Fixed Point Theorems of Non-Lipschitzian Self-mappings in Banach Spaces
(Banach공간 내에서 Lipschitz가 아닌 자기사상의 부동점 정리)

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A thesis submitted in partial fulfillment of the requirement
for the degree of

Master of Education

Graduate School of Education
Pukyong National University

August 2005

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A dissertation

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August 31, 2005

CONTENTS

Banach 공간 내에서 Lipschitz가 아닌 자기사상의 부동점 정리

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요 약

집합 C가 Banach 공간 X의 공집합이 아닌 부분집합이라 할 때, 사상 $T: C \to C$가 Lipschitzian이라 함은 어떤 수열 $\{k_n\}$ 가 존재하여

$$\|T^n x - T^n y\| \leq k_n \|x-y\| \quad \forall\, x, y \in C$$

이다. 특히, $\lim_{n\to\infty} k_n = 1$ 을 만족할 때 사상 T는 점근적비확대라 한다. 더욱, 모든 자연수 n에 대하여 $k_n = k$일 때, T는 균등Lipschitzian이라 하고, $k=1$일 때, T는 비확대라 한다. $K(\neq \emptyset) \subset C \subset X$ 라 하고, 각 $x \in K$에 대하여

$$c_n(x;K) := \sup\{\|T^n x - T^n y\| - \|x-y\| : y \in K\} \vee 0$$

라 하자. 유계이고 볼록인 T-불변인 C의 공집합이 아닌 부분집합 K가 존재하여 각 $x \in K$에 대하여 $c_n(x;C) \to 0$를 만족할 때, 근접한 점근적비확대형사상이라 하자. 특히, $K = C$ 일 때 이것은 잘 알려진 점근적비확대형사상이다. 방정식 $Tx = x$의 해를 부동점라 하고, 그러한 점들의 집합을 $F(T)$로 표기한다.

본 논문에서는 집합 C가 균등정규구조({*uniform normal structure*)를 갖는 Banach 공간 X의 유계이고 볼록인 닫힌 부분집합일 때, 근접한 점근적비확대형인 사상 $T: C \to C$가 부동점을 갖는다는 사실을 밝혔다. 끝으로, 위 정리의 응용으로 여러 가지 잘 알려진 결과들을 도출해 낸다.

1 Introduction

Let C be a nonempty subset of a Banach space X and let $\mathbb{N}$ be the set of natural numbers. A mapping $T : C \to C$ is said to be *Lipschitzian* if for each $n \in \mathbb{N}$, there exists a real number k_n such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \qquad \text{for all } x, y \in C.$$

In particular, T is said to be *asymptotically nonexpansive* (simply *AN*) [7] if $\lim_{n\to\infty} k_n = 1$ and it is said to be *uniformly Lipschitzian* [3] if there exists a real number k such that $k_n = k$ for all $n \in \mathbb{N}$, and T is said to be *nonexpansive* (or *contraction*) if $k_1 = 1$ (or $k_1 < 1$).

In terms of the existence of fixed points, the achievement on metric fixed point theory today largely focuses upon the study of nonexpansive mappings and related classes of mappings, such as asymptotically nonexpansive mappings and uniformly Lipschitzian mappings in Banach spaces. The first existence result for nonexpansive mappings is Kirk's celebrated theorem [13] which depends heavily upon a geometrical property, called *normal structure*, that is, if C is a weakly compact convex subset of a Banach space with normal structure, then every nonexpansive self-mapping T of C has a fixed point, where a nonempty convex subset C of a norm linear space is said to have *normal structure* if each bounded convex subset K of C consisting of more than one point contains a *nondiametral* point, that is, a point $z \in K$ such that $\sup\{\|z - x\| : x \in K\} < \operatorname{diam}(K)$. Note that if X is uniformly convex and $C \subset X$ is closed convex, then C has normal structure. Also, if X is a Banach space and if $C \subset X$ is compact convex, then C has normal structure. As a subsequent result, in 1972, Goebel-Kirk [7] proved that if the space X is assumed to be uniformly convex, then every AN self-mapping T of C has a fixed point. For more detail history and methods on metric fixed point theory, see Geobel-Kirk [8] or Zeidler [20].

In this paper, we concentrate on a class of non-Lipschitzian self-mappings, such as mappings of asymptotically nonexpansive type. We say that T is of *nearly asymptotically nonexpansive type* (simply, *nearly ANT*) if there exists a nonempty bounded convex and T-invariant subset K of C such that $c_n(x) := c_n(x; C) \to 0$ for each $x \in K$. Recall that if $c_n(x) := c_n(x; C) \to 0$ for each $x \in C$, then T is said to be of ANT (see [14]).

The above Geobel-Kirk's result for AN mappings was immediately extended to mappings of ANT in a space with its characteristic of convexity, $\epsilon_o(X) < 1$, by Kirk [14] in 1974. More recently these results have been extended to wider classes of spaces, see for example [2, 3, 6, 12, 17, 18, 19]. In particular, Lim-Xu [17] and Kim-Xu [12] have demonstrated the existence of fixed points for AN mappings in Banach spaces with uniform normal structure, i.e., $\tilde{N}(X) := N(X)^{-1} < 1$, where

$$N(X) = \inf\left\{\frac{\operatorname{diam}(A)}{r_A(A)} : A \subset X \text{ bounded closed convex with } \operatorname{diam}(A) > 0\right\},$$

and see also [3] for some related results. Very recently, the result due to Kim-Xu [12] was extended to mappings of ANT by Li-Sims [16] and Kim [9] independently.

In this paper, we shall prove some fixed point theorems for mappings of nearly ANT in Banach spaces with uniform normal structure (see Theorem 3.2.2).

2 Preliminaries and open question

First, we shall introduce an example of continuous mappings of nearly ANT which are not of ANT, inspired by Example 4.4 in [10].

Example 2.1 [11]. Let $X = \mathbb{R}$, $C = (-\infty, 1]$. First consider a continuous non-Lipschitzian mapping $f : [0, 1/2] \to [0, 1/4]$ defined by

$$f(x) = \begin{cases} \frac{n(2n+1)}{n+1}\left(x - \frac{1}{2n+1}\right), & \frac{1}{2n+1} \le x \le \frac{1}{2n}, \quad n \ge 1; \\ -\frac{(n+1)(2n+1)}{n+2}\left(x - \frac{1}{2n+1}\right), & \frac{1}{2(n+1)} \le x \le \frac{1}{2n+1}, \quad n \ge 1; \\ 0, & x = 0. \end{cases}$$

Note first that for each $n \in \mathbb{N}$, the graph of f on each subinterval $\left[\frac{1}{2(n+1)}, \frac{1}{2n}\right]$ consists of two segments connecting three points $(1/2(n+1), 1/2(n+2))$, $(1/2n+1, 0)$ and $(1/2n, 1/2(n+1))$. For each $x \in C$, we now define

$$Tx = \begin{cases} \frac{x}{1-2x}, & x \leq -\frac{1}{2}; \\ f(x), & x \in [0, 1/2]; \\ -f(-x), & x \in [-1/2, 0]; \\ x^2, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Let us begin with the following lemma due to Casini and Maluta [3] which is very crucial for our further arguments.

Lemma 2.2 [3]. *Let X be a Banach space with $\tilde{N}(X) < 1$. Then, for every bounded sequence $\{x_n\}$, there exists a point $z \in \overline{co}(\{x_n\})$ such that*

(i) $\limsup_{n\to\infty} \|x_n - z\| \leq \tilde{N}(X) \cdot A(\{x_n\})$,

(ii) *for every $y \in X$, $\|z - y\| \leq \limsup_{n\to\infty} \|x_n - y\|$,*

where $\overline{co}(A)$ denotes the closed convex hull of A.

Now let C be a bounded closed convex subset of a Banach space X and let $T : C \to C$ be a *uniformly Lipschitzian* mapping; that is, T satisfies the condition for some constant $k \geq 0$,

$$\|T^n x - T^n y\| \leq k\|x - y\| \quad \forall x, y \in C.$$

A deep result of Casini and Maluta [3] applying Lemma 2.2 is the following.

Theorem 2.3 [3]. *If $k < \sqrt{N(X)}$, then T has a fixed point.*

Let $T : C \to C$ be a mapping of ANT. Suppose T^N is continuous for some integer $N \geq 1$. In [14], Kirk proved that if C is a *compact* convex subset of X, $T : C \to C$ has a fixed point.

What happens if C is weakly compact convex?

In 1981, Alspach [1] gave a counter example which shows that the above question does not holds even if T is nonexpansive. In fact, set

$$K := \left\{ f \in L^1[0,1] : 0 \leq f \leq 1, \ \int_0^1 f = \frac{1}{2} \right\},$$

then K is weakly compact. If $T : K \to K$ is a *baker transform* such that

$$Tf(t) = \begin{cases} (2f(2t)) \wedge 1, & 0 \leq t \leq 1/2; \\ (2f(2t-1)-1) \vee 1, & 1/2 < t \leq 1, \end{cases}$$

then T is an isometry on C but fixed point free. Therefore, we realize some conditions on X are needed for the existence of fixed points. Here we present well known fixed point theorems for Lipschitzian or non-Lipschitzian self-mappings.

Theorem 2.4 [13]. *Let C be a weakly compact convex subset of a Banach space X with normal structure. If $T : C \to C$ is nonexpansive, then T has a fixed point in C.*

Theorem 2.5 [7]. *Let C be a bounded closed convex subset of a uniformly convex Banach space X. If $T : C \to C$ is AN, then T has a fixed point in C.*

Question 2.6. Does normal structure imply the existence of fixed points of AN mappings?

The above question is still open. However, the following result was recently obtained by Kim-Xu [12].

Theorem 2.7 [12]. *Let C be a bounded closed convex subset of a Banach space X with uniform normal structure. If $T : C \to C$ is AN then T has a fixed point in C.*

Theorem 2.7 was immediately extended to mappings of ANT by Li-Sims [16] and Kim [9], independently. Now the following question is naturally raised.

Question 2.8 Does uniform normal structure imply the existence of fixed points of mappings of nearly ANT?

We shall give a positive answer for this question at the following section.

3 Fixed point theorems

First, we exhibit the following easy result.

Lemma 3.1. *Let C be a nonempty closed convex subset of a reflexive Banach space X. If $T : C \to C$ is a continuous mapping of nearly ANT, then there exists a nonempty weakly compact convex and T-invariant subset K of C such that $c_n(x; C) \to 0$ for each $x \in K$.*

Proof. Since T is of partly ANT, there exists a nonempty bounded convex and T-invariant subset A of C such that $c_n(x; C) \to 0$ for each $x \in A$, where

$$c_n(x; A) = \sup_{y \in A}(\|T^n x - T^n y\| - \|x - y\|) \vee 0.$$

Take $K = \overline{A}$, where $\overline{A}$ denotes the closure of A. Clearly, K is T-invariant, and by reflexivity of X, it is weakly compact and convex. Finally we claim that $c_n(x; C) \to 0$ for each $x \in K$. To this end, let $x \in K$. Then there exists a sequence $\{x_m\}$ in A such that $x_m \to x$. Since the supremum of any collection of lower semicontinuous mappings is lower semicontinuous, $c_n(\cdot\,; K)$ is lower semicontinuous for fixed n. Let $\epsilon > 0$ be arbitrarily given. Then there exists a $m_\epsilon \in \mathbb{N}$ such that

$$0 \le c_n(x; C) \le \liminf_{m \to \infty} c_n(x_m; C) < c_n(x_{m_\epsilon}; C) + \epsilon$$

for each $n \in \mathbb{N}$. Since $c_n(x_{m_\epsilon}; C) \to 0$ as $n \to \infty$, this yields

$$0 \le \limsup_{n \to \infty} c_n(x; C) \le \epsilon$$

for arbitrarily given $\epsilon > 0$, and hence $c_n(x; C) \to 0$ for each $x \in K$. □

Now we are ready to answer our first question by applying for two Lemmas 2.2 and 3.1.

Theorem 3.2. *Let X be a Banach space with uniform normal structure (i.e., $\tilde{N}(X) < 1$), C a nonempty closed bounded convex subset of X. If $T : C \to C$ is a mapping of nearly ANT, then T has a fixed point.*

Proof. By Lemma 3.1, there exists a nonempty weakly compact convex and T-invariant subset K of C such that $c_n(x; C) \to 0$ for each $x \in K$. For

$x_0 \in K$, consider the bounded sequence $\{T^n x_0\}$ and let $x_1 \in K$ be the point satisfying Lemma 2.2 for $\{T^n x_0\}$. Repeating this process continuously, we have a sequence $\{x_m\}$ in K satisfying the following properties:

$$\begin{aligned} \limsup_{n\to\infty} \|T^n x_m - x_{m+1}\| &\le \tilde{N}(X) \cdot A(\{T^n x_m\}), \\ \|y - x_{m+1}\| &\le \limsup_{n\to\infty} \|T^n x_m - y\| \quad (y \in X). \end{aligned}$$

Note that for $i, j \in \mathbb{N}$ (we may assume $i > j$)

$$\begin{aligned} \|T^i x_m - T^j x_m\| &\le c_j(x_m; C) + \|T^{i-j} x_m - x_m\| \\ &\le c_j(x_m; C) + \limsup_{n\to\infty} \|T^n x_{m-1} - T^{i-j} x_m\| \\ &\le c_j(x_m; C) + c_{i-j}(x_m; K) + \limsup_{n\to\infty} \|T^n x_{m-1} - x_m\| \end{aligned}$$

and so taking $\sup_{i,j\ge k}$ at first and next $k \to \infty$, we obtain

$$A(\{T^n x_m\}) \le \limsup_{n\to\infty} \|T^n x_{m-1} - x_m\| := r_m.$$

Hence, we get

$$r_{m+1} \le \tilde{N}(X) r_m = [\tilde{N}(X)]^m r_1$$

and clearly $r_m \to 0$ because $\tilde{N}(X) < 1$. For any $k \in \mathbb{N}$, since

$$\begin{aligned} \|x_{m+1} - x_m\| &\le \|x_{m+1} - T^k x_m\| + \|T^k x_m - x_m\| \\ &\le \|x_{m+1} - T^k x_m\| + \limsup_{n\to\infty} \|T^n x_{m-1} - T^k x_m\| \\ &\le \|x_{m+1} - T^k x_m\| + c_k(x_m; C) + \limsup_{n\to\infty} \|T^n x_{m-1} - x_m\|, \end{aligned}$$

taking $\limsup_{k\to\infty}$ yields

$$\begin{aligned}\|x_{m+1} - x_m\| &\leq r_m + r_{m+1} \\ &\leq (1 + \tilde{N}(X))r_m = (1 + \tilde{N}(X))[\tilde{N}(X)]^{m-1}r_1.\end{aligned}$$

So, $\{x_m\}$ is a Cauchy sequence and let $x := \lim_{m\to\infty} x_m \in K$.

On the other hand, note that for any $n \in \mathbb{N}$,

$$\begin{aligned}\|T^n x - x\| &\leq \|T^n x - T^n x_m\| + \|T^n x_m - x_{m+1}\| + \|x_{m+1} - x\| \\ &\leq c_n(x; C) + \|x - x_m\| + \|T^n x_m - x_{m+1}\| + \|x_{m+1} - x\|.\end{aligned}$$

Taking $\limsup_{n\to\infty}$ on both sides, the fact that $c_{n_k}(x; C) \to 0$ yields

$$\limsup_{n\to\infty} \|T^n x - x\| \leq \|x - x_m\| + r_{m+1} + \|x_{m+1} - x\|.$$

Since all terms in the right side converge to 0 as $m \to \infty$, we immediately see

$$\lim_{n\to\infty} \|T^n x - x\| = 0.$$

The continuity of T gives $x \in F(T) \cap K$ and the proof is complete. □

Now let us consider some applications. The following notion was introduced by Lau [15] to study the Chebyshev subset of X. Recall that a Banach space X is called a *U-space* [15] if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\forall x, y \in S_X, \quad \left\|\frac{x+y}{2}\right\| > 1 - \delta \Longrightarrow \langle f, y\rangle > 1 - \epsilon, \quad f \in \nabla_x,$$

where S_X denotes the unit sphere of X and ∇_x denotes the set of norm 1 supporting functionals f of S_X at x for $x \in X$.

It was known in [15] and [5] that

(a) If X is a U-space, then X is uniformly nonsquare, in particular, X is superreflexive [4];

(b) X is a U-space if and only if X^* is a U-space;

(c) Uniformly convex spaces and uniformly smooth spaces are U-spaces;

(d) If X is a U-space, then X has UNS. Further, if X is a Banach space with $\delta(\frac{3}{2}) > \frac{1}{4}$, then X has uniform normal structure (c.f., [5]).

As a direct consequence of Theorem 3.2, we have the following.

Corollary 3.3. *Let X be a U-space and C a nonempty closed convex subset of a Banach space X. If $T : C \to C$ is a mapping of nearly ANT, then T has a fixed point.*

Since uniform smoothness implies uniform normal structure, we have the following result, which was implicitly used in [17].

Corollary 3.4. *Assume X is a uniformly smooth Banach space and C is a nonempty closed convex subset. Then every mapping $T : C \to C$ of nearly ANT has a fixed point.*

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