Generalized Intuitionistic Fuzzy Matrices

일반화된 직관적 퍼지행렬

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일반화된 직관적 퍼지행렬

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요 약

본 논문에서는 일반화된 직관적 퍼지집합(generalized intuitionistic fuzz set)의 개념을 이용하여, 직관적 퍼지행렬보다 일반화된 generalized intuitionistic fuzzy matrix(일반화된 직관적 퍼지행렬)을 정의하고, 그 기본적인 성질을 조사하였다. 또한, 이들 행렬에 대한 행렬연산을 이용하여 정방인 일반화된 직관적 퍼지행렬에 대한 행렬식을 정의하여, 퍼지행렬에서의 Ragab과 Eman[16]의 결과와 직관적 퍼지행렬에서의 Im[8,9] 등의 결과들을 일반화 시켰다.

그리고, 일반화된 직관적 퍼지행렬에 대한 수반행렬(adjoint matrix)을 정의하고 반사성, 대칭성, 순환성 및 추이성과 같은 성질이 유지됨을 보였다. 끝으로, 추이적인 일반화된 직관적 퍼지행렬이나 추이적인 일반화된 직관적 퍼지관계의 연구에서 유용하게 쓰일 다음 결과를 보였다.

정리 3.14 임의의 정방인 일반화된 직관적 퍼지행렬로부터 추이적인 일반화된 직관적 퍼지행렬을 구성할 수 있다.

1 Introduction

In 1965, Zadeh [20] introduced the concept of fuzzy sets which formed the fundamental of fuzzy mathematics. Since then various workers have contributed to the development of the fuzzy theory. In particular, using the idea of fuzzy sets, Kim et al. [12,13] introduced fuzzy matrices as a generalization of matrices over the two element Boolean algebra (matrices with elements having values anywhere in the closed interval [0, 1]). Ragab and Emam [15] further studied some properties of the determinant and adjoint of square fuzzy matrix defined by Thomason [17] and Kim [13], respectively.

Atanassov [1-4] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. Using the idea of "intuitionistic fuzzy sets", Im et al. [8] defined the concept of intuitionistic fuzzy matrices as a natural generalization of fuzzy matrices and in [9] they introduced and studied the determinant of square intuitionistic fuzzy matrices.

Recently, Mondal and Samanta [14] introduced definitions of generalized intuitionistic fuzzy sets, as a generalization of intuitionistic fuzzy sets, generalized intuitionistic fuzzy relations and generalized intuitionistic fuzzy topology and studied some of their properties.

In this paper, using the idea of "generalized intuitionistic fuzzy set", we study the notion of generalized intuitionistic fuzzy matrices as a generalization of fuzzy matrices. We show that some properties of a square generalized intuitionistic fuzzy matrix such as reflexivity, transitivity and circularity are carried over to the adjoint generalized intuitionistic fuzzy matrix. Finally, we prove that $\mathcal{A}(\det\mathcal{A})$ is transitive for generalized intuitionistic fuzzy matrix $\mathcal{A}$. It enables us to construct a transitive generalized intuitionistic fuzzy matrix from a given one and it is useful for studying transitive generalized intuitionistic fuzzy matrix and generalized intuitionistic fuzzy relations [14].

2 GIF matrices

A *generalized intuitionistic fuzzy matrix* (briefly, *GIF matrix*) $\mathcal{A}$ is

$$\mathcal{A} = [(A, B)] = [(a_{ij}, b_{ij})]$$

where A and B are fuzzy matrices, and $a_{ij} \wedge b_{ij} \leq \frac{1}{2}$ for all i, j.

Obviously, every fuzzy matrix $A = [(a_{ij})]$ is an intuitionistic fuzzy matrix of the form $[(a_{ij}, 1 - a_{ij})]$. Every intuitionistic fuzzy matrix $\mathcal{A} = [(a_{ij}, b_{ij})]$ is GIF matrix, since $a_{ij} + b_{ij} \leq 1$ implies $a_{ij} \wedge b_{ij} \not> \frac{1}{2}$ for all i, j.

Let $\mathcal{A} = [(a_{ij}, b_{ij})]$ and $\mathcal{C} = [(c_{ij}, d_{ij})]$ be $m \times n$ GIF matrices and $\mathcal{E} = [(e_{ij}, f_{ij})]$ be an $n \times l$ GIF matrix. Then the matrix operations defined by

(1) $\mathcal{A} + \mathcal{C} = [(a_{ij} \vee c_{ij}, b_{ij} \wedge d_{ij})]$;

(2) $\mathcal{A}\mathcal{E} = \left[\left(\bigvee_{1<k<n}(a_{ik} \wedge e_{kj}), \bigwedge_{1\leq k\leq n}(b_{ik} \vee f_{kj})\right)\right]$;

(3) $\mathcal{A}^T = [(a_{ji}, b_{ji})]$;

(4) $\mathcal{A}^{k+1} = \mathcal{A}^k\mathcal{A}$, $k = 0, 1, 2, \ldots$;

(5) $\mathcal{A} \preceq \mathcal{C}$ if $a_{ij} \leq c_{ij}$ and $b_{ij} \geq d_{ij}$ for all i, j.

Let J be an $n \times n$ fuzzy matrix that have all entries 1 and I be an $n \times n$ identity fuzzy matrix, and $\mathcal{I} = [(I, J - I)]$. Then by the simple calculation

$$\mathcal{A}\mathcal{I} = \mathcal{I}\mathcal{A} = \mathcal{A}.$$

Therefore, $\mathcal{I}$ is the *identity GIF matrix*.

Let P be an $n \times n$ permutation fuzzy matrix and $\mathcal{P} = [(P, J - P)]$. Then by the simple calculation

$$\mathcal{P}\mathcal{P}^T = \mathcal{P}^T\mathcal{P} = \mathcal{I}.$$

Therefore, $\mathcal{P}$ is a *permutation GIF matrix*.

Theorem 2.1 *Let* $\mathcal{A} = [(a_{ij}, b_{ij})]$, $\mathcal{C} = [(c_{ij}, d_{ij})]$ *and* $\mathcal{E} = [(e_{ij}, f_{ij})]$ *be* $n \times n$ *GIF matrices. If* $\mathcal{A} \preceq \mathcal{C}$*, then* $\mathcal{AE} \preceq \mathcal{CE}$.

Proof For each termwise, $(a_{ik} \wedge e_{kj}) \leq (c_{ik} \wedge e_{kj})$ and $(b_{ik} \vee f_{kj}) \geq (d_{ik} \vee (f_{kj})$ where $i, j, k \in \{1, 2, \cdots, n\}$. Hence

$$(a_{i1} \wedge e_{1j}) \vee \cdots \vee (a_{in} \wedge e_{nj}) \leq (c_{i1} \wedge e_{1j}) \vee \cdots \vee (c_{in} \wedge e_{nj}),$$

and

$$(b_{i1} \vee f_{1j}) \wedge \cdots \wedge (b_{in} \vee f_{nj}) \geq (d_{i1} \vee f_{1j}) \wedge \cdots \wedge (d_{in} \vee f_{nj}).$$

Therefore, $\mathcal{AC} \preceq \mathcal{CE}$. □

Theorem 2.2 *Let* $\mathcal{A} = [(A, B)]$ *be a GIF matrix and* $\mathcal{P}$ *be a permutation GIF matrix. Then* $\mathcal{PA}$ *is a row changed matrix of* $\mathcal{A}$ *and* $\mathcal{AP}$ *is a column changed matrix of* $\mathcal{A}$.

Proof Suppose that $\mathcal{A}$ is a GIF matrix and $\mathcal{P}$ is a permutation GIF matrix which is generated by a permutation σ, where

$$\begin{pmatrix} 1 & 2 & \cdots & i & \cdots & (n-1) & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(i) & \cdots & \sigma(n-1) & \sigma(n) \end{pmatrix}.$$

Then

$$\begin{aligned} \mathcal{PA} &= \left[(\vee_{1 \leq k < n}(p_{ik} \wedge a_{kj}), \wedge_{1 \leq k \leq n}(1 - p_{ik} \vee b_{kj}))\right] \\ &= \left[(a_{\sigma(i)j}, b_{\sigma(i)j})\right]. \end{aligned}$$

Therefore, for any i, the i-th row of $\mathcal{PA}$ is a row of $\mathcal{A}$. The case of $\mathcal{AP}$ is similar to the above proof. □

Definition 2.3 The *determinant* det$\mathcal{A}$ of an $n \times n$ GIF matrix $\mathcal{A} = [(a_{ij}, b_{ij})]$ is defined as follows:

$$\det\mathcal{A} = \left(\bigvee_{\sigma \in S_n} a_{1\sigma(1)} \wedge \cdots \wedge a_{n\sigma(n)}, \bigwedge_{\sigma \in S_n} b_{1\sigma(1)} \vee \cdots \vee b_{n\sigma(n)} \right),$$

where S_n denotes the symmetric group of all permutations of the indices $\{1, 2, \cdots, n\}$.

Example 2.4 Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two fuzzy matrices such that

$$A = \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.9 \end{bmatrix}, \quad B = \begin{bmatrix} 0.3 & 0.5 \\ 0.7 & 0.3 \end{bmatrix}.$$

Since $a_{ij} \wedge b_{ij} \leq 1/2$ for all i, j,

$$\mathcal{A} = [(A, B)] = \begin{bmatrix} (0.8, 0.3) & (0.5, 0.5) \\ (0.2, 0.7) & (0.9, 0.3) \end{bmatrix}$$

is a 2×2 GIF matrix. We calculate the determinant $\det\mathcal{A}$ as follows:

$$\begin{aligned} \det\mathcal{A} &= \det[(A, B)] \\ &= (\{0.8 \wedge 0.9\} \vee \{0.5 \wedge 0.2\}, \{0.3 \vee 0.3\} \wedge \{0.5 \vee 0.7\}) \\ &= (0.8 \vee 0.2, 0.3 \wedge 0.7) \\ &= (0.8, 0.3). \end{aligned}$$

Theorem 2.5 *If a GIF matrix $\mathcal{C}$ is obtained from an $n \times n$ GIF matrix $\mathcal{A} = [(A, B)]$ by multiplying $\mathcal{A}$ by $k \in (0, 1]$, then $\det\mathcal{C} = k(\det\mathcal{A})$.*

Proof Suppose that $\mathcal{C} = [(c_{ij}, d_{ij})] = [(ka_{ij}, kb_{ij})]$. Then

$$\begin{aligned} \det\mathcal{C} &= \left[(\textstyle\bigvee_{\sigma \in S_n} c_{1\sigma(1)} \wedge \cdots c_{n\sigma(n)};\ \bigwedge_{\sigma \in S_n} d_{1\sigma(1)} \vee \cdots \vee d_{n(n)})\right] \\ &= \left[(\textstyle\bigvee_{\sigma \in S_n} ka_{1\sigma(1)} \wedge \cdots ka_{n\sigma(n)};\ \bigwedge_{\sigma \in S_n} kb_{1\sigma(1)} \vee \cdots \vee kb_{n(n)})\right] \\ &= \left[(k\textstyle\bigvee_{\sigma \in S_n} a_{1\sigma(1)} \wedge \cdots a_{n\sigma(n)};\ k\bigwedge_{\sigma \in S_n} b_{1\sigma(1)} \vee \cdots \vee b_{n(n)})\right] \\ &= [(k\det A, k\det B)] \\ &= k\,[(\det A, \det B)] \\ &= k(\det\mathcal{A}). \end{aligned}$$

□

Theorem 2.6 *Let $\mathcal{A} = [(A, B)]$ be an $n \times n$ GIF matrix. Then*

$$\det(\mathcal{I}_{ij}\mathcal{A}) = \det\mathcal{A} = \det(\mathcal{A}\mathcal{I}_{ij}),$$

where $\mathcal{I}_{ij}$ is a permutation GIF matrix which is obtained from the identity GIF matrix by interchanging row i and row j.

Proof Let $\mathcal{I}_{ij}\mathcal{A} = [(c_{ij}, d_{ij})]$. Then, for any i, j, the i-th (resp. j-th) row of $\mathcal{I}_{ij}\mathcal{A}$ is the j-th (resp. i-th) row $\mathcal{A}$. In fact, $\mathcal{I}_{ij}$ is a permutation GIF matrix which is generated by a permutation

$$\begin{pmatrix} i & j \\ j & i \end{pmatrix}.$$

Since, for any permutation $\sigma \in S_n$,

$$\begin{pmatrix} i & j \\ j & i \end{pmatrix}\sigma = \tau \in S_n.$$

$$\begin{aligned}
\det(\mathcal{I}_{ij}\mathcal{A}) &= \left(\bigvee_{\sigma\in S_n} c_{1\sigma(1)} \wedge \cdots \wedge c_{n\sigma(n)}, \bigwedge_{\sigma\in S_n} d_{\sigma(1)} \wedge \cdots \wedge d_{n\sigma(n)}\right) \\
&= \left(\bigvee_{\tau\in S_n} a_{1\tau(1)} \wedge \cdots \wedge a_{n\tau(n)}, \bigwedge_{\tau\in S_n} b_{\tau(1)} \wedge \cdots \wedge b_{n\tau(n)}\right) \\
&= \det\mathcal{A}.
\end{aligned}$$

The case of $\mathcal{A}\mathcal{I}_{ij}$ is similar to the above proof. □

Since any permutation GIF matrix is the product of $\mathcal{I}_{ij}$'s for some i, j, we have the following:

Corollary 2.7 *Let $\mathcal{A} = [(A, B)]$ be an $n \times n$ GIF matrix. Then*

$$\det(\mathcal{P}\mathcal{A}\mathcal{Q}) = \det\mathcal{A},$$

where $\mathcal{P}$ and $\mathcal{Q}$ are any permutation GIF matrices.

From Corollary 2.7, we know that $\det(\mathcal{P}\mathcal{A}) = (\det\mathcal{P})(\det\mathcal{A})$ where $\mathcal{P}$ is a permutation GIF matrix and $\mathcal{A}$ is any GIF matrix. However, in general we have the following:

Theorem 2.8 $\det(\mathcal{A}\mathcal{B}) \succeq (\det\mathcal{A})(\det\mathcal{B})$ *for any GIF matrices $\mathcal{A}$ and $\mathcal{B}$.*

Example 2.9 Let

$$\mathcal{A} = \begin{bmatrix} (0.8, 0.35) & (0.42, 0.7) \\ (0.5, 0.6) & (0.25, 0.9) \end{bmatrix} \quad \text{and } \mathcal{B} = \begin{bmatrix} (0.65, 0.35) & (0.8, 0.44) \\ (0.17, 0.6) & (0.36, 0.7) \end{bmatrix}$$

be GIF matrices. Then

$$\mathcal{AB} = \begin{bmatrix} (0.65, 0.35) & (0.8, 0.44) \\ (0.5, 0.6) & (0.5, 0.6) \end{bmatrix}$$

Thus $\det\mathcal{A} = (0.42, 0.7)$, $\det\mathcal{B} = (0.36, 0.6)$, $(\det\mathcal{A})(\det\mathcal{B}) = (0.36, 0.7)$ and $\det(\mathcal{AB}) = (0.5, 0.6)$. Therefore, $\det(\mathcal{AB}) \succeq (\det\mathcal{A})(\det\mathcal{B})$.

3 The adjoint of square GIF matrices

Definition 3.1 The *adjoint matrix* of an $n \times n$ GIF matrix $\mathcal{A} = |(A, B)|$, denoted by $\text{adj}\mathcal{A}$, is defined as follows:

$$\text{adj}\mathcal{A} = |(c_{ij}, d_{ij})| = |(\det A_{ji}, \det B_{ji})|,$$

where $(\det A_{ji}, \det B_{ji})$ is the determinant of the $(n-1) \times (n-1)$ GIF matrix formed by deleting row j and column i from A and B, respectively, in the each operations.

We know that $\det A_{ji}$ can be obtained from $\det A$ by replacing a_{ji} by 1 and all other row-j factors a_{jk}, $k \neq i$, by 0 and $\det B_{ji}$ can be obtained from $\det B$ by replacing b_{ji} by 0 and all other row-j factors b_{jk}, $k \neq i$, by 1. We also can write the elements c_{ij} and d_{ij} of $\text{adj}\mathcal{A}$ as follows:

$$c_{ij} = \left(\bigvee_{\sigma \in S_{n_j n_i}} \wedge_{t \in n_j} a_{t\sigma(t)} \right), \ d_{ij} = \left(\bigwedge_{\sigma \in S_{n_j n_i}} \vee_{t \in n_j} d_{t\sigma(t)} \right),$$

where $n_j = \{1, 2, \cdots, n\} \setminus \{j\}$ and $S_{n_j n_i}$ is the set all permutations of set n_j over the set n_i.

Theorem 3.2 *Let $\mathcal{A} = |(A, B)|$ and $\mathcal{C} = |(C, D)|$ be $n \times n$ GIF matrices. Then*

(1) $\mathcal{A} \preceq \mathcal{C}$ *implies* $\text{adj}\mathcal{A} \preceq \text{adj}\,\mathcal{C}$;

(2) $\text{adj}\mathcal{A} + \text{adj}\mathcal{C} \preceq \text{adj}(\mathcal{A} + \mathcal{C})$;

(3) $\text{adj}(\mathcal{A}^T) = (\text{adj}\mathcal{A})^T$.

Proof (1) Let $\text{adj}\mathcal{A} = |(x_{ij}, y_{ij})|$ and $\text{adj}\mathcal{C} = |(z_{ij}, w_{ij})|$. Then

$$(x_{ij}, y_{ij}) = \left(\bigvee_{\sigma \in S_{n_j n_i}} \wedge_{t \in n_j} a_{t\sigma(t)}, \bigwedge_{\sigma \in S_{n_j n_i}} \vee_{t \in n_j} b_{t\sigma(t)} \right)$$

and

$$(z_{ij}, w_{ij}) = \left(\bigvee_{\sigma\in S_{n_jn_i}} \wedge_{t\in n_j} c_{t\sigma(t)}, \bigwedge_{\sigma\in S_{n_jn_i}} \vee_{t\in n_j} d_{t\sigma(t)} \right).$$

It is clear that $x_{ij} \le z_{ij}$ and $y_{ij} \ge w_{ij}$ since $a_{i\sigma(t)} \le c_{i\sigma(t)}$ and $b_{i\sigma(t)} \ge d_{i\sigma(t)}$ for every $t \neq j$, $\sigma(t) \neq i$.

(2) Since $\mathcal{A}, \mathcal{C} \preceq (\mathcal{A}+\mathcal{C})$, adj$\mathcal{A}$, adj$\mathcal{C} \preceq$ adj$(\mathcal{A}+\mathcal{C})$ and so adj$\mathcal{A}$ + adj$\mathcal{C} \preceq$ adj$(\mathcal{A}+\mathcal{C})$.

(3) Let adj$\mathcal{A} = [(x_{ij}, y_{ij})]$ and adj$(\mathcal{A}^T) = [(z_{ij}, w_{ij})]$. Then

$$(x_{ij}, y_{ij}) = \left(\bigvee_{\sigma\in S_{n_jn_i}} \wedge_{t\in n_j} a_{t\sigma(t)}, \bigwedge_{\sigma\in S_{n_jn_i}} \vee_{t\in n_j} b_{t\sigma(t)} \right)$$

and

$$(z_{ij}, w_{ij}) = \left(\bigvee_{\sigma\in S_{n_jn_i}} \wedge_{\sigma(t)\in n_j} a_{t\sigma(t)}, \bigwedge_{\sigma\in S_{n_jn_i}} \vee_{\sigma(t)\in n_j} b_{t\sigma(t)} \right),$$

which is the element (x_{ji}, y_{ji}). Hence $(\text{adj}\mathcal{A})^T = \text{adj}(\mathcal{A}^T)$. □

Theorem 3.3 *Let $\mathcal{A} = [(A, B)]$ be an $n \times n$ GIF matrix. Then*

(1) $\mathcal{A}(\text{adj}\mathcal{A}) \succeq (\det\mathcal{A})\mathcal{I}_n$;

(2) $(\text{adj}\mathcal{A})\mathcal{A} \succeq (\det\mathcal{A})\mathcal{I}_n$.

Proof (1) Let $\mathcal{A}(\text{adj}\mathcal{A}) = [(x_{ij}, y_{ij})]$. Now, the i-th row of $\mathcal{A}$ is

$$((a_{i1}, b_{i1}), (a_{i2}, b_{i2}), \cdots, (a_{in}, b_{in})).$$

Then the j-th column of adj$\mathcal{A}$ is

$$((\det A_{j1}, \det B_{j1}), (\det A_{j2}, \det B_{j2}), \cdots, (\det A_{jn}, \det B_{jn})).$$

Thus

$$(x_{ij}, y_{ij}) = \left(\bigvee_{1\le k\le n} (a_{ik} \wedge \det A_{jk}), \bigwedge_{1\le k\le n} (b_{ik} \vee \det B_{jk}) \right) \succeq (0, 1).$$

Hence

$$(x_{ii}, y_{ii}) = \left(\bigvee_{1 \le k \le n} (a_{ik} \wedge \det A_{ik}), \bigwedge_{1 \le k \le n} (b_{ik} \vee \det B_{ik}) \right),$$

which is equal to $\det\mathcal{A}$. Therefore, $\mathcal{A}(\mathrm{adj}\mathcal{A}) \succeq (\det\mathcal{A})\mathcal{I}_n$.

(2) The proof is similar to (1). □

Theorem 3.4 *Let $\mathcal{A} = |(A, B)|$ be a GIF matrix with a 0's row in A and a 1's row in B and let O be a zero matrix. Then $(\mathrm{adj}\mathcal{A})\mathcal{A} = |(O, J)|$.*

Proof Let $(\mathrm{adj}\mathcal{A})\mathcal{A} = |(x_{ij}, y_{ij})|$. Then

$$(x_{ij}, y_{pq}) = \left(\bigvee_{1 \le k \le n} (\det A_{ki} \wedge a_{kj}), \bigwedge_{1 \le k \le n} (\det B_{kp} \vee b_{kq}) \right).$$

If the i-th row of A is 0's and the p-th row of B is 1's, then $\mathcal{A}_{ki}$ contains a 0's row where $k \neq i$ and $\mathcal{B}_{kp}$ contains a 1's row where $k \neq p$. So $\det\mathcal{A}_{ki} = 0$ for every $k \neq i$, $\det\mathcal{B}_{kp} = 1$ for every $k \neq p$. If $k = i$, then $a_{ij} = 0$ and if $k = p$, then $b_{ij} = 1$ for every j. Hence

$$\bigvee_{1 \le k \le n} (\det A_{ki} \wedge a_{kj}) = 0 \quad \text{and} \quad \bigwedge_{1 \le k \le n} (\det B_{kp} \vee b_{kq}) = 1.$$

Therefore, $(\mathrm{adj}\mathcal{A})\mathcal{A} = |(O, J)|$. □

Theorem 3.5 *Let $\mathcal{A} = |(A, B)|$ be an $n \times n$ GIF matrix. Then $\det\mathcal{A} = \det(\mathrm{adj}\mathcal{A})$.*

Proof Since

$$\mathrm{adj}\mathcal{A} = \begin{bmatrix} (\det A_{11}, \det B_{11}) & (\det A_{21}, \det B_{21}) & \cdots & (\det A_{n1}, \det B_{n1}) \\ (\det A_{12}, \det B_{12}) & (\det A_{22}, \det B_{22}) & \cdots & (\det A_{n2}, \det B_{n2}) \\ \cdots & \cdots & \cdots & \cdots \\ (\det A_{1n}, \det B_{1n}) & (\det A_{2n}, \det B_{2n}) & \cdots & (\det A_{nn}, \det B_{nn}) \end{bmatrix},$$

$$\det(\mathrm{adj}\mathcal{A})$$

$$\left[\bigvee_{\sigma\in S_n}(\det A_{1\sigma(1)}\wedge\cdots\wedge\det A_{n\sigma(n)}),\ \bigwedge_{\sigma\in S_n}(\det B_{1\sigma(1)}\vee\cdots\vee\det B_{n\sigma(n)})\right]$$

$$\left[\bigvee_{\sigma\in S_n}(\wedge_{1\le i\le n}\det A_{i\sigma(i)}),\ \bigwedge_{\sigma\in S_n}(\vee_{1\le i\le n}\det B_{i\sigma(i)})\right]$$

$$\left[\bigvee_{\sigma\in S_n}(\wedge_{1\le i\le n}(\bigvee_{\theta\in S_{n_in_{\sigma(i)}}}\wedge_{t\in n_i}a_{t\theta(t)})),\ \bigwedge_{\sigma\in S_n}(\vee_{i\le i\le n}(\bigwedge_{\theta\in S_{n_in_{\sigma(i)}}}\vee_{t\in n_i}b_{t\theta(t)}))\right]$$

$$\left[\bigvee_{\sigma\in S_n}((\bigvee_{\theta\in S_{n_1n_{\sigma(1)}}}\wedge_{t\in n_1}a_{t\theta(t)})\wedge\cdots\wedge(\bigvee_{\theta\in S_{n_nn_{\sigma(n)}}}\wedge_{t\in n_n}a_{t\theta(t)})),\right.$$

$$\left.\bigwedge_{\sigma\in S_n}((\bigwedge_{\theta\in S_{n_1n_{\sigma(1)}}}\vee_{t\in n_1}b_{t\theta(t)})\vee\cdots\vee(\bigwedge_{\theta\in S_{n_nn_{\sigma(n)}}}\vee_{t\in n_n}b_{t\theta(t)}))\right]$$

$$\left[\bigvee_{\sigma\in S_n}((\wedge_{t\in n_1}a_{t\theta_1(t)})\wedge\cdots\wedge(\wedge_{t\in n_n}a_{t\theta_n(t)})),\ \bigwedge_{\sigma\in S_n}((\vee_{t\in n_1}b_{t\theta_1(t)})\vee\cdots\vee(\vee_{t\in n_n}b_{t\theta_n(t)}))\right]$$

for some $\theta_1\in S_{n_1n_{\sigma(1)}},\theta_2\in S_{n_2n_{\sigma(2)}},\cdots,\theta_n\in S_{n_nn_{\sigma(n)}}$

$$\left[\bigvee_{\sigma\in S_n}((a_{2\theta_1(2)}\wedge a_{3\theta_1(3)}\wedge\cdots\wedge a_{n\theta_1(n)})\wedge(a_{1\theta_2(1)}\wedge a_{3\theta_2(3)}\wedge\cdots\wedge a_{n\theta_2(n)})\wedge\cdots\wedge\right.$$

$$(a_{1\theta_n(1)}\wedge\cdots\wedge a_{n-1\theta_n(n-1)})),$$

$$\bigwedge_{\sigma\in S_n}((b_{2\theta_1(2)}\vee b_{3\theta_1(3)}\vee\cdots\vee b_{n\theta_1(n)})\vee(b_{1\theta_2(1)}\vee b_{3\theta_2(3)}\vee\cdots\vee b_{n\theta_2(n)})\vee\cdots\vee$$

$$\left.(b_{1\theta_n(1)}\vee\cdots\vee b_{n-1\theta_n(n-1)}))\right]$$

$$\left[\bigvee_{\sigma\in S_n}((a_{1\theta_2(1)}\wedge a_{1\theta_3(1)}\wedge\cdots\wedge a_{1\theta_n(1)})\wedge(a_{2\theta_1(2)}\wedge a_{2\theta_3(2)}\wedge\cdots\wedge a_{2\theta_n(2)})\wedge\cdots\wedge\right.$$

$$(a_{n\theta_1(n)}\wedge a_{n\theta_2(n)}\cdots\wedge a_{n\theta_{n-1}(n)})),$$

$$\bigwedge_{\sigma\in S_n}((b_{1\theta_2(1)}\vee b_{1\theta_3(1)}\vee\cdots\vee b_{1\theta_n(1)})\vee(b_{2\theta_1(2)}\vee b_{2\theta_3(2)}\vee\cdots\vee b_{2\theta_n(2)})\vee\cdots\vee$$

$$\left.(b_{n\theta_1(n)}\vee b_{n\theta_2(n)}\cdots\wedge b_{n\theta_{n-1}(n)}))\right]$$

$$\left[\bigvee_{\sigma\in S_n}(a_{1\theta_{f_1}(1)}\wedge a_{2\theta_{f_2}(2)}\wedge\cdots\wedge a_{n\theta_{f_n}(n)}),\ \bigwedge_{\sigma\in S_n}(b_{1\theta_{f_1}(1)}\vee b_{2\theta_{f_2}(2)}\vee\cdots\vee b_{n\theta_{f_n}(n)})\right]$$

for some $f_i\in\{1,2,\cdots,n\}\setminus\{i\}$, $i=1,2,\cdots,n$.

Since $a_{i\theta_{f_i}(i)} \neq a_{i\sigma(f_i)}$, $a_{i\theta f_i(i)} = a_{i\sigma(i)}$. Therefore,

$$
\begin{aligned}
\det(\text{adj}\mathcal{A}) &= \left[\left(\bigvee_{\sigma \in S_n} a_{1\sigma(1)} \wedge \cdots \wedge a_{n\sigma(n)}, \bigwedge_{\sigma \in S_n} b_{1\sigma(1)} \vee \cdots \vee b_{n\sigma(n)} \right) \right] \\
&= \det\mathcal{A}.
\end{aligned}
$$

□

Example 3.6 Let

$$
\mathcal{A} = \begin{bmatrix} (0.2, 0.8) & (0.4, 0.5) & (0.7, 0.4) \\ (0.1, 0.9) & (0.6, 0.5) & (0.5, 0.3) \\ (0.7, 0.2) & (0.4, 0.4) & (0.4, 0.7) \end{bmatrix}.
$$

Then $\det\mathcal{A} = (0.6, 0.5)$ and

$$
\begin{aligned}
\det\mathcal{A}_{11} &= \det \begin{bmatrix} (0.6, 0.5) & (0.5, 0.3) \\ (0.4, 0.4) & (0.4, 0.7) \end{bmatrix} = (0.4, 0.4), \\
\det\mathcal{A}_{12} &= \det \begin{bmatrix} (0.1, 0.9) & (0.5, 0.3) \\ (0.7, 0.2) & (0.4, 0.7) \end{bmatrix} = (0.5, 0.3), \\
\det\mathcal{A}_{13} &= \det \begin{bmatrix} (0.1, 0.9) & (0.6, 0.5) \\ (0.7, 0.2) & (0.4, 0.4) \end{bmatrix} = (0.6, 0.5), \\
\det\mathcal{A}_{21} &= \det \begin{bmatrix} (0.4, 0.5) & (0.7, 0.4) \\ (0.4, 0.4) & (0.4, 0.7) \end{bmatrix} = (0.4, 0.4), \\
\det\mathcal{A}_{22} &= \det \begin{bmatrix} (0.2, 0.8) & (0.7, 0.4) \\ (0.7, 0.2) & (0.4, 0.7) \end{bmatrix} = (0.7, 0.4), \\
\det\mathcal{A}_{23} &= \det \begin{bmatrix} (0.2, 0.8) & (0.4, 0.5) \\ (0.7, 0.2) & (0.4, 0.4) \end{bmatrix} = (0.4, 0.5), \\
\det\mathcal{A}_{31} &= \det \begin{bmatrix} (0.4, 0.5) & (0.7, 0.4) \\ (0.6, 0.5) & (0.5, 0.3) \end{bmatrix} = (0.6, 0.5), \\
\det\mathcal{A}_{32} &= \det \begin{bmatrix} (0.2, 0.8) & (0.7, 0.4) \\ (0.1, 0.9) & (0.5, 0.3) \end{bmatrix} = (0.2, 0.8),
\end{aligned}
$$

$$\det \mathcal{A}_{33} = \det \begin{bmatrix} (0.2, 0.8) & (0.4, 0.5) \\ (0.1, 0.9) & (0.6, 0.5) \end{bmatrix} = (0.2, 0.8).$$

Thus

$$\operatorname{adj}\mathcal{A} = \begin{bmatrix} (0.4, 0.4) & (0.4, 0.4) & (0.6, 0.5) \\ (0.5, 0.3) & (0.7, 0.4) & (0.2, 0.8) \\ (0.6, 0.5) & (0.4, 0.5) & (0.2, 0.8) \end{bmatrix}.$$

Therefore, $\det(\operatorname{adj}\mathcal{A}) = (0.6, 0.5) = \det\mathcal{A}$.

Definition 3.7 An $m \times n$ GIF matrix $\mathcal{A} = [(A, B)]$ is said to be *constant* if $a_{ik} = a_{jk}$ and $b_{ik} = b_{jk}$ for all i, j, k, that is, its rows are equal to each other.

Theorem 3.8 *Let $\mathcal{A} = [(A, B)]$ be an $n \times n$ constant GIF matrix. Then*

(1) $(\operatorname{adj}\mathcal{A})^T$ *is constant;*

(2) $\mathcal{A}(\operatorname{adj}\mathcal{A})$ *is constant;*

(3) $\det A$ *is the minimum element in A and* $\det B$ *is the maximum element in B.*

Proof (1) Let $\mathcal{C} = [(c_{ij}, d_{ij})] = \operatorname{adj}\mathcal{A}$. Then

$$c_{ij} = \bigvee_{\sigma \in S_{n_j n_i}} \left(\wedge_{t \in n_j} a_{t\sigma(t)} \right), \quad c_{ik} = \bigvee_{\sigma \in S_{n_k n_i}} \left(\wedge_{t \in n_k} a_{t\sigma(t)} \right)$$

and

$$d_{ij} = \bigvee_{\sigma \in S_{n_j n_i}} \left(\wedge_{t \in n_j} b_{t\sigma(t)} \right), \quad d_{ik} = \bigvee_{\sigma \in S_{n_k n_i}} \left(\wedge_{t \in n_k} b_{t\sigma(t)} \right).$$

We know that $c_{ij} = c_{ik}$ and $d_{ij} = d_{ik}$ since the numbers $\sigma(t)$ of columns cannot be changed in the paired expansions of c_{ij} and c_{ik}, and d_{ij} and d_{ik}. Therefore, $(\operatorname{adj}\mathcal{A})^T$ is constant.

(2) Let $\mathcal{A}(\operatorname{adj}\mathcal{A}) = [(X, Y)]$. Since $\mathcal{A}$ is constant, $A_{jk} = A_{ik}$ and $B_{jk} = B_{ik}$ for every $i, j \in \{1, 2, \cdots, n\}$. Then

$$x_{ij} = \bigvee_{1\leq k\leq n} (a_{ik} \wedge \det(A_{jk})) = \det A$$

and

$$y_{ij} = \bigwedge_{1\leq k\leq n} (b_{ik} \wedge \det(B_{jk})) = \det B.$$

Therefore, $\mathcal{A}(\mathrm{adj}\mathcal{A})$ is constant.

(3) Now,

$$\det A = \bigvee_{\sigma\in S_n} a_{1\sigma(1)} \wedge a_{2\sigma(2)} \wedge \cdots \wedge a_{n\sigma(n)} = a_{1\sigma(1)} \wedge a_{2\sigma(2)} \wedge \cdots \wedge a_{n\sigma(n)}$$

and

$$\det B = \bigwedge_{\sigma\in S_n} b_{1\sigma(1)} \vee b_{2\sigma(2)} \vee \cdots \vee b_{n\sigma(n)} = b_{1\sigma(1)} \vee b_{2\sigma(2)} \vee \cdots \vee b_{n\sigma(n)}$$

for any $\sigma \in S_n$. Taking σ the identity permutation, we have

$$\det A = a_{11} \wedge a_{22} \wedge \cdots \wedge a_{nn} \quad \text{and} \quad \det B = b_{11} \vee b_{22} \vee \cdots \wedge b_{nn}$$

which are the minimum and maximum elements in A and B, respectively. □

Definition 3.9 Let $\mathcal{A} = [(A, B)]$ be an $n \times n$ GIF matrix. Then $\mathcal{A}$ is called

(1) *symmetric* if $\mathcal{A} = \mathcal{A}^T$;

(2) *reflexive* if $\mathcal{A} \succeq \mathcal{I}_n$;

(3) *transitive* if $\mathcal{A}^2 \preceq \mathcal{A}$;

(4) *idempotent* if $\mathcal{A}^2 = \mathcal{A}$;

(5) *circular* if $(\mathcal{A}^2)^T \preceq \mathcal{A}$.

Lemma 3.10 *Let $\mathcal{A}$ be an $n \times n$ reflexive GIF matrix. Then* $\mathrm{adj}\mathcal{A} = \mathcal{A}^c$ *where $\mathcal{A}^c$ is idempotent and $c \leq n - 1$.*

Proof The proof is similar to that of Proposition 4 in [18]. □

Theorem 3.11 *Let $\mathcal{A} = |(A, B)|$ be an $n \times n$ reflexive GIF matrix. Then*

(1) $\text{adj}\mathcal{A}$ *is reflexive;*

(2) $\text{adj}\mathcal{A} \succeq \mathcal{A}$;

(3) $\text{adj}\mathcal{A}^2 = (\text{adj}\mathcal{A})^2 = \text{adj}\mathcal{A}$;

(4) *If $\mathcal{A}$ is idempotent, then* $\text{adj}\mathcal{A} = \mathcal{A}$;

(5) $\text{adj}(\text{adj}\mathcal{A}) = \text{adj}\mathcal{A}$;

(6) $\mathcal{A}(\text{adj}\mathcal{A}) = \text{adj}\mathcal{A} = (\text{adj}\mathcal{A})\mathcal{A}$.

Proof (1) Let $\mathcal{C} = |(C, D)| = \text{adj}\mathcal{A}$. Then, for each i,

$$c_{ii} = \bigvee_{\sigma \in S_{n_i}} (\wedge_{t \in n_i} a_{t\sigma(t)}) \quad \text{and} \quad d_{ii} = \bigwedge_{\sigma \in S_{n_i}} (\vee_{t \in n_i} b_{t\sigma(t)}).$$

Taking the identity permutation $\sigma(t) = t$, we have

$$c_{ii} \geq a_{11} \wedge a_{22} \wedge \cdots \wedge a_{(i-1)(i-1)} \wedge a_{(i+1)(i+1)} \wedge \cdots \wedge a_{nn} = 1$$

and

$$d_{ii} \leq b_{11} \vee b_{22} \vee \cdots \vee b_{(i-1)(i-1)} \vee b_{(i+1)(i+1)} \vee \cdots \vee b_{nn} = 0.$$

Therefore, $\text{adj}\mathcal{A}$ is reflexive since $c_{ii} = 1$ and $d_{ii} = 0$.

(2) Let $\mathcal{C} = |(C, D)| = \text{adj}\mathcal{A}$. Then, for each i, j,

$$c_{ij} = \bigvee_{\sigma \in S_{n_j n_i}} (\wedge_{t \in n_j} a_{t\sigma(t)}) \quad \text{and} \quad d_{ij} = \bigwedge_{\sigma \in S_{n_j n_i}} (\vee_{t \in n_j} b_{t\sigma(t)}).$$

Taking the permutation $\sigma(i) = j$ and $\sigma(h) = h$, where $h \neq i$, then the permutation σ in $S_{n_j n_i}$ is the following form:

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & i & \cdots & (j-1) & (j+1) & \cdots & n \\ 1 & 2 & 3 & \cdots & j & \cdots & (j-1) & (j+1) & \cdots & n \end{pmatrix}.$$

Then, each

$$a_{11} \wedge a_{22} \wedge \cdots \wedge a_{ij} \wedge \cdots \wedge a_{(j-1)(j-1)} \wedge a_{(j+1)(j+1)} \wedge \cdots \wedge a_{nn}$$

and

$$b_{11} \vee b_{22} \vee \cdots \vee b_{ij} \vee \cdots \vee b_{(j-1)(j-1)} \vee b_{(j+1)(j+1)} \vee \cdots \vee b_{nn}$$

are terms of c_{ij} and d_{ij}, respectively. Hence

$$c_{ij} \geq a_{11} \wedge a_{22} \wedge \cdots \wedge a_{ij} \wedge \cdots \wedge a_{(j-1)(j-1)} \wedge a_{(j-1)(j+1)} \wedge \cdots \wedge a_{nn} = a_{ij}$$

and

$$d_{ij} \leq b_{11} \vee b_{22} \vee \cdots \vee b_{ij} \vee \cdots \vee b_{(j-1)(j-1)} \vee b_{(j-1)(j+1)} \vee \cdots \vee b_{nn} = b_{ij}.$$

Therefore, $\mathcal{C} = \text{adj}\mathcal{A} \succeq \mathcal{A}$.

(3) Since $\mathcal{A}$ is reflexive, we get $\mathcal{A}^2$ is also reflexive and $\text{adj}\mathcal{A}^2 = (\mathcal{A}^2)^c = (\mathcal{A}^c)^2 = (\text{adj}\mathcal{A})^2$. But since $\mathcal{A}^c$ is idempotent, we have $(\text{adj}\mathcal{A})^2 = \text{adj}\mathcal{A}$.

(4) We have by Lemma 3.10 that $\text{adj}\mathcal{A} = \mathcal{A}^c (c \leq n-1)$. But we have also that $\mathcal{A}$ is idempotent. So $\mathcal{A}^c = \mathcal{A}$. Thus $\text{adj}\mathcal{A} = \mathcal{A}$.

(5) Since $\mathcal{A}$ is reflexive, we get $\text{adj}\mathcal{A}$ is idempotent by Lemma 3.10 and reflexive by (1). So that by (4) $\text{adj}(\text{adj}\mathcal{A}) = \text{adj}\mathcal{A}$.

(6) Let $\mathcal{E} = [(e_{ij}, f_{ij})] = \mathcal{A}(\text{adj}\mathcal{A})$ and $\mathcal{G} = [(g_{ij}, h_{ij})] = (\text{adj}\mathcal{A})\mathcal{A}$. Then

$$e_{ij} = \bigvee_{1 \leq k \leq n} a_{ik} \wedge \det\mathcal{A}_{jk} \geq a_{ii} \wedge \det\mathcal{A}_{ji} = \det\mathcal{A}_{ji} = c_{ij}$$

and

$$f_{ij} = \bigwedge_{1 \leq k \leq n} b_{ik} \vee \det\mathcal{A}_{jk} \leq b_{ii} \vee \det\mathcal{A}_{ji} = \det\mathcal{A}_{ji} = d_{ij}.$$

Similarly, $g_{ij} \geq c_{ij}$ and $h_{ij} \leq d_{ij}$. Thus we have $\mathcal{A}(\text{adj}\mathcal{A}) \succeq \text{adj}\mathcal{A}$ and $(\text{adj}\mathcal{A})\mathcal{A} \succeq \text{adj}\mathcal{A}$. But by (3), (2) and Theorem 2.1 we see that $\text{adj}\mathcal{A} = (\text{adj}\mathcal{A})(\text{adj}\mathcal{A}) \succeq \mathcal{A}\text{adj}\mathcal{A}$. So that $\mathcal{A}(\text{adj}\mathcal{A}) = \text{adj}\mathcal{A}$. Also $\text{adj}\mathcal{A} = (\text{adj}\mathcal{A})(\text{adj}\mathcal{A}) \succeq (\text{adj}\mathcal{A})\mathcal{A}$ so that $(\text{adj}\mathcal{A})\mathcal{A} = \text{adj}\mathcal{A}$. Thus we get $\mathcal{A}(\text{adj}\mathcal{A}) = (\text{adj}\mathcal{A})\mathcal{A} = \text{adj}\mathcal{A}$. □

Example 3.12 Let $\mathcal{A} = [(A, B)]$ be a 3×3 reflexive GIF matrix as follows:

$$\mathcal{A} = \begin{bmatrix} (1.0, 0.0) & (0.4., 0.5) & (0.7, 0.4) \\ (0.1, 0.9) & (1.0, 0.0) & (0.5, 0.3) \\ (0.8, 0.3) & (0.5, 0.6) & (1.0, 0.0) \end{bmatrix}.$$

Since

$$\begin{aligned} \det\mathcal{A}_{11} &= \det\begin{bmatrix} (1.0, 0.0) & (0.5, 0.3) \\ (0.5, 0.6) & (1.0, 0.0) \end{bmatrix} & \det\mathcal{A}_{12} &= \det\begin{bmatrix} (0.1, 0.9) & (0.5, 0.3) \\ (0.8, 0.3) & (1.0, 0.0) \end{bmatrix} \\ &= (1.0, 0.0), & &= (0.5, 0.3), \end{aligned}$$

$$\begin{aligned} \det\mathcal{A}_{13} &= \det\begin{bmatrix} (0.1, 0.9) & (1.0, 0.0) \\ (0.8, 0.3) & (0.5, 0.6) \end{bmatrix} & \det\mathcal{A}_{21} &= \det\begin{bmatrix} (0.4, 0.5) & (0.7, 0.4) \\ (0.5, 0.6) & (1.0, 0.0) \end{bmatrix} \\ &= (0.8, 0.3), & &= (0.5, 0.5), \end{aligned}$$

$$\begin{aligned} \det\mathcal{A}_{22} &= \det\begin{bmatrix} (1.0, 0.0) & (0.7, 0.4) \\ (0.8, 0.3) & (1.0, 0.0) \end{bmatrix} & \det\mathcal{A}_{23} &= \det\begin{bmatrix} (1.0, 0.0) & (0.4, 0.5) \\ (0.8, 0.3) & (0.5, 0.6) \end{bmatrix} \\ &= (1.0, 0.0), & &= (0.5, 0.3), \end{aligned}$$

$$\begin{aligned} \det\mathcal{A}_{31} &= \det\begin{bmatrix} (0.4, 0.5) & (0.7, 0.4) \\ (1.0, 0.0) & (0.5, 0.3) \end{bmatrix} & \det\mathcal{A}_{32} &= \det\begin{bmatrix} (1.0, 0.0) & (0.7, 0.4) \\ (0.1, 0.9) & (0.5, 0.3) \end{bmatrix} \\ &= (0.7, 0.4), & &= (0.5, 0.3), \end{aligned}$$

$$\det\mathcal{A}_{33} = \det\begin{bmatrix} (1.0, 0.0) & (0.4, 0.5) \\ (0.1, 0.9) & (1.0, 0.0) \end{bmatrix} = (1.0, 0.0),$$

we have

$$\mathrm{adj}\mathcal{A} = \begin{bmatrix} (1.0, 0.0) & (0.5, 0.5) & (0.7, 0.4) \\ (0.5, 0.3) & (1.0, 0.0) & (0.5, 0.3) \\ (0.8, 0.3) & (0.5, 0.3) & (1.0, 0.0) \end{bmatrix}.$$

Therefore, adj$\mathcal{A}$ is reflexive and adj$\mathcal{A} \succeq \mathcal{A}$.

Now,

$$\mathcal{A}(\text{adj}\mathcal{A}) = (\text{adj}\mathcal{A})\mathcal{A}$$

$$= \begin{bmatrix} (1.0, 0.0) & (0.4, 0.5) & (0.7, 0.4) \\ (0.1, 0.9) & (1.0, 0.0) & (0.5, 0.3) \\ (0.8, 0.3) & (0.5, 0.6) & (1.0, 0.0) \end{bmatrix} \cdot \begin{bmatrix} (1.0, 0.0) & (0.5, 0.5) & (0.7, 0.4) \\ (0.5, 0.3) & (1.0, 0.0) & (0.5, 0.3 \\ (0.8, 0.3) & (0.5, 0.3) & (1.0, 0.0) \end{bmatrix}$$

$$= \begin{bmatrix} (1.0, 0.0) & (0.5, 0.5) & (0.7, 0.4) \\ (0.5, 0.3) & (1.0, 0.0) & (0.5, 0.3 \\ (0.8, 0.3) & (0.5, 0.3) & (1.0, 0.0) \end{bmatrix} \cdot \begin{bmatrix} (1.0, 0.0) & (0.4, 0.5) & (0.7, 0.4) \\ (0.1, 0.9) & (1.0, 0.0) & (0.5, 0.3) \\ (0.8, 0.3) & (0.5, 0.6) & (1.0, 0.0) \end{bmatrix}$$

$$= \begin{bmatrix} (1.0, 0.0) & (0.5, 0.5) & (0.7, 0.4) \\ (0.5, 0.3) & (1.0, 0.0) & (0.5, 0.3) \\ (0.8, 0.3) & (0.5, 0.3) & (1.0, 0.0) \end{bmatrix} = \text{adj}\mathcal{A}.$$

□

Theorem 3.13 *For an $n \times n$ GIF matrix $\mathcal{A} = [(A, B)]$ we have the following:*

(1) *If $\mathcal{A}$ is symmetric, then* adj$\mathcal{A}$ *is symmetric.*

(2) *If $\mathcal{A}$ is transitive, then* adj$\mathcal{A}$ *is transitive.*

(3) *If $\mathcal{A}$ is circular, then* adj$\mathcal{A}$ *is circular.*

Proof (1) Let $\mathcal{C} = \text{adj}\mathcal{A} = [(C, D)]$. Since $\mathcal{A}$ is symmetric, we have

$$c_{ij} = \bigvee_{\sigma \in S_{n_j n_i}} \wedge_{t \in n_j} a_{t\sigma(t)} = \bigvee_{\sigma \in S_{n_i n_j}} \wedge_{t \in n_i} a_{\sigma(t)t} = c_{ji}$$

and

$$d_{ij} = \bigwedge_{\sigma \in S_{n_j n_i}} \vee_{t \in n_j} b_{t\sigma(t)} = \bigwedge_{\sigma \in S_{n_i n_j}} \vee_{t \in n_i} b_{\sigma(t)t} = d_{ji} .$$

(2) Let $\mathcal{G} = [(g_{hk}, e_{hk})] = \mathcal{A}_{ij}$. We can determine the elements of $\mathcal{G}$ in terms of the elements of A, B as follow:

$$(g_{hk}, c_{hk}) = \begin{cases} (a_{hk}, b_{hk}) & \text{if } h < i,\ k < j, \\ (a_{(h+1)k}, b_{(h+1)k}) & \text{if } h \geq i,\ k < j, \\ (a_{k(k+1)}, b_{k(k+1)}) & \text{if } h < i,\ k \geq j, \\ (a_{(h+1)(k+1)}, b_{(h+1)(k+1)}) & \text{if } h \geq i,\ k \geq j, \end{cases}$$

where $\mathcal{A}_{ij} = [(A_{ij}, B_{ij})]$ denotes the $(n-1) \times (n-1)$ GIF matrix obtained from $\mathcal{A}$ by deleting the i-th row and column j.

Now we show that $\mathcal{A}_{st}\mathcal{A}_{tu} \preceq \mathcal{A}_{su}$, for every $t \in \{1, 2, \cdots, n\}$. Let $\mathcal{R} = \mathcal{A}_{st} = [(r_{ij}, s_{ij})]$, $\mathcal{P} = \mathcal{A}_{tu} = [(p_{ij}, q_{ij})]$, $\mathcal{F} = \mathcal{A}_{su} = [(f_{ij}, l_{ij})]$ and $\mathcal{W} = \mathcal{A}_{st}\mathcal{A}_{tu} = [(w_{ij}, y_{ij})]$. Note that $\mathcal{A}$ is transitive. Then

$$w_{ij} = \bigvee_{1 \leq k \leq n-1} r_{ik} \wedge p_{kj}$$

$$= \begin{cases} \bigvee_{1 \leq k \leq n-1} (a_{ik} \wedge a_{kj}) \leq a_{ij} = f_{ij} & \text{if } i < s,\ k < t,\ j < u, \\ \bigvee_{1 \leq k \leq n-1} (a_{ik} \wedge a_{k(j+1)}) \leq a_{i(j+1)} = f_{ij} & \text{if } i < s,\ k < t,\ j \geq u, \\ \bigvee_{1 \leq k \leq n-1} (a_{i(k+1)} \wedge a_{(k+1)j}) \leq a_{ij} = f_{ij} & \text{if } i < s,\ k \geq t,\ j < u, \\ \bigvee_{1 \leq k \leq n-1} (a_{i(k+1)} \wedge a_{(k+1)(j+1)}) \leq a_{i(j+1)} = f_{ij} & \text{if } i < s,\ k \geq t,\ j \geq u, \\ \bigvee_{1 \leq k \leq n-1} (a_{(i+1)k} \wedge a_{kj}) \leq a_{(i+1)j} = f_{ij} & \text{if } i \geq s,\ k < t,\ j < u, \\ \bigvee_{1 \leq k \leq n-1} (a_{(i+1)(k+1)} \wedge a_{(k+1)j}) \leq a_{(i+1)j} = f_{ij} & \text{if } i \geq s,\ k \geq t,\ j < u, \\ \bigvee_{1 \leq k \leq n-1} (a_{(i+1)(k+1)} \wedge a_{(k+1)(j+1)}) \leq a_{(i+1)(j+1)} = f_{ij} & \text{if } i \geq s,\ k \geq t,\ j \geq u. \\ \bigvee_{1 \leq k \leq n-1} (a_{(i+1)k} \wedge a_{k(j+1)}) \leq a_{(i+1)(j+1)} = f_{ij} & \text{if } i \geq s,\ k < t,\ j \geq u. \end{cases}$$

and

$$
\begin{aligned}
y_{ij} &= \bigwedge_{1\le k\le n-1} s_{ik} \vee q_{kj} \\
&\begin{cases}
\bigwedge_{1\le k\le n-1} (b_{ik}\wedge b_{kj}) \ge b_{ij} = l_{ij} & \text{if } i<s,\ k<t,\ j<u, \\
\bigwedge_{1\le k\le n-1} (b_{ik}\wedge b_{k(j+1)}) \ge b_{i(j+1)} = l_{ij} & \text{if } i<s,\ k<t,\ j\ge u, \\
\bigwedge_{1\le k\le n-1} (b_{i(k+1)}\wedge b_{(k+1)j}) \ge b_{ij} = l_{ij} & \text{if } i<s,\ k\ge t,\ j<u, \\
\bigwedge_{1\le k\le n-1} (b_{i(k+1)}\wedge b_{(k+1)(j+1)}) \ge b_{i(j+1)} = l_{ij} & \text{if } i<s,\ k\ge t,\ j\ge u, \\
\bigwedge_{1\le k\le n-1} (b_{(i+1)k}\wedge b_{kj}) \ge b_{(i+1)j} = l_{ij} & \text{if } i\ge s,\ k<t,\ j<u, \\
\bigwedge_{1\le k\le n-1} (b_{(i+1)(k+1)}\wedge b_{(k+1)j}) \ge b_{(i+1)j} = l_{ij} & \text{if } i\ge s,\ k\ge t,\ j<u, \\
\bigwedge_{1\le k\le n-1} (b_{(i+1)(k+1)}\wedge b_{(k+1)(j+1)}) \ge b_{(i+1)(j+1)} = l_{ij} & \text{if } i\ge s,\ k\ge t,\ j\ge u, \\
\bigwedge_{1\le k\le n-1} (b_{(i+1)k}\wedge b_{k(j+1)}) \ge b_{(i+1)(j+1)} = l_{ij} & \text{if } i\ge s,\ k<t,\ j\ge u.
\end{cases}
\end{aligned}
$$

Thus $w_{ij} \le f_{ij}$, $y_{ij} \ge l_{ij}$ in every case and therefore $\mathcal{A}_{st}\mathcal{A}_{tu} \preceq \mathcal{A}_{su}$ for every $t \in \{1, 2, \cdots, n\}$. By Theorem 2.8 we get $(\det\mathcal{A}_{st})(\det\mathcal{A}_{tu}) \preceq \det(\mathcal{A}_{st}\mathcal{A}_{tu}) \preceq \det\mathcal{A}_{su}$. This means $c_{ts}c_{ut} \le c_{us}$ and $d_{ts}d_{ut} \ge d_{us}$, i.e, $c_{ut}c_{ts} \le c_{us}$ and $d_{ut}d_{ts} \ge d_{us}$ for every $t \in \{1, 2, \cdots, n\}$. Hence $\mathcal{C} = \mathrm{adj}\mathcal{A}$ is transitive.

(3) Similarly, as in (2) we can show that $\mathcal{A}_{st}\mathcal{A}_{tu} \preceq \mathcal{A}_{us}^T$ for every $t \in \{1, 2, \cdots, n\}$ so that $(\det\mathcal{A}_{st})(\det\mathcal{A}_{tu}) \preceq \det\mathcal{A}_{us}^T = \det\mathcal{A}_{us}$. Thus $c_{st}c_{tu} \le c_{us}$, $d_{st}d_{tu} \ge d_{us}$ and $\mathcal{C} = \mathrm{adj}\mathcal{A}$ is circular.

Theorem 3.14 *If $\mathcal{A} = |(A, B)|$ is an $n \times n$ GIF matrix, then GIF matrix $\mathcal{A}(\mathrm{adj}\mathcal{A})$ is transitive.*

Proof Let $\mathcal{C} = \mathcal{A}(\mathrm{adj}\mathcal{A})$, i.e.,

$$c_{ij} = \bigvee_{1\le k\le n} a_{ik} \wedge \det\mathcal{A}_{jk} = a_{if} \wedge \det\mathcal{A}_{jf}$$

and

$$d_{pq} = \bigwedge_{1\le k\le n} b_{pk} \vee \det\mathcal{A}_{qk} = b_{pf} \vee \det\mathcal{A}_{qf}$$

for some $f \in \{1, 2, \cdots, n\}$, and

$$\begin{aligned} c_{ij}^{(2)} &= \bigvee_{1\le s\le n} c_{is} \wedge c_{sj} = \bigvee_{1\le s\le n} \left[\left(\bigvee_{1\le l\le n} a_{il} \wedge \det\mathcal{A}_{sl} \right) \wedge \left(\bigvee_{1\le t\le n} a_{st} \wedge \det\mathcal{A}_{jt} \right) \right] \\ &= \bigvee_{1\le s\le n} a_{ih} \wedge \det\mathcal{A}_{sh} \wedge a_{su} \wedge \det\mathcal{A}_{ju} \le a_{ih} \wedge \det\mathcal{A}_{ju} \\ &\le a_{if} \wedge \det\mathcal{A}_{if} = c_{ij} \end{aligned}$$

and

$$\begin{aligned} d_{pq}^{(2)} &= \bigwedge_{1\le s\le n} d_{ps} \vee d_{sq} = \bigwedge_{1\le s\le n} \left[\left(\bigwedge_{1\le l\le n} b_{pl} \vee \det\mathcal{A}_{sl} \right) \vee \left(\bigwedge_{1\le t\le n} b_{st} \vee \det\mathcal{A}_{qt} \right) \right] \\ &= \bigwedge_{1\le s\le n} b_{ph} \vee \det\mathcal{A}_{sh} \vee b_{su} \vee \det\mathcal{A}_{ju} \ge b_{ih} \vee \det\mathcal{A}_{ju} \\ &\ge b_{pf} \vee \det\mathcal{A}_{qf} = d_{ij} \end{aligned}$$

for some $h, u \in \{1, 2, \cdots, n\}$. Thus $(\mathcal{A}(\text{adj}\mathcal{A}))^2 \preceq \mathcal{A}(\text{adj}\mathcal{A})$. □

Example 3.15 Let $\mathcal{A} = [(A, B)]$ be a 3×3 GIF matrix as follows:

$$\mathcal{A} = \begin{bmatrix} (0.5, 0.5) & (0.7, 0.4) & (0.8, 0.3) \\ (0.3, 0.8) & (0.6, 0.2) & (0.4, 0.7) \\ (0.9, 0.1) & (0.2, 0.9) & (1.0, 0.0) \end{bmatrix}.$$

Then we have

$$(\mathcal{A}(\text{adj}\mathcal{A}))^2 = \begin{bmatrix} (0.6, 0.3) & (0.7, 0.4) & (0.5, 0.5) \\ (0.4, 0.7) & (0.6, 0.3) & (0.4, 0.7) \\ (0.6, 0.2) & (0.7, 0.4) & (0.6, 0.3) \end{bmatrix} \begin{bmatrix} (0.6, 0.3) & (0.7, 0.4) & (0.5, 0.5) \\ (0.4, 0.7) & (0.6, 0.3) & (0.4, 0.7) \\ (0.6, 0.2) & (0.7, 0.4) & (0.6, 0.3) \end{bmatrix}$$

$$= \begin{bmatrix} (0.6, 0.3) & (0.6, 0.4) & (0.5, 0.5) \\ (0.4, 0.7) & (0.6, 0.3) & (0.4, 0.7) \\ (0.6, 0.3) & (0.6, 0.4) & (0.6, 0.3) \end{bmatrix} \preceq \mathcal{A}(\text{adj}\mathcal{A}).$$

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