Fixed Point Theorems of Non-Lipschitzian Self-mappings

(비-Lipschitz 자기사상들의 부동점정리)

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A dissertation

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비-Lipschitz 자기사상들의 부동점정리

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요 약

집합 X 는 실 Banach 공간이고 $K(\neq \varnothing) \subset C \subset X$ 라 하고, 각 $x \in K$ 에 대하여

$$c_n(x,K) := sup\{\|T^n x - T^n y\| - \|x-y\| : y \in K\} \vee 0$$

라 하자. 잘 알려진 비-Lipschitz 사상으로 $T: C \to C$ 가 점근적비확대형(*asymptotically nonexpansive type; in brief, ANT*)이라 함은 각 $x \in C$ 에 대하여 $c_n(x,C) \to 0$ 을 만족함을 뜻한다. 특히, $c_n := sup_{x \in C} c_n(x,K) \to 0$ 일 때, T 는 강점근적비확대형(*strongly ANT*)이라하고, 유계이고 볼록인 T-불변(*invariant*)인 C 의 공집합이 아닌 부분집합 K 가 존재하여 각 $x \in K$ 에 대하여 $c_n(x,K) \to 0$ 를 만족할 때, 부분적으로 점근적비확대형 사상(mappings of *partly ANT*)이라 한다. 방정식 $Tx = x$ 의 해(*solution*)를 부동점(*fixed point*)라 하고, 그러한 점들의 집합을 $F(T)$ 로 표기한다.

본 논문의 제 2장에서는 예제를 통하여 위에 소개한 여러 가지 유형의 비-Lipschitz 사상들을 상호 비교하고, 더욱, Banach 공간의 기하학적인 몇몇 성질들을 소개한다. 3장에서는 집합 C 가 균등정규구조({*uniform normal structure*)를 갖는 Banach 공간 X 의 유계이고 볼록인 닫힌 부분집합일 때, 부분적으로 점근적비확대형인 사상 $T: C \to C$ 가 부동점을 갖는다는 사실을 밝혔다(정리 3.2.2). 1994년 Lim-Xu [50]는 Banach 공간 X 의 어떤 기하학적인 계수, 즉 Maulta 상수 $D(X) < 1$ 에 대하여 점근적비확대사상 $T: C \to C$ 에 대하여 부동점을 갖는가? 라고 질문한 후, 만약 사상 T 가 집합 C 위에서 약점근적정규(*weakly asymptotic regularity*)라면, 위 문제는 긍정적이라는 것을 밝혔다. 그리고 이 결과는 Kim-Kim [37]에 의하여 곧바로 점근적비확대형 연속사상으로 확장되었다. 4장에서는 약균등정규구조(*weak uniform normal structure*)를 갖는 reflexive Banach 공간 내에서 부분적으로 점근적비확대형인 사상의 부동점 정리 4.3.1을 증명한다. 마지막 5장에서는 쌍곡형 거리공간에서 연속인 강점근적비확대형 사상에 대한 반복적인 부동점을 구축한다 (정리 5.2.5).

Chapter 1

Introduction

For sets K and X with $K \subseteq X$ and a mapping $T : K \to X$, every solution of the equation $Tx = x$ is called a *fixed point* of T, and the set of all such points is denoted by $F(T)$. *Fixed point theory* entails

(i) the study of conditions on K and (or) T which assure that T always has at least one fixed point, as well as

(ii) the study of methods of approximating fixed points when they do exists and

(iii) the study of the structure of $F(T)$.

Fixed point theory is a major branch of nonlinear functional analysis because of its wide applicability. Numerous questions in physical and biological sciences lead to various nonlinear differential and integral equations which in turn can often be reduced to a operator equation of the form $F(x) = 0$ where x is an element of a Banach space with the additive identity 0. Moreover, finding a solution to the equation $F(x) = 0$ reduces to finding a fixed point for the mapping T defined by

$$Tx = x - F(x).$$

More generally, it suffices to find a fixed point of T where T is defined in any way such that a fixed point of T is a zero of F, e.g.,

$$Tx = x - \lambda g(F(x)) \quad (\lambda \neq 0;\ g(u) = 0 \ \Rightarrow \ u = 0).$$

There are three major branches of fixed point theory in functional analysis, and each branch has its celebrated theorems as follows:

(i) Set (or order) theoretic fixed point theory; Zemelo [68] (or see p. 5: Dunford-Schwartz [21]), Bourbaki-Kneser([9, 44]), Tarski [63], Caristi [18], Amann [3] and so on.

(ii) Topological fixed point theory; Bohl [8], Brouwer [11], Schauder [61], Leray-Schauder [47], Sadovskii [59] and so on.

(iii) Metric fixed point theory; Banach's contraction principle [6], Browder-Göhde-Kirk ([12, 29, 39]), Sadovskii [59], Caristi [18] and so on.

This manuscript is largely related to study some fixed point theorems on metric fixed point theory. A mapping T defined on a metric space (M, d) and taking values in a metric space (N, d) is said to have *Lipschitz constant* k, or to be k-*lipschitzian* if there exists a real number $k \geq 0$ such that

$$d(Tx, Ty) \leq kd(x, y) \qquad (x, y \in M).$$

If $k < 1$, then T is said to be a *contraction mapping*, or k-*contraction*, and if $k = 1$, then T is said to be *nonexpansive*.

The first footstone on metric fixed point theory is a Banach's achievement in 1922 for contraction mappings, which is well known as *Banach's contraction principle* (see [6]), that is, if (M, d) is a complete metric space and if $T : M \to M$ is a contraction mapping with Lipschitz constant $k < 1$, then T has exactly one fixed point, say $z \in M$, Moreover, the sequence $\{x_n\}$ of successive approximations

$$x_{n+1} = Tx_n, \quad x_0 \in M \qquad (n = 0, 1, 2, \ldots)$$

converges strongly to z, for an arbitrary choice of an initial point x_0 in M. Very recently, Kirk introduced a notion of *asymptotic contraction* on a metric space (M, d). A mapping $T : M \to M$ is said to be an *asymptotic contraction* [42] if, for each $n \in \mathbb{N}$, there exists a function $\phi_n : \mathbb{R}^+ := [0, \infty) \to \mathbb{R}^+$ such that

$$d(T^n x, T^n y) \leq \phi_n(d(x, y)) \qquad (x, y \in M) \tag{1.1}$$

and if $\phi_n \to \phi \in \Phi$ uniformly on the range of d, where Φ denotes the collection of all functions φ from $\mathbb{R}^+$ to $\mathbb{R}^+$ satisfying the properties:

(i) φ is continuous;

(ii) $\phi(0) = 0$ and $\phi(t) < t$ for all $t > 0$.

He also proved that if (M, d) is complete, $T : M \to M$ is an asymptotic contraction for which the mapping ϕ_n in (1.1) are continuous, and if some orbit of T is bounded, then T has a unique fixed point z, and moreover the Picard sequence $\{T^n x\}$ converges to z for each $x \in M$. See also Xu [65] for a simple proof.

The class of nonexpainsive mappings can be viewed as natural extensions of contraction mappings. However, fixed point theory of nonexpansive mappings differs sharply from that of contraction mappings in the sense that an additional structure is needed on the underlying space to assure the existence of fixed points. The first existence result for nonexpansive mappings is Kirk's celebrated theorem [39] which depends heavily upon a geometrical property, called *normal structure*, that is, if C is a weakly compact convex subset of a Banach space with normal structure, then every nonexpansive self-mapping T of C has a fixed point, where a nonempty convex subset C of a norm linear space is said to have *normal structure* if each bounded convex subset K of C consisting of more than one point contains a *nondiametral* point, that is, a point $z \in K$ such that $\sup\{\|z - x\| : x \in K\} < \text{diam}(K)$. Note that if X is uniformly convex and $C \subset X$ is closed convex, then C has normal structure. Also, if X is a Banach space and if $C \subset X$ is compact convex, then C has normal structure.

Let C be a nonempty subset of a Banach space X and let $\mathbb{N}$ be the set of natural numbers. A mapping $T : C \to C$ is said to be *Lipschitzian* if for each $n \in \mathbb{N}$, there exists a real number k_n such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \qquad \text{for all } x, y \in C.$$

In particular, T is said to be *asymptotically nonexpansive* (simply *AN*) [26] if $\lim_{n\to\infty} k_n = 1$ and it is said to be *uniformly Lipschitzian* [19] if there exists a real number k such that $k_n = k$ for all $n \in \mathbb{N}$, and T is said to be *nonexpansive* (or *contraction*) if $k_1 = 1$ (or $k_1 < 1$).

In terms of the existence of fixed points, the achievement on metric fixed point theory today largely focuses upon the study of nonexpansive mappings and related classes of mappings, such as asymptotically nonexpansive mappings and uniformly Lipschitzian mappings in Banach spaces. For an example, in 1972, Goebel-Kirk [26] proved that if the space X is assumed to be uniformly convex, then every *AN* self-mapping T of C has a fixed point. For more detail history and methods on metric fixed point theory, see Geobel-Kirk [27] or Zeidler [67].

In this paper, we concentrate on a class of non-Lipschitzian self-mappings, such as mappings of asymptotically nonexpansive type. We say that a mapping $T : C \to C$ is of *asymptotically nonexpansive type* (in brief, *ANT*) [40] if for each $x \in C$, $\lim_{n\to\infty} c_n(x) = 0$, where

$$c_n(x) := \sup\{\|T^n x - T^n y\| - \|x - y\| : y \in C\} \vee 0,$$

where $a \vee b = \max\{a, b\} = \frac{|a-b|+a+b}{2}$. T is said to be of *strongly asymptotically nonexpansive type* (in brief, *strongly ANT*) if $\lim_{n\to\infty} c_n = 0$, where $c_n = \sup_{x\in C} c_n(x)$.

The above Geobel-Kirk's result for AN mappings was immediately extended to mappings of ANT in a space with its characteristic of convexity, $\epsilon_o(X) < 1$, by Kirk [40] in 1974. More recently these results have been extended to wider classes of spaces, see for example [15, 19, 25, 38, 50, 51, 64]. In particular, Lim-Xu [50] and Kim-Xu [38] have demonstrated the existence of fixed points for AN mappings in Banach spaces with uniform normal structure, see also [19] for some related results. Very recently, the result due to Kim-Xu [38] was extended to mappings of ANT by Li-Sims [48] and Kim [34] independently.

In Chapter 2, we first define some kind of non-Lipschitzian mappings, such as self-mappings of partly (or nearly) ANT (see Definition 2.1.2). Next we shall present many examples for such non-Lipschitzian self-mappings for our arguments and also introduce some geometrical properties for Banach spaces.

In Chapter 3, we shall prove some fixed point theorems for mappings of partly ANT in Banach spaces with uniform normal structure (see Theorem 3.2.2).

On the other hand, fixed point theorems due to Lim-Xu [50] for AN mappings defined on a weakly compact convex subset C in a Banach space X with either a weakly continuous duality mapping or for which $D(X) < 1$ having an additional condition, i.e., weak asymptotic regularity on C for T, where $D(X)$ is Maluta's constant (see [52]), were carried over continuous mappings of ANT by Kim-Kim [37].

In Chapter 4 of this paper, we modify some results in [37] and carry over these to a class of continuous mappings of partly ANT in a Banach space with weak uniform normal structure (see Theorem 4.3.1). Some applications for our main theorem are also added.

In 1955, M. A. Krasnoselskii [45] proved that if K is a compact and convex subset of a uniformly convex Banach space X and if $T : K \to K$ is nonexpansive then for any $x_0 \in K$, the sequence $\{f^n(x_0)\}$ of iterates of $f = \frac{1}{2}(I + T)$ always converges to a fixed point of T. It was quickly noted by Schaefer [58] that one obtains the same conclusion if the mapping f is replaced with $f_\alpha = (1-\alpha)I + \alpha T$ for some $\alpha \in (0, 1)$, subsequently Edelstein [22] observed that the assumption of uniform convexity (in Schaefer's modification) could be replaced with the assumption that X is strictly convex norm.

Major generalization of the Krasnoselskii process occurred in 1976 and 1978 when, respectively, Ishikawa [31] and Edelstein-ÓBrien [23] proved, in different ways, that the uniform convexity hypothesis can be removed completely. Each proved even more. Ishikawa generalized the iteration scheme while Edelstein-ÓBrien proved that the convergence is uniform on K. Later, Kirk [41] carried over a well-known iteration scheme due to Krasnoselskii [45] for approximation of fixed points of nonexpansive mappings in Banach spaces to a wider class of spaces containing convex metric spaces of hyperbolic type.

We suppose that (M, d) is a metric space containing a family L of metric lines such that distinct points $x, y \in M$ lie on exactly one number $l(x, y)$ of L.

This metric line determines a unique metric segment joining x and y. We denote this segment by $S[x, y]$. For each $\alpha \in [0, 1]$ there is a unique point z in $S[x, y]$ for which

$$d(x, z) = \alpha d(x, y) \text{ and } d(z, y) = (1 - \alpha)d(x, y).$$

Adopting the notation of [28] or [57], we shall denote this point by $(1-\alpha)x \oplus \alpha y$.

In [62] Takahashi introduced a notion of convex metric spaces as follows: A metric space (X, d) is said to be (Takahashi) convex if there is a function $\alpha : [0, 1] \times X \times X \to X$ such that $\alpha(0, x, y) = \alpha(1, y, x) = x$, and

$$d(\alpha(t, x_1, x_2), y) \leq (1 - t)d(x_1, y) + td(x_2, y).$$

As an aside, we remark that if it is assumed that for any $\bar{x} \in X$ the function $(x, t) \mapsto \alpha(t, x, \bar{x})$ is *continuous* and also that the metric d only satisfies

$$d(\alpha(t, x_1, x_2), y) \leq \max\{d(x_1, y), d(x_2, y)\},$$

then it is possible to define a c-structure on X so that X becomes a Φ-space in the sense of [30]. Such spaces are rich enough to yield abstract version of a number of classical fixed-point, minimax results, selection theorems, etc. (cf., [30, 7]). These include an abstract version of Schauder's theorem.

Spaces which are Takahashi-convex provide examples of spaces which are of 'hyperbolic type' as introduced in [41]. For more examples, see [57] and [28].

In Chapter 5 of this paper, we first introduce some definitions needed for our arguments and a crucial result (originally due to Kirk [41] for nonexpansive mappings) for a non-Lipschitzian self-mapping (see Proposition 5.1.1). We shall next show that if K is a compact convex subset of a metric space (M, d) of pre-hyperbolic type, and if $T : K \to K$ is a continuous mapping of strongly ANT and asymptotically regular on K, then T has an iterative fixed point (see Theorem

5.2.5). This improves the result due to Kirk [41] for nonexpansive mappings. Finally, we give that the fixed point set of any mapping of ANT is a 1-local retract of M under some conditions for M. (see Theorem 5.2.12).

Chapter 2

Non-Lipschitzian self-mappings and geometrical properties

2.1 Non-Lipschitzian self-mappings

Let C be a nonempty subset of a Banach space X and let $T : C \to C$ be a mapping. Note first that every nonexpansive mapping is AN. Also every AN mapping is Lipschitzian (hence uniformly continuous) and, furthermore, a mapping of strongly ANT (hence, a mapping of ANT) if C is assumed to be bounded. However, all mappings of strongly ANT may be non-Lipschitzian. In particular, if $T^n x \to 0$ uniformly on C, then T is of strongly ANT. Also, for all $x \in K$, if $T^n x \in F(T) = \{z\}$ for some $n \geq 1$, T is of ANT.

Here are some examples to show the relations between Lipschitzian mappings and non-Lipschitzian mappings.

Example 2.1.1. (a) Let $C = [-1/\pi, 1/\pi] \subset \mathbb{R}$ and $|k| < 1$. For each $x \in K$ we define $Tx = kx \sin \frac{1}{x}$ if $x \neq 0$, and $T0 = 0$. Note that $T^n x \to 0$ uniformly on C. Hence, $T : C \to C$ is a continuous mapping of ANT which is not Lipschitzian.

(b) Let $C = [0, 1] \subseteq \mathbb{R}$ and define $Tx = \frac{1}{4}$ if $x = \frac{1}{4}$, $Tx = 1$ for $x \in [0, \frac{1}{2}] \setminus \frac{1}{4}$, and $Tx = \frac{1}{2}$ for $x \in (\frac{1}{2}, 1]$. Note that for all $x \in C$, $T^n x = \frac{1}{4} \in F(T) = \{\frac{1}{4}\}$ for $n \geq 3$. Then $T : C \to C$ is a discontinuous mapping of ANT which is not nonexpansive.

(c) Let $C = [0, 1] \subseteq \mathbb{R}$ and let φ be the Cantor ternary function. Define $T : C \to C$ by

$$Tx = \begin{cases} x/2, & 0 \leq x \leq 1/2; \\ \varphi((1-x)/2), & 1/2 < x \leq 1. \end{cases}$$

Note that $T^n x \to 0$ uniformly on C. Therefore, T is a discontinuous mapping of strongly ANT but not AN because φ is not Lipschizian on $[0, \frac{1}{2}]$ (This is a slight modification of Example 1.1 due to Miyadera [53]).

For little variation for non-Lipschitzian self-mappings we assume that C is convex. A Set satisfying $T(K) \subset K$ is said to be *invariant under* T or T-*invariant*. Let K be a nonempty subset of C and for each $x \in K$, we then set

$$c_n(x; K) = \sup_{y \in K}(\|T^n x - T^n y\| - \|x - y\|) \vee 0.$$

Definition 2.1.2. $T : C \to C$ is of *partly asymptotically nonexpansive type* (simply, *partly ANT*) if there exists a nonempty bounded convex and T-invariant subset K of C such that $c_n(x; K) \to 0$ for each $x \in K$. We also say that T is of *nearly asymptotically nonexpansive type* (simply, *nearly ANT*) if there exists a nonempty bounded convex and T-invariant subset K of C such that $c_n(x) := c_n(x; C) \to 0$ for each $x \in K$. Obviously mappings of nearly ANT are of partly ANT. Recall that if $c_n(x) := c_n(x; C) \to 0$ for each $x \in C$, then T is said to be of ANT (see [40]).

Remark 2.1.3. (i) If $c_n(x; K) \to 0$ then $c_{n\pm p}(x; K) \to 0$ for fixed $p \in \mathbb{N}$.

(ii) If C is bounded, then the following implications holds immediately.

$$\text{(strongly ANT)} \Rightarrow \text{(ANT)} \Rightarrow \text{(nearly ANT)} \Rightarrow \text{(partly ANT)}.$$

First, we shall give an example of non-Lipschitzian mappings of nearly ANT which are not of ANT. inspired by Example 4.4 in [35]. This example also satisfies all assumptions of our main result (see Theorem 4.3.1 below).

Example 2.1.4. Let $X = \mathbb{R}$, $C = (-\infty, 1]$. First consider a continuous non-Lipschitzian mapping $f : [0, 1/2] \to [0, 1/4]$ defined by

$$f(x) = \begin{cases} \frac{n(2n+1)}{n+1}\left(x - \frac{1}{2n+1}\right), & \frac{1}{2n+1} \le x \le \frac{1}{2n}, \quad n \ge 1; \\ -\frac{(n+1)(2n+1)}{n+2}\left(x - \frac{1}{2n+1}\right), & \frac{1}{2(n+1)} \le x \le \frac{1}{2n+1}, \quad n \ge 1; \\ 0, & x = 0. \end{cases}$$

Note first that for each $n \in \mathbb{N}$, the graph of f on each subinterval $\left[\frac{1}{2(n+1)}, \frac{1}{2n}\right]$ consists of two segments connecting three points $(1/2(n+1), 1/2(n+2))$, $(1/2n+1, 0)$ and $(1/2n, 1/2(n+1))$. For each $x \in C$, we now define

$$Tx = \begin{cases} \frac{x}{1-2x}, & x \leq -\frac{1}{2}; \\ f(x), & x \in [0, 1/2]; \\ -f(-x), & x \in [-1/2, 0]; \\ x^2, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Obviously, $|T^n z| \leq \frac{1}{2(n+1)}$ for $|z| \leq \frac{1}{2}$, and so $T^n z \to 0$ uniformly on $[-1/2, 1/2]$. Also, since $|Tz| \leq 1/2$ for $z \leq -1/2$, we have $T^n z \to 0$ uniformly on $(-\infty, -1/2]$. We thus obtain $T^n z \to 0$ uniformly on $(-\infty, 1/2]$. It is obvious that T is not of ANT because $c_n(1) = 1$ for each n. However, if we take $K := [-1/2, 0]$, it is clearly T-invariant. For this closed interval K of C, T is of nearly ANT, i.e., $c_n(x) \to 0$ for each $x \in K$. Indeed, since

$$\begin{aligned} c_n(x) &= \sup_{y \in C}[|T^n x - T^n y| - |x - y|] \vee 0 \\ &= \sup_{y \in (-\infty, 1/2]} [|T^n x - T^n y| - |x - y|] \vee \sup_{y \in [1/2, 1]} [|T^n x - T^n y| - |x - y|] \vee 0, \end{aligned}$$

and $T^n z \to 0$ uniformly on $(-\infty, 1/2]$, we have

$$c_n(x) = \sup_{y \in [1/2, 1]} [|T^n x - T^n y| - |x - y|] \vee 0.$$

For each $n \in \mathbb{N}$, there exists $y_n \in [1/2, 1]$ such that $c_n(x) = [|T^n x - T^n y_n| - |x - y_n|] \vee 0$. If $y_n = 1$, since $0 \leq T^n x \leq 1/4$, we have $|T^n x - 1| = 1 - T^n x \leq 1 - x = |x - 1|$, and so $|T^n x - 1| - |x - 1| \leq 0$. If $y_n < 1$, since $T^n x \to 0$ uniformly, for sufficiently large n,

$$\begin{aligned} |T^n x - T^n y_n| - |x - y_n| &= T^n y_n - T^n x - (y_n - x) \\ &= (T^n y_n - y_n) + (x - T^n x) < 0. \end{aligned}$$

Therefore, in any case, $c_n(x) \to 0$ for each $x \in K$, and hence T is of nearly ANT on C. Finally, note that every sequence $\{T^n x\}$ converges uniformly to $0 \in F(T) \cap K$ for each $x \in K$.

Finally we shall introduce an example of non-Lipschitzian mappings of partly ANT, but are not of nearly ANT. The following example is a slight modification of Example 4.3 in [35].

Example 2.1.5. Let $X = C = \mathbb{R}$ and and let $|k| < 1$. For each $x \in C$, we define

$$Tx = \begin{cases} kx \sin \frac{1}{x}, & x \neq 0, \ |x| \leq 1/\pi; \\ 0, & x = 0; \\ \pi|x| - 1, & |x| > 1/\pi. \end{cases}$$

Then, clearly $c_n(1) = c_n(1; C) \geq T^n 1 - 1 \to \infty$, and so T is not of ANT. Note similarly that $c_n(x) = c_n(x, C) \to \infty$ for all fixed $x \in C$. Therefore T is not of nearly ANT. But if we take $K = [-1/\pi, 1/\pi]$, then K is T-invariant and so T is of partly ANT. Indeed, it suffices to show that $c_n(x; K) \to 0$ for each $x \in K$. For fixed $x \in K$, set

$$H_n(y) = |T^n x - T^n y| - |x - y| \qquad \forall y \in K.$$

Then $H_n(\cdot)$ is also continuous on K, and so it achieves its maximum in K, i.e., there exists a $y_n \in K$ such that $c_n(x; K) = H_n(y_n) \vee 0$. Since $T^n x \to 0$ uniformly on K, we have $c_n(x; K) \to 0$ for each $x \in K$.

2.2 Geometrical properties

Let X be a Banach space. Next, let us introduce normal structure coefficient of X introduced by Bynum [17]. For $A \subset X$, $\text{diam}(A)$ and $r_A(A)$ denote the *diameter* and the *self-Chebyshev radius* of A, respectively, i.e.,

$$\begin{aligned} \text{diam}(A) &= \sup_{x,y \in A} \|x - y\|, \\ r_A(A) &= \inf_{x \in A} (\sup_{y \in A} \|x - y\|) \end{aligned}$$

Recall that X has *uniform normal structure* (simply UNS) if $N(X) > 1$, where

$$N(X) = \inf \left\{ \frac{\text{diam}(A)}{r_A(A)} : A \subset X \text{ bounded closed convex with } \text{diam}(A) > 0 \right\}.$$

Obviously, if $N(X) > 1$, then X has normal structure.

Recall that if X is a non-Schur Banach space, then the weakly convergent sequence coefficient of X, denoted by $WCS(X)$, is defined by

$$\begin{aligned} WCS(X) &= \sup\{M > 0 : \text{for each weakly convergent sequence } \{x_n\}, \\ &\exists \;\; y \in \overline{co}(\{x_n\}) \text{ such that } M \cdot \limsup_{n \to \infty} \|x_n - y\| \le A(\{x_n\})\}, \end{aligned}$$

where $\overline{co}(K)$ denotes the closed convex hull of a set K and $A(\{x_n\})$ denotes the asymptotic diameter of $\{x_n\}$, i.e.,

$$A(\{x_n\}) = \lim_{n \to \infty} \sup\{\|x_i - x_j\| : i, j \ge n\}.$$

It is easy to give a sharp expression $WCS(X)$ as follows;

$$WCS(X) = \sup\{M : x_n \rightharpoonup u \ \Rightarrow \ M \cdot \limsup_{n\to\infty} \|x_n - u\| \le D(\{x_n\})\},$$

where $D(\{x_n\}) := \limsup_{m\to\infty}\limsup_{n\to\infty} \|x_n - x_m\|$ and "$\rightharpoonup$" means the weak convergence. For more details, see [17] and [36].

Note that $N(X) = BS(X) = A(X)$ for any Banach space (see Lim [49]), furthermore, if X is reflexive, then $1 \le N(X) \le BS(X) \le WCS(X) \le 2$ (cf., [17]), where $BS(X)$ means the *bounded sequence coefficient* of X, i.e.,

$$\begin{aligned} BS(X) \ = \ \sup\Big\{M : & \text{ for any bounded sequence } \{x_n\} \text{ in } X, \\ & \exists \ y \in \overline{co}(\{x_n\}) \text{ such that } M \cdot \limsup_{n\to\infty} \|x_n - y\| \le A(\{x_n\})\Big\}. \end{aligned}$$

While $N(X)$ and $BS(X)$ can be defined in every Banach space, $WCS(X)$ is well defined only in infinite dimensional reflexive spaces, where, by Eberlein-Šmulian theorem, we can assure the existence of weakly convergent sequences which do not converge.

It is well-known [17] that if $WCS(X) > 1$, then X has *weak normal structure*. This means that any weakly compact convex subset C of X with $\operatorname{diam}(C) > 0$ has a nondiametral point. The coefficient $WCS(X)$ plays important roles in fixed point theory. A space X such that $WCS(X) > 1$ is said to have *weak uniform normal structure*.

Let X be a Banach space. Finally recall that Maluta's constant $D(X)$ [52] of X is defined by

$$D(X) = \sup\left\{\frac{\limsup d(x_{n+1}, co(\{x_1, x_2, \cdots x_n\}))}{diam(\{x_n\})}\right\},$$

where the supremum is taken over all bounded nonconstant sequences $\{x_n\}$ in X.

We remark the following properties for Maluta's constant given in [52].

Lemma 2.2.1. *Let X be a Banach space. Then*

(a) $D(X) \leq \tilde{N}(X) := 1/N(X)$.

(b) $D(X) = \sup\{D(Y) : Y \subset X \text{ separable}\}$.

(c) $D(X) = 0$ *if and only if* X *is finite-dimensional.*

(d) If X *is reflexive, then* $D(X) \leq 1/WCS(X)$.

(e) If $D(X) < 1$, *then the Banach space* X *is reflexive and has normal structure.*

Remark 2.2.2. (i) The property (a) says that if X has uniform normal structure, then $D(X) < 1$. However, the converse does not hold (see Example 5.1 and Corollary 5.2 in [52]).

(ii) In view of (d), Maluta asked whether $D(X) = 1/WCS(X)$ holds true for every infinite dimensional reflexive Banach space X. In 1985, Amir [4] gave a partial solution for this question as follows; if X satisfies Opial's condition, then $D(X) \geq 1/WCS(X)$. Five years later, this question was completely solved by Prus [55].

(iii) The converse of (c) also does not hold (see Example 4.1 in [52] as follows; $X = (\sum \bigoplus \ell_n)_2$ is reflexive and has normal structure although $D(X) = 1$).

Chapter 3

Fixed point theorems for mappings of partly ANT in Banach spaces with uniform normal structure

3.1 Preliminaries

Let us begin with the following lemma due to Casini and Maluta [19] which is very crucial for our further arguments.

Lemma 3.1.1 [19]. *Let* X *be a Banach space with* $\tilde{N}(X) < 1$. *Then, for every bounded sequence* $\{x_n\}$, *there exists a point* $z \in \overline{co}(\{x_n\})$ *such that*

(i) $\limsup_{n\to\infty} \|x_n - z\| \leq \tilde{N}(X) \cdot A(\{x_n\})$;

(ii) *for every* $y \in X$, $\|z - y\| \leq \limsup_{n\to\infty} \|x_n - y\|$.

Now let C be a bounded closed convex subset of a Banach space X and let $T : C \to C$ be a *uniformly Lipschitzian* mapping; that is, T satisfies the condition for some constant $k \geq 0$.

$$\|T^n x - T^n y\| \leq k\|x - y\| \quad \forall x, y \in C.$$

A deep result of Casini and Maluta [19] applying Lemma 2.1.1 is the following.

Theorem 3.1.2 [19]. *If $k < \sqrt{N(X)}$, then T has a fixed point.*

Let $T : C \to C$ be a mapping of ANT. Suppose T^N is continuous for some integer $N \geq 1$. In [40], Kirk proved that if C is a *compact* convex subset of X, $T : C \to C$ has a fixed point.

What happens if C is weakly compact convex?

In 1981, Alspach [2] gave a counter example which shows that the above question does not holds even if T is nonexpansive. In fact, set

$$K := \left\{ f \in L^1[0,1] : 0 \leq f \leq 1, \int_0^1 f = \frac{1}{2} \right\},$$

then K is weakly compact. If $T : K \to K$ is a *baker transform* such that

$$Tf(t) = \begin{cases} (2f(2t)) \wedge 1, & 0 \leq t \leq 1/2; \\ (2f(2t-1) - 1) \vee 1, & 1/2 < t \leq 1, \end{cases}$$

then T is an isometry on C but fixed point free. Therefore, we realize some conditions on X are needed for the existence of fixed points. Here we present well known fixed point theorems for Lipschitzian or non-Lipschitzian self-mappings.

Theorem 3.1.3 [39]. *Let C be a weakly compact convex subset of a Banach space X with normal structure. If $T : C \to C$ is nonexpansive, then T has a fixed point in C.*

Theorem 3.1.4 [26]. *Let C be a bounded closed convex subset of a uniformly convex Banach space X. If $T : C \to C$ is AN, then T has a fixed point in C.*

Question 3.1.5. *Does normal structure imply the existence of fixed points of AN mappings?*

The above question is still open. However, the following result was recently obtained by Kim-Xu [38].

Theorem 3.1.6 [38]. *Let C be a bounded closed convex subset of a Banach space X with uniform normal structure. If $T : C \to C$ is AN then T has a fixed point in C.*

Theorem 3.1.5 was immediately extended to mappings of ANT by Li-Sims [48] and Kim [34], independently. Now the following question is naturally raised.

Question 3.1.7. *Does uniform normal structure imply the existence of fixed points of mappings of partly ANT?*

We shall give a positive answer for this question at the following section.

3.2 Uniform normal structure and fixed point theorems

First, we exhibit the following easy result.

Lemma 3.2.1. *Let C be a nonempty closed convex subset of a reflexive Banach space X. If $T : C \to C$ is a continuous mapping of partly ANT, then there exists a nonempty weakly compact convex and T-invariant subset K of C such that $c_n(x; K) \to 0$ for each $x \in K$.*

Proof. Since T is of partly ANT, there exists a nonempty bounded convex and T-invariant subset A of C such that $c_n(x; A) \to 0$ for each $x \in A$, where

$$c_n(x; A) = \sup_{y \in A}(\|T^n x - T^n y\| - \|x - y\|) \vee 0.$$

For each fixed $x \in A$ and $n \in \mathbb{N}$, since T^n is continuous, we easily get $c_n(x; A) = c_n(x; \overline{A})$ and so $c_n(x; \overline{A}) \to 0$ for each $x \in A$, where $\overline{A}$ denotes the closure of A. Take $K = \overline{A}$. Clearly, K is T-invariant, and by reflexivity of X, it is weakly compact and convex. Finally we claim that $c_n(x; K) \to 0$ for each $x \in K$. To this end, let $x \in K$. Then there exists a sequence $\{x_m\}$ in A such that $x_m \to x$. Since the supremum of any collection of lower semicontinuous mappings is lower semicontinuous, $c_n(\cdot\,; K)$ is lower semicontinuous for fixed n. Let $\epsilon > 0$ be arbitrarily given. Then there exists a $m_\epsilon \in \mathbb{N}$ such that

$$0 \le c_n(x; K) \le \liminf_{m \to \infty} c_n(x_m; K) < c_n(x_{m_\epsilon}; K) + \epsilon$$

for each $n \in \mathbb{N}$. Since $c_n(x_{m_\epsilon}; K) \to 0$ as $n \to \infty$, this yields

$$0 \leq \limsup_{n \to \infty} c_n(x; K) \leq \epsilon$$

for arbitrarily given $\epsilon > 0$, and hence $c_n(x; K) \to 0$ for each $x \in K$. □

Now we are ready to answer our first question by applying for two Lemmas 3.1.1 and 3.2.1.

Theorem 3.2.2. *Let X be a Banach space with uniform normal structure (i.e., $\tilde{N}(X) < 1$), C a nonempty closed bounded convex subset of X. If $T : C \to C$ is a mapping of partly ANT, then T has a fixed point.*

Proof. By Lemma 3.2.1, there exists a nonempty weakly compact convex and T-invariant subset K of C such that $c_n(x; K) \to 0$ for each $x \in K$. For $x_0 \in K$, consider the bounded sequence $\{T^n x_0\}$ and let $x_1 \in K$ be the point satisfying Lemma 3.1.1 for $\{T^n x_0\}$. Repeating this process continuously, we have a sequence $\{x_m\}$ in K satisfying the following properties:

$$\begin{aligned} \limsup_{n \to \infty} \|T^n x_m - x_{m+1}\| &\leq \tilde{N}(X) \cdot A(\{T^n x_m\}), \\ \|y - x_{m+1}\| &\leq \limsup_{n \to \infty} \|T^n x_m - y\| \quad (y \in X). \end{aligned}$$

Note that for $i, j \in \mathbb{N}$ (we may assume $i > j$)

$$\begin{aligned} \|T^i x_m - T^j x_m\| &\leq c_j(x_m; K) + \|T^{i-j} x_m - x_m\| \\ &\leq c_j(x_m; K) + \limsup_{n \to \infty} \|T^n x_{m-1} - T^{i-j} x_m\| \\ &\leq c_j(x_m; K) + c_{i-j}(x_m; K) + \limsup_{n \to \infty} \|T^n x_{m-1} - x_m\| \end{aligned}$$

and so taking $\sup_{i,j\geq k}$ at first and next $k \to \infty$, we obtain

$$A(\{T^n x_m\}) \leq \limsup_{n\to\infty} \|T^n x_{m-1} - x_m\| := r_m.$$

Hence, we get

$$r_{m+1} \leq \tilde{N}(X) r_m = [\tilde{N}(X)]^m r_1$$

and clearly $r_m \to 0$ because $\tilde{N}(X) < 1$. For any $k \in \mathbb{N}$, since

$$\begin{aligned}
\|x_{m+1} - x_m\| &\leq \|x_{m+1} - T^k x_m\| + \|T^k x_m - x_m\| \\
&\leq \|x_{m+1} - T^k x_m\| + \limsup_{n\to\infty} \|T^n x_{m-1} - T^k x_m\| \\
&\leq \|x_{m+1} - T^k x_m\| + c_k(x_m; K) + \limsup_{n\to\infty} \|T^n x_{m-1} - x_m\|,
\end{aligned}$$

taking $\limsup_{k\to\infty}$ yields

$$\begin{aligned}
\|x_{m+1} - x_m\| &\leq r_m + r_{m+1} \\
&\leq (1 + \tilde{N}(X)) r_m = (1 + \tilde{N}(X))[\tilde{N}(X)]^{m-1} r_1.
\end{aligned}$$

So, $\{x_m\}$ is a Cauchy sequence and let $x := \lim_{m\to\infty} x_m \in K$.

On the other hand, note that for any $n \in \mathbb{N}$,

$$\begin{aligned}
\|T^n x - x\| &\leq \|T^n x - T^n x_m\| + \|T^n x_m - x_{m+1}\| + \|x_{m+1} - x\| \\
&\leq c_n(x; K) + \|x - x_m\| + \|T^n x_m - x_{m+1}\| + \|x_{m+1} - x\|.
\end{aligned}$$

Taking $\limsup_{n\to\infty}$ on both sides, the fact that $c_{n_k}(x; K) \to 0$ yields

$$\limsup_{n\to\infty} \|T^n x - x\| \leq \|x - x_m\| + r_{m+1} + \|x_{m+1} - x\|.$$

Since all terms in the right side converge to 0 as $m \to \infty$, we immediately see

$$\lim_{n\to\infty} \|T^n x - x\| = 0.$$

The continuity of T gives $x \in F(T) \cap K$ and the proof is complete. □

Now let us consider some applications. The following notion was introduced by Lau [46] to study the Chebyshev subset of X. Recall that a Banach space X is called a U*-space* [46] if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\forall x, y \in S_X, \quad \left\|\frac{x+y}{2}\right\| > 1 - \delta \Longrightarrow \langle f, y\rangle > 1 - \epsilon, \quad f \in \nabla_x,$$

where S_X denotes the unit sphere of X and ∇_x denotes the set of norm 1 supporting functionals f of S_X at x for $x \in X$.

It was known in [46] and [24] that

(a) If X is a U*-space*, then X is uniformly nonsquare, in particular, X is superreflexive [20];

(b) X is a U*-space* if and only if X^* is a U*-space*;

(c) Uniformly convex spaces and uniformly smooth spaces are U-spaces;

(d) If X is a U*-space*, then X has UNS. Further, if X is a Banach space with $\delta(\frac{3}{2}) > \frac{1}{4}$, then X has uniform normal structure (c.f., [24]).

As a direct consequence of Theorem 3.2.2, we have the following.

Corollary 3.2.3. *Let X be a U-space and C a nonempty convex subset of a Banach space X. If $T : C \to C$ is a mapping of partly ANT, then T has a fixed point.*

Since uniform smoothness implies uniform normal structure, we have the following result, which was implicitly used in [50].

Corollary 3.2.4. *Assume X is a uniformly smooth Banach space and C is a nonempty convex subset. Then every mapping $T : C \to C$ of partly ANT has a fixed point.*

Chapter 4

Fixed point theorems for mappings of partly ANT in Banach spaces with weak uniform normal structure

4.1 Preliminaries

First recall that all the Banach spaces with uniform normal structure (that is, $\tilde{N}(X) < 1$) satisfies its Maluta's contant $D(X) < 1$, but the converse does not hold (see (a) of Lemma 1.2.1 and Remark 1.2.1). The following question is therefore very naturally raised from Theorem 2.2.2.

Question 4.1.1. *Is the assumption $\tilde{N}(X) < 1$ in Theorem 3.2.2 replaced by $D(X) < 1$?*

In finite dimensional reflexive Banach spaces, it is well known (see Remark 2.2.1) that $D(X) < 1$ if and only if $WCS(X) > 1$. Furthermore, if $D(X) < 1$, X has normal structure and hence the fixed point property for nonexpansive mappings, that is, for every weakly compact convex subset C of X, every nonexpansive map $T : C \to C$ has a fixed point. However, it is still open whether $D(X) < 1$ implies the fixed point property for AN mappings. In 1994. Lim-Xu [50] gave a partial answer for this question as follows:

Theorem 4.1.2 [50]. *Suppose that X is a Banach space such that $D(X) < 1$, and C is a closed bounded convex subset of X. If a mapping $T : C \to C$ is AN, and weakly asymptotically regular on C. i.e., $T^{n+1}x - T^n x \rightharpoonup 0$ $\forall x \in C$, then T has a fixed point.*

Immediately, Theorem 4.1.2 was extended to all mappings of ANT by Kim-Kim (see Corollary 3.3 in [37]). In fact, under the assumption of weakly asymptotic regularity of T, the conditions for X and T can be weakened, in other words, Theorem 4.1.2 can be extended to mappings of partly ANT with $WCS(X) > 1$.

4.2 Some properties of ultra filters

We will begin this section by introducing some concepts of ultrafilters and giving some important results concerning ultrafilters. For more details the reader may consult [1, 32].

Recall that a *filter* $\mathcal{F}$ on a nonempty set I is a nonempty collection of subsets of I satisfying

(a) if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$;

(b) if $A \in \mathcal{F}$ and $A \subset B$ then $B \in \mathcal{F}$.

Obviously, $I \in \mathcal{F}$ for any filter on I. If $\mathcal{F}$ is a filter on I and $\emptyset \in \mathcal{F}$, then $\mathcal{F} = 2^I$ and is called the *improper filter*. Let $i \in I$ be fixed. $\mathcal{F}_i = \{A \subset I : i \in A\}$ is a filter on I and is called a *trivial filter*.

Now let $\mathcal{P}$ be the family of all proper filters on I, i.e.,

$$\mathcal{P} = \{\mathcal{F} : \mathcal{F} \text{ is a filter on } I, \mathcal{F} \neq 2^I\}.$$

Since $\mathcal{P}$ is an *inductive* set (every increasing chain has an upper bound), it follows from Zorn's lemma that $\mathcal{P}$ has a maximal element. In other words, there exists some $\mathcal{F} \in \mathcal{P}$ such that if $\mathcal{D} \in \mathcal{P}$ and $\mathcal{F} \subset \mathcal{D}$, then $\mathcal{F} = \mathcal{D}$. Such a maximal element of $\mathcal{P}$ is called an *ultrafilter* on I.

Lemma 4.2.1. (i) *A filer $\mathcal{U}$ on I is an ultrafilter if and only if for every $A \subset I$, either A or $I \setminus A$ belongs to $\mathcal{U}$.* (ii) *The ultrafilter $\mathcal{U}$ on I is trivial if and only if there exists a finite set $A \in \mathcal{U}$.*

Let X be a Hausdorff topological space and let $\{x_i\}_{i \in I}$ be a collection of elements of X indexed by a set I, and consider a filter $\mathcal{F}$ on I. We then say that $\{x_i\}_{i \in I}$ *converges to* $x \in X$ *over* $\mathcal{F}$ if the set $\{i \in I : x_i \in V\}$ is in $\mathcal{F}$ for any neighborhood V of x. The limit will be denoted by $\lim_{i,\mathcal{F}} x_i$ or $\lim_{\mathcal{F}} x_i$. Note that if $\mathcal{F}$ is proper, the limit over $\mathcal{F}$ is unique, and if, moreover, C is a closed subset of X and $\{x_i\} \subset C$, then $\lim_{\mathcal{F}} x_i \in C$.

If $\mathcal{F}_{i_0}$ is the trivial filter generated by $i_0 \in I$, then $\lim_{\mathcal{F}_{i_0}} x_i = x_{i_0}$. Trivial filters give no information on asymptotic behavior of sets, so we will generally avoid them. Here we give some properties for ultrafilters given by [1].

Lemma 4.2.2. *Let $\mathcal{U}$ be a nontrivial ultrafilter on $\mathbb{N}$ and suppose $\{x_n\}$ converges to x in the topology of the space X. Then $\lim_{\mathcal{U}} x_n = x$. If X is a metric space and $\lim_{\mathcal{U}} x_n = x$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = x$.*

The next result is interesting because it shows how ultrafilters can be used to characterize compactness of a topological space.

Lemma 4.2.3. *Let K be a nonempty subset of a Hausdorff topological space. Then, K is compact if and only if any set $\{x_i\}_{i\in I} \subset K$ is convergent in K over any ultrafilter $\mathcal{U}$ on I.*

The following result (cf., [1]) is to show how limits over ultrafilters cooperate well with the linear structure.

Lemma 4.2.4. *Let X ba a topological linear space and $\mathcal{U}$ an ultrafilter on a set I. Suppose that $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ are two subsets of X and $\lim_{\mathcal{U}} x_i = x$*

and $\lim_{\mathcal{U}} y_i = y$ *both exists. Then*

$$\lim_{\mathcal{U}}(x_i + y_i) = x + y \quad and \quad \lim_{\mathcal{U}}(\alpha x_i) = \alpha x$$

for any scalar α.

4.3 Weak uniform normal structure and fixed point theorems

Let C be a nonempty subset of a Banach space X, and let $T : C \to C$ be a mapping. Suppose there exists a nonempty subset K of C and the weak limit $w\text{-}\lim_{\mathcal{U}} T^n x$ exists in K for each $x \in K$, where $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$. We then can define a mapping $S : K \to K$ by

$$Sx = w\text{-}\lim_{\mathcal{U}} T^n x, \qquad \forall x \in K.$$

Note first that if K is weakly compact and T-invariant, then the weak limit $w\text{-}\lim_{\mathcal{U}} T^n x$ always exists in K for each $x \in K$. Also, it is obvious to see that $F(T) \cap K \subset F(S)$. What are conditions on X and T for which the converse inclusion remains true? Our purpose is to find some conditions on X and T to answer the above question.

Under the same hypotheses as Lemma 2.2.1, the mapping $S : K \to K$ defined as above is then well defined and nonexpansive. In fact, for $x, y \in K$,

$Sx = w\text{-}\lim_{\mathcal{U}} T^n x$ and $Sy = w\text{-}\lim_{\mathcal{U}} T^n y$. By Lemma 4.2.4,

$$Sx - Sy = w\text{-}\lim_{\mathcal{U}}(T^n x - T^n y).$$

Then there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $T^{n_k}x - T^{n_k}y \rightharpoonup Sx - Sy$ as $k \to \infty$. Since the norm $\|\cdot\|$ is weakly lower semicontinuous, and $c_n(x; K) \to 0$ for each $x \in K$, we have

$$\begin{aligned}
\|Sx - Sy\| &\leq \liminf_{k\to\infty} \|T^{n_k}x - T^{n_k}y\| \\
&\leq \limsup_{k\to\infty}[\|T^{n_k}x - T^{n_k}y\| - \|x - y\|] + \|x - y\| \\
&\leq \lim_{k\to\infty} c_{n_k}(x; K) + \|x - y\| = \|x - y\|.
\end{aligned}$$

Now we are ready to present a partial answer for the above question, that is, a sufficient condition for $F(S) \subset F(T) \cap K$.

Theorem 4.3.1. *Let C be a nonempty closed convex subset of a reflexive Banach space X with $WCS(X) > 1$. If $T : C \to C$ is a continuous mapping of partly ANT and weakly asymptotically regular on C, then there exist a nonempty weakly compact convex and T-invariant subset K of C and a nonexpansive mapping $S : K \to K$ such that $F(T) \cap K = F(S) \neq \emptyset$.*

Proof. By Lemma 3.2.1, there exists a nonempty weakly compact convex and T-invariant subset K of C such that $c_n(x; K) \to 0$ for each $x \in K$. Then we can define a nonexpansive mapping $S : K \to K$ as above. Now to complete the proof, we claim that $F(S) \subset F(T) \cap K$. To this end, let $x \in F(S)$, that is, $w\text{-}\lim_{\mathcal{U}} T^n x = x \in K$. Then there exists a subsequence $\{T^{n_k}x\}$ of the sequence $\{T^n x\}$ such that $T^{n_k}x \rightharpoonup x$ as $k \to \infty$. By the well known property of $WCS(X)$,

$$\limsup_{k\to\infty} \|T^{n_k}x - x\| \leq \frac{1}{WCS(X)} D(\{T^{n_k}x\}). \tag{4.1}$$

By weakly asymptotic regularity of T, it follows that $T^{n_k+m}x \rightharpoonup x$ as $k \to \infty$ for any $m \geq 0$. On the other hand, for each $i, j \in \mathbb{N}$ with $i > j$, the weak lower semicontinuity of the norm $\|\cdot\|$ immediately yields that

$$\begin{aligned}
&\|T^{n_j}x - T^{n_i}x\| \\
\leq\ & \left(\|T^{n_j}x - T^{n_j}(T^{n_i-n_j}x)\| - \|x - T^{n_i-n_j}x\|\right) + \|x - T^{n_i-n_j}x\| \\
\leq\ & c_{n_j}(x, K) + \|x - T^{n_i-n_j}x\| \quad (T^{n_k+m}x \rightharpoonup x \text{ as } k \to \infty, \text{ with } m = n_i - n_j) \\
\leq\ & c_{n_j}(x, K) + \liminf_{k\to\infty} \|T^{n_k+m}x - T^{n_i-n_j}x\| \\
\leq\ & c_{n_j}(x, K) + c_{n_i-n_j}(x, K) + \limsup_{k\to\infty} \|x - T^{n_k}x\|.
\end{aligned}$$

Taking $\limsup_{i\to\infty}$ first and next $\limsup_{j\to\infty}$ on both sides, since $c_n(x, K) \to 0$ for each $x \in K$, this yields

$$D(T^{n_i}x) \leq \limsup_{k\to\infty} \|x - T^{n_k}x\|,$$

and this together with (4.1) gives

$$(WCS(X) - 1) \limsup_{k\to\infty} \|T^{n_k}x - x\| \leq 0,$$

which in turn implies that $x = \lim_{k\to\infty} T^{n_k}x$. By the continuity and the weak asymptotic regularity of T, we have $Tx = x$, i.e., $x \in F(T)$. □

Remark 4.3.2. If C is weakly compact and convex, then the reflexivity of X in Theorem 4.3.1 can be removed and the hypotheses for T can be immediately replaced by mappings which are of ANT and weakly asymptotically regular on C. It is still open whether Theorem 3.3.1 holds true with no condition of weakly asymptotic regularity of T, or not.

Applying for the similar method of the proof as in Theorem 4.3.1, we can observe the following.

Theorem 4.3.3. *Let C be a nonempty bounded subset of a reflexive Banach space X. Let $T : C \to C$ be a continuous mapping of ANT which is weakly asymptotically regular on C. Suppose there exists a nonempty closed convex subset K of C with the following property*

$$(\omega) \qquad x \in K \quad \Longrightarrow \quad \omega_w(x) \subset K,$$

where $\omega_w(x)$ is the weak ω-limit set of T at x, i.e.,

$$\omega_w(x) = \{y \in X : y = w\text{-}\lim_{k\to\infty} T^{n_k}x \text{ for some } n_k \uparrow \infty\}.$$

Then T has a fixed point in K.

Proof. Since X is reflexive, K is weakly compact convex and $WSC(X) > 1$. Since the sequence $\{T^n x\}$ belongs to C, and $\overline{co}(C)$ is weakly compact, the weak limit $w\text{-}\lim_{\mathcal{U}} T^n x$ always exists in $\overline{co}(C)$ for each $x \in K$. Define $Sx = w\text{-}\lim_{\mathcal{U}} T^n x$ for each $x \in K$. Then, there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $T^{n_k} x \rightharpoonup Sx$ as $k \to \infty$. By property of (ω), it follows that $Sx \in \omega_w(x) \subset K$. Therefore, $S : K \to K$ is well defined, and clearly nonexpansive. Repeating the method of proof in Theorem 4.3.1, we can easily derive the conclusion. □

Corollary 4.3.4. *Let C be a nonempty bounded subset of a reflexive Banach space X. Let $T : C \to C$ be an AN mapping which is weakly asymptotically regular on C. Suppose there exists a nonempty closed convex subset K of C with the property (ω). Then T has a fixed point in K.*

Let C be a weakly compact convex subset of a Banach space X. Consider a family $\mathcal{F}$ of subsets K of C which are nonempty, closed, convex, and satisfy the property (ω). The weak compactness of C now allows one to use Zorn's lemma to obtain a minimal element (say) $K \in \mathcal{F}$. Here, as a direct consequence of Theorem 4.3.1 or 4.3.3, we give the following result, which generalizes the one due to Kim-Kim [37] for continuous mappings of ANT.

Corollary 4.3.5. *Let C be a nonempty weakly compact convex subset of a Banach space X. If $T : C \to C$ is a continuous mapping of ANT. and weakly asymptotically regular on C, then $F(T)$ is a nonempty nonexpansive retract of C.*

Proof. Note first that T is of partly ANT with $K = C$. Since C is weakly compact and convex, all assumptions of Theorem 4.3.1 or 4.3.3 are fulfilled, and hence $F(T) = F(S) \neq \emptyset$. Since S is nonexpansive, it follows from [14] that $F(S)$ is a nonempty nonexpansive retract of C. □

Chapter 5

Iterative fixed points for mappings of strongly ANT in metric spaces of pre-hyperbolic type

5.1 Preliminaries and definitions

Recall that a metric space M is said to be of *pre-hyperbolic* type if for each $x, y \in M$ there is a specified metric segment $S[x, y]$ joining x and y, which has the property that if $p \in M$, $\alpha \in (0, 1)$ and if m is the point of $S[x, y]$ which satisfies $d(x, m) = \alpha d(x, y)$, then

$$\text{(A)} \qquad d(p, m) \leq \alpha d(p, y) + (1 - \alpha) d(p, x).$$

Also, a subset C of M is *convex* if $S[x, y] \subset C$ whenever $x, y \in C$.

As stronger concept, a metric space M is said to be of *hyperbolic* type if for each $x, y \in M$ there is a specified metric segment $S[x, y]$ joining x and y for which the following property holds: Let $p, q, r \in M$ and $\alpha \in (0, 1)$, and suppose m_1 and m_2 are points of $S[p, r]$ and $S[p, q]$ respectively, which satisfy

$$d(m_1, p) = \alpha d(p, r) \text{ and } d(m_2. p) = \alpha d(p, q).$$

Then

$$\text{(H)} \qquad d(m_1, m_2) \leq \alpha d(r, q).$$

Obviously, (H) implies (A) (cf. [41]). There is an important consequence of condition (H). If M is of hyperbolic type and if $m_1 = (1 - \alpha)p \oplus \alpha q$ and $m_2 = (1 - \alpha)s \oplus \alpha r$, for $p, q, r, s \in M$ and $\alpha \in (0.1)$, then (H) in fact implies

$$\text{(H}'\text{)} \qquad d(m_1, m_2) \leq (1 - \alpha)d(p, s) + \alpha d(q, r).$$

Indeed,

$$\begin{aligned} d(m_1, m_2) &\leq d(m_1, (1 - \alpha)p \oplus \alpha r) + d((1 - \alpha)p \oplus \alpha r), m_2) \\ &\leq (1 - \alpha)d(p, s) + \alpha d(q. r) \quad \text{(using (H))}. \end{aligned}$$

We remark that the term 'hyperbolic type' is used in the above context because condition (H) with strict inequality is characteristic of hyperbolic geometry (see

[66]). At the same time, all normed linear spaces are of hyperbolic type. (As a matter of fact, if equality always holds in (H), then the resulting condition *characterizes* normed linear spaces among an appropriate class of metric spaces ([5]).) So are all Hadamard manifolds, that is, all finite-dimensional connected, simply connected, complete Riemannian manifolds of nonpositive curvature (cf., [16, pp. 305]). An infinite-dimensional example is provided by the Hilbert ball equipped with the hyperbolic metric (see [28, pp, 104]). For other results in this setting we refer, for example, to Reich [56] (and citations therein), Shafrir [60] and Reich-Shafrir [57].

Here we shall introduce a metric version of non-Lipschitzian self-mappings introduced in Chapter 1. Let C be a nonempty subset of a metric space (M, d) and let $T : C \to C$ be a mapping. Then T is said to be of *asymptotically nonexpansive type* (in brief, *ANT*) [41, 43] if for each $x \in C$, $\lim_{n\to\infty} c_n(x) = 0$, where

$$c_n(x) = \sup_{y\in C}[d(T^n x, T^n y) - d(x, y)] \vee 0.$$

Also, T is said to be of *strongly asymptotically nonexpansive type* (in brief, *strongly ANT*) if $\lim_{n\to\infty} c_n = 0$, where $c_n = \sup_{x\in C} c_n(x)$. In particular, T is said to be *Lipschitzian* if there exisits a positive number L such that $d(Tx, Ty) \leq Ld(x, y)$ for all $x, y \in K$. In particular, if $L = 1$, T is said to be *nonexpansive* and it is said to be *asymptotically nonexpansive* (cf., [26]) if each iterate T^n is Lipschitzian with Lipschitz constants $L_n \to 1$ as $n \to \infty$.

Let C be a nonempty closed convex subset of a metric space (M,d) of pre-hyperbolic type. Suppose $T : C \to C$ is a mapping of strongly ANT. Let $\alpha \in (0,1)$. Select $x_0 \in C$. For all $k = 0,1,2,\cdots$, set $x_{0,k} \equiv x_0 \in C$. For each $n = 0,1,2\cdots$, denote $y_{n,k} = T^k x_{n,k}$ and take $x_{n+1,k}$ to be the point of $S[x_{n,k}, y_{n,k}]$ for which $d(x_{n,k}, x_{n+1,k}) = \alpha d(x_{n,k}, y_{n,k})$. Since T is of strongly ANT, we have

$$\limsup_{k\to\infty} d(y_{n+1,k}, y_{n,k}) \le \limsup_{k\to\infty} d(x_{n+1,k}, x_{n,k}) \tag{5.1}$$

for $n = 0,1,\cdots$. On the other hand, since

$$\begin{aligned} d(y_{n+1,k}, x_{n+1,k}) &\le d(y_{n+1,k}, y_{n,k}) + d(y_{n,k}, x_{n+1,k}) \\ &\le c_k + d(x_{n+1,k}, x_{n,k}) + d(y_{n,k}, x_{n+1,k}) \\ &= c_k + d(y_{n,k}, x_{n,k}); \end{aligned}$$

taking lim sup on both sides as $k \to \infty$ yields

$$\limsup_{k\to\infty} d(y_{n+1,k}, x_{n+1,k}) \le \limsup_{k\to\infty} d(y_{n,k}, x_{n,k}) \tag{5.2}$$

for $n = 0,1,\cdots$.

In what follows we shall let $M(x_0,\alpha)$ denote the procedure for defining $\{x_{n,k}\}$ as described above. Mimicking the proof of [41], we easily have the following.

Proposition 5.1.1. *Suppose C is a nonempty bounded closed convex subset of a metric space (M,d) of pre-hyperbolic type. Let $T : C \to C$ be a* *mapping of strongly ANT and let $\alpha \in (0,1)$. Suppose double sequences $\{x_{n,k}\}$ and $\{y_{n,k}\}$ in K satisfy (5.1), (5.2) and $x_{n+1,k}$ is the point of $S[x_{n,k}, y_{n,k}]$ for which $d(x_{n,k}, x_{n+1,k}) = \alpha d(x_{n,k}, y_{n,k})$ for all $n, k \in \mathbb{N}$. Then*

$$\limsup_{k\to\infty} d(y_{i+n,k}, x_{i,k}) \geq (1-\alpha)^{-n}[\limsup_{k\to\infty} d(y_{i+n,k}, x_{i+n,k}) - \limsup_{k\to\infty} d(y_{i,k}, x_{i,k})] + (1+n\alpha)\limsup_{k\to\infty} d(y_{i,k}, x_{i,k}) \tag{5.3}$$

for all $i, n \in \mathbb{N}$. Further, if in addition

$$\limsup_{k\to\infty} d(y_{n+1,k}, x_{n+1,k}) = \limsup_{k\to\infty} d(y_{n,k}, x_{n,k}) \tag{5.4}$$

for $n \in \mathbb{N}$, then for all $i, n \in \mathbb{N}$,

$$\limsup_{k\to\infty} d(y_{i+n,k}, x_{i,k}) = (1+n\alpha)\limsup_{k\to\infty} d(y_{i,k}, x_{i,k}) \tag{5.5}$$

Proof. The proof employs the method of [41], which was proved by induction on n. First observe that (5.3) is trivially true for all i if $n = 0$. We make the inductive assumption that (5.3) holds for a given integer n and for all i. Replacing i with $i+1$ in (5.3), we obtain

$$\limsup_{k\to\infty} d(y_{i+n+1,k}, x_{i+1,k}) \geq (1-\alpha)^{-n}[\limsup_{k\to\infty} d(y_{i+n+1,k}, x_{i+n+1,k}) - \limsup_{k\to\infty} d(y_{i+1,k}, x_{i+1,k})] + (1+n\alpha)\limsup_{k\to\infty} d(y_{i+1,k}, x_{i+1,k}). \tag{5.6}$$

As M is of pre-hyperbolic type, (A) implies

$$\begin{aligned} d(y_{i+n+1,k}, x_{i+1,k}) &\leq (1-\alpha)d(y_{i+n+1,k}, x_{i,k}) + \alpha d(y_{i+n+1,k}, y_{i,k}) \\ &\leq (1-\alpha)d(y_{i+n+1,k}, x_{i,k}) + \alpha\sum_{j=0}^{n} d(y_{i+j+1,k}, y_{i+j,k}) \end{aligned}$$

for $k \in \mathbb{N}$. Taking lim sup on both sides as $k \to \infty$ and using (5.1) yield

$$\begin{aligned} & \limsup_{k\to\infty} d(y_{i+n+1,k}, x_{i+1,k}) \qquad (5.7) \\ \leq \; & (1-\alpha) \limsup_{k\to\infty} d(y_{i+n+1,k}, x_{i,k}) + \alpha \sum_{j=0}^{n} \limsup_{k\to\infty} d(x_{i+j+1,k}, x_{i+j,k}). \end{aligned}$$

Combining (5.6) and (5.7), we have

$$\begin{aligned} & \limsup_{k\to\infty} d(y_{i+n+1,k}, x_{i,k}) \\ \geq \; & (1-\alpha)^{-(n+1)} [\limsup_{k\to\infty} d(y_{i+n+1,k}, x_{i+n+1,k}) - \limsup_{k\to\infty} d(y_{i+1,k}, x_{i+1,k})] \\ & +(1-\alpha)^{-1}(1+n\alpha) \limsup_{k\to\infty} d(y_{i+1,k}, x_{i+1,k}) \\ & -\alpha(1-\alpha)^{-1} \sum_{j=0}^{n} \limsup_{k\to\infty} d(x_{i+j+1,k}, x_{i+j,k}). \end{aligned}$$

Since $d(x_{i+j+1,k}, x_{i+j,k}) = \alpha d(y_{i+j,k}, x_{i+j,k})$, this with (5.2) yields

$$\begin{aligned} & \limsup_{k\to\infty} d(y_{i+n+1,k}, x_{i,k}) \\ \geq \; & (1-\alpha)^{-(n+1)} [\limsup_{k\to\infty} d(y_{i+n+1,k}, x_{i+n+1,k}) - \limsup_{k\to\infty} d(y_{i+1,k}, x_{i+1,k})] \\ & +(1-\alpha)^{-1}(1+n\alpha) \limsup_{k\to\infty} d(y_{i+1,k}, x_{i+1,k}) \\ & -\alpha^2(1-\alpha)^{-1}(n+1) \limsup_{k\to\infty} d(y_{i,k}, x_{i,k}) \\ = \; & (1-\alpha)^{-(n+1)} [\limsup_{k\to\infty} d(y_{i+n+1,k}, x_{i+n+1,k}) - \limsup_{k\to\infty} d(y_{i,k}, x_{i,k})] \\ & +[(1-\alpha)^{-1}(1+n\alpha) - (1-\alpha)^{-(n+1)}] \limsup_{k\to\infty} d(y_{i+1,k}, x_{i+1,k}) \\ & +[(1-\alpha)^{-(n+1)} - \alpha^2(1-\alpha)^{-1}(n+1)] \limsup_{k\to\infty} d(y_{i,k}, x_{i,k}). \end{aligned}$$

Since $[(1-\alpha)^{-1}(1+n\alpha)-(1-\alpha)^{-(n+1)}]<0$, by (5.2), we obtain

$$\begin{aligned}
&\limsup_{k\to\infty} d(y_{i+n+1,k}, x_{i,k}) \\
\geq\ & (1-\alpha)^{-(n+1)}[\limsup_{k\to\infty} d(y_{i+n+1,k}, x_{i+n+1,k}) - \limsup_{k\to\infty} d(y_{i,k}, x_{i,k})] \\
&+[(1-\alpha)^{-1}(1+n\alpha)-(1-\alpha)^{-(n+1)}]\limsup_{k\to\infty} d(y_{i,k}, x_{i,k}) \\
&+[(1-\alpha)^{-(n+1)}-\alpha^2(1-\alpha)^{-1}(n+1)]\limsup_{k\to\infty} d(y_{i,k}, x_{i,k}) \\
=\ & (1-\alpha)^{-(n+1)}[\limsup_{k\to\infty} d(y_{i+n+1,k}, x_{i+n+1,k}) - \limsup_{k\to\infty} d(y_{i,k}, x_{i,k})] \\
&+[(1+(n+1)\alpha]\limsup_{k\to\infty} d(y_{i,k}, x_{i,k}).
\end{aligned}$$

This completes the proof of (5.3).

For the proof of (5.5), note at first that (5.4) and (5.3) immediately imply

$$\limsup_{k\to\infty} d(y_{i+n,k}, x_{i,k}) \geq (1+n\alpha)\limsup_{k\to\infty} d(y_{i,k}, x_{i,k})$$

for all $i, n \in \mathbb{N}$. Next, since (5.4) implies

$$\begin{aligned}
\limsup_{k\to\infty} d(y_{i+n,k}, x_{i,k}) &\leq \limsup_{k\to\infty} d(y_{i+n,k}, x_{i+n,k}) + \limsup_{k\to\infty} d(x_{i+n,k}, x_{i,k}) \\
&\leq \limsup_{k\to\infty} d(y_{i+n,k}, x_{i+n,k}) + \sum_{j=0}^{n-1}\limsup_{k\to\infty} d(x_{j+i+1,k}, x_{j+i,k}) \\
&= \limsup_{k\to\infty} d(y_{i,k}, x_{i,k}) + \alpha\sum_{j=0}^{n-1}\limsup_{k\to\infty} d(y_{j+i,k}, x_{j+i,k}) \\
&= (1+n\alpha)\limsup_{k\to\infty} d(y_{i,k}, x_{i,k}),
\end{aligned}$$

and the proof of (5.5) is complete. □

Remark 5.1.2. Compare Proposition 5.1.1 with Proposition 1 in [41]. Also see Lemma 9.4 of [27] for a Banach space version of Proposition 5.1.1.

5.2 Iterative fixed points

In this section, we first apply Proposition 5.1.1 for the study of the existence theorem of an iterative fixed point for mappings of strongly ANT in a metric space of pre-hyperbolic type.

Lemma 5.2.1. *Let C be a nonempty bounded closed convex subset of a metric space (M,d) of pre-hyperbolic type. Suppose $T: C \to C$ is a mapping of strongly ANT. Let $x_0 \in C$, and for $\alpha \in (0,1)$, let $\{x_{n,k}\}$ be the double sequence defined by the process $M(x_0, \alpha)$. Then*

$$\lim_{n\to\infty} \limsup_{k\to\infty} d(x_{n,k}, T^k x_{n,k}) = 0. \tag{5.8}$$

Proof. We employs the routine proof in [41]. Let $y_{n,k} = T^k x_{n,k}$, where $x_{0,k} := x_0$. Since K is bounded, there is a number ρ such that

$$\limsup_{k\to\infty} d(y_{i+n,k}, x_{i,k}) \le \rho$$

for all $i, n \in \mathbb{N}$. Suppose

$$\lim_{n\to\infty} \limsup_{k\to\infty} d(y_{n,k}, x_{n,k}) := r > 0.$$

Choose an integer N so that $N \ge d/r\alpha$ and let $\epsilon > 0$ satisfy $\epsilon(1-\alpha)^{-N} < r$. By (5.2), since the sequence $(\limsup_{k\to\infty} d(y_{n,k}, x_{n,k}))$ is decreasing to r as $n \to \infty$, there exists $i \in \mathbb{N}$ so that

$$0 \le \limsup_{k\to\infty} d(y_{i,k}, x_{i,k}) - \limsup_{k\to\infty} d(y_{i+N,k}, x_{i+N,k}) \le \epsilon.$$

Combined with (5.3), these choices of N, ϵ and i yield

$$\begin{aligned} d + r &\leq (1 + N\alpha)r \leq (1 + N\alpha) \limsup_{k\to\infty} d(y_{i,k}, x_{i,k}) \\ &\leq \limsup_{k\to\infty} d(y_{i+N,k}, x_{i,k}) + (1-\alpha)^{-N} [\limsup_{k\to\infty} d(y_{i,k}, x_{i,k}) \\ &\quad - \limsup_{k\to\infty} d(y_{i+N,k}, x_{i+N,k})] \quad \text{(by (5.3))} \\ &\leq d + (1-\alpha)^{-N}\epsilon < d + r. \end{aligned}$$

This contradiction establishes the theorem. □

Remark 5.2.2. Fix $\alpha \in (0,1)$. For each $k \in \mathbb{N}$, let $S_{\alpha,k} : C \to C$ be defined by

$$S_{\alpha,k}x = (1-\alpha)x \oplus \alpha T^k x$$

for all $x \in C$. Let $x_{n,k} = S^n_{\alpha,k}x$ and $y_{n,k} = T^k \circ S^n_{\alpha,k}x$. Following Lemma 5.2.1 we can prove that

$$\lim_{n\to\infty} \limsup_{k\to\infty} d(S^{n+1}_{\alpha,k}x, S^n_{\alpha,k}x) = \lim_{n\to\infty} \limsup_{k\to\infty} d(S^n_{\alpha,k}x, T^k \circ S^n_{\alpha,k}x) = 0 \qquad (5.9)$$

for $x \in K$. Note that

$$\limsup_{k\to\infty} d(S_{\alpha,k}x, S_{\alpha,k}y) \leq d(x,y).$$

Asymptotic regularity is a fundamentally important concept in metric fixed point theory. It was formally introduced by Browder-Petryshyn [13].

Definition 5.2.3. Let C be a nonempty subset of a metric space (M, d). A mapping $T : C \to C$ is said to be *asymptotically regular* on C if

$$\lim_{n \to \infty} d(T^n x, T^{n+1} x) = 0$$

for each $x \in C$.

Remark 5.2.4. Furthermore, if $T : C \to C$ is nonexpansive, then it is well-known in [41] that (5.8) can be simply expressed as

$$\lim_{n \to \infty} d(x_n, T(x_n)) = 0, \tag{5.10}$$

where for fixed $x_0 \in C$, $x_{n+1} = (1 - \alpha)x_n \oplus \alpha T x_n$ for $n \geq 0$.

Theorem 5.2.4. *Let C be a nonempty bounded closed convex subset of a metric space (M, d) of pre-hyperbolic type. Suppose $T : C \to C$ is nonexpansive. For a fixed $\alpha \in (0, 1)$ set $S_\alpha := (1 - \alpha)I \oplus \alpha T$, where I is the identity operator of M. Then S_α is nonexpansive with the same fixed point set of T. Moreover, S_α is asymptotically regular on C.*

Proof. For $x_0 \in C$, since $x_{n+1} = S_\alpha x_n = S_\alpha^{n+1} x_0$, by (5.10), we have

$$d(S_\alpha^{n+1} x_0, S_\alpha^n x_0) = d(S_\alpha x_n, x_n) = \alpha d(x_n, T x_n) \to 0.$$

Hence S_α is asymptotic regular on C. □

The following is an immediate consequence of Lemma 5.2.1.

Theorem 5.2.5. Let C be a compact convex subset of a metric space (M, d) of pre-hyperbolic type. Let $T : C \to C$ be a continuous mapping of strongly ANT. If T is asymptotically regular on C, then T has an iterative fixed point in C.

Proof. Let $x_0 \in C$, and for $\alpha \in (0, 1)$, let $\{x_{n,k}\}$ be the double sequence defined by the process $M(x_0, \alpha)$. Since C is compact, for each $k = 0, 1, 2 \cdots$ there exists a subsequence $\{x_{n_j,k}\}$ of the sequence $\{x_{n,k}\}$ such that $\lim_{j\to\infty} x_{n_j,k} = x_k$. Then we first show that

$$\lim_{j\to\infty} \limsup_{k\to\infty} d(x_k, x_{n_j,k}) = 0. \tag{5.11}$$

Indeed, given $\epsilon > 0$ and $k = 0, 1, 2 \cdots$, there is a $N \geq k$ such that

$$\limsup_{k\to\infty} d(x_k, x_{n_j,k}) \leq \sup_{i\geq k} d(x_{n_j,i}, x_i) < d(x_{n_j,N}, x_N) + \frac{\epsilon}{2}.$$

Taking lim sup on both sides as $j \to \infty$ implies

$$\lim_{j\to\infty} \limsup_{k\to\infty} d(x_k, x_{n_j,k}) < \epsilon,$$

and so (5.11) is obtained. On the other hand, note that

$$\begin{aligned} d(x_k, T^k x_k) &\leq d(x_k, x_{n_j,k}) + d(x_{n_j,k}, T^k x_{n_j,k}) + d(T^k x_{n_j,k}, T^k x_k) \\ &\leq c_k + 2d(x_k, x_{n_j,k}) + d(x_{n_j,k}, T^k x_{n_j,k}) \end{aligned}$$

for all k, j. Taking lim sup on both sides as first $k \to \infty$ and next $j \to \infty$, this combined with (5.8) and (5.10) yields $\limsup_{k\to\infty} d(x_k, T^k x_k) = 0$ and so

$$\lim_{k\to\infty} d(x_k, T^k x_k) = 0.$$

Taking a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ converging to $x \in C$, we readily see that $x \in F(T)$. In fact, since

$$\begin{aligned} d(x, T^{k_i}x) &\leq d(x, x_{k_i}) + d(x_{k_i}, T^{k_i}x_{k_i}) + d(T^{k_i}x_{k_i}, T^{k_i}x) \\ &\leq c_{k_i} + 2d(x, x_{k_i}) + d(x_{k_i}, T^{k_i}x_{k_i}), \end{aligned}$$

it implies that $x = \lim_{i\to\infty} T^{k_i}x$. Since T is both asymptotically regular and continuous,

$$d(x, T(x)) = \lim_{i\to\infty} d(T^{k_i}x, T^{k_i+1}x) = 0,$$

and so $x = Tx$. □

Since all asymptotically nonexpansive mapping are uniformly continuous mappings of strongly ANT, as a direct consequence of Theorem 5.2.5, we have

Corollary 5.2.6. Let C be a compact convex subset of a metric space (M, d) of pre-hyperbolic type. Let $T : C \to C$ be asymptotically nonexpansive. If T is asymptotically regular on K, then T has an iterative fixed point in C.

As a direct consequence of Theorem 5.2.4 and Corollary 5.2.6, we have

Corollary 5.2.7 [41]. Let C be a compact convex subset of a metric space (M, d) of pre-hyperbolic type. If $T : C \to C$ is a nonexpansive mapping, then T has an iterative fixed point in C.

Proof. Apply for S_α in Theorem 5.2.5 and $F(S_\alpha) = F(T)$. □

Theorem 5.2.8. Let C be a compact convex subset of a metric space (M, d) of pre-hyperbolic type. Let $T : C \to C$ be a mapping of strongly ANT. Suppose $x_0 \in K$ satisfies

$$\limsup_{k\to\infty} d(x_0, T^k x_0) = \inf\{\limsup_{k\to\infty} d(x, T^k x) : x \in K\}. \tag{5.12}$$

Then if $x_0 \neq Tx_0$ and if $\alpha \in (0,1)$, the sequence $\{x_{n,k}\}$ defined by $M(x_0, \alpha)$ is unbounded.

Proof. Since $x_0 \neq Tx_0$, note

$$r := \limsup_{k\to\infty} d(x_0, T^k x_0) > 0.$$

Let $y_{n,k} = T^k x_{n,k}$. As noted at (5.2), we immediately have

$$\limsup_{k\to\infty} d(x_{n,k}, y_{n,k}) \le \limsup_{k\to\infty} d(x_0, T^k x_0)$$

for $n = 0, 1, \cdots$. In view of (5.12), this in turn implies

$$\begin{aligned} \limsup_{k\to\infty} d(x_0, T^k x_0) &\le \limsup_{k\to\infty} d(x_{n,k}, T^k x_{n,k}) \\ &= \limsup_{k\to\infty} d(x_{n,k}, y_{n,k}) \\ &\le \limsup_{k\to\infty} d(x_0, T^k x_0) \end{aligned}$$

for $n = 0, 1, \cdots$. This yields (5.4) of Proposition 5.1.1. The conclusion now follows from (5.5). □

Definition 5.2.9 [62]. Let (X, d) be a metric space and $I = [0,1]$. A mapping $W : X \times X \times I \to X$ is said to be a *convex structure* on X if for each

$(x, y, \lambda) \in X \times X \times I$ and $u \in X$,

$$d(W(x, y, \lambda), u) \leq \lambda d(u, x) + (1 - \lambda) d(u, y).$$

A metric space (X, d) together with a convex structure W is called a *convex metric space*. A nonempty subset C of X is said to be *convex* if $W(x, y, \lambda) \in C$ for all $(x, y, \lambda) \in C \times C \times I$.

Note that metric spaces which are convex in the sense of Takahashi is of pre-hyperbolic type. As a direct consequence of Theorem 5.2.5, we obtain the following.

Corollary 5.2.10. *Let C be a compact convex subset of a convex metric space (M, d). Let $T : C \to C$ be a continuous mapping of strongly ANT. If T is asymptotically regular on C, then T has an iterative fixed point in C.*

Penot [54] observed that if (M, d) is a bounded metric space which possesses a convexity structure which is compact and normal, then every nonexpansive mapping $T : M \to M$ has a fixed point. In extending Penot's result to commutative families of nonexpansive mappings, Khamsi [33] introduced the following concept of one-local retracts.

Definition 5.2.11 [33]. A subset $A \subseteq M$ is said to be a 1-*local retract* of M if every family $\{B_i; i \in I\}$ of closed balls centered at points of A has the property

$$\bigcap_{i \in I} B_i \neq \emptyset \;\Rightarrow\; (\bigcap_{i \in I} B_i) \cap A \neq \emptyset.$$

Khamsi proved in [33] that under Penot's assumptions the common fixed point set of any commutative family of nonexpansive mappings of $M \to M$ is not only nonempty, but is in fact a 1-local retract of M.

Theorem 5.2.12. *Suppose (M,d) be a complete metric space of prehyperbolic type, and suppose each closed convex subset H of M has the fixed point property for mapping of ANT. Then the fixed point set of any mapping of ANT $T : M \to M$ is a (nonempty) 1-local retract of M.*

Proof. By assumption $F(T) \neq \emptyset$. Suppose $x_i \in F(T)$ and $r_i \geq 0$ for $i \in I$, and suppose $S_0 := (\cap_{i\in I} B(x_i; r_i)) \neq \emptyset$. For each $x \in M$ and $i \in I$, let

$$r(\{T^n(x)\}; x_i) = \limsup_{n\to\infty} d(T^n(x), x_i),$$

and let

$$S_1 = \{x \in M : r(\{T^n(x)\}, x_i) \leq r_i\}.$$

It is easy to see that $S_0 \subseteq S_1$: indeed, if $x \in S_0$ then

$$\begin{aligned} r(\{T^n(x)\}; x_i) &= \limsup_{n\to\infty} d(T^n(x), x_i) \\ &= \limsup_{n\to\infty} d(T^n(x), T^n(x_i)) \\ &\leq \limsup_{n\to\infty} [c_n(x) + d(x, x_i)] \\ &= d(x, x_i) \leq r_i. \end{aligned}$$

Thus $S_1 \neq \emptyset$. We now show that S_1 is a *closed convex* subset of M. Indeed, suppose $\{u_m\} \subseteq S_1$ with $u_m \to x$ as $m \to \infty$. Note that for each i

$$r(\{T^n(u_m)\}; x_i) \leq r_i.$$

Thus for each m,

$$\begin{aligned} & \limsup_{n\to\infty} d(T^n(x), x_i) \\ \leq\ & \limsup_{n\to\infty} d(T^n(x), T^n(u_m)) + \limsup_{n\to\infty} d(T^n(u_m), x_i) \\ \leq\ & \limsup_{n\to\infty} [c_n(x) + d(x, u_m)] + r(\{T^n(u_m)\}; x_i) \\ =\ & d(x, u_m) + r(\{T^n(u_m)\}; x_i) \\ \leq\ & d(x, u_m) + r_i. \end{aligned}$$

Since $u_m \to x$ as $m \to \infty$, $x \in S_1$ and hence S_1 is closed. For convexity of S_1, suppose $x, y \in S_1$, let $z = (1-\alpha)x \oplus \alpha y$, where $\alpha \in [0, 1]$, and let $i \in I$. Then, as in the preceding argument combined with the fact that M is of prehyperbolic type,

$$\begin{aligned} r(\{T^n(z)\}; x_i) &= \limsup_{n\to\infty} d(T^n(x), x_i) \\ &\leq \limsup_{n\to\infty} [c_n(z) + d(z, x_i)] = d(z, x_i) \\ &\leq (1-\alpha) d(x, x_i) + \alpha d(y, x_i) \leq r_i. \end{aligned}$$

Therefore $z \in S_1$ and thus S_1 is convex.

Finally, since $T : S_1 \to S_1$, it must be the case that $S_1 \cap F(T) \neq \emptyset$. But $S_1 \cap F(T) = S_0 \cap F(T)$. To see this, note that $S_0 \cap F(T) \subseteq S_1 \cap F(T)$ since $S_0 \subseteq S_1$. Conversely, suppose $x \in S_1 \cap F(T)$. Then for each $i \in I$,

$$d(x, x_i) = r(\{x\} : x_i) = r(\{T^n(x)\}; x_i) \leq r_i.$$

□

References

[1] A. G. Aksoy and M. A. Khamsi, Nonstandard Methods in Fixed Point Theory, Springer-Verlag, 1990.

[2] D. E. Alspach, *A fixed point free nonexpansive map*, Proc. Amer. Math. Soc. **82** (1981), 423–424.

[3] H. Amann, Order Structure and Fixed Points, Ruhr Universität Lecture Notes, Bochum, 1977.

[4] D. Amir, *On Jung's constant and related constants in normed linear spaces*, Pacific J. Math. **118** (1985), 1–15.

[5] E. Z. Andalafte and L. M. Blumenthal, *Metric characterizations of Banach and Euclidean spaces*, Fund. Math. **55** (1964), 23–55.

[6] S. Banach, *Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales*, Fund. Math. **3** (1922), 133-181.

[7] C. Bardaro and R. Ceppitelli. *A general best approximation theorem with applications in H-metrizable spaces*, Atti Sem. Fix. Univ. Modena. **XLIII** (1995), 33–40.

[8] P. Bohl, *Uber die Bewegung eines mechanischen Systems in der Nähe einer Gleichgewichtslage*, J. für Reine und Angewandte Mathematik **127** (1904), 179–276.

[9] N. Bourbaki, Topologie Générale, Hermann, Paris, 1940.

[10] Brodskii and D. P. Milman, *On the center of a a convex set*, Dokl. Akad. Nauk SSSR. **59** (1948), 837–840.

[11] L. E. J. Brouwer, *Über Abbildungen von Mannigfaltigkeiten*, Math. Ann. **71** (1912), 97–115.

[12] F. E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci. USA. **54** (1965), 1041–1044.

[13] F. Browder and W. Petryshyn, *The solution by iteration of nonlinear functional equations in Banach spaces*, Bull. Amer. Math. Soc. **72** (1966), 571–575.

[14] R. E. Bruck, *A common fixed point theorem for a commuting family of nonexpansive mappings*, Pacific J. Math. **53** (1974), 59–71.

[15] R. Bruck, T. Kuczumov and S. Reich, *Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property*, Coll. Math. **65** (1993), 169–179.

[16] H. Busemann, *Spaces with nonpositive curvature*, Acta Math. **80** (1948), 259–310.

[17] W. L. Bynum, *Normal structure coefficients for Banach spaces*, Pacific J. Math. **86** (1980), 427–436.

[18] J. Caristi, *Fixed point theorems for mappings satisfying inwardness conditions*, Trans. Amer. Math. Soc. **215** (1976), 241–251.

[19] E. Casini and E. Maluta, *Fixed points of uniformly Lipschitzian mappings in space with uniform normal structure*, Nonlinear Analysis, TMA. **9** (1985), 103–108.

[20] M. M. Day, Normed Linear Spaces, 2nd ed., Springer, 1973.

[21] N. Dunford and J. Schwartz, Linear Operators, Part I, Pure and Applied Mathematics VII, Interscience Publ. Inc., New York, 1957.

[22] M. A. Edelstein, *A remark on a theorem of M. A. Krasnoselskii*, Amer. Math. Monthly **13** (1966), 509–510.

[23] M. A. Edelstein and R. C. ÓBrien, *Nonexpansive mappings, asymptotic regularity, and successive approximations*, J. London Math. Soc. **17** (1978), 547–554.

[24] J. Gao and K. S. Lau, *On two classes of Banach spaces with uniform normal structure*, Studia Math. **99** (1991), 41–56.

[25] J. Garcia-Falset, B. Sim and M.A. Smyth, *The demiclosedness principle for mappings of asymptotically nonexpansive type*, Houston J. Math. **22** (1996), 101–108.

[26] K. Goebel and W. A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. **35** (1972), 171–174.

[27] K. Goebel and W. A. Kirk, Topics in Metric Fixed Point Theory, Cambridge Univ. Press, Cambridge, 1990.

[28] K. Goebel and S. Reich, Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings, Marcel Dekker, New York and Basel, 1984.

[29] D. Göhde. *Zum Prinzip der Kontrativen Abbildung*, Math. Nach. **30** (1965), 251–258.

[30] C. D. Horvath, *Contractibility and generalized convexity*, J. Math. Anal. Appl. **156** (1991), 341–357.

[31] S. Ishikawa, *Fixed points and iteration of a nonexpansive mapping in a Banach space*, Proc. Amer. Math. Soc. **59** (1976), 65–71.

[32] J. L. Kelly, General Topology, Van Nostrand, 1955.

[33] M. A. Khamsi, *One-local retract and common fixed point for commuting mappings in metric spaces*, Nonlinear Analysis, TMA. **27** (1996), 1307-1313.

[34] T. H. Kim, *Fixed point theorems for non-Lipschitzian self-mappings and geometric properties of Banach spaces*, Fixed Point Theory and Applications(Y. J. Cho, eds.), Vol. 3 (2002), Nova Sci. Publ. Inc., 125–135.

[35] T. H. Kim and J. W. Choi, *Asymptotic behavior of almost-orbits of non-Lipschitzian mappings in Banach spaces*, Math. Japonica **38** (1993), 191–197.

[36] T. H. Kim and E. S. Kim, *Iterate fixed points of non-Lipschitzian self-mappings*, Kodai Math. J. **18** (1995), 275–283.

[37] T. H. Kim and E. S. Kim, *Fixed point theorems for non-Lipschitzian mappings in Banach spaces*, Math. Japonica **45** (1997), 61–67.

[38] T. H. Kim and H. K. Xu, *Remarks on asymptotically nonexpansive mappings*, Nonlinear Analysis, TMA. **22** (1994), 1345–1355.

[39] W. A. Kirk, *A fixed point theorem for mappings which do not increase distance*, Amer. Math. Monthly **72** (1965), 1004–1006.

[40] W. A. Kirk, *Fixed point theorems for non-Lipschitzian mappings of asymptotically nonexpansive type*, Israel J. Math. **17** (1974), 339–346.

[41] W. A. Kirk, *Krasnoselskii's iteration process in hyperbolic spaces* Numer. Func. Optimiz. **4** (1981/82), 371–381.

[42] W. A. Kirk. *Fixed points of asymptotic contractions.* J. Math. Anal. Appl. **277** (2003), 645 650.

[43] W. A. Kirk and R. Torrejón, *Asymptotically nonexpansive semigroups in Banach spaces*, Nonlinear Analysis, TMA. **3** (1979), 111–121.

[44] H. Kneser, *Eine direkte Ableitung des Zornschen Lemmas aus dem Auswahlaxiom*, Math. Z. **53** (1950), 110–113.

[45] M. A. Krasnoselskii, *Two observations about the method of successive approximations.* Uspehi Math. Nauk. **10** (1955), 123–127.

[46] K. S. Lau, *Best approximation by closed sets in Banach spaces*, J. Approx. Theory **23** (1978), 29–36.

[47] J. Leray and J. Schauder, *Topologie et équations fonctionelles*, Ann. SCi. Ecole Norm. Sup. **51** (1934), 45–78.

[48] G. Li and B. Sims, *Fixed point theorems for mappings of asymptotically nonexpansive type*. Nonlinear Analysis, TMA. **50** (2002), 1085–1095.

[49] T. C. Lim, *On the normal structure coefficient and the bounded sequence coefficient*, Proc. Amer. Math. Soc. **88** (1983), 262–264.

[50] T. C. Lim and H. K. Xu, *Fixed point theorems for asymptotically nonexpansive mappings*, Nonlinear Analysis, TMA. **22** (1994), 1345–1355.

[51] P. K. Lin, K. K. Tan and H. K. Xu, *Demiclosedness principle and asymptotic behavior for asymptotically nonexpansive mappings*, Nonlinear Analysis, TMA. **24** (1995), 929–946.

[52] E. Maluta, *Uniformly normal structure and related coefficients*, Pacific J.Math. **111** (1984), 357–369.

[53] I. Miyadera, *Nonlinear ergodic theorems for semigroups of non-Lipschitzian mappings in Hilbert spaces*, Taiwanese J. Math. **4** (2000), 261–274.

[54] J. Penot, *Fixed point theorems without convexity*, Bull. Soc. Math. France, Memoire **60** (1979), 129–152.

[55] S. Prus, *On Bynum's fixed point theorem*, Atti Sem. Mat. Fis. Univ. Modena **38** (1990), 535–545.

[56] S. Reich, *Fixed point theory in hyperbolic spaces*, Fixed Point Theory and Applications (J-B. Baillon and M. A. Théra, eds.), Longman Sceintific & Technical, Essex, 1991, pp. 351-358.

[57] S. Reich and I. Shafrir, Nonexpansive iterations in hyperbolic spaces, Nonlinear Analysis, TMA. **15** (1990), 537–558.

[58] B. N. Sadovskii, *A fixed-point principle*, Funktsional'nyi Analiz i Ego Prilozheniya, **1** (1967), 74–76.

[59] H. Schaefer, *Über die Methoden sukzessiver Approximationen*, Jahresberichte Deutsch. Math. Verein. **59** (1957), 131–140.

[60] J. Schauder, *Der Fixpunktsatz in Funktionalräumen*, Studia Math. **2** (1930), 171–180.

[61] I. Shafrir, *The approximate fixed point property in Banach and hyperbolic spaces*, Israel J. Math. **71** (1990), 211–223.

[62] W. Takahashi, *A convexity in metric spaces and nonexpansive mappings I*, Kodai Math. Sem. Rep. **2** (1970), 142–149.

[63] A. Tarski, *A lattice-theoretic fixed point theorem and its applications*, Pacific. J. Math. **5** (1955), 285–309.

[64] H. K. Xu, *Existence and convergence for fixed points of mappings of asymptotically nonexpansive type*, Nonlinear Analysis, TMA. **16** (1991), 1139–1146.

[65] H. K. Xu, *A simple proof of a theorem of Kirk*, submitted to J. Math. Anal. Appl.

[66] W. H. Young, *On the analytical basis of non-euclidean geometry*, Amer. J. Math. **33** (1911), 249–286.

[67] E. Zeidler, I: Fixed-Point Theorems, in *Nonlinear Functional Analysis and its Applications*, Springer-Verlag, New York, 1986.

[68] E. Zermelo, *Neuer Beweis für die Möglichkeit einer Wohlordnung*, Math. Ann. **65** (1908), 107–128.

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