \beta (\|x - Tx, a\| + \|y - Ty, a\|)$$

for all $x, y \in K$ *and* $a \in X$*, where* $\alpha, \beta \geq 0$ *and* $\alpha + 4\beta < 2$*, then* T *has a fixed point in* K*.*

Proof. Define $\phi(p,q,r) = (\frac{\alpha}{2} + 2\beta)\max\{2p, q, r\}$ for all $(p,q,r) \in \mathbb{R}^3_+$. Obviously, $\phi \in \Phi$ with $h = c = \alpha + 4\beta < 2$. By applying Theorem 4.2, T has a fixed point in K. $\square$

REMARK 4.1. We note that if K is the whole Banach space X, Corollary 4.2 reduces to a 2-norm version of the result due to Iśeki [8].

COROLLARY 4.3. *Let* X *be a 2-Banach space, let* K *be a nonempty* 1-*local retract of* $\overline{conv}(K)$*. Let* $T : K \to K$ *be a* k*-lipschitzian involution. If* $0 \leq k < 2$*, then* T *has a fixed point in* K*.*

Proof. Apply Corollary 4.2 with $\alpha = k$ and $\beta = 0$. $\square$

REMARK 4.2. Note that Corollary 4.3 is just the 2-Banach spaces' version of the result obtained by Kim-Kirk [11] in Banach spaces, where they say T satisfies *Goebel's Lipschitz condition* if $0 \leq k < 2$. In particular, if K is convex, it reduces to the well-known results of [4] and [5] in Banach spaces.

Finally we give an example of mappings in 2-Banach spaces which is a k-lipschitzian involution and satisfies the property (4.1) but not (4.2). This means the condition (4.1) in Theorem 4.2 is more general than (4.2).

EXAMPLE 4.1. Let $X := \mathbb{R}^2$ be a 2-dimensional Euclidean 2-Banach space with 2-norm $\|\cdot,\cdot\|_1$ as in Example 2.1. Let $1 < k \leq 3$ and $K := \mathbb{R} \times \{0\}$. From Proposition 3.2, note that K is closed and convex in $(X, \|\cdot,\cdot\|_1)$. For any $x = (x_1, 0) \in K$, we define a mapping $T : K \to K$ by

$$Tx := \begin{cases} -kx & \text{if } 0 \leq x_1; \\ -x/k & \text{if } x_1 < 0. \end{cases}$$

Then T is obviously a k-lipschitzian involution on K and $Fix(T) = \{0\}$. First, we claim that T satisfies (4.1). Define

$$\phi(p, q, r) = \frac{k^2+1}{k^2+k} \max\{2p, q, r\}$$

for every $(p, q, r) \in \mathbb{R}_+$. Then, $\phi : \mathbb{R}^3_+ \to \mathbb{R}_+$ is obviously continuous on $\mathbb{R}^3_+$. Also, $\phi(1,1,1) = \frac{k^2+1}{k^2+k} \cdot 2 \,(:= c) < 2$ because $k > 1$. Now let $s \geq 0$, $t \geq 0$ and

$$s \leq \phi(t, 2t, s) = \frac{k^2+1}{k^2+k} \max\{2t, 2t, s\}.$$

Then it should be $\max\{2t, 2t, s\} = 2t$. Otherwise we should obtain $s \leq \frac{k^2+1}{k^2+k} s < s$, which is a contradiction. This immediately yields $s \leq ht$ with $h = \frac{k^2+1}{k^2+k} \cdot 2 = c < 2$. Thus $\phi \in \Phi$. Next let us show that T is k-lipschizian and satisfies (4.1). Let $x = (x_1, 0), y = (y_1, 0) \in K$. From now on, consider the following three cases: Let $a = (a_1, a_2) \in X$. (i) If $x_1, y_1 \geq 0$,

$$\|Tx - Ty, a\|_1 = \|-k(x-y), a\|_1 = k\|x-y, a\|_1. \tag{4.3}$$

Clearly, T is k-lipschtzian by (4.3). Now let us see 2-norm of the right side in (4.3), i.e.,

$$\begin{aligned} \|x-y,a\|_1 &= \left\{ \begin{vmatrix} x_1-y_1 & 0 \\ a_1 & a_2 \end{vmatrix}^2 \right\}^{1/2} \\ &= |a_2| \cdot |x_1-y_1|. \end{aligned}$$

If $x_1 \geq y_1$ at first, then $|x_1-y_1| = x_1-y_1 \leq x_1$ because $y_1 \geq 0$. Then (4.3) yields

$$\begin{aligned} \|Tx-Ty,a\|_1 \leq k|a_2|x_1 &= \frac{k}{1+k}\big((1+k)|a_2|x_1\big) = \frac{k}{1+k}\|x-Tx,a\|_1 \\ &\leq \frac{k^2+1}{k^2+k}\max\{2\|x-y,a\|_1, \|x-Tx,a\|_1, \|y-Ty,a\|_1\} \\ &= \phi(\|x-y,a\|_1, \|x-Tx,a\|_1, \|y-Ty,a\|_1). \end{aligned}$$

Next, if $x_1 < y_1$, then (4.3) similarly becomes

$$\begin{aligned} \|Tx-Ty,a\|_1 \leq k|a_2|y_1 &= \frac{k}{1+k}\|y-Ty,a\|_1 \\ &\leq \phi(\|x-y,a\|_1, \|x-Tx,a\|_1, \|y-Ty,a\|_1). \end{aligned}$$

In other words, (4.1) is satisfied in any case. (ii) if $x_1, y_1 < 0$, we have

$$\|Tx-Ty,a\|_1 = (1/k)\|x-y,a\|_1$$

for every $a \in X$. That is, T is $1/k$-lipschitzian (hence k-lipschtzian because $1 < k \leq 3$). Also,

$$\begin{aligned} \|Tx-Ty,a\|_1 = (1/k)\|x-y,a\|_1 &\leq \frac{k^2+1}{k^2+k} \cdot 2\|x-y,a\|_1 \\ &\leq \phi(\|x-y,a\|_1, \|x-Tx,a\|_1, \|y-Ty,a\|_1) \end{aligned}$$

and so (4.1) is satisfied. Finally, (iii) if either $0 \leq x_1,\ y_1 < 0$ or $x_1 < 0,\ y_1 \geq 0$ (for the latter, may be exchange x_1 and y_1), since

$$\begin{aligned} (4.4) \qquad \|Tx - Ty, a\|_1 &= \| -kx - (-y/k), a\|_1 \\ &= \left\{ \begin{vmatrix} -kx_1 - (-y_1/k) & 0 \\ a_1 & a_2 \end{vmatrix}^2 \right\}^{1/2} \\ &= |a_2|(kx_1 - y_1/k) \end{aligned}$$

and $(-y_1)/k \leq k(-y_1)$, we have

$$\|Tx - Ty, a\|_1 \leq k|a_2|(x_1 - y_1) = k\|x - y, a\|_1$$

for every $a \in X$, i.e., T is k-lipschitzian. On the other hand, First, if $x \geq -y$, (4.4) yields

$$\begin{aligned} \|Tx - Ty, a\|_1 &= |a_2|(kx_1 - y_1/k) \leq |a_2|(k + 1/k)x_1 = \frac{k^2+1}{k}|a_2|x_1 \\ &= \frac{k^2+1}{k(k+1)} \cdot (k+1)|a_2|x_1 = \frac{k^2+1}{k^2+k}\|x - Tx, a\|_1 \\ &\leq \phi(\|x - y, a\|_1, \|x - Tx, a\|_1, \|y - Ty, a\|_1). \end{aligned}$$

Next, if $x < -y$, since $1 < k \leq 3$, a simple calculation gives

$$\begin{aligned} \|Tx - Ty, a\|_1 = |a_2|(kx_1 - y_1/k) &\leq \frac{k^2+1}{k^2+k} \cdot 2\|x - y, a\|_1 \\ &\leq \phi(\|x - y, a\|_1, \|x - Tx, a\|_1, \|y - Ty, a\|_1). \end{aligned}$$

Hence T always satisfies (4.1) in all cases. At last, we claim that T does not satisfy (4.2). Otherwise for some $k \in [2, 3]$, $x = (0, 0)$ and $y = (1, 0)$, the condition (4.2) implies

$$\|Tx - Ty, a\|_1 = \|(-k, 0), a\|_1 = k|a_2|$$

$$\leq \alpha \|x - y, a\| + \beta (\|x - Tx, a\| + \|y - Ty, a\|)$$

$$= \alpha |a_2| + \beta (1 + k)|a_2| \leq (\alpha + 4\beta)|a_2|$$

$$< 2 |a_2| \leq k |a_2|$$

for every $a = (a_1, a_2) \in X$, which gives a contradiction. This contradiction means T does not satisfy (4.2) for all $k \in [2, 3]$.

REMARK 4.2. The above example with $k = 2$ in $\mathbb{R}$ with the usual norm is originally due to Assad-Sessa [1].

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