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Kaehler submanifolds with totally real bisectional curvature tensor

by

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Graduate School of Education
Pukyong National University

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전실양축단면곡률을 갖는 Kaehler 부분다양체

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전실양측단면곡률을 갖는 Kaehler 부분다양체

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요 약

본 논문에서는 전실양측단면곡률이 아래로 유계인 완비 Kaehler 부분다양체에 대해 연구하여, 다음 정리를 증명하였다.

정리 1. 양의 상수 c를 정칙단면곡률로 갖는 (n+p)-차원 복소공간형의 n(≥ 3)-차원 완비 Kaehler 부분다양체 M의 모든 전실양측단면곡률이 $\frac{c}{4(n^2-1)}n(2n-1)$ 이상이면, M은 전측지적 부분다양체이다.

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KAEHLER SUBMANIFOLDS WITH TOTALLY REAL BISECTIONAL CURVATURE TENSOR

1. Introduction

The theory of Kaehler submanifolds is one of fruitful fields in Riemannian geometry and we have many studies [1], [2] and [9] etc. For the curvatures of a Kaehler manifold M, we can consider two kinds of sectional curvature which are related to almost complex structure J and different from usual sectional curvatures, holomorphic sectional curvatures and totally real bisectional curvatures. The pinching problem for these three kinds, the sectional curvature, the holomorphic sectional curvature and the totally real bisectional curvature, is an interesting topic in Kaehler geometry.

For a complex submanifold $M = M^n$ of a complex space form $M' = M^{n+p}(c)$, the set $B(M)$ of the totally real bisectional curvatures satisfies $B(M) \leqq \frac{c}{2}$ by the Gauss equation. It is easily seen that a totally geodesic complex submanifold $M = M^n(c)$ of $M' = M^{n+p}(c)$ satisfies $B(M) = \frac{c}{2}$ again by the Gauss equation. On the other hand, a complex quadric $M = Q^n$ of $M' = M^{n+p}(c)$, $c > 0$, satisfies $0 \leqq B(M) \leqq \frac{c}{2}$ by Kobayashi and Nomizu [7]. By paying attention to this fact, the following theorem by Ros for holomorphic sectional curvatures.

Theorem 1.1([10]). *Let $M = M^n$ be an n-dimensional complete Kaehler submanifold of an $(n+p)$-dimensional complex space form $M' = M^{n+p}(c)$ of constant holomorphic sectional curvature $c(>0)$. If every holomorphic sectional curvature of M is greater than $\frac{c}{2}$, then M is totally geodesic.*

Ogiue gave also the following theorem.

Theorem 1.2([8]). *Let $M = M^n$ be an n-dimensional complete Kaehler submanifold of an $(n+p)$-dimensional complex space form $M' = M^{n+p}(c)$ of constant holomorphic sectional curvature $c(>0)$. If every Ricci curvature of M is greater than $\frac{c}{2}n$, then M is totally geodesic.*

The purpose of this paper is to consider the similar problem for totally real bisectional curvatures. In this paper, we proved the following two results.

Theorem 4.4. *Let $M = M^n$ be an $n(\geqq 3)$-dimensional complete Kaehler submanifold of an $(n+p)$-dimensional complex space form $M' = M^{n+p}(c)$ of constant holomorphic sectional curvature $c(>0)$. If every totally real bisectional curvature of M is greater than $\frac{c}{4(n^2-1)}n(2n-1)$, then M is totally geodesic.*

Corollary 4.5. *Let $M = M^n$ be an $n(\geqq 3)$-dimensional complete Kaehler submanifold of an $(n+p)$-dimensional complex space form $M' = M^{n+p}(c)$ of constant holomorphic sectional curvature $c(> 0)$. If every sectional curvature of M is greater than $\frac{c}{8(n^2-1)}n(2n-1)$, then M is totally geodesic.*

2. Kaehler manifolds

This section is concerned with recalling basic formulas on Kaehler manifolds. Let M be a complex $n(\geqq 2)$-dimensional Kaehler manifold equipped with Kaehler metric tensor g and almost complex structure J. We can choose a local field $\{E_j, E_{j^*}\} = \{E_1, \cdots, E_n, E_{1^*}, \cdots, E_{n^*}\}$ of orthonormal frames on a neighborhood of M, where $E_{j^*} = JE_j$ and $j^* = n+j$. Here and in the sequel, the Latin small indices $j, k, \cdots$ run from 1 to n. We set $U_j = \frac{1}{\sqrt{2}}(E_j - iE_{j^*})$ and $\overline{U}_j = \frac{1}{\sqrt{2}}(E_j + iE_{j^*})$, where i denotes the imaginary unit. Then $\{U_j\}$ constitutes a local field of unitary frames on the neighborhood of M. With respect to the Kaehler metric, we have $g(U_j, \overline{U}_k) = \delta_{jk}$.

Now let $\{\omega_j\}$ be the canonical form with respect to the local field $\{U_j\}$ of unitary frames on the neighborhood of M. Then $\{\omega_j\} = \{\omega_1, \cdots, \omega_n\}$ consists of complex valued 1-forms of type (1,0) on M such that $\omega_j(U_k) = \delta_{jk}$ and $\omega_1, \cdots, \omega_n, \bar{\omega}_1, \cdots, \bar{\omega}_n$ are linearly independent. The Kaehler metric g of M can be expressed as $g = 2\sum_j \omega_j \otimes \bar{\omega}_j$. Associated with the frame field $\{U_j\}$, there exist complex-valued 1-forms ω_{jk}, which are usually called *complex connection forms* on M such that they satisfy the structure equations of M

$$(2.1)\qquad \begin{aligned} &d\omega_i + \sum_k \omega_{ik} \wedge \omega_k = 0, \qquad \omega_{ij} + \bar{\omega}_{ji} = 0, \\ &d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \\ &\Omega_{ij} = \sum_k K_{\bar{i}jk\bar{l}}\, \omega_k \wedge \bar{\omega}_l, \end{aligned}$$

where Ω_{ij} (resp. $K_{\bar{i}jk\bar{l}}$) the curvature form (resp. the components of the Riemannian curvature tensor R) of M. From the structure equations, the components of the curvature tensor satisfy

$$(2.2)\qquad K_{\bar{i}jk\bar{l}} = \bar{K}_{\bar{j}il\bar{k}},$$

$$(2.3)\qquad K_{\bar{i}jk\bar{l}} = K_{\bar{i}kj\bar{l}} = K_{\bar{l}jk\bar{i}} = K_{\bar{l}kj\bar{i}}.$$

Next, relative to the frame field chosen above, the Ricci tensor S of M can be expressed as follows :

$$(2.4)\qquad S = \sum_{i,j} (S_{i\bar{j}}\omega_i \otimes \bar{\omega}_j + S_{\bar{i}j}\bar{\omega}_i \otimes \omega_j),$$

where $S_{i\bar{j}} = \sum_k K_{\bar{k}ki\bar{j}} = S_{\bar{j}i} = \overline{S}_{\bar{i}j}$. The scalar curvature r of M is also given by

$$r = 2\sum_j S_{j\bar{j}}. \tag{2.5}$$

An n-dimensional Kaehler manifold M is said to be *Einstein* if the Ricci tensor S satisfies the condition

$$S_{i\bar{j}} = \frac{r}{2n}\delta_{ij}. \tag{2.6}$$

The components $K_{\bar{i}jk\bar{l}m}$ and $K_{\bar{i}jk\bar{l}\bar{m}}$ (resp. $S_{i\bar{j}k}$ and $S_{i\bar{j}\bar{k}}$) of the covariant derivative of the Riemannian curvature tensor R (resp. the Ricci tensor S) are given by

$$\begin{aligned} \sum_m (K_{\bar{i}jk\bar{l}m}\omega_m + K_{\bar{i}jk\bar{l}\bar{m}}\bar{\omega}_m) &= dK_{\bar{i}jk\bar{l}} \\ &\quad - \sum_m (K_{\bar{m}jk\bar{l}}\bar{\omega}_{mi} + K_{\bar{i}mk\bar{l}}\omega_{mj} + K_{\bar{i}jm\bar{l}}\omega_{mk} + K_{\bar{i}jk\bar{m}}\bar{\omega}_{ml}), \end{aligned} \tag{2.7}$$

$$\sum_k (S_{i\bar{j}k}\omega_k + S_{i\bar{j}\bar{k}}\bar{\omega}_k) = dS_{i\bar{j}} - \sum_k (S_{k\bar{j}}\omega_{ki} + S_{i\bar{k}}\bar{\omega}_{kj}). \tag{2.8}$$

The second Bianchi identity is given as follows :

$$K_{\bar{i}jk\bar{l}m} = K_{\bar{i}jm\bar{l}k}. \tag{2.9}$$

And hence we have

$$S_{i\bar{j}k} = S_{k\bar{j}i} = \sum_m K_{\bar{j}ik\bar{m}m}. \tag{2.10}$$

Lastly, a Kaehler manifold of constant holomorphic sectional curvature is called a *complex space form*. The components $K_{\bar{i}jk\bar{l}}$ of the Riemannian curvature tensor R of an n-dimensional complex space form of constant holomorphic sectional curvature c is given by

$$K_{\bar{i}jk\bar{l}} = \frac{c}{2}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}). \tag{2.11}$$

3. Complex submanifolds

This section is recalled complex submanifolds of a Kaehler manifold. First of all, the basic formulas for the theory of complex submanifolds are prepared.

Let $M' = M^{n+p}$ be an $(n+p)$-dimensional Kaehler manifold with Kaehler structure (g', J'). Let M be an n-dimensional complex submanifold of M' and let g be the induced Kaehler metric tensor on M from g'. We can choose a local field $\{U_A\} = \{U_i, U_x\} = \{U_1, \cdots, U_{n+p}\}$ of unitary frames on a neighborhood of M' in such a way that, restricted to M, $U_1, \cdots, U_n$ are tangent to M and the others are normal to M. Here and in the sequel, the following convention on the range of indices is used throughout this paper, unless otherwise stated :

$$
\begin{aligned}
A, B, \cdots &= 1, \cdots, n, n+1, \cdots, n+p, \\
i, j, \cdots &= 1, \cdots, n, \\
x, y, \cdots &= n+1, \cdots, n+p.
\end{aligned}
$$

With respect to the frame field, let $\{\omega_A\} = \{\omega_i, \omega_x\}$ be its dual frame fields. Then the Kaehler metric tensor g' of M' is given $g' = 2\sum_A \omega_A \otimes \bar{\omega}_{A.}$ The canonical forms $\omega_{A,}$ the connection forms ω_{AB} of the ambient space M' satisfy the structure equations

$$
\begin{aligned}
& d\omega_A + \sum \epsilon_C \omega_{AC} \wedge \omega_C = 0, \qquad \omega_{AB} + \bar{\omega}_{BA} = 0, \\
(3.1) \qquad & d\omega_{AB} + \sum_C \omega_{AC} \wedge \omega_{CB} = \Omega'_{AB,} \\
& \Omega'_{AB} = \sum_{C,D} K'_{\overline{A}BC\overline{D}} \omega_C \wedge \bar{\omega}_{D,}
\end{aligned}
$$

where Ω'_{AB} (resp. $K'_{\overline{A}BC\overline{D}}$) denotes the curvature form (resp. the components of the Riemannian curvature tensor R') of M'.

Restricting these forms to the submanifold M, we have

$$
(3.2) \qquad \omega_x = 0,
$$

and the induced Kaehler metric tensor g of M is given by $g = 2\sum_j \omega_j \otimes \bar{\omega}_{j.}$ Then $\{U_j\}$ is a local unitary frame field with respect to the induced metric and $\{\omega_j\}$ is a local dual frame filed due to $\{U_j\}$, which consists of complex-valued 1-forms of type (1,0) on M. Moreover, $\omega_1, \cdots, \omega_n, \bar\omega_1, \cdots, \bar\omega_n$ are linearly independent, and $\{\omega_j\}$ is the canonical forms on M. It follows from (3.2) and Cartan's lemma that the exterior derivatives of (3.2) give rise to

$$\omega_{xi} = \sum_j h^x_{ij}\omega_j, \qquad h^x_{ij} = h^x_{ji}. \tag{3.3}$$

The quadratic form $\alpha = \sum_{i,j,x} h^x_{ij}\omega_i \otimes \omega_j \otimes U_x$ with values in the normal bundle on M in M' is called the *second fundamental form* of the submanifold M. From the structure equations for M', it follows that the structure equations for M are similarly given by

$$\begin{aligned} &d\omega_i + \sum_k \omega_{ik} \wedge \omega_k = 0, \qquad \omega_{ij} + \bar\omega_{ji} = 0,\\ &d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_k = \Omega_{ij},\\ &\Omega_{ij} = \sum_{k,l} K_{\bar i j k \bar l}\omega_k \wedge \bar\omega_l. \end{aligned} \tag{3.4}$$

For the Riemannian curvature tensors R and R' of M and M', respectively, it follows from (3.1), (3.3) and (3.4) that

$$K_{\bar i j k \bar l} = K'_{\bar i j k \bar l} - \sum_x h^x_{jk}\bar h^x_{il}. \tag{3.5}$$

The components $S_{i\bar j}$ of the Ricci tensor S and the scalar curvature r on M are given by

$$S_{i\bar j} = \sum_k K'_{\bar k k i \bar j} - {h_{i\bar j}}^2, \tag{3.6}$$

$$r = 2(\sum_{j,k} K'_{\bar k k j \bar j} - h_2), \tag{3.7}$$

where ${h_{i\bar j}}^2 = {h_{\bar j i}}^2 = \sum_{m,x} h^x_{im}\bar h^x_{mj}$ and $h_2 = \sum_j {h_{j\bar j}}^2$.

Now the components h^x_{ijk} and $h^x_{ij\bar k}$ of the covariant derivative of the second fundamental form on M are given by

$$\begin{aligned} &\sum_k (h^x_{ijk}\omega_k + h^x_{ij\bar k}\bar\omega_k)\\ &\quad = dh^x_{ij} - \sum_k (h^x_{jk}\omega_{ki} + h^x_{ik}\omega_{kj}) + \sum_y h^y_{ij}\omega_{xy}. \end{aligned} \tag{3.8}$$

Then, substituting dh^x_{ij} in this definition into the exterior derivative

$$d\omega_{xi} = \sum_j (dh^x_{ij} \wedge \omega_j + h^x_{ij} d\omega_j)$$

of (3.3) and using (3.1) $\sim$ (3.4) and (3.6), we have

$$h^x_{ijk} = h^x_{ikj}, \qquad h^x_{ij\bar{k}} = -K'_{\bar{x}ij\bar{k}}. \tag{3.9}$$

In particular, let the ambient space $M' = M^{n+p}(c)$ be an $(n+p)$-dimensional complex space form of constant holomorphic sectional curvature c. Then, by (2.11) and (3.5) $\sim$ (3.7), we get

$$K_{\bar{i}jk\bar{l}} = \frac{c}{2}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) - \sum_x h^x_{jk}\bar{h}^x_{il}, \tag{3.10}$$

$$S_{i\bar{j}} = \frac{c}{2}(n+1)\delta_{ij} - {h_{i\bar{j}}}^2, \tag{3.11}$$

$$r = cn(n+1) - 2h_2, \tag{3.12}$$

$$h^x_{ij\bar{k}} = 0. \tag{3.13}$$

4. Totally real bisectional curvatures

In this section, we are concerned with the totally real bisectional curvature of a Kaehler manifold.

Let M be an n-dimensional Kaehler manifold. A plane section P of the tangent space T_xM of M at any point x is said to be *non-degenerate*, provided that the restriction of $g_x|_{T_xM}$ to P is non-degenerate. It is easily seen that P is non-degenerate if and only if it has a basis $\{X, Y\}$ such that $g(X,X)g(Y,Y) - g(X,Y)^2 \neq 0$.

If the non-degenerate plane P is invariant by the complex structure J, then it said to be *holomorphic*. For the non-degenerate plane P spanned by X and Y in P, the *sectional curvature* $K(P)$ of P is usually defined by

$$K(P) = K(X,Y) = \frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2}.$$

The sectional curvature $K(P)$ of the non-degenerate holomorphic plane P is called the *holomorphic sectional curvature,* which is denoted by $H(P)$. The Kaehler manifold M is said to be *of constant holomorphic sectional curvature* if its holomorphic sectional curvature $H(P)$ is constant for any non-degenerate holomorphic plane P and any point on M. Then M is called a *complex space form*, which is denoted by $M^n(c)$ provided that it is of constant holomorphic sectional curvature c and of complex dimension n. It is seen by Wolf [11] that the standard models of complex space forms are the following three kinds : the complex projective space $CP^n(c)$, the complex Euclidean space C^n or the complex hyperbolic space $CH^n(c)$, according as $c > 0$, $c = 0$ or $c < 0$.

Let (M, g) be an n-dimensional Kaehler manifold with almost complex structure J. In their paper [4], Bishop and Goldberg introduced the notion for totally real bisectional curvature $B(X, Y)$ on a Kaehler manifold.

A plane section P in the tangent space T_pM at any point p in M is said to be *totally real* if P is orthogonal to JP. For an orthonormal basis $\{X, Y\}$ of the totally real plane section P, any vectors X, JX, Y and JY are mutually orthogonal. It implies that for orthogonal vectors X and Y in P, it is totally real if and only if two holomorphic plane sections spanned by X, JX and Y, JY are orthogonal.

Houh [6] showed that an $n(\geqq 3)$-dimensional Kaehler manifold has constant totally real bisectional curvature c if and only if it has constant holomorphic sectional curvature $2c$. On the other hand, Goldberg and Kobayashi [5] introduced the notion of holomorphic bisectional curvature $H(X, Y)$ which is determined by two holomorphic planes $\mathrm{Span}\{X, JX\}$ and $\mathrm{Span}\{Y, JY\}$, and asserted that the complex projective space $CP^n(c)$ is the only compact Kaehler manifold with positive holomorphic bisectional curvature and constant scalar curvature. If we compare the notion of $B(X, Y)$ with the holomorphic bisectional curvature $H(X, Y)$ and the holomorphic sectional curvature $H(X)$, then the holomorphic bisectional curvature $H(X, Y)$ turns out to be totally real bisectional curvature $B(X, Y)$ (resp. holomorphic sectional curvature $H(X)$), when two holomorphic planes $\mathrm{Span}\{X, JX\}$ and $\mathrm{Span}\{Y, JY\}$ are orthogonal to each other (resp. coincides with each other). From this, it follows that the positiveness of $B(X, Y)$ is weaker than the positiveness of $H(X, Y)$, because $H(X, Y) > 0$ implies that both of $B(X, Y)$ and $H(X)$ are positive but we do know whether or not $B(X, Y) > 0$ implies $H(X, Y) > 0$.

Furthermore, Goldberg and Kobayashi [5] showed that a complete Kaehler manifold M with constant scalar curvature and positive holomorphic bisectional curvature is Einstein. In order to get this result, they should have verified that the Ricci tensor is positive definite. In that proof, they used that the fact that the holomorphic sectional curvature $H(X)$ is positive, which necessarily from the condition $H(X, Y) > 0$. But the condition $B(X, Y) > 0$ curries less information than the condition of $H(X, Y) > 0$, and it gives us no meanings to use Goldberg and Kobayashi's method to derive the fact that M is Einstein. That is, we can not use the condition $H(X, Y) > 0$. The totally real bisectional curvature $B(X, Y)$ can be also consider for non-degenerate totally real planes $\mathrm{Span}\{X, Y\}$ in any Kaehler manifold. In their paper [3], Barros and Romero asserted that above mentioned Houh's result can be extended to indefinite Kaehler manifolds. Aiyama, Kwon and Nakagawa [1] have also studied the classification problem of space-like complex submanifolds of indefinite complex hyperbolic space $CH^{n+p}_{0+p}(c)$ with bounded scalar curvature.

By being motivated by these results, the classification problems with bounded totally real bisectional curvature are presented. The problems are here introduced.

Let (M, g) be an n-dimensional Kaehler manifold with almost complex structure J. In the sequel, we only consider the definite totally real planes, unless otherwise stated.

Definition 4.1. For a totally real plane section P spanned by orthonormal vectors X and Y, the *totally real bisectional curvature* $B(X,Y)$ is defined by

$$B(X,Y) = g(R(X,JX)JY,Y). \tag{4.1}$$

Then, using the first Bianchi identity to (4.1) and the fundamental properties of the Riemannian curvature tensor of Kaehler manifolds, we get

$$\begin{aligned} B(X,Y) &= g(R(X,Y)Y,X) + g(R(X,JY)JY,X) \\ &= K(X,Y) + K(X,JY), \end{aligned} \tag{4.2}$$

where $K(X,Y)$ means the sectional curvature of the plane spanned by X and Y.

Example 4.2. Let $M^n(c)$ be an n-dimensional complex space form of constant holomorphic sectional curvature c. Then, $M^n(c)$ has constant totally real bisectional curvature $\frac{c}{2}$. In fact, if a plane Span$\{X,Y\}$ is totally real, then we have

$$B(X,Y) = \frac{g(R(X,JX)JY,Y)}{g(X,X)g(Y,Y)} = \frac{c}{2},$$

which follows easily from the form of the curvature tensor of $M^n(c)$.

Example 4.3. Let Q^n be a complex quadric in a complex projective space $CP^{n+1}(c)$ of constant holomorphic sectional curvature c. In $CP^{n+1}(c)$ with homogeneous coordinates $z^0, z^1, \cdots, z^{n+1}$, the complex quadric Q^n is complex hypersurface defined by the equation

$$(z^0)^2 + (z^1)^2 + \cdots + (z^{n+1})^2 = 0.$$

Let g be the Fubini-Study metric on $CP^{n+1}(c)$ of constant holomorphic sectional curvature c. Its restriction g to Q^n is a Kaehler metric. Then, it is seen in Kobayashi and Nomizu [7] that Q^n is an Einstein hypersurface whose Ricci tensor S satisfies

$$S = \frac{c}{2}ng,$$

and its totally real bisectional curvature $B(Q^n)$ satisfies

$$0 \leqq B(Q^n) \leqq \frac{c}{2}.$$

In the rest of this section, we suppose that X and Y are orthonormal vectors in a non-degenerate totally real plane section such that $g(X,X) = g(Y,Y) = \pm 1$. If we put $X' = \frac{1}{\sqrt{2}}(X+Y)$ and $Y' = \frac{1}{\sqrt{2}}(X-Y)$, then it is easily seen that

$$g(X',X') = g(Y',Y') = \pm 1, \quad g(X',Y') = 0.$$

Thus we get

$$\begin{aligned} B(X',Y') &= g(R(X',JX')JY',Y') \\ &= \frac{1}{4}\{H(X) + H(Y) + 2B(X,Y) - 4K(X,JY)\}, \end{aligned}$$

where $H(X) = K(X,JX)$ means the holomorphic sectional curvature of the holomorphic plane spanned by X and JX. Hence we have

$$4B(X',Y') - 2B(X,Y) = H(X) + H(Y) - 4K(X,JY). \tag{4.3}$$

If we put $X'' = \frac{1}{\sqrt{2}}(X + JY)$ and $Y'' = \frac{1}{\sqrt{2}}(JX + Y)$, then we get by the definiteness of the plane Span$\{X,Y\}$

$$g(X'',X'') = g(Y'',Y'') = \pm 1, \quad g(X'',Y'') = 0.$$

Using the similar method as in (4.3), we have

$$4B(X'',Y'') - 2B(X,Y) = H(X) + H(Y) - 4K(X,Y). \tag{4.4}$$

Summing up (4.3) and (4.4) and taking account of (4.2), we obtain

$$2B(X',Y') + 2B(X'',Y'') = H(X) + H(Y). \tag{4.5}$$

Now let $M = M^n$ be an $n(\geqq 3)$-dimensional complex submanifold of an $(n+p)$-dimensional Kaehler manifold $M' = M^{n+p}(c)$ of constant holomorphic sectional curvature c.

Assume that the totally real bisectional curvatures on M is bounded from below (resp. above) by a constant a (resp. b), and let $a(M)$ and $b(M)$ be the infimum and the supremum of the set $B(M)$ of the totally real bisectional curvatures on M, respectively. By definition, we see $a \leqq a(M)$ (resp. $b \geqq b(M)$). From (4.5), we have

$$H(X) + H(Y) \geqq 4a \text{ (resp. } \leqq 4b). \tag{4.6}$$

For an orthonormal frame field $\{E_1, \cdots, E_n, E_1^*, \cdots, E_n^*\}$ on a neighborhood of M, the holomorphic sectional curvature $H(E_j)$ of the holomorphic plane spanned by E_j can be expressed as

$$H(E_j) = g(R(E_j, JE_j)JE_j, E_j) = R_{jj^*j^*j} = K_{\bar{j}j j\bar{j}}. \tag{4.7}$$

On the other hand, it is easily seen that the plane sections $\mathrm{Span}\{E_j, JE_j\}$, and $\mathrm{Span}\{E_k, JE_k\}$, $j \neq k$, are orthogonal and the totally real bisectional curvature $B(E_j, E_k)$ is given by

$$B(E_j, E_k) = g(R(E_j, JE_j)JE_k, E_k) = K_{\bar{j}jk\bar{k}}, \quad j \neq k. \tag{4.8}$$

From the inequality (4.6) for $X = E_j$ and $Y = E_k$, we have

$$K_{\bar{j}jj\bar{j}} + K_{\bar{k}kk\bar{k}} \geqq 4a \text{ (resp. } \leqq 4b), \quad j \neq k. \tag{4.9}$$

Thus we have

$$\sum_{j<k}(K_{\bar{j}jj\bar{j}} + K_{\bar{k}kk\bar{k}}) \geqq 2an(n-1) \text{ (resp. } \leqq 2bn(n-1)), \tag{4.10}$$

which implies that

$$\sum_j K_{\bar{j}jj\bar{j}} \geqq 2an \text{ (resp. } \leqq 2bn), \tag{4.11}$$

where the equality holds if and only if $K_{\bar{j}jj\bar{j}} = 2a$ (resp. $= 2b$) for any index j.

Since the scalar curvature r is given by

$$r = 2\sum_{j,k} K_{\bar{j}jk\bar{k}} = 2(\sum_j K_{\bar{j}jj\bar{j}} + \sum_{j\neq k} K_{\bar{j}jk\bar{k}}),$$

we have by (4.10)

$$r \geqq 2\sum_j K_{\bar{j}jj\bar{j}} + 2an(n-1) \text{ (resp. } \leqq 2\sum_j K_{\bar{j}jj\bar{j}} + 2bn(n-1)),$$

from which it follows that

$$\sum_j K_{\bar{j}j j\bar{j}} \leqq \frac{r}{2} - an(n-1) \text{ (resp. } \geqq \frac{r}{2} - bn(n-1)), \tag{4.12}$$

where the equality holds if and only if $K_{\bar{j}jk\bar{k}} = a$ (resp. $= b$) for any distinct indices j and k. In this case, M is locally congruent to $M^n(a)$ (resp. $M^n(b)$) due to Houh [6]. Also (4.9) gives us

$$\sum_{k(\neq j)} (K_{\bar{j}jj\bar{j}} + K_{\bar{k}kk\bar{k}}) \geqq 4a(n-1) \text{ (resp. } \leqq 4b(n-1))$$

for each j, so that

$$(n-2)K_{\bar{j}jj\bar{j}} + \sum_k K_{\bar{k}kk\bar{k}} \geqq 4a(n-1) \text{ (resp. } \leqq 4b(n-1)).$$

From this inequality together with (4.12), it follows that

$$\begin{aligned} (n-2)K_{\bar{j}jj\bar{j}} &\geqq a(n-1)(n+4) - \frac{r}{2} \\ \text{(resp. } &\leqq b(n-1)(n+4) - \frac{r}{2}) \end{aligned} \tag{4.13}$$

for any index j, so that the holomorphic sectional curvature $K_{\bar{j}jj\bar{j}}$ is bounded from below (resp. above) for $n \geqq 3$. Moreover, the equality holds for some index j if and only if M is locally congruent to $M^n(2a)$ (resp. $M^n(2b)$).

By applying Theorem 1.1, the following theorem is proved.

Theorem 4.4. *Let $M = M^n$ be an $n(\geqq 3)$-dimensional complete Kaehler submanifold of an $(n+p)$-dimensional complex space form $M' = M^{n+p}(c)$ of constant holomorphic sectional curvature $c(> 0)$. If every totally real bisectional curvature of M is greater than $\frac{c}{4(n^2-1)}n(2n-1)$, then M is totally geodesic.*

Proof. By the assumption $B(M) \geqq a$ and (4.13), we have

$$(n-2)H(M) \geqq a(n-1)(n+4) - \frac{r}{2},$$

where $H(M)$ is the set of the holomorphic sectional curvatures of M.

Since we see $r = cn(n+1) - 2h_2$ by (3.12), we obtain

$$H(M) \geqq \frac{1}{2(n-2)}\{2a(n-1)(n+4) - cn(n+1)\} \equiv a_0.$$

Thus we have by (3.10)

$$K_{\bar{j}j j\bar{j}} = c - \sum_x h^x_{jj}\bar{h}^x_{jj} \geqq a_0, \qquad K_{\bar{i}ij\bar{j}} = \frac{c}{2} - \sum_x h^x_{ij}\bar{h}^x_{ij} \geqq a \tag{4.14}$$

for any distinct indices i and j. Since the Ricci curvature $S_{j\bar{j}}$ of M is given by (3.11)

$$S_{j\bar{j}} = \frac{c}{2}(n+1) - \lambda_j, \qquad \lambda_j = \sum_{m,x} h^x_{jm}\bar{h}^x_{jm}$$

and

$$\lambda_j = \sum_x h^x_{jj}\bar{h}^x_{jj} + \sum_{m(\neq j),x} h^x_{jm}\bar{h}^x_{jm} \leqq (c - a_0) + (\frac{c}{2} - a)(n-1)$$

by (4.14), from which together with the Ricci curvatures it follows that

$$S_{j\bar{j}} \geqq a_0 + a(n-1).$$

Given constants a and a_0, we obtain

$$S_{j\bar{j}} > \frac{c}{2}n$$

for any index j. By Theorem B, it completes the proof. □

Remark. We should here remark that $\frac{c}{4(n^2-1)}n(2n-1) < \frac{c}{2}$ for $n \geqq 3$ and $c > 0$.

As a direct consequence of Theorem 4.4 combined with the equation (4.2), we can prove

Corollary 4.5. *Let $M = M^n$ be an $n(\geqq 3)$-dimensional complete Kaehler submanifold of an $(n+p)$-dimensional complex space form $M' = M^{n+p}(c)$ of constant holomorphic sectional curvature $c(> 0)$. If every sectional curvature of M is greater than $\frac{c}{8(n^2-1)}n(2n-1)$, then M is totally geodesic.*

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