# Generalization of Certain Subclasses of Analytic Functions with Negative Coefficients
# (음의 계수를 갖는 해석함수들의 어떤 부분족들의 일반화에 관한 연구)


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# Generalization of Certain Subclasses of Analytic Functions with Negative Coefficients

A dissertation

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# CONTENTS

# 음의 계수를 갖는 해석함수들의 어떤 부분족들의 일반화

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## 요 약

기하함수 이론은 지금까지 많은 학자들에 의하여 다양하게 연구되어 왔다. 특히, 1976년 Silverman은 음의 계수를 갖는 어떤 해석함수들의 부분족들을 소개하고 여러 기하학적 성질을 조사하였다.

본 논문에서는 Salagean에 의하여 소개된 미분 연산자를 이용하여 정의된 해석함수들의 부분족 $T_j^*(A,B,n,m,\alpha)$들을 소개하고, $T_j^*(A,B,n,m,\alpha)$에 속하는 해석함수들에 대하여 계수부등식, 왜곡정리, closure 정리를 연구하여 기존에 알려진 여러 결과들을 확장하였다. 또한, 족 $T_j^*(A,B,n,m,\alpha)$에 대하여 성형, 볼록성 및 close-to-convexity 반경과 변형된 대합에 관한 성질들을 조사하였다.

더욱이, 족 $T_j^*(A,B,n,m,\alpha)$에 속하는 함수들에 대하여 어떤 분수 연산자들의 왜곡성질들을 조사하였으며, 적분보존성질들을 연구하였다.

## 1. Introduction

Let $S_j$ denote the class of functions f(z) of the form

$$f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k \quad (j \in N = \{1, 2, \cdots\}) \tag{1.1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. We use $\Omega$ to denote the class of analytic functions $w(z)$ in $U$ satisfies the condition $w(0) = 0$ and $|w(z)| \leq |z|$ for $z \in U$.

For a function $f(z)$ in $S_j$, we define

$$D^0 f(z) = f(z), \tag{1.2}$$

$$D^1 f(z) = Df(z) = zf'(z) \tag{1.3}$$

and

$$D^n f(z) = D(D^{n-1} f(z)) \quad (n \in N). \tag{1.4}$$

The differential operator $D^n$ was introduced by Salagean [14]. With the help of the differential operator $D^n$, we say that a function $f(z)$ belonging to $S_j$ is in the class $S_j(A, B, n, m, \alpha)$ if and only if

$$\frac{D^{n+m} f(z)}{D^n f(z)} \prec \frac{1 + [B + (A - B)(1 - \alpha)]z}{1 + Bz} \quad (n, m \in N_0 = N \cup \{0\}) \tag{1.5}$$

for some $-1 \leq A < B \leq 1, 0 < B \leq 1, 0 \leq \alpha < 1$, and for all $z \in U$.

Equivalently, a function $f(z)$ of $S_j$ belongs to the class $S_j(A, B, n, m, \alpha)$ if and only if there exists a function $w(z) \in \Omega$ such that

$$\frac{D^{n+m} f(z)}{D^n f(z)} = \frac{1 + [B + (A - B)(1 - \alpha)]w(z)}{1 + Bw(z)}, \quad z \in U. \tag{1.6}$$

It is easy to see that the condition (1.6) is equivalent to

$$\left|\frac{\frac{D^{n+m}f(z)}{D^n f(z)}-1}{B\frac{D^{n+m}f(z)}{D^n f(z)}-[B+(A-B)(1-\alpha)]}\right|<1, \quad z\in U. \tag{1.7}$$

We note that $S_1(-1,1,0,1,\alpha)=S^*(\alpha)$ and $S_1(-1,1,1,1,\alpha)=K^*(\alpha)$, are the class of starlike functions of order $\alpha$ and the class of convex functions of order $\alpha$, respectively, were introduced by Robertson [12], and $S_1(-1,1,n,1,\alpha)=S_n(\alpha)$, is the class of functions defined by Salagean [14].

Let $T_j$ denote the subclass of $S_j$ consisting of functions $f(z)$ of the form

$$f(z)=z-\sum_{k=j+1}^{\infty} a_k z^k \quad (a_k\geq 0; j\in N). \tag{1.8}$$

Further, we define the class $T_j^*(A,B,n,m,\alpha)$ by

$$T_j^*(A,B,n,m,\alpha)=S_j(A,B,n,m,\alpha)\cap T_j. \tag{1.9}$$

We note that, by specializing the parameters $j,A,B,n,m,$ and $\alpha$, we obtain the following subclasses studied by various authors:

(1) $T_j^*(-1,1,n,m,\alpha)$=$T_j^*(n,m,\alpha)$ (Sekine [17]);

(2) $T_1^*(-1,1,n,1,\alpha)$=$T(n,\alpha)$ (Hur and Oh [7]);

(3) $T_1^*(-1,1,0,1,\alpha)$=$T^*(\alpha)$ and $T_1^*(-1,1,1,1,\alpha)$=$C(\alpha)$ (Silverman [18]);

(4) $T_1^*(-\beta,\beta,0,1,\alpha)$=$S^*(\alpha,\beta)$ and $T_1^*(-\beta,\beta,1,1,\alpha)$=$C^*(\alpha,\beta)$ $(0\leq\alpha<1, 0<\beta\leq 1)$ (Gupta and Jain [6]);

(5) $T_j^*(-1,1,0,1,\alpha)$=$T_\alpha(j)$ and $T_j^*(-1,1,1,1,\alpha)$=$C_\alpha(j)$ (Chatterjea [3] and Srivastava, Owa and Chatterijea [20]);

(6) $T_1^*(A,B,0,1,0)$=$T_1^*(A,B)$ and $T_1^*(A,B,1,1,0)$=$C_1(A,B)$ (Goel and Shoi [5]);

(7) $T_1^*(A,B,0,1,\alpha)$=$T_1^*(A,B,\alpha)$ and $T_1^*(A,B,1,1,\alpha)$=$C_1(A,B,\alpha)$ (Aouf [1]);

(8) $T_1^*(-\beta,\mu\beta,0,1,\alpha)=S^*(\alpha,\beta,\mu)$ and $T_1^*(-\beta,\mu\beta,1,1,\alpha)=C^*(\alpha,\beta,\mu)$ $(0 \le \alpha < 1, 0 < \beta \le 1, 0 \le \mu \le 1)$ (Owa and Aouf [10]);

(9) $T_j^*(-\beta,\beta,0,1,\alpha)=S_j^*(\alpha,\beta)$ and $T_j^*(-\beta,\beta,1,1,\alpha)=C_j^*(\alpha,\beta)$ where $S_j^*(\alpha,\beta)$ representes the class of functions $f(z) \in T_j$ satisfying the condition

$$\left|\frac{\frac{zf'(z)}{f(z)}-1}{\frac{zf'(z)}{f(z)}+1-2\alpha}\right| < \beta \quad (z \in U; 0 \le \alpha < 1; 0 < \beta \le 1)$$

and $C_j^*(\alpha,\beta)$ represents the class of functions $f(z) \in T_j$ such that $zf'(z) \in S_j^*(\alpha,\beta)$.

(10) $T_j^*(-\beta,\beta,n,m,\alpha)=S_j^*(\alpha,\beta,n,m)$, where $S_j^*(\alpha,\beta,n,m)$ represents the class of functions $f(z) \in T_j$ satisfying the condition

$$\left|\frac{\frac{D^{n+m}f(z)}{D^nf(z)}-1}{\frac{D^{n+m}f(z)}{D^nf(z)}+1-2\alpha}\right| < \beta \quad (z \in U),$$

where $0 \le \alpha < 1$ and $0 < \beta \le 1$.

In the present paper, Coefficient estimates, distortion theorems, closure theorems and radii of close-to-convexity, starlikeness and convexity for the class $T_j^*(A,B,n,m,\alpha)$ are determined. We also prove results involving the modified Hadamard product of two functions associated with the class $T_j^*(A,B,n,m,\alpha)$. Also we obtain several interesting distortion theorems for certain fractional operators of functions in the class $T_j^*(A,B,n,m,\alpha)$. We also obtain class preserving integral operator of the form

$$F(z) = \frac{d+1}{z^d}\int_0^z t^{d-1}f(t)dt, \ (d > -1)$$

for the class $T_j^*(A,B,n,m,\alpha)$. Conversely when $F(z) \in T_j^*(A,B,n,m,\alpha)$, radius of univalence of $f(z)$ defined by the above equation is obtained.

## 2. Coefficient Estimates

**Theorem 1.** Let the function $f(z)$ be defined by (1.8) with $j = 1$. Then $f(z) \in T_1^*(A,B,n,m,\alpha)$ $(-1 \le A < B \le 1, 0 < B \le 1, n \in N_0, m \in N_0,$ and $0 \le \alpha < 1)$ if and only if

$$(2.1) \qquad \sum_{k=2}^{\infty} c_k a_k \leq (B-A)(1-\alpha),$$

where

$$(2.2) \qquad c_k = k^n[(1+B)(k^m-1)+(B-A)(1-\alpha)].$$

The result is sharp.

**Proof.** Let $|z|=1$. Then

$$\begin{aligned}
&|D^{n+m}f(z) - D^n f(z)| - |BD^{n+m}f(z) - [B+(A-B)(1-\alpha)]D^n f(z)| \\
&= |-\sum_{k=2}^{\infty} k^n(k^m-1)a_k z^k| - |(B-A)(1-\alpha)z \\
&- \sum_{k=2}^{\infty} k^n[(k^m-1)B+(B-A)(1-\alpha)]a_k z^k| \\
&\leq \sum_{k=2}^{\infty} k^n[(1+B)(k^m-1)+(B-A)(1-\alpha)]a_k - (B-A)(1-\alpha) \leq 0.
\end{aligned}$$

Hence, by the principle of maximum modulus, $f(z) \in T_1^*(A,B,n,m,\alpha)$.

Conversely, suppose that

$$\begin{aligned}
&\left| \frac{\frac{D^{n+m}f(z)}{D^n f(z)} - 1}{B\frac{D^{n+m}f(z)}{D^n f(z)} - [B+(A+B)(1-\alpha)]} \right| \\
&= \left| \frac{\sum_{k=2}^{\infty} k^n(k^m-1)a_k z^k}{(B-A)(1-\alpha)z - \sum_{k=2}^{\infty} k^n[(k^m-1)B+(B-A)(1-\alpha)]a_k z^k} \right| \\
&< 1, \quad z \in U.
\end{aligned}$$

Since $|Re(z)| \leq |z|$ for all $z$, we have

$$(2.3) \quad Re\left\{ \frac{\sum_{k=2}^{\infty} k^n(k^m-1)a_k z^k}{(B-A)(1-\alpha)z - \sum_{k=2}^{\infty} k^n[(k^m-1)B+(B-A)(1-\alpha)]a_k z^k} \right\} < 1.$$

Choose valuse of $z$ on the real axis so that $\frac{D^{n+m}f(z)}{D^n f(z)}$ is real. Upon clearing the denominator in (2.3)and letting $z \to 1^-$ through real values, we obtain

$$\sum_{k=2}^{\infty} k^n(k^m - 1)a_k \leq \{(B-A)(1-\alpha)$$
$$- \sum_{k=2}^{\infty} k^n[(k^m-1)B + (B-A)(1-\alpha)]a_k\},$$

which implies that

$$\sum_{k=2}^{\infty} k^n[(1+B)(k^m-1) + (B-A)(1-\alpha)]a_k \leq (B-A)(1-\alpha),$$

which gives (2.1). The result is sharp for the function

$$f(z) = z - \frac{(B-A)(1-\alpha)}{c_k} z^k \quad (k \geq 2). \tag{2.4}$$

**Theorem 2.** Let the function $f(z)$ be defined by (1.8). Then $f(z) \in T_j^*(A, B, n, m, \alpha)$ $(-1 \leq A < B \leq 1, 0 < B \leq 1, n \in N_0, m \in N_0,$ and $0 \leq \alpha < 1)$ if and only if

$$\sum_{k=j+1}^{\infty} c_k a_k \leq (B-A)(1-\alpha), \tag{2.5}$$

where $c_k$ is given by (2.2). The result is sharp for the function

$$f(z) = z - \frac{(B-A)(1-\alpha)}{c_k} z^k \quad (k \geq j+1). \tag{2.6}$$

**Proof.** Putting $a_k = 0$ $(k = 2, 3, \cdots, j)$ in Theorem 1, we can prove the assertion of Theorem 2.

**Corollary 1.** Let the function $f(z)$ defined by (1.8) be in the class $T_j^*(A, B, n, m, \alpha)$. Then

$$a_k \le \frac{(B-A)(1-\alpha)}{c_k} \quad (k \ge j+1). \tag{2.7}$$

The equality in (2.7) is attained by the function $f(z)$ given by (2.6).

**Corollary 2.** $T_j^*(A,B,n+1,m,\alpha) \subset T_j^*(A,B,n,m,\alpha)$ and $T_j^*(A,B,n,m+1,\alpha) \subset T_j^*(A,B,n,m,\alpha)$.

## 3. Distortion Theorems

**Theorem 3.** Let the function $f(z)$ defined by (1.8) be in the class $T_j^*(A,B,n,m,\alpha)$. Then we have

$$|D^i f(z)| \ge |z| - \frac{(B-A)(1-\alpha)(j+1)^i}{c_{j+1}}|z|^{j+1} \tag{3.1}$$

and

$$|D^i f(z)| \le |z| + \frac{(B-A)(1-\alpha)(j+1)^i}{c_{j+1}}|z|^{j+1} \tag{3.2}$$

for $z \in U$, where $0 \le i \le n$. The result is sharp.

**Proof.** Note that $f(z) \in T_j^*(A,B,n,m,\alpha)$ if and only if $D^i f(z) \in T_j^*(A,B,n-i,m,\alpha)$, and that

$$D^i f(z) = z - \sum_{k=j+1}^{\infty} k^i a_k z^k. \tag{3.3}$$

Using Theorem 2, we know that

$$\frac{c_{j+1}}{(j+1)^i} \sum_{k=j+1}^{\infty} k^i a_k \le \sum_{k=j+1}^{\infty} c_k a_k \le (B-A)(1-\alpha), \tag{3.4}$$

that is,

(3.5)
$$\sum_{k=j+1}^{\infty} k^i a_k \leq \frac{(B-A)(1-\alpha)(j+1)^i}{c_{j+1}}.$$

It follows from (3.3) and (3.5) that

(3.6)
$$\begin{aligned} |D^i f(z)| &\geq |z| - |z|^{j+1} \sum_{k=j+1}^{\infty} k^i a_k \\ &\geq |z| - \frac{(B-A)(1-\alpha)(j+1)^i}{c_{j+1}} |z|^{j+1} \end{aligned}$$

and

(3.7)
$$\begin{aligned} |D^i f(z)| &\leq |z| + |z|^{j+1} \sum_{k=j+1}^{\infty} k^i a_k \\ &\leq |z| + \frac{(B-A)(1-\alpha)(j+1)^i}{c_{j+1}} |z|^{j+1}. \end{aligned}$$

Finally, we note that the equalities in (3.1) and (3.2) are attained for the function $f(z)$ defined by

(3.8)
$$D^i f(z) = z - \frac{(B-A)(1-\alpha)(j+1)^i}{c_{j+1}} z^{j+1}.$$

This completes the proof of Theorem 3.

**Corollary 3.** Let the function $f(z)$ defined by (1.8) be in the class $T_j^*(A, B, n, m, \alpha)$. Then we have

(3.9)
$$|f(z)| \geq |z| - \frac{(B-A)(1-\alpha)}{c_{j+1}} |z|^{j+1}$$

and

$$|f(z)| \leq |z| + \frac{(B-A)(1-\alpha)}{c_{j+1}}|z|^{j+1} \tag{3.10}$$

for $z \in U$. The equalities in (3.9) and (3.10) are attained for the function

$$f(z) = z - \frac{(B-A)(1-\alpha)}{c_{j+1}} z^{j+1}. \tag{3.11}$$

**Proof.** Taking $i = 0$ in Theorem 3, we can easily show (3.9) and (3.10).

**Corollary 4.** Let the function $f(z)$ defined by (1.8) be in the class $T_j^*(A, B, n, m, \alpha)$. Then we have

$$|f'(z)| \geq 1 - \frac{(B-A)(1-\alpha)(j+1)}{c_{j+1}}|z|^j \tag{3.12}$$

and

$$|f'(z)| \leq 1 + \frac{(B-A)(1-\alpha)(j+1)}{c_{j+1}}|z|^j \tag{3.13}$$

for $z \in U$. The equalities in (3.12) and (3.13) are attained for the function $f(z)$ given by (3.11).

**Proof.** Note that $Df(z) = zf'(z)$. Hence, taking $i = 1$ in Theorem 3, we have Corollary 4.

## 4. Closure Theorems

**Theorem 4.** The class $T_j^*(A, B, n, m, \alpha)$ is closed under convex linear combinations.

**Proof.** Let the functions

$$f_\nu(z) = z - \sum_{k=j+1}^{\infty} a_{k,\nu} z^k \quad (a_{k,\nu} \geq 0; \nu = 1, 2) \tag{4.1}$$

be in the class $T_j^*(A, B, n, m, \alpha)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z) = \lambda f_1(z) + (1-\lambda) f_2(z) \quad (0 \le \lambda \le 1) \tag{4.2}$$

is in the class $T_k^*(A, B, n, m, \alpha)$. Since

$$h(z) = z - \sum_{k=j+1}^{\infty} \{\lambda a_{k,1} + (1-\lambda) a_{k,2}\} z^k, \tag{4.3}$$

with the aid of Theorem 2, we have

$$\sum_{k=j+1}^{\infty} c_k \{\lambda a_{k,1} + (1-\lambda) a_{k,2}\} \le (B-A)(1-\alpha), \tag{4.4}$$

which implies that $h(z) \in T_j^*(A, B, n, m, \alpha)$.

As a consequence of Theorem 4, there exists the extreme points of the class $T_j^*(A, B, n, m, \alpha)$.

**Theorem 5.** Let $f_j(z) = z$ and

$$f_k(z) = z - \frac{(B-A)(1-\alpha)}{c_k} z^k \quad (n, m \in N_0, k \ge j+1) \tag{4.5}$$

for $-1 \le A < B \le 1$, $0 < B \le 1$, and $0 \le \alpha < 1$. Then $f(z)$ is in the class $T_j^*(A, B, n, m, \alpha)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=j}^{\infty} \lambda_k f_k(z), \tag{4.6}$$

where $\lambda_k \ge 0$ for $k \ge j$ and $\sum_{k=j}^{\infty} \lambda_k = 1$.

**Proof.** Suppose

$$(4.7)\qquad f(z)=\sum_{k=j}^{\infty}\lambda_k f_k(z)=\lambda_j f_j(z)+\sum_{k=j+1}^{\infty}\lambda_k f_k(z)$$

$$=\left[1-\sum_{k=j+1}^{\infty}\lambda_k\right]z+\sum_{k=j+1}^{\infty}\lambda_k\left\{z-\frac{(B-A)(1-\alpha)}{c_k}z^k\right\}$$

$$=z-\sum_{k=j+1}^{\infty}\frac{(B-A)(1-\alpha)\lambda_k}{c_k}z^k.$$

Then it follows that

$$(4.8)\qquad \sum_{k=j+1}^{\infty}\frac{c_k}{(B-A)(1-\alpha)}\frac{(B-A)(1-\alpha)\lambda_k}{c_k}=\sum_{k=j+1}^{\infty}\lambda_k=1-\lambda_j\le 1.$$

So by Theorem 2, $f(z)\in T_j^*(A,B,n,m,\alpha)$.

Conversely, assume that the function $f(z)$ defined by (1.8) belongs to the class $T_j^*(A,B,n,m,\alpha)$. Then

$$(4.9)\qquad a_k\le\frac{(B-A)(1-\alpha)}{c_k}\quad(k\ge j+1).$$

Set

$$(4.10)\qquad \lambda_k=\frac{c_k}{(B-A)(1-\alpha)}a_k\quad(k\ge j+1),$$

and

$$(4.11)\qquad \lambda_j=1-\sum_{k=j+1}^{\infty}\lambda_k.$$

Hence, we can see that $f(z)$ can be expressed in the form (4.6). This completes the proof of Theorem 5.

**Corollary 5.** The extreme points of the class $T_j^*(A,B,n,m,\alpha)$ are the functions $f_k(z)$ $(k \geq j)$ given by Theorem 5.

## 5. Modified Hadamard Product

Let the functions $f_\nu(z)$ $(\nu = 1,2)$ be defined by (4.1). The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$f_1 * f_2(z) = z - \sum_{k=j+1}^{\infty} a_{k,1}a_{k,2}z^k. \tag{5.1}$$

**Theorem 6.** Let the functions $f_\nu(z)$ $(\nu = 1,2)$ defined by (4.1) be in the class $T_j^*(A,B,n,m,\alpha)$. Then we have

$$f_1 * f_2(z) \in T_j^*(A,B,n,m,\beta(A,B,j,n,m,\alpha)), \tag{5.2}$$

where

$$\beta(A,B,j,n,m,\alpha) = \frac{(j+1)^n - (B-A)[(1+B)(j+1)^m - (1+A)][\frac{(1-\alpha)(j+1)^n}{c_{j+1}}]^2}{(j+1)^n - [\frac{(B-A)(1-\alpha)(j+1)^m}{c_{j+1}}]^2}. \tag{5.3}$$

The result is sharp.

**Proof.** Employing the technique used earlier by Schild and Silverman [16], we need to find the largest $\beta = \beta(A,B,j,n,m,\alpha)$ such that

$$\sum_{k=j+1}^{\infty} \frac{k^n[(1+B)(k^m-1)+(B-A)(1-\beta)]}{(B-A)(1-\beta)} a_{k,1}a_{k,2} \leq 1. \tag{5.4}$$

By virtue of the Cauchy-Schwarz inequality, it follows from (2.5) that

$$\sum_{k=j+1}^{\infty} \frac{c_k}{(B-A)(1-\alpha)} \sqrt{a_{k,1}a_{k,2}} \leq 1. \tag{5.5}$$

Thus we need to find the largest $\beta$ such that

$$(5.6)\qquad \frac{k^n[(1+B)(k^m-1)+(B-A)(1-\beta)]}{(B-A)(1-\beta)} a_{k,1}a_{k,2} \leq \frac{c_k}{(B-A)(1-\alpha)}\sqrt{a_{k,1}a_{k,2}}$$

or, equivalently, that

$$(5.7)\qquad \sqrt{a_{k,1}a_{k,2}} \leq \frac{c_k}{(1-\alpha)} \frac{(1-\beta)}{k^n[(1+B)(k^m-1)+(B-A)(1-\beta)]} \quad (k \geq j+1).$$

In view of (5.5), it is sufficient to find the largest $\beta$ such that

$$(5.8)\qquad \frac{(B-A)(1-\alpha)}{c_k} \leq \frac{c_k}{(1-\alpha)} \frac{(1-\beta)}{k^n[(1+B)(k^m-1)+(B-A)(1-\beta)]}.$$

The ineqality (5.8) yields

$$(5.9)\qquad \beta \leq \frac{k^n - (B-A)[(1+B)k^m-(1+A)][\frac{(1-\alpha)k^n}{c_k}]^2}{k^n - [\frac{(B-A)(1-\alpha)k^n}{c_k}]^2}.$$

The right-hand side of (5.9) is an increasing function of $k$ $(k \geq j+1)$. Therefore, by setting $k=j+1$ in (5.9), we get

$$(5.10)\qquad \beta \leq \frac{(j+1)^n - (B-A)[(1+B)(j+1)^m-(1+A)][\frac{(1-\alpha)(j+1)^n}{c_{j+1}}]^2}{(j+1)^n - [\frac{(B-A)(1-\alpha)(j+1)^n}{c_{j+1}}]^2},$$

which evidently proves the assertion (5.2) under the constraint (5.3).

Finally, by taking the functions

$$(5.11)\qquad f_\nu(z) = z - \frac{(B-A)(1-\alpha)}{c_{j+1}} z^{j+1} \quad (\nu = 1,2),$$

we can see that the result in Theorem 6 is sharp.

**Theorem 7.** Let the functions $f_\nu(z)$ $(j = 1, 2)$ defined by (4.1) be in the class $T_j^*(A, B, n, m, \alpha)$. Then the function

$$h(z) = z - \sum_{k=j+1}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^k \tag{5.12}$$

belongs to the class $T_j^*(A, B, n, m, \gamma(A, B, n, m, \alpha))$, where

$$\gamma(A, B, n, m, \alpha) = \frac{(j+1)^n - 2(B-A)[(1+B)(j+1)^m - (1+A)][\frac{(1-\alpha)(j+1)^n}{c_{j+1}}]^2}{(j+1)^n - 2[\frac{(B-A)(1-\alpha)(j+1)^n}{c_{j+1}}]^2}. \tag{5.13}$$

The result is sharp for the functions $f_\nu(z)$ $(\nu = 1, 2)$ defined by (5.11).

**Proof.** By virtue of Theorem 2, we obtain

$$\sum_{k=j+1}^{\infty} \left[\frac{c_k}{(B-A)(1-\alpha)}\right]^2 a_{k,1}^2 \leq \left[\sum_{k=j+1}^{\infty} \frac{c_k}{(B-A)(1-\alpha)} a_{k,1}\right]^2 \leq 1 \tag{5.14}$$

and

$$\sum_{k=j+1}^{\infty} \left[\frac{c_k}{(B-A)(1-\alpha)}\right]^2 a_{k,2}^2 \leq \left[\sum_{k=j+1}^{\infty} \frac{c_k}{(B-A)(1-\alpha)} a_{k,2}\right]^2 \leq 1. \tag{5.15}$$

It follows from (5.14) and (5.15) that

$$\sum_{k=j+1}^{\infty} \frac{1}{2}\left[\frac{c_k}{(B-A)(1-\alpha)}\right]^2 (a_{k,1}^2 + a_{k,2}^2) \leq 1. \tag{5.16}$$

Therefore, we need to find the largest $\gamma = \gamma(A, B, j, n, m, \alpha)$ such that

(5.17)

$$\frac{k^n[(1+B)(k^m-1)+(B-A)(1-\gamma)]}{(B-A)(1-\gamma)} \leq \frac{1}{2}\left[\frac{c_k}{(B-A)(1-\alpha)}\right]^2 \quad (k \geq j+1),$$

that is,

$$\gamma \leq \frac{k^n - 2(B-A)[(1+B)k^m-(1+A)][\frac{(1-\alpha)k^n}{c_k}]^2}{k^n - 2[\frac{(B-A)(1-\alpha)k^n}{c_k}]^2}, \quad (k \leq j+1). \tag{5.18}$$

Since

$$\Phi(k) = \frac{k^n - 2(B-A)[(1+B)k^m-(1+A)][\frac{(1-\alpha)k^n}{c_k}]^2}{k^n - 2[\frac{(B-A)(1-\alpha)k^n}{c_k}]^2} \tag{5.19}$$

is an increasing function of $k (k \geq j+1)$, we readily have

(5.20)

$$\gamma \leq \Phi(j+1) = \frac{(j+1)^n - 2(B-A)[(1+B)(j+1)^m-(1+A)][\frac{(1-\alpha)(j+1)^n}{c_{j+1}}]^2}{(j+1)^n - 2[\frac{(B-A)(1-\alpha)(j+1)^n}{c_{j+1}}]^2}$$

and Theorem 7 follows at once.

## 6. Radii of Close-to-Convexity, Starlikeness and Convexity

**Theorem 8.** Let the function $f(z)$ defined by (1.8) be in the class $T_j^*(A, B, n, m, \alpha)$. Then $f(z)$ is close-to-convex of order $\rho$ $(0 \leq \rho < 1)$ in $|z| < R_1$, where

$$R_1 = \inf_k \left[\frac{(1-\rho)c_k}{k(B-A)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \geq j+1). \tag{6.1}$$

The result is sharp with the extremal function $f(z)$ given by (2.6).

**Proof.** It is sufficient to show that $|f'(z)-1| \leq 1-\rho$ $(0 \leq \rho < 1)$ for $|z| < R_1$. We have

$$|f'(z)-1| \le |-\sum_{k=j+1}^{\infty} ka_k z^{k-1}| \le \sum_{k=j+1}^{\infty} ka_k|z|^{k-1}.$$

Thus $|f'(z)-1| \le 1-\rho$ if

$$\sum_{k=j+1}^{\infty} \Big(\frac{k}{1-\rho}\Big) a_k |z|^{k-1} \le 1. \tag{6.2}$$

Hence, by Theorem 2, (6.2) will be true if

$$(\frac{k}{1-\rho})|z|^{k-1} \le \frac{c_k}{(B-A)(1-\alpha)},$$

or if

$$|z| \le \Big[\frac{(1-\rho)c_k}{k(B-A)(1-\alpha)}\Big]^{\frac{1}{k-1}} \quad (k \ge j+1). \tag{6.3}$$

The theorem follows easily from (6.3).

**Theorem 9.** Let the function $f(z)$ defined by (1.8) be in the class $T_j^*(A, B, n, m, \alpha)$. Then $f(z)$ is starlike of order $\rho$ $(0 \le \rho < 1)$ in $|z| < R_2$, where

$$R_2 = \inf_k \Big[\frac{(1-\rho)c_k}{(k-\rho)(B-A)(1-\alpha)}\Big]^{\frac{1}{k-1}} \quad (k \ge j+1). \tag{6.4}$$

The result is sharp with the extremal function $f(z)$ given by (2.6).

**Proof.** We must show that $\left|\frac{zf'(z)}{f(z)} - 1\right| \le 1-\rho$ $(0 \le \rho < 1)$ for $|z| < R_2$. We have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \frac{\sum_{k=j+1}^{\infty}(k-1)a_k|z|^{k-1}}{1-\sum_{k=j+1}^{\infty} a_k|z|^{k-1}}.$$

Thus $|\frac{zf'(z)}{f(z)} - 1| \le 1-\rho$ if

$$\sum_{k=j+1}^{\infty} \frac{(k-\rho)a_k|z|^{k-1}}{1-\rho} \le 1. \tag{6.5}$$

Hence, by Theorem 2, (6.5) will be true if

$$\frac{(k-\rho)|z|^{k-1}}{1-\rho} \le \frac{c_k}{(B-A)(1-\alpha)},$$

or if

$$|z| \le \left[\frac{(1-\rho)c_k}{(k-\rho)(B-A)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \ge j+1). \tag{6.6}$$

The theorem follows easily from (6.6).

**Corollary 6.** Let the function $f(z)$ defined by (1.8) be in the class $T_j^*(A,B,n,m,\alpha)$. Then $f(z)$ is convex of order $\rho$ $(0 \le \rho < 1)$ in $|z| < R_3$, where

$$R_3 = \inf_k \left[\frac{(1-\rho)c_k}{k(k-\rho)(B-A)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \ge j+1). \tag{6.7}$$

The result is sharp with the extremal function $f(z)$ given by (2.6).

## 7. Fractional Calculus

Many essentially equivalent definitions of fractional calculus (That is, fractional derivatives and fractional integrals) have been given in the literature (c.f., e.g., [2], [4, Chapter 13], [8], [11], [13], [15], [19, p.28 et seq.], and [22]). We find it to be convenient to restrict ourselves to the following definitions used recently by Owa [9] (and by Srivastava and Owa [21]).

**Definition 1.** The fractional integral of order $\lambda$ is defined, for a function $f(z)$, by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta \quad (\lambda > 0), \tag{7.1}$$

where $f(z)$ is an analytic function in a simply connected region of the $z$-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

**Definition 2.** The fractional derivative of order $\lambda$ is defined, for a function $f(z)$, by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta \quad (0 \leq \lambda < 1), \tag{7.2}$$

where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by, as in Definition 1.

**Definition 3.** Under the hypotheses of Definition 2, the fractional derivative of order $n+\lambda$ is defined by

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z) \quad (0 \leq \lambda < 1; n \in N_0 = N \cup \{0\}). \tag{7.3}$$

**Theorem 10.** Let the function $f(z)$ defined by (1.8) be in the class $T_j^*(A, B, n, m, \alpha)$. Then we have

$$|D_z^{-\lambda}(D^i f(z))| \geq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left\{ 1 - \frac{\Gamma(j+2)\Gamma(2+\lambda)(B-A)(1-\alpha)(j+1)^i}{\Gamma(j+2+\lambda)c_{j+1}} |z|^j \right\} \tag{7.4}$$

and

$$|D_z^{-\lambda}(D^i f(z))| \leq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left\{ 1 + \frac{\Gamma(j+2)\Gamma(2+\lambda)(B-A)(1-\alpha)(j+1)^i}{\Gamma(j+2+\lambda)c_{j+1}} |z|^j \right\} \tag{7.5}$$

for $\lambda > 0$, $0 \leq i \leq n$, and $z \in U$. The result is sharp.

**Proof.** Let

$$\begin{aligned} F(z) &= \Gamma(2+\lambda) z^{-\lambda} D_z^{-\lambda}(D^i f(z)) \\ &= z - \sum_{k=j+1}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\lambda)}{\Gamma(k+1+\lambda)} k^i a_k z^k \\ &= z - \sum_{k=j+1}^{\infty} \Psi(k) k^i a_k z^k, \end{aligned} \tag{7.6}$$

where

$$\Psi(k) = \frac{\Gamma(k+1)\Gamma(2+\lambda)}{\Gamma(k+1+\lambda)} \quad (k \geq j+1). \tag{7.7}$$

Since

$$0 < \Psi(k) \leq \Psi(j+1) = \frac{\Gamma(j+2)\Gamma(2+\lambda)}{\Gamma(j+2+\lambda)}, \tag{7.8}$$

by using (3.5) and (7.8), we can see that

$$\begin{aligned} |F(z)| &\geq |z| - \Psi(j+1)|z|^{j+1} \sum_{k=j+1}^{\infty} k^i a_k \\ &\geq |z| - \frac{\Gamma(j+2)\Gamma(2+\lambda)(B-A)(1-\alpha)(j+1)^i}{\Gamma(j+2+\lambda)c_{j+1}} |z|^{j+1} \end{aligned} \tag{7.9}$$

and

$$\begin{aligned} |F(z)| &\leq |z| + \Psi(j+1)|z|^{j+1} \sum_{k=j+1}^{\infty} k^i a_k \\ &\leq |z| + \frac{\Gamma(j+2)\Gamma(2+\lambda)(B-A)(1-\alpha)(j+1)^i}{\Gamma(j+2+\lambda)c_{j+1}} |z|^{j+1}, \end{aligned} \tag{7.10}$$

which prove the inequalities of Theorem 10. Further, equalities are attained for the function

(7.11)

$$D^{-\lambda}(D^i f(z)) = \frac{z^{1+\lambda}}{\Gamma(2+\lambda)} \left\{ 1 - \frac{\Gamma(j+2)\Gamma(2+\lambda)(B-A)(1-\alpha)(j+1)^i}{\Gamma(j+2+\lambda)c_{j+1}} z^j \right\}$$

or

$$D^i f(z) = z - \frac{(B-A)(1-\alpha)(j+1)^i}{c_{j+1}} z^{j+1}. \tag{7.12}$$

Thus we completes the proof of Theorem 10.

Taking $i = 0$ in Theorem 10, we have

**Corollary 7.** Let the function $f(z)$ defined by (1.8) be in the class $T_j^*(A, B, n, m, \alpha)$. Then we have

$$|D_z^{-\lambda} f(z)| \geq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)}\left\{1 - \frac{\Gamma(j+2)\Gamma(2+\lambda)(B-A)(1-\alpha)}{\Gamma(j+2+\lambda)c_{j+1}}|z|^j\right\} \tag{7.13}$$

and

$$|D_z^{-\lambda} f(z)| \leq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)}\left\{1 + \frac{\Gamma(j+2)\Gamma(2+\lambda)(B-A)(1-\alpha)}{\Gamma(j+2+\lambda)c_{j+1}}|z|^j\right\} \tag{7.14}$$

for $\lambda > 0$ and $z \in U$. The equalities in (7.13) and (7.14) are attained for the function $f(z)$ given by (3.11).

**Theorem 11.** Let the function $f(z)$ defined by (1.8) be in the class $T_j^*(A, B, n, m, \alpha)$. Then we have

$$|D_z^{\lambda}(D^i f(z))| \geq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)}\left\{1- \frac{\Gamma(j+2)\Gamma(2-\lambda)(B-A)(1-\alpha)(j+1)^i}{\Gamma(j+2-\lambda)c_{j+1}}|z|^j\right\} \tag{7.15}$$

and

$$|D_z^{\lambda}(D^i f(z))| \leq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)}\left\{1+ \frac{\Gamma(j+2)\Gamma(2-\lambda)(B-A)(1-\alpha)(j+1)^i}{\Gamma(j+2-\lambda)c_{j+1}}|z|^j\right\}. \tag{7.16}$$

for $0 \leq \lambda < 1$, $0 \leq j \leq n-1$, and $z \in U$. The result is sharp.

**Proof.** Let

$$G(z) = \Gamma(2-\lambda)z^{\lambda}D_z^{\lambda}(D^i f(z)) \tag{7.17}$$
$$= z - \sum_{k=j+1}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)} k^i a_k z^k$$
$$= z - \sum_{k=j+1}^{\infty} \Upsilon(k) k^{j+1} a_k z^k,$$

where

$$\Upsilon(k) = \frac{\Gamma(k)\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)} \quad (k \geq j+1). \tag{7.18}$$

Noting

$$0 < \Upsilon(k) \leq \Upsilon(j+1) = \frac{\Gamma(j+1)\Gamma(2-\lambda)}{\Gamma(j+2-\lambda)}. \tag{7.19}$$

With the aid of (3.5) and (7.19), we have

$$|G(z)| \geq |z| - \Upsilon(j+1)|z|^{j+1} \sum_{k=j+1}^{\infty} k^{i+1} a_k \tag{7.20}$$
$$\geq |z| - \frac{\Gamma(j+2)\Gamma(2-\lambda)(B-A)(1-\alpha)(j+1)^i}{\Gamma(j+2-\lambda)c_{j+1}} |z|^{j+1}$$

and

$$|G(z)| \leq |z| + \Upsilon(j+1)|z|^{j+1} \sum_{k=j+1}^{\infty} k^{i+1} a_k \tag{7.21}$$
$$\leq |z| + \frac{\Gamma(j+2)\Gamma(2-\lambda)(B-A)(1-\alpha)(j+1)^i}{\Gamma(j+2-\lambda)c_{j+1}} |z|^{j+1}.$$

Thus (7.15) and (7.16) follows from (7.20) and (7.21), respectively. Further, since the equalities in (7.15) and (7.16) are attained by the function $f(z)$ defined by

$$|D_z^{\lambda}(D^i f(z))| \leq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)}\Big\{1- \frac{\Gamma(j+2)\Gamma(2-\lambda)(B-A)(1-\alpha)(j+1)^i}{\Gamma(j+2-\lambda)c_{j+1}}|z|^j\Big\} \tag{7.22}$$

or by the function $f(z)$ defined by (7.12). Thus we complete the proof of Theorem 11.

Making $i = 0$ in Theorem 11, we have

**Corollary 8.** Let the function $f(z)$ defined by (1.8) be in the class $T_j^*(A, B, n, m, \alpha)$. Then we have

$$|D_z^{\lambda} f(z)| \geq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)}\Big\{1 - \frac{\Gamma(j+2)\Gamma(2-\lambda)(B-A)(1-\alpha)}{\Gamma(j+2-\lambda)c_{j+1}}|z|^j\Big\} \tag{7.23}$$

and

$$|D_z^{\lambda} f(z)| \leq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)}\Big\{1 + \frac{\Gamma(j+2)\Gamma(2-\lambda)(B-A)(1-\alpha)}{\Gamma(j+2-\lambda)c_{j+1}}|z|^j\Big\} \tag{7.24}$$

for $0 \leq \lambda < 1$ and $z \in U$. The equalities in (7.23) and (7.24) are attained for the function $f(z)$ given by (3.11)

## 8. Fractional Integral Operator

We need the following definition of fractional integral operator given by Srivastava, Saigo and Owa [23].

**Definition 4.** For real numbers $\beta > 0$, $\gamma$ and $\eta$, the fractional integral operator $I_{0,z}^{\beta,\gamma,\eta}$ is defined by

$$I_{0,z}^{\beta,\gamma,\eta} f(z) = \frac{z^{-\beta-\gamma}}{\Gamma(\beta)}\int_0^z (z-t)^{\beta-1} F(\beta+\gamma, -\eta; \beta; 1-\frac{t}{z}) f(t)dt \tag{8.1}$$

where $f(z)$ is an analytic function in a simply connected region of the $z$-plane containing the origin with the order

$$f(z) = O(|z|^{\epsilon}), z \to 0,$$

where $\epsilon > Max(0, \gamma - \eta) - 1$,

$$F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k (1)_k} z^k, \tag{8.2}$$

where $(\nu)_k$ is the Pochhammer symbol defined by

$$(\nu)_k = \frac{\Gamma(\nu + k)}{\Gamma(\nu)} = \begin{cases} 1 & (k = 0) \\ \nu(\nu+1)\cdots(\nu+k-1) & (k \in N) \end{cases}, \tag{8.3}$$

and the multiplicity of $(z-t)^{\beta-1}$ is removed by requiring $\log(z-t)$ to be real when $z - t > 0$.

**Remark.** For $\gamma = -\beta$, we note that

$$I_{0,z}^{\beta,-\beta,\eta} f(z) = D_z^{-\beta} f(z).$$

In order to prove our result for the fractional integral operator, we have to recall here the following lemma due to Srivastava, Saigo and Owa [23].

**Lemma 1.** If $\beta > 0$ and $k > \gamma - \eta - 1$, then

$$I_{0,z}^{\beta,\gamma,\eta} z^k = \frac{\Gamma(k+1)\Gamma(k-\gamma+\eta+1)}{\Gamma(k-\gamma+1)\Gamma(k+\beta+\eta+1)} z^{k-\gamma}. \tag{8.4}$$

With the aid of the above Lemma, we prove

**Theorem 12.** Let $\beta > 0$, $\gamma < 2$, $\beta + \eta > -2$, $\gamma - \eta < 2$ and $\gamma(\beta+\eta) \leq \beta(j+1)$, and $j \in N$. If the function $f(z)$ defined by (1.8) is in the class $T_j^*(A, B, n, m, \alpha)$, then

$$\left| I_{0,z}^{\beta,\gamma,\eta} f(z) \right| \geq \frac{\Gamma(2-\gamma+\eta)|z|^{1-\gamma}}{\Gamma(2-\gamma)\Gamma(2+\beta+\eta)} \left\{ 1 - \frac{(B-A)(1-\alpha)(2-\gamma+\eta)_j (2)_j}{c_{j+1}(2-\gamma)_j (2+\beta+\eta)_j} |z|^j \right\} \tag{8.5}$$

and

$$(8.6)\qquad \left|I_{0,z}^{\beta,\gamma,\eta}f(z)\right| \leq \frac{\Gamma(2-\gamma+\eta)|z|^{1-\gamma}}{\Gamma(2-\gamma)\Gamma(2+\beta+\eta)}\Big\{1+ \frac{(B-A)(1-\alpha)(2-\gamma+\eta)_j(2)_j}{c_{j+1}(2-\gamma)_j(2+\beta+\eta)_j}|z|^j\Big\}$$

for $z \in U_0$, where

$$U_0 = \begin{cases} U & (\gamma \leq 1) \\ U-\{0\} & (\gamma > 1) \end{cases}.$$

The equalities in (8.5) and (8.6) are attained by the function $f(z)$ given by (3.11).

**Proof.** By using Lemma 1, we have

$$(8.7)\qquad I_{0,z}^{\beta,\gamma,\eta}f(z) = \frac{\Gamma(2-\gamma+\eta)}{\Gamma(2-\gamma)\Gamma(2+\beta+\eta)}z^{1-\gamma} - \sum_{k=j+1}^{\infty}\frac{\Gamma(k+1)\Gamma(k-\gamma+\eta+1)}{\Gamma(k-\gamma+1)\Gamma(k+\beta+\eta+1)}a_k z^{k-\gamma}.$$

Letting

$$(8.8)\qquad \begin{aligned} H(z) &= \frac{\Gamma(2-\gamma)\Gamma(2+\beta+\eta)}{\Gamma(2-\gamma+\eta)}z^{\gamma}I_{0,z}^{\beta,\gamma,\eta}f(z) \\ &= z - \sum_{k=j+1}^{\infty} h(k)a_k z^k, \end{aligned}$$

where

$$(8.9)\qquad h(k) = \frac{(2-\gamma+\eta)_{k-1}(1)_k}{(2-\gamma)_{k-1}(2+\beta+\eta)_{k-1}} \quad (k \geq j+1),$$

we can see that $h(k)$ is non-increasing for integers $k \geq j+1$, and we have

$$(8.10)\qquad 0 < h(k) \leq h(j+1) = \frac{(2-\gamma+\eta)_j(2)_j}{(2-\gamma)_j(2+\beta+\eta)_j}.$$

Therefore, by using Theorem 2 and (8.10), we have

$$(8.11) \qquad \begin{aligned} |H(z)| &\geq |z| - h(j+1)|z|^{j+1} \sum_{k=j+1}^{\infty} a_k \\ &\geq |z| - \frac{(B-A)(1-\alpha)(2-\gamma+\eta)_j(2)_j}{c_{j+1}(2-\gamma)_j(2+\beta+\eta)_j}|z|^{j+1} \end{aligned}$$

and

$$(8.12) \qquad \begin{aligned} |H(z)| &\leq |z| + h(j+1)|z|^{j+1} \sum_{k=j+1}^{\infty} a_k \\ &\leq |z| + \frac{(B-A)(1-\alpha)(2-\gamma+\eta)_j(2)_j}{c_{j+1}(2-\gamma)_j(2+\beta+\eta)_j}|z|^{j+1}. \end{aligned}$$

This completes the proof of Theorem 12.

**Remark.** Taking $\gamma = -\beta = -\lambda$ in Theorem 12, we get Corollary 7.

## 9. Integral Operators

**Theorem 13.** Let the function $f(z)$ defined by (1.8) be in the class $T_j^*(A, B, n, m, \alpha)$ and let $d$ be a real number such that $d > -1$. Then the function $F(z)$ defined by

$$(9.1) \qquad F(z) = \frac{d+1}{z^d} \int_0^z t^{d-1} f(t) dt$$

also belongs to the class $T_j^*(A, B, n, m, \alpha)$.

**Proof.** From the representation of $F(z)$, it follows that

$$F(z) = z - \sum_{k=j+1}^{\infty} b_k z^k,$$

where

$$b_k = \left(\frac{d+1}{d+k}\right) a_k.$$

Therefore

$$\sum_{k=j+1}^{\infty} c_k b_k = \sum_{k=j+1}^{\infty} c_k \Big(\frac{d+1}{d+k}\Big) a_k$$
$$\leq \sum_{k=j+1}^{\infty} c_k a_k \leq (B-A)(1-\alpha),$$

since $f(z) \in T_j^*(A,B,n,m,\alpha)$. Hence, by Theorem 1, $F(z) \in T_j^*(A,B,n,m,\alpha)$.

**Theorem 14.** Let $d$ be a real number such that $d > -1$. If $F(z) \in T_j^*(A,B,n,m,\alpha)$, then the function $f(z)$ defined by (1.8) is univalent in $|z| < R^*$, where

$$R^* = \inf_k \Big[\frac{(d+1)c_k}{(d+k)(B-A)(1-\alpha)}\Big]^{\frac{1}{k-1}} \quad (k \geq j+1). \tag{9.2}$$

The result is sharp.

**Proof.** Let $F(z) = z - \sum_{k=j+1}^{\infty} a_k z^k (a_k \geq 0)$. It follows from (9.1) that

$$f(z) = \frac{z^{1-d}[z^d F(z)]'}{(d+1)} \quad (d > -1)$$
$$= z - \sum_{k=j+1}^{\infty} \Big(\frac{d+k}{d+1}\Big) a_k z^k.$$

To prove the result it suffices to show that $|f'(z)-1| \leq 1$ for $|z| < R^*$.
Now

$$|f'(z)-1| = |-\sum_{k=j+1}^{\infty} \frac{k(d+k)}{d+1} a_k |z|^{k-1}|$$
$$\leq \sum_{k=j+1}^{\infty} \frac{k(d+k)}{d+1} a_k |z|^{k-1}.$$

Thus $|f'(z)-1| < 1$ if

$$\sum_{k=j+1}^{\infty} \frac{k(d+k)}{d+1} a_k |z|^{k-1} < 1. \tag{9.3}$$

But Theorem 1 confirms that

$$\sum_{k=j+1}^{\infty} \frac{c_k}{(B-A)(1-\alpha)} a_k \leq 1.$$

Hence (9.3) will be satisfied if

$$\frac{k(d+k)|z|^{k-1}}{d+1} \leq \frac{c_k}{(B-A)(1-\alpha)} \quad (k \geq j+1),$$

or if

$$(9.5) \qquad |z| \leq \left[\frac{(d+1)c_k}{(d+k)(B-A)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \geq j+1).$$

The required result follows now from (9.5). The result is sharp for the function

$$(9.6) \qquad f(z) = z - \frac{(B-A)(1-\alpha)(d+k)}{c_k(d+1)} z^k \quad (k \geq j+1).$$

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