GENERALIZED *b*-CLOSED SETS IN TOPOLOGICAL SPACES

위상공간상의 일반화된 *b*-폐집합

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A Dissertation

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요 약

본 논문에서는 g-폐집합을 기초로 gs-폐집합과 gp-폐집합보다 약한 gb-폐집합을 정의하여, 그 기본적인 성질을 조사하고, semi-pre- $T_{1/2}$공간상에서 모든 gb-폐집합이 b-폐집합이 됨을 보였다. 또한, gb-폐집합을 이용하여 b-연속함수보다 약한 gb-연속함수를 정의하여 gb-폐포를 이용한 이 함수의 특성을 찾고, gb-irresolute함수와의 관계를 조사하였다.

그리고 gb-개집합을 이용하여 위상공간상에서의 GBO-연결성을 정의하고 이들과의 동치조건을 조사하였으며, 전사인 gb-연속함수에 의한 GBO-연결공간의 상이 연결공간임을 보였다. 또한, 두 개의 gb-개집합들의 적이 gb-개집합이 된다는 결과를 이용하여 두 개의 적공간이 GBO-연결이면, 각각의 상공간 역시 GBO-연결이 됨을 보였다.

1. Introduction

The initiation of the study of generalized closed sets was done by Levine [25] in 1970 as he considered sets whose closure belongs to every open supersets. He called them generalized closed (briefly g-closed) and studied their most fundamental properties. The spaces in which the concept of g-closed sets and closed sets coincide are called $T_{1/2}$-spaces. In 1977, Dunham [20] showed that $T_{1/2}$-spaces are precisely the spaces in which singletons are open or closed. In 1990, Balachandran et al. [10] introduced the concept of a new class of maps, namely g-continuous maps, which includes the class of continuous maps, and a class of gc-irresolute maps defined as an analogy of irresolute maps. Moreover they introduce the concept of GO-connectedness of topological spaces and prove product theorem for GO-connected spaces, i.e. if the product space of two non-empty spaces is GO-connected, then each factor space is GO-connected.

The generalization of generalized closed sets and generalized continuity was intensively studied in recent years by Balachandran, Devi, Maki, Arya, Nour, Arokiarani and Sundaram, et al.

Bhattacharya and Lahiri [14] introduced the notion of semi-generalized closed sets by replacing the closure operator in the original Levin's definition with semi-closure operator and by replacing openness of the superset with semi-openness. Arya and Nour [9] defined the notion of generalized semi-closed sets (briefly gs-closed sets). Although g-closed and sg-closed sets are independent notions, they both imply gs-closedness and the reverse implications fails to be always true. Maki et al. [27, 31] defined and investigated the concept of gp-closed sets and used this notion to obtain a characterization of p-normal spaces. This notion is generalization of preclosed sets which were further studied by Dontchev and Maki [17], Arokiarani et al. [8] Noiri et al. [31] and Park et al. [33]. In 1995, Dontchev [18] defined the concepts of generalized semi-preclosed sets and semi-pre-$T_{1/2}$-spaces. He showed that the notions of sp-closed sets and gs-

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closed sets are independent from each other. Moreover, he investigated the characterizations of semi-pre-$T_{1/2}$, semi-$T_{1/2}$ and $T_{1/2}$-spaces.

The aim of this paper is to continue the study of the above mentioned classes of sets by introducing the notion of generalized b-closed sets (briefly, gb-closed sets) via the concept of b-open sets due to Andrijević [6]. The class of gb-closed sets contains properly the classes of g-closed, gs-closed, sg-closed and gp-closed sets and is contained in the class of gsp-closed sets. And generalized b-continuous functions are defined and investigated. Moreover, we introduce the concept of GBO-connectedness of topological spaces and prove product theorem for GBO-connected spaces as follows:

Theorem 5.8. *If the product space of two non-empty spaces is GBO-connected, then each factor space is GBO-connected.*

2. Preliminaries

In recent years a number of generalization of open sets have been considered.

Definition 2.1. A subset A of a space (X, τ) is called:

(1) *α-set* [30] if $A \subset \text{int}(\text{cl}(\text{int}(A))$,

(2) *semi-open* [24] if $A \subset \text{cl}(\text{int}(A))$,

(3) *preopen* [28] if $A \subset \text{int}(\text{cl}(A))$,

(4) *semi-preopen* [3] if $A \subset \text{cl}(\text{int}(\text{cl}(A)))$,

(5) *b-open* [3] if $A \subset \text{cl}(\text{int}(A)) \cup \text{int}(\text{cl}(A))$.

We denote the classes of these sets in a space (X, τ) by τ_α, $SO(X)$, $PO(X)$, $SPO(X)$ and $BO(X)$, respectively. All of them are larger then τ and closed under forming arbitrary unions. Njaståd [28] showed that τ_α is a topology on X. In general, $SO(X)$ need not be a topology on X, but the intersection of a semi-open set and an open set is semi-open. The same result holds for $PO(X)$, $SPO(X)$ and $BO(X)$. The complement of a semi-open set is called semi-closed. Thus A is semi-closed if and only if $\text{int}(\text{cl}(A)) \subset A$. The notions of preclosed, semi-preclosed and b-open sets are similarly defined. For a subset A of a space X the semi-closure (resp. preclosure, semi-preclosure, b-closure) of A, denoted by scl(A) (resp. pcl(A), spcl(A), bcl(A)) is the intersection of all semi-closed (resp. preclosed, semi-preclosed, b-closed) subsets of X containing A. Dually, the semi-interior (resp. preinterior, semi-preinterior, b-interior) of A, (resp. pint(A), spint(A), bint(A)) is the union of all semi-open(resp. preopen, semi-preopen, b-open) subsets of X contained in A. It is obvious that $PO(X) \cup SO(X) \subset BO(X) \subset SPO(X)$ and we shall show that the inclusions cannot be replaced with equalities.

Example 2.2 [6]. Consider the set R of real numbers with the usual topology, and let A =[0,1]∪((1,2)∩Q) where Q stands for the set of rational numbers. Then A is b-open but neither semi-open nor preopen. On the other hand, let T= [0,1)∩Q. Then T is semi-preopen but not b-open.

Theorem 2.3 [6]. *For a subsets A of a space (X, τ), the following are equivalent:*

(a) *A is b-open.*

(b) *$A = pint(A) \cup sint(A)$.*

(c) *$A \subset pcl(pint(A))$.*

Theorem 2.4 [6]. *Let A be a subset of a space (X, τ). Then:*

(a) *$bcl(A) = scl(A) \cap pcl(A)$.*

(b) *$bint(A) = sint(A) \cup pint(A)$.*

Definition 2.5. A subset A of a space (X, τ) is called:

(a) *generalized closed* (briefly, *g-closed*) [25] if $\mathrm{cl}(A) \subset U$ whenever $A \subset U$ and U is open in X,

(b) *semi-generalized closed* (briefly, *sg-closed*) [14] if $\mathrm{scl}(A) \subset U$ whenever $A \subset U$ and U is semi-open in X,

(c) *generalized semi-closed* (briefly, *gs-closed*) [9] if $\mathrm{scl}(A) \subset U$ whenever $A \subset U$ and U is open in X,

(d) *generalized preclosed* (briefly, *gp-closed*) [13] if $\mathrm{pcl}(A) \subset U$ whenever $A \subset U$ and U is open in X,

(e) *generalized semi-preclosed* (briefly, *gsp-closed*) [16] if $\mathrm{spcl}(A) \subset U$ whenever $A \subset U$ and U is open in X.

Definition 2.6 [16]. A space (X, τ) is called:

(a) $T_{1/2}$ if every generalized closed is closed,

(b) semi-$T_{1/2}$ if every *sg*-closed is semi-closed,

(c) semi-pre-$T_{1/2}$ if every *gsp*-closed is semi-pre-closed.

Theorem 2.7 [16]. *For a space (X, τ), the following implications hold:*

$$T_1 \Rightarrow T_{1/2} \Rightarrow T_0$$

Theorem 2.8 [16]. *A space (X, τ) is semi-$T_{1/2}$ if and only if every singleton is (semi)-open or semi-closed.*

Remark 2.9 [16]. Every $T_{1/2}$ space is semi-$T_{1/2}$. Then the reverse is not usually true. The following example shows this: let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, X\}$, then X is semi-$T_{1/2}$ but not $T_{1/2}$.

Theorem 2.10 [16]. *For a space (X, τ), the following conditions are equivalent:*

(a) *X is semi-pre-$T_{1/2}$.*

(b) *Every singleton of X is closed or semi-preopen.*

(c) *Every singleton of X is closed or preopen.*

(b) *Every nowhere dense singleton of X is closed.*

(c) *Every non-preopen singleton is closed.*

Remark 2.11 [16]. Every $T_{1/2}$ space is semi-pre-$T_{1/2}$. But a semi-pre-$T_{1/2}$ space need not be $T_{1/2}$. The following example shows this: For the real line with the indiscrete topology, none of the singletons in this space is either semi-open or semi-closed. Thus it is not even semi-$T_{1/2}$.

In remark 2.9, X need not semi-pre-$T_{1/2}$. Hence the concepts of semi-pre-$T_{1/2}$ and semi-$T_{1/2}$ spaces are independent from each other.

Theorem 2.12 [16]. *For a space (X, τ), the following conditions are eqivalent:*

(a) *X is $T_{1/2}$.*

(b) *X is semi-$T_{1/2}$ and semi-pre-$T_{1/2}$.*

Definition 2.13 [16]. A function $f : (X, \tau) \to (Y, \sigma)$ is called:

(a) precontinuous [28] (resp. β-continuous [1], *gp*-continuous [8], *gsp*-continu ous [16]) if $f^{-1}(V)$ is preclosed (resp. β-closed, *gp*-closed, *gsp*-closed) in (X, τ) for every closed set V of (Y, σ),

(b) β-irresolute [26](resp. *gp*-irresolute [8], *gsp*-irresolute [16]) if $f^{-1}(V)$ is β-closed, (resp. *gp*-closed, *gsp*-closed) in (X, τ) for every β-closed (resp. *gp*-closed, *gsp*-closed) set V of (Y, σ),

(c) pre-β-closed [26] if $f(V)$ is semi-preclosed in (Y, σ) for every semi-preclosed set V of (X, τ).

Remark 2.14. From above definition, for a function $(X,\tau) \to (Y,\sigma)$, we have the following diagram of implications:

$$\begin{array}{ccccc} & & \beta\text{-irresoluteness} & & gsp\text{-irresoluteness} \\ & & \downarrow & & \downarrow \\ \text{precontinuity} & \to & \beta\text{-continuity} & \to & gsp\text{-continuity} \end{array}$$

Theorem 2.15 [16]. *Let $f : (X,\tau) \to (Y,\sigma)$ be a gsp-irresolute function. If (X,τ) is a semi-pre-$T_{1/2}$-space, then f is β-irresolute.*

Proof. Let V be a semi-preclosed subset of (Y,σ). Then V is gsp-closed. Since f is gsp-irresolute, then $f^{-1}(V)$ is gsp-closed in (X,τ). Since X is semi-pre-$T_{1/2}$, then $f^{-1}(V)$ is semi-preclosed in (X,τ). Hence f is β-irresolute. □

Theorem 2.16 [16]. *Let $f : (X,\tau) \to (Y,\sigma)$ be a continuous and pre-β-closed function. Then for every gsp-closed set A of (X,τ), $f(A)$ is gsp-closed in (Y,σ).*

The composition of two gsp-continuous funtions need not be gsp-continuous. For, consider the following example:

Example 2.17 [19]. Let $X = \{a,b,c,d,e\}$, $\tau = \{\phi,\{a,b\},\{c,d\},\{a,b,c,d\},X\}$ and let $\sigma = \{\phi,\{a,d,e\},X\}$. Let $f : (X,\tau) \to (X,\sigma)$ be the identity function. Then f is gsp-continuous. Let $\nu = \{\phi,\{e\},X\}$. Clearly the identity funtion $g : (X,\sigma) \to (X,\nu)$ is also gsp-continuous, since $\{a,b,c,d\}$ is gsp-closed in (X,τ). But the composition function $g \circ f : (X,\tau) \to (X,\nu)$ is not gsp-continuous, since $\{a,b,c,d\}$ is closed in (X,ν) but not gsp-closed in (X,τ).

However the following theorem holds.

Theorem 2.18 [19]. *Let $f : (X,\tau) \to (Y,\sigma)$ and $g : (Y,\sigma) \to (Z,\nu)$ be two funtions. Then:*

(a) *If g is continuous and f is gsp-continuous, then $g \circ f$ is gsp-continuous.*

(b) *If g is gsp-irresolute and f is gsp-irresolute, then $g \circ f$ is gsp-irresolute.*

(c) *If g is gsp-continuous and f is gsp-irresolute, then $g \circ f$ is gsp-continuous.*

(d) *Let (Y, σ) be semi-pre-$T_{1/2}$-space. If g is gsp-continuous and f is β-irresolute, then $g \circ f$ is β-continuous.*

Theorem 2.19 [16]. *Let $f : (X, \tau) \to (Y, \sigma)$ be an onto, gsp-irresolute and pre-β-closed function. If (X, τ) is a semi-pre-$T_{1/2}$ space, then (Y, σ) is also semi-pre-$T_{1/2}$.*

3. Generalized b-closd sets

Definition 3.1. A subset A of a space (X, τ) is called *generalized b-closed* (briefly, *gb-closed*) if $\mathrm{bcl}(A) \subset U$ whenever $A \subset U$ and U is b-open in X.

Remark 3.2. From Definitions 2.2 and 3.1, we have the following diagram of implications:

$$\begin{array}{ccccc} & & gs\text{-closed} & & \\ & \nearrow & & \searrow & \\ g\text{-closed} & & & & gb\text{-closed} \rightarrow gsp\text{-closed} \\ & \searrow & & \nearrow & \\ & & gp\text{-closed} & & \end{array}$$

The reverses in the remark above need not be true as the following examples show.

Example 3.3. Let $X = \{a, b, c, d, e\}$ and $\tau = \{X, \phi, \{a\}, \{e\}, \{c, d\}, \{a, e\}, \{a, c, d\}, \{c, d, e\}, \{a, c, d, e\}, \{b, c, d, e\}\}$. Then $\{a, d, e\}$ is gsp-closed but not gb-closed in (X, τ).

Example 3.4. Let $X = \{a, b, c, d, e\}$ and $\tau = \{X, \phi, \{a, b\}, \{c, d\}, \{a, b, c, d\}\}$. Then $\{b, c\}$ is gb-closed but not gs-closed in (X, τ).

Example 3.5. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Then $\{a\}$ is gb-closed but not gp-closed in (X, τ).

Example 3.6. Let $X = \{a, b, c, d, e\}$ and $\tau = \{X, \phi, \{a\}, \{a, b, c\}\}$. Then $\{a, b, d\}$ is gb-closed but not b-closed in (X, τ).

The union (or intersection) of two gb-closed sets need not be gb-closed.

Example 3.7. Let (X,τ) be a topological space given in Example 3.3. Put $A=\{c,d\}$ and $B=\{e\}$. Then A and B are gb-closed but $A \cup B$ is not gb-closed in X.

Example 3.8. Let $X=\{a,b,c,d,e\}$ and $\tau=\{X,\phi,\{a\},\{a,b,c\}\}$. Put $A=\{a,b,c\}$ and $B=\{a,c,e\}$. Then A and B are gb-closed but $A \cap B$ is not gb-closed in (X,τ).

Theorem 3.9. *If A is gb-closed in (X,τ), then bcl$(A)\setminus A$ contains no nonempty closed set of X.*

Proof. Suppose that F is nonempty closed subset of $\text{bcl}(A)\setminus A$. Then $F \subset \text{bcl}(A)\setminus A$. Therefore $F \subset \text{bcl}(A)$ and $F \subset X\setminus A$. Since $X \setminus F$ is open and A is a gb-closed, $\text{bcl}(A) \subset X\setminus F$. Therefore $F \subset X \setminus \text{bcl}(A)$. Thus $F \subset \text{bcl}(A) \cap (X \setminus \text{bcl}(A)) = \phi$. This is a contradiction. □

However, the converse of above theorem need not be true as is seen from the following example.

Example 3.10. Let $X=\{a,b,c,d\}$ and $\tau=\{X,\phi,\{a\},\{c\},\{a,b\},\{a,c\},\{a,c,d\},\{a,b,c\}\}$. Put $A=\{a,c\}$. Then $\text{bcl}(A)\setminus A=\{b,d\}$ contains closed set $\{d\}$ but A is not gb-closed in (X,τ).

Corollary 3.11. *Let A be gb-closed in X. Then A is b-closed if and only if bcl$(A)\setminus A$ is closed.*

Proof. If A is b-closed then $\text{bcl}(A)\setminus A=\phi$ is closed.

Conversely, since $\text{bcl}(A)\setminus A$ is gb-closed set and $\text{bcl}(A)\setminus A$ is closed subset of itself, by above theorem $\text{bcl}(A)\setminus A=\phi$ and hence A be b-closed. □

Theorem 3.12. *Let A be a gb-closed set of X and $A \subset B \subset$ bcl(A). Then B is gb-closed in X.*

Proof. Let $B \subset U$ and U is an open set and $\text{bcl}(A) \subset U$. Then $\text{bcl}(B) \subset \text{bcl}(A) \subset U$. Hence B is gb-closed. □

Theorem 3.13. *Let $BO(X,\tau) = \tau$. Let B be gb-closed relative to A and A be gb-closed and open in X. Then B is gb-closed relative to X.*

Proof. Let $B \subset G$ and G be open in X. Then $B \subset A \cap G$ and $A \cap G$ is open in A. Since B is gb-closed relative to A, $\mathrm{bcl}_A(B) = \mathrm{bcl}(B) \cap A \subset A \cap G$. Therefore $A \cap \mathrm{bcl}(B) \subset G$ and so $A \subset G \cup (X \setminus \mathrm{bcl}(B))$. By hypothesis $G \cup (X \setminus \mathrm{bcl}(B))$ is a open set. Then $\mathrm{bcl}(B) \subset \mathrm{bcl}(A) \subset G \cup (X \setminus \mathrm{bcl}(B))$. Hence $\mathrm{bcl}(B) \subset G$. □

Corollary 3.14. *Let $BO(X,\tau) = \tau$. If A is gb-closd and F is b-closed, then $A \cap F$ is gb-closed.*

Proof. Since F is b-closed, $A \cap F$ is a b-closed set of A and then $A \cap F$ is gb-closed in A. Hence, by Theorem 3.13, $A \cap F$ is gb-closed in X. □

Lemma 3.15. *Let $A \subset Y \subset X$ and Y be an open subspace of a space X. Then we have $bcl_Y(A) = bcl_X(A) \cap Y$, where $bcl_Y(A)$ is b-closure of A in subspace Y.*

Theorem 3.16. *Let $A \subset Y \subset X$ and Y be open in X. If A is gb-closed relative to X, then A is gb-closed relative to Y.*

Proof. Let $A \subset G$ and G be an open set of Y. Then there exists an open set H of X such that $H \cap Y = G$. Since $B \subset H$, H is an open set of X and B is a gb-closed set of X. Therefore $\mathrm{bcl}(B) \subset H$. By Lemma 3.15, $\mathrm{bcl}_Y(A) = \mathrm{bcl}(A) \cap A \subset H \cap A \subset G$. Hence A is a gb-closed relative to Y. □

Theorem 3.17. *Let A be subset of a space (X,τ). Then A is gb-closed if and only if $\tau \subset BF(X,\tau)$, where $BF(X,\tau)$ is the family of all b-closed sets.*

Proof. Suppose that $\tau \subset BF(X,\tau)$. Let $A \subset G$ and G be an open set. Then $\mathrm{bcl}(A) \subset \mathrm{bcl}(G) = G$ and so A is gb-closed. Conversely, let $G \in \tau$. Since $G \subset G$ and G is gb-closed, $\mathrm{bcl}(G) \subset G$ and then $G \in BF(X,\tau)$. Hence $\tau \subset BF(X,\tau)$. □

Definition 3.18. A subset A of (X,τ) is called gb-open if its complement $X \setminus A$ is gb-closed.

Theorem 3.19. *Let A be subset of a space (X, τ). Then A is gb-open if and only if $F \subset bint(A)$ whenever $F \subset A$ and F is closed in X.*

Proof. Let F be closed in X and $F \subset A$. Then $X \setminus A \subset X \setminus F$ and $X \setminus F$ is open in X. Since $X \setminus A$ is *gb*-closed, $\text{bcl}(X \setminus A) \subset X \setminus F$. Hence $F \subset \text{bint}(A)$. Conversely, let U be open in X and $X \setminus A \subset U$. By hypothesis, $X \setminus U \subset \text{bint}(A)$ and $\text{bcl}(X \setminus A) \subset U$. Therefore $X \setminus A$ is *gb*-closed, that is, A is *gb*-open. □

Definition 3.20. Two subsets A and B of a space (X, τ) is said to be *b*-separated if $\text{bcl}(B) \cap A = \phi = \text{bcl}(A) \cap B$.

Theorem 3.21. *Let $BO(X, \tau) = \tau$. Then:*

(a) *If A and B are b-separated b-open sets, then $A \cup B$ is gb-open.*

(b) *If A and B are b-separated gb-closed sets, then $A \cap B$ is gb-closed.*

Proof. We prove only (a). Let F be a closed set and $F \subset A \cup B$. Then $F \cap \text{bcl}(A) \subset \{(A \cup B) \cap \text{bcl}(A)\} = A \cap (B \cap \text{bcl}(A)) = A$. Thus $F \cap \text{bcl}(A) \subset \text{bint}(A)$. Similarly, $F \cap \text{bcl}(B) \subset \text{bint}(B)$. Hence $F \subset \{F \cap (A \cup B)\} \subset \{(F \cap \text{bcl}(A)) \cup (F \cap \text{bcl}(B))\} \subset \{\text{bint}(A) \cup \text{bint}(B)\} \subset \text{bint}(A \cup B)$. Hence $A \cup B$ is *gb*-open. □

The intersection of *gb*-open sets is generally not *gb*-open as the following example shows.

Example 3.22. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Put $A = \{a, c, d\}$ and $B = \{b, c, d\}$. Then A and B are *gb*-open but $A \cap B$ is not *gb*-open in (X, τ).

Theorem 3.23. *Let A be a gb-open set and $bint(A) \subset B \subset A$. Then B is gb-open set.*

Proof. Let F is a closed set and $F \subset B$. Since A is *gb*-open and $F \subset \text{bint}(A)$, $F \subset \text{bint}(A) \subset \text{bint}(B)$. Hence B is *gb*-open. □

Theorem 3.24. *Let $BO(X, \tau) = \tau$ and $A \subset X$. Then A is gb-closed if and only if $bcl(A) \setminus A$ is gb-open.*

Proof. Let $F \subset \text{bcl}(A) \setminus A$ and F be a closed set. By Theorem 3.8, we have $F = \phi$. Hence $F \subset \text{bint}(\text{bcl}(A) \setminus A)$. By definition $\text{bcl}(A) \setminus A$ is *gb*-open.

Conversely, let $A \subset O$ and O be an open set. Now, $\text{bcl}(A) \cap (X \setminus O) \subset \text{bcl}(A) \cap (X \setminus A) = \text{bcl}(A) \setminus A$. Since $\text{bcl}(A) \cap (X \setminus O)$ is closed and $\text{bcl}(A) \setminus A$ is gb-open, $\text{bcl}(A) \cap (X \setminus O) \subset \text{bint}(\text{bcl}(A) \setminus A) = \phi$. Thus $\text{bcl}(A) \cap (X \setminus O) = \phi$, that is, $\text{bcl}(A) \subset O$. Hence A is gb-closed. □

Definition 3.25. The gb-closure of subset A of a space X, denoted by $gb\text{-cl}^*(A)$, is $gb\text{-cl}^*(A) = \cap\{F : A \subset F \text{ and } F \text{ is } gb\text{-closed}\}$.

Theorem 3.26. (a) *For subsets A, B of (X, τ), the following implications hold:*

i) $A \subset gb\text{-cl}^*(A) \subset bcl(A)$.

ii) $gb\text{-cl}^*(\phi) = \phi$ and $gb\text{-cl}^*(X) = X$.

iii) $gb\text{-cl}^*(A) \cap gb\text{-cl}^*(B) \subset gb\text{-cl}^*(A \cap B)$.

iv) $gb\text{-cl}^*(gb\text{-cl}^*(A)) = gb\text{-cl}^*(A)$.

v) If A is gb-closed in X, then $gb\text{-cl}^*(A) = A$.

(b) *If $BO(X, \tau)$ is closed under finite intersections, then $GBO(X, \tau)$ is closed under finite intersections.*

(c) *Let $\tau_b^* = \{U : gb\text{-cl}^*(X \setminus U) = X \setminus U\}$. If $BO(X, \tau)$ is closed under finite intersections, τ_b^* is a topology for X.*

Proof. (a) i) and ii) are trivial.

iii) Let F be a gb-closed set of $A \cup B$. Then $A, B \subset F$ and so $gb\text{-cl}^*(A)$, $gb\text{-cl}^*(B) \subset F$. Thus $gb\text{-cl}^*(A) \cup gb\text{-cl}^*(B) \subset gb\text{-cl}^*(A \cup B)$.

iv) Let F be a b-closed set and $A \subset F$. Then $gb\text{-cl}^*(A) \subset F$ and hence $gb\text{-cl}^*(gb\text{-cl}^*(A)) \subset gb\text{-cl}^*(A)$. But from (a), $gb\text{-cl}^*(A) \subset gb\text{-cl}^*(gb\text{-cl}^*(A))$. Thus $gb\text{-cl}^*(gb\text{-cl}^*(A)) = gb\text{-cl}^*(A)$.

(b) Let $A \subset GBO(\tau)$ and $B \subset GBO(\tau)$. Let U be an open set containing $X \setminus (A \cap B)$. Since $X \setminus A \subset U$ and $X \setminus B \subset U$, we have $\text{bcl}(X \setminus A) \subset U$ and $\text{bcl}(X \setminus B) \subset U$. Since $BO(X)$ is closed under finite intersections, $\text{bcl}(X \setminus (A \cap B) = \text{bcl}(X \setminus A) \cup \text{bcl}(X \setminus B) \subset U$. Therefore $X \setminus (A \cap B)$ is gb-closed in (X, τ) and hence $A \cap B \in GBO(\tau)$.

(c) i) By (a), clear.

ii) Let $U_i \in \tau_b^*$ for all $i \in I$. Then $gb\text{-cl}^*(X \setminus U_i) = X \setminus U_i$. By (a), $X \setminus (\cup_i U_i) \subset gb\text{-cl}^*(X \setminus (\cup_i U_i))$. And $gb\text{-cl}^*(X \setminus (\cup_i U_i)) = gb\text{-cl}^*(\cap_i (X \setminus U_i)) \subset$

$gb\text{-cl}^*(X \setminus U_i) = X \setminus U_j$ for all $j \in I$. Hence $gb\text{-cl}^*(X \setminus (\cup_i U_i)) \subset \cap_i (X \setminus U_i) = X \setminus (\cup_i U_i)$.

iii) Let $U, V \in \tau_b^*$. Then $gb\text{-cl}^*(X \setminus U) = X \setminus U$ and $gb\text{-cl}^*(X \setminus V) = X \setminus V$. Since $BO(X, \tau)$ is closed under finite intersections, $gb\text{-cl}^*(X \setminus (U \cap V)) = gb\text{-cl}^*((X \setminus U) \cup (X \setminus V)) = gb\text{-cl}^*(X \setminus U) \cup gb\text{-cl}^*(X \setminus V) = (x \setminus U) \cup (X \setminus V) = X \setminus (U \cap V)$. Therefore, by i)-iii), τ_b^* is a topology for X. □

Theorem 3.27. (a) *For a space (X, τ), every gb-closed set is preclosed (i.e. $GBO(\tau) = BO(\tau)$) if and only if $\tau_b^* = BO(\tau)$ holds.*

(b) *For a space (X, τ), every gb-closed set is closed (i.e. $GBO(\tau) = \tau$) if and only if $\tau_b^* = \tau$ holds.*

Proof. (a) Since $GBO(\tau) = BO(\tau)$ holds by assumption, we have $gb\text{-cl}^*(E) = \text{bcl}(E)$ for every subset E of (X, τ). We need to show $\tau_b^* \subset BO(\tau)$. Let $V \in \tau_b^*$. Then, $X \setminus V = gb\text{-cl}^*(X \setminus V) = \text{bcl}(X \setminus V)$ and so $X \setminus V$ is b-closed, that is, $V \in BO(\tau)$. Conversely, let V be a gb-closed set. Then by assumptions, $gb\text{-cl}^*(V) = V$ and hence $X \setminus V \in \tau_b^* = BO(\tau)$. Therefore, every gb-closed set is b-closed.

(b) By $\tau \subset BO(\tau) \subset \tau_b^*$, this is shown similarly to the proof of (a).

Theorem 3.28. *For a space (X, τ), the following conditions are equivalent:*

(a) *X is a semi-pre-$T_{1/2}$.*

(b) *Every singleton of X is closed or b-open.*

(c) *Every gb-closed is b-closed.*

Proof. (a)⇔(b): Since preopen is b-open, by Theorem 2.10, trivial.

(b)⇒(c): Let $A (\subset X)$ be gb-closed. We need to show that A is b-closed or eqivalently that $\text{bcl}(A) = A$. The inclusion $A \subset \text{bcl}(A)$ is trivial. For the reverse, let $x \in \text{bcl}(A)$. By assumption $\{x\}$ is either closed or b-open. We consider these two cases.

Case 1. Let $\{x\}$ be closed. By Theorem 3.9 $\text{bcl}(A) \setminus A$ does not contain $\{x\}$. Since $x \in \text{bcl}(A)$, then $x \in A$.

Case 2. Let $\{x\}$ be preopen. Clearly $\{x\}$ is b-open and since $x \in \text{bcl}(A)$, then $\{x\} \cap A \neq \phi$. Thus $x \in A$.

This shows that in both cases $x \in A$ or equivalently $\text{bcl}(A) \subset A$.

(c)⇒(b): Assume that for some $x \in X$ the set $\{x\}$ is not closed. Then $X \setminus \{x\}$ is not open. Thus the only open set containing $X \setminus \{x\}$ is X itself and hence $X \setminus \{x\}$ is trivially gb-closed. By (c) it is b-closed or eqivalently $\{x\}$ is b-open. □

Theorem 3.29. *For a space (X, τ), the following conditions are equivalent:*

(a) *X is a semi-pre-$T_{1/2}$.*

(b) *Every non b-open singleton is closed.*

Proof. It follows from Theorem 3.28. □

4. Generalized b-continuous functions

Definition 4.1. A function $f : X \to Y$ is called:

(a) b-continuous [20] (resp. b-irresolute) if for every open (resp. b-open) set G of Y, $f^{-1}(G)$ is b-open in X.

(b) pre-β-closed if for every b-closed set F of X, $f(F)$ is b-closed in Y.

Definition 4.2. A function $f : X \to Y$ is called:

(a) generalized b-continuous (briefly, gb-continuous) if for every closed set F of Y, $f^{-1}(F)$ is gb-closed in X.

(b) generalized b-irresolute (briefly, gb-irresolute) if for every gb-closed set F of Y, $f^{-1}(F)$ is gb-closed in X.

Theorem 4.3. *Every b-continuous function is gb-continuous.*

Proof. Since every b-closed set is gb-closed, it is obvious. □

Remark 4.4. For a function $f : X \to Y$ from Definitions 2.13, 4.1 and 4.2, we have the following diagram of implications:

$$\begin{array}{ccccc} gp\text{-irresoluteness} & & gb\text{-irresoluteness} & & gsp\text{-irresoluteness} \\ \downarrow & & \downarrow & & \downarrow \\ gp\text{-cotinuity} & \to & gb\text{-continuity} & \to & gsp\text{-continuity} \end{array}$$

Example 4.5. Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}\}$, $Y = \{p, q\}$ and $\sigma = \{X, \phi, \{p\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ be defined by $f(c) = q$, $f(a) = f(b) = p$. Then f is gb-continuous, but it is not b-continuous.

Lemma 4.6. *Let A be a subset of a space (X, τ). Then $x \in gb\text{-cl}^*(A)$ if and only if for each gb-open set U containing x, $U \cap A \neq \phi$.*

Proof. Suppose that $x \in gb\text{-cl}^*(A)$. If there exists a gb-open set U containing x such that $U \cap A = \phi$. Then $A \subset X \backslash U$. Since $X \backslash U$ is gb-closed, $gb\text{-cl}^*(A) \subset X \backslash U$ and so $x \notin gb\text{-cl}^*(A)$. This is a contradiction.

Conversely, suppose that $x \notin gb\text{-cl}^*(A)$. Then there exists a gb-closed set F such that $A \subset F$ and $x \notin F$. Thus $x \in X \setminus F$ and $X \setminus F$ is a gb-open set but $X \setminus F \cap A = \phi$. This is a contradiction by hypothesis. □

Theorem 4.7. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function.*

(a) *f is gb-continuous.*

(b) *The inverse image of each open set of Y is gb-open in X.*

(c) *$f(gb\text{-}cl^*(A)) \subset cl(f(A))$ for each $A \subset X$.*

(d) *For each $x \in X$ and each V is open set containing $f(x)$, there exists a gb-open set U containing x of X such that $f(U) \subset V$.*

Then we have the following implications:

$$(a) \Leftrightarrow (b) \Rightarrow (c) \Leftrightarrow (d)$$

Proof. (a)⇔(b): From definition, it is clear.

(a)⇒(c): Let A be any subset of X. Since $A \subset f^{-1}(\text{cl}(f(A)))$, by (a), $gb\text{-cl}^*(A) \subset gb\text{-cl}^*(f^{-1}(\text{cl}(f(A)))) = f^{-1}(\text{cl}(f(A)))$. Hence $f(gb\text{-cl}^*(A)) \subset f(f^{-1}(\text{cl}(f(A)))) \subset \text{cl}(f(A))$.

(c)⇒(d): Let $x \in X$ and V be any open set containing $f(x)$. Let $A = f^{-1}(X \setminus V)$ then $x \notin A$. Since $f(gb\text{-cl}^*(A)) \subset \text{cl}(f(A)) \subset X \setminus V$. Then $gb\text{-cl}^*(A) \subset f^{-1}(f(gb\text{-cl}^*(A)) \subset f^{-1}(X \setminus V) = A$. Therefore $gb\text{-cl}^*(A) = A$. Since $x \notin gb\text{-cl}^*(A)$, there exists a gb-open set U containing x such that $U \cap A = \phi$. Thus $f(U) \subset f(X \setminus A) \subset V$.

(d)⇒(c): Let $y \in f(gb\text{-cl}^*(A))$ and V be any open neighborhood of y. Then there exists $x \in X$ and there exists a gb-open set U such that $f(x) = y$, $x \in U$, $x \in gb\text{-}cl^*(A)$ and $f(U) \subset V$. Since $x \in gb\text{-cl}^*(A)$, $U \cap A \neq \phi$. Hence $f(A) \cap V \neq \phi$ and so $y = f(X) \in \text{cl}((f(A))$. Hence $f(gb\text{-cl}^*(A)) \subset \text{cl}(f(A))$. □

Theorem 4.8. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function.*

(a) *f is gb-irresolute.*

(b) *The inverse image of each gb-open set of Y is gb-open in X.*

(c) *$f(gb\text{-}cl^*(A)) \subset gb\text{-}cl^*(f(A))$ for each $A \subset X$.*

(d) *For each $x \in X$ and each V is gb-open set containing $f(x)$, there exists a gb-open set U containing x of X such that $f(U) \subset V$.*

Then we have the following implications:

$$(a) \Leftrightarrow (b) \Rightarrow (c) \Leftrightarrow (d)$$

Proof. Similarly to Theorem 4.7. □

Theorem 4.9. *Let $f : (X,\tau) \to (Y,\sigma)$ be a gb-continuous function and $\tau = BO(X,\tau)$. If H is a gb-closed and open subspace of (X,τ), then $f \mid H : (H, \tau \mid H) \to (Y,\sigma)$ is gb-continuous.*

Proof. Let F be a closed set of Y. By using assumption and Corollary 3.13 $f^{-1}(F) \cap H$ is a *gb*-closed set of X. By Theorem 3.14, $f^{-1}(F) \cap H = (f \mid H)^{-1}(F)$ is a *gb*-closed set of H. Thus $f \mid H$ is *gb*-continuous. □

Theorem 4.10. *Let $X = H \cup G$ be a space with topology τ and Y be a space with topology σ. Let $f : (G, \tau \mid G) \to (Y,\sigma)$ and $g : (H, \tau \mid H) \to (Y,\sigma)$ are gb-continuous functions such that $f(x) = g(x)$ for all $x \in G \cap H$. Suppose that $BO(X,\tau) = \tau$ and G, H are gb-closed and open in X. Then the combination $h : (X,\tau) \to (Y,\sigma)$ is gb-continuous.*

Proof. Let F be any closed set of Y and $h^{-1}(F) = f^{-1}(F) \cup g^{-1}(F) = C \cup D$ where $C = f^{-1}(F)$, $D = g^{-1}(F)$. Since C is a *gb*-closed set of H, by hypothesis, C is *gb*-closed in X. Similarly, D is *gb*-closed in X. By hypothesis, $C \cup D$ is a *gb*-closed set. Thus $h^{-1}(F)$ is *gb*-closed in X. Hence h is *gb*-continuous. □

Theorem 4.11. *If $f : (X,\tau) \to (Y,\sigma)$ is gb-irresolute and (X,τ) is a semi-pre-$T_{1/2}$-space. Then f is b-irresolute.*

Proof. Let V be a *b*-closed subset of Y. Then V is *gb*-closed. Since f is *gb*-irresolute, $f^{-1}(V)$ is *gb*-closed in X. And, since X is semi-pre-$T_{1/2}$, $f^{-1}(V)$ is *b*-closed in X. Hence f is *b*-irresolute. □

Theorem 4.12. *Let $f : (X,\tau) \to (Y,\sigma)$ be continuous and pre-b-closed. Then for every gb-closed set A of (X,τ), $f(A)$ is gb-closed in (Y,σ).*

Proof. Let A be *gb*-closed in X. Let $f(A) \subset O$, where O is open in Y. Since A is *gb*-closed and $f^{-1}(O)$ is open in X, bcl(A) $\subset f^{-1}(O)$ and so f(bcl(A)) $\subset O$. Therefore bcl(A) $\subset$ bcl(f(bcl(A))) $=$ f(bcl(A)) $\subset O$, since f is pre-*b*-closed. This implies that $f(A)$ is *gb*-closed in Y. □

Theorem 4.13. *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \nu)$ be two funtions. Then:*

(a) *If g is continuous and f is gb-continuous, then $g \circ f$ is gb-continuous.*

(b) *If g is gb-irresolute and f is gb-irresolute, $g \circ f$ is gb-irresolute.*

(c) *If g is gb-continuous and f is gb-irresolute, then $g \circ f$ is gb-continuous.*

(d) *Let (Y, σ) be semi-pre-$T_{1/2}$-space. If g is gb-continuous and f is b-irresolute, then $g \circ f$ is b-continuous.*

Proof. Obvious. □

Theorem 4.14. *Let $f : (X, \tau) \to (Y, \sigma)$ be a gb-irresolute and pre-b-closed surjection. If (X, τ) is a semi-pre-$T_{1/2}$-space, then (Y, σ) is also semi-pre-$T_{1/2}$.*

Proof. Let A be a gb-closed subset of Y. Since f is gb-irresolute, $f^{-1}(A)$ is gb-closed in X. And, since X is semi-pre-$T_{1/2}$ space, $f^{-1}(A)$ is b-closed in X. By the rest of the assumption, it follows that A is b-closed in Y or equivalently Y is semi-pre-$T_{1/2}$. □

5. *GBO*-connectedness

Definition 5.1. A space X is said to be *GBO-connected* if X cannot be written as a disjoint union of two non-empty *gb*-open sets. A subset of X is *GBO*-connected if it is *GBO*-connected as a subspace.

Theorem 5.2. *For a space X, the following are equivalent:*

(a) *X is GBO-connected.*

(b) *The only subsets of X which are both gb-open and gb-closed are the empty set ϕ and X.*

(c) *Each gb-continuous function of X into a discrete space Y with at least two points is a constant function.*

Proof. (a)$\Rightarrow$(b): Let U be a *gb*-open and *gb*-closed subset of X. Then $X \setminus U$ is both *gb*-open and *gb*-closed. Since X is the disjoint union of the *gb*-open sets U and $X \setminus U$, one of these must be empty, that is $U = \phi$ or $U = X$.

(b)$\Rightarrow$(a): Suppose that $X = A \cup B$ where A and B are disjoint non-empty *gb*-open subsets of X. Then A is both *gb*-open and *gb*-closed. By assumption, $A = \phi$ or X. Hence X is *GBO*-connected.

(b)$\Rightarrow$(c): Let $f : X \to Y$ be a *gb*-continuous function. Then X is covered by *gb*-open and *gb*-closed covering $\{f^{-1}(y) : y \in Y\}$. By assumption, $f^{-1}(y) = \phi$ or X for each $y \in Y$. If $f^{-1}(y) = \phi$ for all $y \in Y$ then f fails to be a function. Then there exists only one point $y \in Y$ such that $f^{-1}(y) \neq \phi$ and hence $f^{-1}(y) = X$ which shows that f is a constant function.

(c)$\Rightarrow$(b): Let U be both *gb*-open and *gb*-closed in X. Suppose $U \neq \phi$. Let $f : X \to Y$ be a *gb*-continuous function defined by $f(U) = \{y\}$ and $f(X \setminus U) = \{w\}$ for some distinct points y and w in Y. By assumption, f is constant. Hence we have $U = X$. □

It is obvious that every *GBO*-connected space is connected. The following example shows that the converse is not true.

Example 5.3. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}\}$. Then (X, τ) is connected. However, since every subset of X is both gb-open and gb-closed, (X, τ) is not GBO-connected by Theorem 5.2.

Theorem 5.4. *In space (X, τ) with at least two points, if $\tau = BF(X, \tau)$, then X is not GBO-connected.*

Proof. Let $\tau = BF(X, \tau)$. Then every subset of X is gb-closed. In fact, let $A \subset X$ and let $U \in \tau$ such that $A \subset U$. Then $\text{bcl}(A) \subset \text{bcl}(U) = U$ and hence A is gb-closed. There is a proper non-empty subset of X which is both gb-open and gb-closed in X. By Theorem 5.2, X is not GBO-connected. □

Theorem 5.5. *If $f : X \to Y$ is a gb-continuous surjection and X is GBO-connected, then Y is connected.*

Proof. Suppose that Y is not connected. Let $Y = A \cup B$ where A and B are disjoint non-empty open sets in Y. Since f is gb-continuous and onto, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non-empty and gb-open in X. This contradicts that fact that X is GBO-connected. Hence Y is connected. □

With reference to Theorem 5.5 we have the following:

Theorem 5.6. *Let $f : (X, \tau) \to (Y, \sigma)$ be a gb-continuous function and $\sigma = BO(Y, \sigma)$. If H is an open, gb-closed and GBO-connected subset of X, then $f(H)$ is a connected subset of Y.*

Proof. By Theorem 4.8, the restriction $f|H$ of f is gb-continuous. By Theorem 5.5, the image of the GBO-connected space $(H, \tau|H)$ under $f|H : (H, \tau|H) \to (f(H), \sigma|f(H))$ is connected. Thus $(f(H), \sigma|f(H))$ is connected and hence $f(H)$ is connected subset of Y. □

Lemma 5.7. *Let A and B be subsets of X and Y respectively. If A is gb-open in (X, τ) and B is gb-open in (Y, σ), then $A \times B$ is gb-open in $(X \times Y, \tau \times \sigma)$.*

Proof. Let F be a closed subset of $(X \times Y, \tau \times \sigma)$ such that $F \subset A \times B$. For each $(x, y) \in F$, $\text{cl}\{x\} \times \text{cl}\{y\} = \text{cl}(\{x\} \times \{y\}) = \text{cl}(\{x, y\}) \subset \text{cl}(F) = F \subset A \times B$. Two closed sets $\text{cl}\{x\}$ and $\text{cl}\{y\}$ are contained in A and B respectively. It follows from assumption that $\mathrm{cl}\{x\} \subset \mathrm{bint}(A)$ and $\mathrm{cl}\{y\} \subset \mathrm{bint}(B)$ hold. This implies that, for each $(x, y) \in F$, $(x, y) \in \mathrm{bint}(A) \times \mathrm{bint}(B) \subset \mathrm{bint}(A \times B)$ and so $F \subset \mathrm{bint}(A \times B)$. Hence $A \times B$ is gb-open. □

Theorem 5.8. *If the product space of two non-empty spaces is GBO-connected, then each factor space is GBO-connected.*

Proof. Let $(X \times Y, \tau \times \sigma) \to (X, \tau)$ be projection function and let $(X, \times Y, \tau \times \sigma)$ be a GBO-connected space. We first show that inverse image of every gb-closed under the projection p is gb-closed. Let F be gb-closed in X. Since $X \setminus F$ and Y are gb-open, by Lemma 5.7, $p^{-1}(X \setminus F) = (X \setminus F) \times Y$ is gb-open. And, since $p^{-1}(F) = F \times Y = X \times Y \setminus ((X \setminus F) \times Y) = X \times Y \setminus (p^{-1}(X \setminus F))$, $p^{-1}(F)$ is gb-closed.

Next we show that each factor space is GBO-connected. Suppose that X is not GBO-connected. By Theorem 5.2, there exists non-empty proper subset A of X which is both gb-open and gb-closed in X. Then $p^{-1}(A) = A \times Y$ is non-empty proper subset of $X \times Y$ which is both gb-open and gb-closed in $X \times Y$. This contradicts the fact that $X \times Y$ is GBO-connected. Hence X is GBO-connected. The proof for a space Y is similar to the case of X. □

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