Fixed Point Theorems for Mapping of Asymptotically Nonexpansive Type

점근적비확대형 사상에 대한 부동점 정리

Ad[illegible]is[illegible] [illegible]e Hwa Ki[illegible]

by

S[illegible]ng[illegible] J[illegible]g

A thesis submitted in partial fulfillment
of the requirement for the degree of

Master of Education

Graduate School of Education
Pukyong National University

August 2004

Fixed Point Theorems for Mapping of Asymptotically Nonexpansive Type

A dissertation

by

Su Jung Jung

Approved by :

(Chairman) Nak Eun Cho, Ph. D.

(Member) Jin Mun Jeong, Ph. D.

(Member) Tea Hwa Kim, Ph. D.

June 16, 2004

Contents

점근적비확대형 사상에 대한 부동점 정리

정 수 정

부경대학교교육대학원 수학교육전공

요 약

1994년 Lim-Xu [18]는 X의 어떤 기하학적인 계수, 즉 Maulta 상수 $D(X)$에 대하여 다음 질문을 던졌다. $D(X) < 1$가 점근적비확대사상에 대하여 부동점을 갖는가? 라고 질문한 후, 만약 사상 T가 집합 C 위에서 약점근적정규(weakly asymptotic regularity)라면, 위 문제는 긍정적이라는 것을 밝혔다. 그리고 이 결과는 Kim-Kim [12]에 의하여 곧바로 점근적비확대형의 연속사상으로 확장되었다.

본 논문에서는 Kim-Kim의 결과를 다시 한번 살펴보고 더 일반적인 다음 결과를 증명한다.

[정리] 집합 C는 $D(X) < 1$을 만족하는 바나흐 공간 X의 유계이고 닫힌 볼록집합이라 하자. 만약 집합 C의 어떤 약 컴팩트이고 볼록인 집합 K가 존재하고, 사상 $T: C \to C$가 점근적비확대형이고 C 위에서 약점근적정규라면, 어떤 비확대사상(nonexpansive mapping) $S: K \to K$가 존재하여 $F(T) \cap K = F(S) \neq \varnothing$를 만족한다. 여기서, $F(T) = \{x \in X : Tx = x\}$는 사상 T의 부동점들의 집합을 뜻한다.

끝으로, 위 정리로부터 여러 가지 잘 알려진 결과들을 도출해 낸다.

1 Introduction

Let C be a nonempty subset of a Banach space X and let $\mathbf{N}$ be the set of natural numbers. Let $T : C \to C$ be a mapping. Sets satisfying $T(K) \subset K$ are said to be *invariant under* T or T-*invariant*. T is said to be *Lipschitzian* if for each $n \in \mathbf{N}$, there exists a real number k_n such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \qquad \text{for all } x, y \in C.$$

In particular, T is said to be *asymptotically nonexpansive* (simply *a.n.*) [8] if $\lim_{n\to\infty} k_n = 1$ and it is said to be *nonexpansive* if $k_n = 1$ for all $n \in \mathbf{N}$. We say that T is of *asymptotically nonexpansive type* (simply *a.n.t.*) [15] if $c_n(x) := c_n(x; C) \to 0$ for each $x \in C$, where

$$c_n(x; C) = \sup_{y \in C}(\|T^n x - T^n y\| - \|x - y\|) \vee 0.$$

Note first that if $c_n(x) \to 0$ then $c_{n+p}(x) \to 0$ for fixed $p \in \mathbf{N}$.

In 1965, Kirk[14] proved that if C is a weakly compact convex subset of a Banach space with normal structure, then every nonexpansive self-mapping T of C has a fixed point, where a nonempty convex subset C of a norm linear space is said to have *normal structure* if each bounded convex subset K of C consisting of more than one point contains a nondiametral point, that is, a point $z \in K$ such that $\sup\{\|z - x\| : x \in K\} < \operatorname{diam}(K)$. Seven years later, in 1972, Goebel-Kirk [8] proved that if the space X is assumed to be uniformly convex, then every a.n. self-mapping T of C has a fixed point. This was immediately extended to mappings of a.n.t. in a space with its characteristic of convexity, $\epsilon_o(X) < 1$, by Kirk [15] in 1974. More recently these results have been extended to wider classes of spaces, see for example [4, 6, 7, 13, 18, 19, 22]. In particular, Lim-Xu [18] and Kim-Xu [13] have demonstrated the existence of fixed points for a.n. mappings in Banach spaces with uniform normal structure, see also [6] for some related results. Very recently, the result due to Kim-Xu [13] was extended to mappings of a.n.t. by Li-Sims [17].

On the other hand, in 1994, Lim-Xu [18] asked whether the Maluta's constant $D(X) < 1$ (cf., [20]) implies the fixed point property for a.n. mappings. they gave fixed point theorems for a.n. mappings defined on a weakly compact convex subset C in a Banach space for which $D(X) < 1$ having an additional condition, i.e., weak asymptotic regularity on C for T, and this was immediately carried over continuous mappings of a.n.t. by Kim-Kim [12]. In this paper, we first provide important properties concerning ultrafilters and some geometrical coefficients of a Banach space X, and next review some results given in [12] relating to an above open question raised by Lim-Xu [18] and prove the following result (see Theorem 3.3): Let C be a nonempty bounded closed convex subset of a Banach space X with $D(X) < 1$ and let $T : C \to C$ be a continuous mapping of a.n.t. which is weakly asymptotically regular on C. Then, if there exists a nonempty weakly compact convex and T-invariant subset K of C, there exist a nonexpansive mapping $S : K \to K$ such that $F(T) \cap K = F(S) \neq \emptyset$. Also, some applications of this theorem are also given.

2 Preliminaries

We will begin this section by introducing some concepts of ultrafilters and giving some important results concerning ultrafilters. For more details the reader may consult [1, 10].

Recall that a *filter* $\mathcal{F}$ on a nonempty set I is a nonempty collection of subsets of I satisfying

(a) if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$;

(b) if $A \in \mathcal{F}$ and $A \subset B$ then $B \in \mathcal{F}$.

Obviously, $I \in \mathcal{F}$ for any filter on I. If $\mathcal{F}$ is a filter on I and $\emptyset \in \mathcal{F}$, then $\mathcal{F} = 2^I$ and is called the *improper filter*. Let $i \in I$ be fixed. $\mathcal{F}_i = \{A \subset I : i \in A\}$ is a filter on I and is called a *trivial filter*.

Now let $\mathcal{P}$ be the family of all proper filters on I, i.e.,

$$\mathcal{P} = \{\mathcal{F} : \mathcal{F} \text{ is a filter on } I, \mathcal{F} \neq 2^I\}.$$

Since $\mathcal{P}$ is an *inductive* set (every increasing chain has an upper bound), it follows from Zorn's lemma that $\mathcal{P}$ has a maximal element. In other words, there exists some $\mathcal{F} \in \mathcal{P}$ such that if $\mathcal{D} \in \mathcal{P}$ and $\mathcal{F} \subset \mathcal{D}$, then $\mathcal{F} = \mathcal{D}$. Such a maximal element of $\mathcal{P}$ is called an *ultrafilter* on I.

Lemma 2.1. (i) A filer $\mathcal{U}$ on I is an ultrafilter if and only if for every $A \subset I$, either A or $I \setminus A$ belongs to $\mathcal{U}$. (ii) The ultrafilter $\mathcal{U}$ on I is trivial if and only if there exists a finite set $A \in \mathcal{U}$.

Let X be a Hausdorff topological space and let $(x_i)_{i \in I}$ be a collection of elements of X indexed by a set I, and consider a filter $\mathcal{F}$ on I. We then say that $(x_i)_{i \in I}$ *converges to* $x \in X$ *over* $\mathcal{F}$ if the set $\{i \in I : x_i \in V\}$ is in $\mathcal{F}$ for any neighborhood V of x. The limit will be denoted by $\lim_{i,\mathcal{F}} x_i$ or $\lim_{\mathcal{F}} x_i$. Note that if $\mathcal{F}$ is proper, the limit over $\mathcal{F}$ is unique, and if, moreover, C is a closed subset of X and $\{x_i\} \subset C$, then $\lim_{\mathcal{F}} x_i \in C$.

If $\mathcal{F}_{i_0}$ is the trivial filter generated by $i_0 \in I$, then $\lim_{\mathcal{F}_{i_0}} x_i = x_{i_0}$. Trivial filters give no information on asymptotic behavior of sets, so we will generally avoid them. Here we give some properties for ultrafilters given by [1]. For the convenience of our argument, we shall provide the detailed proof.

Lemma 2.2. Let $\mathcal{U}$ be a nontrivial ultrafilter on $\mathbf{N}$ and suppose (x_n) converges to x in the topology of the space X. Then $\lim_{\mathcal{U}} x_n = x$. If X is a metric space and $\lim_{\mathcal{U}} x_n = x$, then there exists a subsequence (x_{n_k}) of (x_n) such that $\lim_{k \to \infty} x_{n_k} = x$.

Proof. Let V be any neighborhood of x. Since the set $A = \{i : x_i \notin V\}$ is a finite subset of $\mathbf{N}$ and $\mathcal{U}$ is nontrivial, $A \notin \mathcal{U}$. By (i) of Lemma 2.1, $\mathbf{N} \setminus A = \{i : x_i \in V\} \in \mathcal{U}$, and so $\lim_{\mathcal{U}} x_n = x$.

Now suppose that (X, d) is a metric space and $\lim_{\mathcal{U}} x_n = x$. Then for each $k \in \mathbf{N}$, $A_k := \{i : x_i \in U_{1/k}(x)\} \in \mathcal{U}$, where $U_r(x) := \{y \in X : d(y, x) < r\}$, the open ball centered at x with its radius r. Since $\mathcal{U}$ is nontrivial, by (ii) of Lemma 2.1, A_k is infinite for every k. After choosing a $n_1 \in \mathbf{N}$ such that $x_{n_1} \in A_1$, we can take $n_2 > n_1$ and $x_{n_2} \in A_2$ because A_2 is infinite. Repeating this process we easily get a subsequence (x_{n_k}) of (x_n) such that $d(x_{n_k}, x) < 1/k$ for each $k \in \mathbf{N}$. ∎

The next result is interesting because it shows how ultrafilters can be used to characterize compactness of a topological space.

Lemma 2.3. Let K be a nonempty subset of a Hausdorff topological space. Then, K is compact if and only if any set $(x_i)_{i \in I} \subset K$ is convergent in K over any ultrafilter $\mathcal{U}$ on I.

The following result (cf., [1]) is to show how limits over ultrafilters cooperate well with the linear structure. For the convenience's sake, we shall give the detailed proof.

Lemma 2.4. Let X ba a topological linear space and $\mathcal{U}$ an ultrafilter on a set I. Suppose that $(x_i)_{i \in I}$ and $(y_i)_{i \in I}$ are two subsets of X and $\lim_{\mathcal{U}} x_i = x$ and $\lim_{\mathcal{U}} y_i = y$ both exists. Then

$$\lim_{\mathcal{U}}(x_i + y_i) = x + y \quad \text{and} \quad \lim_{\mathcal{U}}(\alpha x_i) = \alpha x$$

for any scalar α.

Proof. Let V be a neighborhood of $x+y$. Since the space X is linear and addition "+" is continuous, there exists neighborhoods V_x and V_y of x and y, respectively, such that $x+y \in V_x + V_y \subset V$. Since $\lim_{\mathcal{U}} x_i = x$ and $\lim_{\mathcal{U}} y_i = y$ both exists, we have $I_z := \{i \in I : z_i \in V_z\} \in \mathcal{U}$ for $z = x$ or y. Then,

$$\{i \in I : x_i + y_i \in V_x + V_y\} = I_x \cap I_y \in \mathcal{U}.$$

Since $\{i \in I : x_i + y_i \in V\} \supset \{i \in I : x_i + y_i \in V_x + V_y\}$ and hence in $\mathcal{U}$. Therefore $\lim_{\mathcal{U}}(x_i + y_i) = x + y$. The proof of $\lim_{\mathcal{U}}(\alpha x_i) = \alpha x$ is similar. ■

3 Some geometrical coefficients and open questions

Let X be a Banach space. First, let us introduce normal structure coefficient of X introduced by Bynum [5]. For $A \subset X$, $\operatorname{diam}(A)$ and $r_A(A)$ denote the *diameter* and the *self-Chebyshev radius* of A, respectively, i.e.,

$$\begin{aligned} \operatorname{diam}(A) &= \sup_{x,y \in A} \|x - y\|, \\ r_A(A) &= \inf_{x \in A} (\sup_{y \in A} \|x - y\|) \end{aligned}$$

Recall that X has *uniform normal structure* (simply UNS) if $N(X) > 1$, where

$$N(X) = \inf \left\{ \frac{\operatorname{diam}(A)}{r_A(A)} : A \subset X \text{ bounded closed convex with } \operatorname{diam}(A) > 0 \right\}.$$

Obviously, if $N(X) > 1$, then X has normal structure.

Recall that if X is a non-Schur Banach space, then the weakly convergent sequence coefficient of X, denoted by $WCS(X)$, is defined by

$$\begin{aligned} WCS(X) &= \sup\{M > 0 : \text{for each weakly convergent sequence } (x_n), \\ &\qquad \exists\ y \in \overline{co}(x_n) \text{ such that } M \cdot \limsup_{n\to\infty} \|x_n - y\| \le A(x_n)\}, \end{aligned}$$

where $\overline{co}(K)$ denotes the closed convex hull of a set K and $A(x_n)$ denotes the asymptotic diameter of (x_n), i.e.,

$$A(x_n) = \lim_{n\to\infty} \sup\{\|x_i - x_j\| : i, j \ge n\}.$$

It is easy to give a sharp expression $WCS(X)$ as follows;

$$WCS(X) = \sup\{M : x_n \rightharpoonup u \ \Rightarrow\ M \cdot \limsup_{n\to\infty} \|x_n - u\| \le D(x_n)\},$$

where $D(x_n) := \limsup_{m\to\infty} \limsup_{n\to\infty} \|x_n - x_m\|$ and "$\rightharpoonup$" means the weak convergence. For more details, see [5] and [11].

Note that if X is reflexive, then $1 \le N(X) \le BS(X) < WCS(X) < 2$ (cf., [5]), where $BS(X)$ means the *bounded sequence coefficient* of X, i.e.,

$$\begin{aligned} BS(X) &= \sup\Big\{M : \text{ for any bounded sequence } \{x_n\} \text{ in } X, \\ &\qquad \exists\ y \in \overline{co}(\{x_n\}) \text{ such that } M \cdot \limsup_{n\to\infty} \|x_n - y\| \le A(\{x_n\})\Big\}. \end{aligned}$$

While $N(X)$ and $BS(X)$ can be defined in every Banach space, $WCS(X)$ is well defined only in infinite dimensional reflexive spaces, where, by Eberlein-Šmulian theorem, we can assure the existence of weakly convergent sequences which do not converge.

It is well-known (see [5]) that if $WCS(X) > 1$, then X has *weak normal structure*. This means that any weakly compact convex subset C of X with $\mathrm{diam}(C) > 0$ has a nondiametral point. The coefficients $WCS(X)$ play important roles in fixed point theory. A space X such that $WCS(X) > 1$ is said to have *weak uniform normal structure*.

Let X be a Banach space. Recall that Maluta's constant $D(X)$ [20] of X is defined by

$$D(X) = \sup \left\{ \frac{\limsup d(x_{n+1}, co(x_1, x_2, \cdots x_n))}{diam(x_n)} \right\},$$

where the supremum is taken over all bounded nonconstant sequences (x_n) in X.

We remark the following properties for Maluta's constant given in [20].

Lemma 3.1. Let X be a Banach space. Then

(a) $D(X) \leq \tilde{N}(X) = 1/N(X)$.

(b) $D(X) = \sup\{D(Y) : Y \subset X \text{ separable}\}$.

(c) $D(X) = 0$ if and only if X is finite-dimensional.

(d) If X is reflexive, then $D(X) \leq 1/WCS(X)$.

(e) If $D(X) < 1$, then the Banach space X is reflexive and has normal structure.

Remark 3.1. (i) The property (a) says that if X has uniform normal structure, then $D(X) < 1$. However, the converse does not hold (see Example 5.1 and Corollary 5.2 in [20]). (ii) In view of (d), Maluta asked if $D(X) = 1/WCS(X)$ for every infinite dimensional reflexive space X. In 1985, Amir [2] gave a partial solution for this question, that is, if X satisfies Opial's condition, i.e., for any sequence (x_n) in X converging weakly to x_0,

$$\liminf_{n\to\infty} \|x_n - x_0\| < \liminf_{n\to\infty} \|x_n - x\| \qquad \forall x \neq x_0$$

(see [9]), then $D(X) \geq 1/WCS(X)$. Five years later, this question was completely solved by Prus [21]. (iii) The converse of (e) also does not hold (see Example 4.1 in [20], $X = (\sum \bigoplus \ell_n)_2$ is reflexive and has normal structure although $D(X) = 1$).

Note that, by (e) of Lemma 3.1, if $D(X) < 1$, X has normal structure and hence the fixed point property for nonexpansive mappings, that is, for every weakly compact convex subset C of X, every nonexpansive map $T : C \to C$

has a fixed point. However, it is still open whether $D(X) < 1$ implies the fixed point property for a.n. mappings. In 1994, Lim-Xu [18] gave a partial answer for this question under an additional assumption as follows:

Theorem LX. [18] Suppose that X is a Banach space such that $D(X) < 1$, that C is a closed bounded convex subset of X. If a mapping $T : C \to C$ is a.n. and weakly asymptotically regular on C, i.e., $T^{n+1}x - T^n x \rightharpoonup 0 \ \forall x \in C$, then T has a fixed point.

Immediately, Theorem LX was extended to all mappings of a.n.t. by Kim-Kim (see Corollary 3.3 in [12]).

Let C be a nonempty subset of a Banach space X, and let $T : C \to C$ be a mapping. Suppose there exists a nonempty subset K of C and the weak limit $w\text{-}\lim_{\mathcal{U}} T^n x$ exists in K for each $x \in K$, where $\mathcal{U}$ is a free ultrafilter on $\mathbf{N}$. We then can define a mapping $S : K \to K$ by

$$Sx = w\text{-}\lim_{\mathcal{U}} T^n x, \qquad \forall x \in K.$$

Note first that if K is weakly compact and T-invariant, then the weak limit $w\text{-}\lim_{\mathcal{U}} T^n x$ always exists in K for each $x \in K$ by Lemma 2.3. We can next see that $F(T) \cap K \subset F(S)$. What are conditions on X and T for which the converse inclusion remains true? Our purpose is to find some conditions on X and T to answer the above question.

First, we exhibit the following easy result:

Lemma 3.2. Let C be a nonempty subset of a Banach space X and let K be a nonempty weakly compact convex subset of C. If $T : C \to C$ is a continuous mapping of a.n.t., and S is defined as in above, then S is nonexpansive.

Proof. Let $x, y \in K$, $Sx = w\text{-}\lim_{\mathcal{U}} T^n x$ and $Sy = w\text{-}\lim_{\mathcal{U}} T^n y$. By Lemma 2.4, we have $Sx - Sy = w\text{-}\lim_{\mathcal{U}}(T^n x - T^n y)$. By Lemma 2.2, there exists a subsequence (n_k) of (n) such that $T^{n_k}x - T^{n_k}y \rightharpoonup Sx - Sy$ as $k \to \infty$. Since the norm $\|\cdot\|$ is weakly lower semicontinuous and $c_n(x) \to 0$ for each $x \in C$, we have

$$\begin{aligned} \|Sx - Sy\| &\leq \liminf_{k\to\infty} \|T^{n_k}x - T^{n_k}y\| \\ &\leq \limsup_{k\to\infty}[\|T^{n_k}x - T^{n_k}y\| - \|x - y\|] + \|x - y\| \\ &\leq \lim_{k\to\infty} c_{n_k}(x) + \|x - y\| = \|x - y\|. \end{aligned}$$

■

Note that if X has weak normal structure, by the classical fixed point theorem of Kirk [14], $F(S) \neq \emptyset$ and, furthermore, it is a nonexpansive retract of K (see Bruck [3]).

Now we will present a partial answer of the above question, that is, a sufficient condition for $F(S) \subset F(T)$, with a slight modification of the proof in Lemma 3.1 of [12]. Here, we shall give the detailed proof for convenience sake.

Theorem 3.3. Let C be a nonempty bounded subset of a Banach space X with $D(X) < 1$. Let $T : C \to C$ be a continuous mapping of a.n.t. and weakly asymptotically regular on C, and suppose there exists a weakly compact convex and T-invariant subset of C. Then there exists a nonexpansive mapping $S : K \to K$ such that $F(T) \cap K = F(S) \neq \emptyset$.

Proof. By Lemma 3.2, $S : K \to K$ is nonexpansive. Now it suffices to show that $Fix(S) \subset Fix(T) \cap K$. Let $x \in F(S)$, that is, $w\text{-}\lim_{\mathcal{U}} T^n x = x \in K$. By Lemma 2.2, there exists a subsequence $(T^{n_k}x)$ of the sequence $(T^n x)$ such that $T^{n_k}x \rightharpoonup x$ as $k \to \infty$. Note that X is reflexive by (e) by Lemma 3.1. By (ii) of Remark 3.1, $D(X) = 1/WCS(X)$ and so we can apply the well known property of $WCS(X)$,

$$\limsup_{k \to \infty} \|T^{n_k}x - x\| \leq \frac{1}{WCS(X)} D(T^{n_k}x). \qquad (1)$$

By weakly asymptotic regularity of T, it follows that $T^{n_k+m}x \rightharpoonup x$ as $k \to \infty$ for any $m > 0$. On the other hand, for each $i, j \in \mathbf{N}$ with $i > j$, the weak lower semicontinuity of the norm $\|\cdot\|$ and $c_n(x)$ for each $x \in C$ immediately yield that

$$\|T^{n_j}x - T^{n_i}x\|$$

$$
\begin{aligned}
&< \left(\|T^{n_j}x - T^{n_j}(T^{n_i-n_j}x)\| - \|x - T^{n_i-n_j}x\| \to \infty\right) + \|x - T^{n_i-n_j}x\| \\
&\leq c_{n_j}(x) + \|x - T^{n_i-n_j}x\| \quad (T^{n_k+m}x \rightharpoonup x \text{ as } k \to \infty, \text{ with } m = n_i - n_j) \\
&< c_{n_j}(x) + \liminf_{k\to\infty} \|T^{n_k+m}x - T^{n_i-n_j}x\| \\
&\leq c_{n_j}(x) + c_{n_i-n_j}(x) + \limsup_{k\to\infty} \|x - T^{n_k}x\|.
\end{aligned}
$$

Taking $\limsup_{i\to\infty}$ first and next $\limsup_{j\to\infty}$ on both sides, this implies that

$$D(T^{n_i}x) < \limsup_{k\to\infty} \|x - T^{n_k}x\|,$$

and this together with (1) yields

$$(WCS(X) - 1) \limsup_{k\to\infty} \|T^{n_k}x - x\| \leq 0,$$

which in turn implies that $x = \lim_{k\to\infty} T^{n_k}x$. By the continuity and the weak asymptotic regularity of T, we have $Tx = x$, i.e., $x \in Fix(T)$. ∎

Remark 3.2. (i) Note that if C is weakly compact convex, the reflexivity of X can also be removed in Theorem 3.3. (ii) As a direct consequence of the proof of Theorem 3.3, we notice that, under the same assumptions of C, X and T, if $(T^{n_k}x)$ is a subsequence of $(T^n x)$ converging weakly to $x \in K$, then $\lim_{k\to\infty} T^{n_k}x = x$. However, if the whole sequence $(T^n x)$ converges weakly, the weakly asymptotic regularity on C for T is abundant.

As a slight modification of the proof of Theorem 3.3., we can prove the following result.

Lemma 3.4. Let C be a nonempty bounded closed convex subset of a reflexive Banach space X with $WCS(X) > 1$. If $T : C \to C$ is a continuous mapping of a.n.t., then $w\text{-}\lim_{n\to\infty} T^n x = x \in K \Rightarrow \lim_{n\to\infty} T^n x = x \in F(T)$.

4 Some applications

In this section, we first observe the following result by using the similar method of the proof as in Theorem 3.3.

Theorem 4.1. Let C be a nonempty bounded subset of a Banach space X with $D(X) < 1$. Let $T : C \to C$ be a continuous mapping of a.n.t. which is weakly asymptotically regular on C. Suppose there exists a nonempty closed convex subset K of C with the following property

$$(\omega), \qquad x \in K \quad \Longrightarrow \quad \omega_w(x) \subset K.$$

where $\omega_w(x)$ is the weak ω-limit set of T at x, i.e.,

$$\omega_w(x) = \{y \in X : y = w\text{-}\lim_{k\to\infty} T^{n_k}x \text{ for some } n_k \uparrow \infty\}.$$

Then there exists a nonexpansive mapping $S : K \to K$ such that $F(T) \cap K = F(S) \neq \emptyset$.

Proof. By (ii) of Remark 3.1, K is weakly compact convex and $WSC(X) > 1$. Since the sequence $(T^n x)$ belongs to C, and $\overline{co}(C)$ is weakly compact, the weak limit $w\text{-}\lim_{\mathcal{U}} T^n x$ always exists in $\overline{co}(C)$ for each $x \in K$ by Lemma 2.3.

Define $Sx = w\text{-}\lim_{\mathcal{U}} T^n x$ for each $x \in K$. Then, by Lemma 2.2, there exists a subsequence (n_k) of (n) such that $T^{n_k}x \rightharpoonup Sx$ as $k \to \infty$. By property of (ω), it follows that $Sx \in \omega_w(x) \subset K$. Therefore, $S : K \to K$ is well defined, and also nonexpansive. Repeating the method of proof in Theorem 3.3, we can easily obtain the conclusion. ■

It is clear that if C is a nonempty bounded subset of a Banach space X, and if $T : C \to C$ is a.n. with its Lipschitz constant of T^n, $k_n \geq 1$, then T is an uniformly Lipschitzian mapping of a.n.t. Indeed, for each $x \in C$,

$$\begin{aligned} c_n(x) &= \sup_{y \in C}(\|T^n x - T^n y\| - \|x - y\|) \vee 0 \\ &\leq (k_n - 1)\mathrm{diam}(C) \to 0. \end{aligned}$$

Therefore, we have the following easy result.

Corollary 4.1. Let C be a nonempty bounded subset of a Banach space X with $D(X) < 1$. Let $T : C \to C$ be an a.n. mapping which is weakly asymptotically regular on C. Suppose there exists a nonempty closed convex subset K of C with the following property (ω). Then T has a fixed point in K.

Let C be a weakly compact convex subset of a Banach space X. Consider a family $\mathcal{F}$ of subsets K of C which are nonempty, closed, convex, and satisfy the following property (ω). The weak compactness of C now allows one to use Zorn's lemma to obtain a minimal element (say) $K \in \mathcal{F}$. Therefore, as a direct consequence of Theorem 3.3 or 4.1, we have the following result due to Kim-Kim [12].

Corollary 4.2. Let C be a nonempty bounded closed convex subset of a Banach space X with $D(X) > 1$. If $T : C \to C$ is a continuous mapping of a.n.t. and weakly asymptotically regular on C, then $F(T)$ is a nonempty nonexpansive retract of C.

Proof. Since C is weakly compact and convex, we can easily apply for Theorem 3.3 or 4.1, and hence $F(T) = F(S) \neq \emptyset$. Since S is nonexpansive, it follows from [3] that $F(S)$ is a nonempty nonexpansive retract of C. ■

Recall that a Banach space X is said to be *uniformly convex in every direction* [9] if $\delta_z(\epsilon) > 0$ for all $\epsilon > 0$ and all $z \in X$ with $\|z\| = 1$, where $\delta_z(\cdot)$ means the *modulus od convexity of* X *in the direction* z, that is,

$$\delta_z(\epsilon) = \{1 - \|x + y\|/2 : \|x\| \leq 1, \|y\| \leq 1, x - y = \epsilon z\}.$$

Zizler [23] has shown that a space X may be uniformly convex in every direction while failing to be uniformly convex. Obviously, such spaces are always strictly convex.

Corollary 4.3. Suppose that X is a reflexive Banach space which is uniformly convex in every direction and for which $WCS(X) > 1$ and that C is a closed bounded convex subset of X. Then, if $T : C \to C$ is a continuous mapping of a.n.t., T has a fixed point.

Proof. Use the same argument presented in the proof of Theorem 5 in [18] and Lemma 3.4. ■

References

[1] A. G. Aksoy and M. A. Khamsi, *Nonstandard Methods in Fixed Point Theory*, Springer-Verlag, 1990.

[2] D. Amir, On Jung's constant and related constants in normed linear spaces, *Pacific J. Math.*, 118 (1985), 1–15.

[3] R. E. Bruck, A common fixed point theorem for a commuting family of nonexpansive mappin, *Pacific J. Math.*, 53 (1974), 59–71.

[4] R. Bruck, T. Kuczumov and S. Reich, Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property, *Coll. Math.*, 65 (1993), 169–179.

[5] W. L. Bynum, Normal structure coefficients for Banach spaces, *Pacific J. Math.*, 86(1980), 427-436.

[6] E. Casini and E. Maluta, Fixed points of uniformly Lipschitzian mappings in space with uniform normal structure, *Nonlinear Analysis, TMA.*, 9 (1985), 103–108.

[7] J. Garcia-Falset, B. Sim and M.A. Smyth, The demiclosedness principle for mappings of asymptotically nonexpansive type, *Houston J. Math.*, 22 (1996), 101–108.

[8] K. Goebel and W. A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, *Proc. Amer. Math. Soc.*, 35 (1972), 171–174.

[9] K. Goebel and W. A. Kirk, *Topics in metric fixed point theory*, Cambridge University Press, (1990).

[10] J. L. Kelly, *General Topology*. Van Nostrand. 1955.

[11] T. H. Kim and E. S. Kim, Iterate fixed points of non-Lipschitzian self-mappings, *Kodai Math. J.*, 18 (1995), 275–283.

[12] T. H. Kim and E. S. Kim, Fixed point theorems for non-Lipschitzian mappings in Banach spaces, *Math. Japonica*, 45(1) (1997), 61–67.

[13] T. H. Kim and H. K. Xu, Remarks on asymptotically nonexpansive mappings, *Nonlinear Analysis, TMA.*, 22 (1994), 1345–1355.

[14] W. A. Kirk, A fixed point theorem for mappings which do not increase distance, *Amer. Math. Monthly,* 72 (1965), 1004–1006.

[15] W. A. Kirk, Fixed point theorems for non-Lipschitzian mappings of asymptotically nonexpansive type, *Israel J. Math.,* 17 (1974), 339–346.

[16] G. Li and B. Sims, Fixed point theorems for mappings of asymptotically nonexpansive type, *Nonlinear Analysis, TMA.,* 50 (2002), 1085–1095.

[17] P. K. Lin, K. K. Tan and H. K. Xu, Demiclosedness principle and asymptotic behavior for asymptotically nonexpansive mappings, *Nonlinear Analysis, TMA*. 24 (1995), 929–946.

[18] T. C. Lim and H. K Xu, Fixed point theorems for asymptotically nonexpansive mappings, *Nonlinear Analysis, TMA.,* 22(11) (1994), 1345–1355.

[19] E. Maluta, Uniformly normal structure and related coefficients, *Pacific J.Math.*, 111 (1984), 357–369.

[20] S. Prus, On Bynum's fixed point theorem, *Atti Sem. Mat. Fis. Univ. Modena,* 38 (1990), 535–545.

[21] H. K. Xu, Existence and convergence for fixed points of mappings of asymptotically nonexpansive type, *Nonlinear Analysis, TMA.*, 16 (1991), 1139–1146.

[22] V. Zizler, On some rotundity and smoothness properties of Banach spaces, *Rozprawy Mat.*, 87 (1971).