Mann and Ishikawa with Errors for Mappings of Strongly Asymptotically Nonexpansive Type in Banach Spaces

Banach 공간에서의 강접근적 비확대 사상에 대한 오류항을 찾는 Mann과 Ishikawa형 구조의 수렴성

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by

[illegible]

A thesis submitted in partial fulfillment
of the requirement for the degree of

Master of Education

Graduate School of Education
Pukyong National University

August 2003

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Mappings of Strongly Asymptotically Nonexpansive Type in Banach Spaces

A Dissertation

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June 2003

CONTENTS

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요 약

집합 C가 고른볼록(uniformly convex) 성질을 갖는 Banach 공간 X의 공집합이 아닌 볼록인 닫힌집합이라 하자. 그리고 $S, T : C \to C$인 한 쌍의 (S, T)에 대하여 $x_1 \in C$으로부터 생성된 다음과 같은 수정된 Ishikawa형 반복구조를 생각하자.

$$x_{n+1} = \alpha_n x_n + \beta_n T^n [\, \alpha_n' S^n x_n + \beta_n' T^n x_n + \gamma_n' v_n \,] + \gamma_n u_n \quad (n \geq 1) \qquad (*)$$

여기서, 수열 $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha_n'\}$, $\{\beta_n'\}$, $\{\gamma_n'\}$은 모두 구간 $[0,1]$내에 있고, $\{u_n\}$, $\{v_n\}$은 C내에 있는 유계수열로 다음 두 조건을 만족한다고 하자.

(i) $\alpha_n + \beta_n + \gamma_n = 1 = \alpha_n' + \beta_n' + \gamma_n' \quad (n \geq 1)$

(ii) $\sum \gamma_n < \infty, \quad \sum \gamma_n' < \infty$

본 논문에서는 $S, T : C \to C$가 완전연속(completely continuous)인 강점근적비확대형사상(mappings of strongly asymptotically nonexpansive type)이라 할 때, (i)과 (ii)의 매개변수 조건하에서 (*)와 같이 생성된 오류항을 갖는 Ishikawa 형 반복수열 $\{x_n\}$가 S와 T의 어떤 공통부동점(common fixed point)에 강수렴한다는 사실을 밝혔다.

특히, $S = I$인 경우에는 위의 결과는 최근에 연구된 Kim-Kim의 결과를 낳는다.

1. Introduction

Let C be a nonempty closed convex subset of a Banach space X and let T be a mapping of C into itself. Then T is said to be *asymptotically nonexpansive* [3] if there exists a sequence $\{L_n(T)\}$ of real numbers with $\lim_{n\to\infty} L_n(T) = 1$ such that

$$\|T^n x - T^n y\| \le L_n(T)\|x - y\|$$

for $x, y \in C$ and $n = 1, 2, \cdots$. In particular, if $L_n(T) = 1$ for all $n \ge 1$, T is said to be *nonexpansive.* The weaker definition (cf., Kirk [9]) requires that

$$\limsup_{n\to\infty} \sup_{y\in C}(\|T^n x - T^n y\| - \|x - y\|) \le 0$$

for each $x \in C$, and that T^N be continuous for some $N \ge 1$. Consider a definition somewhere between these two: T is said to be *asymptotically nonexpansive in the intermediate sense* [1] provided T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \le 0.$$

Recently, for a mappings T of C into itself, Huang [6] considered the following Ishikawa iteration process with errors (cf., Liu [12]) $\{x_n\}$ in C defined by

$$\begin{aligned} & x_1 \in C, \\ (1) \qquad & x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n + u_n, \quad n \in \mathbb{N} \\ & y_n = (1 - \beta_n)x_n + \beta_n T^n x_n + v_n, \quad n \in \mathbb{N}, \end{aligned}$$

where $\{u_n\}$ and $\{v_n\}$ are two sequences in C satisfying $\sum_{n=1}^{\infty} \|u_n\| < \infty$ and $\sum_{n=1}^{\infty} \|v_n\| < \infty$, and $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences of real numbers in $[0, 1]$ satisfying that $\{\alpha_n\}$ is bounded away from 0 and 1 and $\{\beta_n\}$ is bounded away from 1. He proved that if X is a uniformly convex Banach space, C is a nonempty *bounded* closed convex subset of X, and if $T : C \to C$ is a completely continuous asymptotically nonexpansive mapping, then the Ishikawa (and Mann) iteration process with errors defined by (1) converges strongly to some fixed point of T. These results generalize the result due to Rhoades [15], proving the strong convergence of the Ishikawa (and Mann) iterates of an asymptotically nonexpansive mappings which extended the result of Schu [17] to uniformly convex Banach spaces. However, there is no assurance that $\{x_n\}$ remains in C in Definition 3 of Huang [6]. For handling this defection and eliminating the boundedness assumption on C, consider a more general iterative scheme of the type (cf., Xu [19]) emphasizing the randomness of errors as follows:

$$x_1 \in C,$$

$$(2) \qquad x_{n+1} = \alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n,$$

$$y_n = \alpha'_n x_n + \beta'_n T^n x_n + \gamma'_n v_n,$$

where $\{\alpha_n\}$, $\{\beta_n\}$,$\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$,$\{\gamma'_n\}$ are real sequence in [0,1] and

$\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$ are two bounded sequences in C such that

(i) $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1$ for all $n \geq 1$,

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \gamma'_n < \infty$.

If $\gamma_n = \gamma'_n = 0$ for all $n \geq 1$, then the Ishikawa iteration process with errors reduces to the Ishikawa iteration process [7], while setting $\beta'_n = 0$ and $\gamma'_n = 0$ for all $n \geq 1$ reduces to the Mann iteration process with errors which is a generalized case of the Mann iteration process [13].

Recently in [11], they proved strong convergence theorems of the Ishikawa (and Mann) iteration process with errors defined by (2) with certain conditions in addition to (i) and (ii) for an asymptotically nonexpansive mapping in the intermediate sense, which generalize the recent theorems due to Huang [6]. Their proof is also much simpler than the one given by Huang [6].

In this paper, we consider the following iteration scheme generated by a pair of mappings (T, S) instead of (2) with non-Lipschitzian self mappings S and T under the same parameter conditions.

$$x_1 \in C,$$

$$x_{n+1} = \alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n, \tag{3}$$

$$y_n = \alpha'_n S^n x_n + \beta'_n T^n x_n + \gamma'_n v_n,$$

Note that our iterative scheme yields the one given by Kim-Kim [11] only when $S = I$, where I denotes the identity mapping of C.

2. PRELIMINARIES

Let X be a real Banach space with norm $\|\cdot\|$ and let X^* be its dual. The value of $x^* \in X^*$ at $x \in X$ will be denoted by $\langle x, x^* \rangle$. When $\{x_n\}$ is a sequence in X, then $x_n \to x$ (resp. $x_n \rightharpoonup x$, $x_n \overset{*}{\rightharpoonup} x$) will denote strong (resp. weak, weak*) convergence of the sequence $\{x_n\}$ to x.

A Banach space X is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$, where the modulus $\delta(\epsilon)$ of convexity of X is defined by

$$\delta(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \leq 1,\ \|y\| \leq 1,\ \|x-y\| \geq \epsilon\right\}.$$

Let $S(X) = \{x \in X : \|x\| = 1\}$. Then the norm of X is said to be *Gâteaux differentiable* (and E is said to be *smooth*) if

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{4}$$

exists for each x, y in $S(X)$. It is said to be *Fréchet differentiable* if for each $x \in S(X)$, the limit in (4) is attained uniformly for $y \in S(X)$. The norm is said to be *uniformly Gâteaux differentiable* if for each $y \in S(X)$, the limit in (4) is approached uniformly for x varies over $S(X)$. Finally, it is said to be *uniformly Fréchet differentiable* (or X is said to be *uniformly smooth*) if the limit is attained uniformly for $(x, y) \in S(X) \times S(X)$.

We associate with each $x \in X$ the set

$$J_\phi(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|\|x^*\| \text{ and } \|x^*\| = \phi(\|x\|)\},$$

where $\phi : [0,\infty) \to [0,\infty)$ is a continuous and strictly increasing function with $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$. Then $J_\phi : X \to 2^{X^*}$ is said to be the *duality* mapping. Suppose that J_ϕ is single-valued. Then J_ϕ is said to be *weakly sequentially continuous* if for each $\{x_n\} \in X$ with $x_n \rightharpoonup x$, $J_\phi(x_n) \overset{*}{\rightharpoonup} J_\phi(x)$. For abbreviation, we set $J := J_\phi$. In our proof, we assume without loss of generality that J is normalized. For the relations between the duality mapping J and the above geometric properties of X, we summarize the following

REMARK 2.1.

(a) If X is smooth, then the duality mapping J is single-valued and norm(strong)-to-weak* continuous.

(b) If X is uniformly smooth, it is norm-to-norm uniformly continuous on every bounded subset of X; if the norm of X has uniformly Gâteaux differentiable, then J is norm-to-weak* uniformly continuous on every bounded subset of X.

(c) The norm of X is uniformly Fréchet differentiable if and only if X^* is uniformly convex.

For more detailed properties, see [2].

Let X be a real Banach space, C a subset of X (not necessarily convex), and $T : C \to C$ a self-mapping of C. nonexpansive mapping. First, as the weaker definition (cf. Kirk [9]), T is said to be of *asymptotically nonexpansive type* (in brief, ANT) if for each $x \in C$, $\lim_{n\to\infty} c_n(x) = 0$, where

$$c_n(x) = 0 \vee \sup_{y\in C}(\|T^n x - T^n y\| - \|x - y\|) \leq 0,$$

and next, as the stronger sense, it is said to be of *strongly asymptotically nonexpansive type* (in brief, strongly ANT) if $\lim_{n\to\infty} c_n = 0$, where $c_n = \sup_{x\in C} c_n(x)$.

Recall that T is said to be *Lipschitzian* if $\exists\ L > 0$ such that $\|Tx - Ty\| \leq L\|x - y\|$ for all $x, y \in C$. In particular, if $L = 1$, T is said to be *nonexpansive* and it is said to be *asymptotically nonexpansive* (in brief, AN) [3] if each iterate T^n is Lipschitzian with Lipschitz constants $L_n(T) \to 1$ as $n \to \infty$. As an easy observation, we have the following

REMARK 2.2. (a) all nonexpansive mappings are AN.

(b) Every AN mapping is uniformly continuous and of strongly ANT (hence, a mapping of ANT).

(c) Any mapping of strongly ANT may be non-Lipschitzian.

(d) All mappings $T : C \to C$ with the property $T^n x \to 0$ uniformly on C are of strongly ANT.

(e) For all $x \in C$, if $T^n x \in F(T) = \{z\}$ for some $n \geq 1$, T is a mapping of ANT.

For investigating the relations between the above concepts, we here give the following example.

EXAMPLE 2.1.

(a) Let $C = [-1/\pi, 1/\pi] \subseteq \mathbb{R}$ and $|k| < 1$. For each $x \in C$ we define $Tx = kx \sin \frac{1}{x}$ if $x \neq 0$, and $T0 = 0$. Note that $T^n x \to 0$ uniformly on C. Hence, $T : C \to C$ is a continuous mapping of ANT which is not Lipschitzian.

(b) Let $C = [0,1] \subseteq \mathbb{R}$ and define $Tx = \frac{1}{4}$ if $x = \frac{1}{4}, 1$, $Tx = 1$ for $x \in [0, \frac{1}{2}] \setminus \frac{1}{4}$, and $Tx = \frac{1}{2}$ for $x \in (\frac{1}{2}, 1]$. Note that for all $x \in C$, $T^n x = \frac{1}{4} \in F(T) = \{\frac{1}{4}\}$ for $n \geq 3$. Then $T : K \to K$ is a discontinuous mapping of ANT which is not nonexpansive.

(c) [14] Let $C = [0,1] \subseteq \mathbb{R}$ and let φ be the Cantor ternary function. Define $T : C \to C$ by

$$T(x) = \begin{cases} x/2 & \text{if } 0 \leq x \leq 1/2, \\ \varphi((1-x)/2) & \text{if } 1/2 < x \leq 1. \end{cases}$$

Note that $T^n x \to 0$ uniformly on C. Therefore, T is a discontinuous mapping of strongly ANT but not AN because φ is not Lipschitzian continuous on $[0, \frac{1}{2}]$.

Finally we introduce the relation between continuous compact and completely continuity. Let X, Y be Banach spaces. $T : X \to Y$ is said to be *compact* if it maps every bounded subset of X into a relatively compact subset of Y. Equivalently T is compact if and only if for every bounded sequence $\{x_n\}$ in X, $\{T(x_n)\}$ has a convergent subsequence in Y. Also, $T : X \to Y$ is said to be *completely continuous* if for any sequence $\{x_n\}$ converging weakly to x_0, the sequence $\{Tx_n\}$ converges strongly to Tx_0. That is, $x_n \rightharpoonup x_0 \Rightarrow Tx_n \to Tx_0$.

Let X be a reflexive Banach space. Then it is well known [8] that if $T : X \to Y$ is completely continuous, then T is continuous and compact. In general the converse does not hold. However, if T is linear, the converse remains true.

3. Strong convergence theorems

We first begin with the following:

LEMMA 3.1 [18]. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers such that $\sum_{n=1}^{\infty} b_n < \infty$ and*

$$a_{n+1} \leq a_n + b_n$$

for all $n \geq 1$. Then $\lim_{n\to\infty} a_n$ exists.

LEMMA 3.2 [5]. *Let X be a uniformly convex Banach space. Let $x, y \in X$. If $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \epsilon > 0$, then $\|\lambda x + (1-\lambda)y\| \leq 1 - 2\lambda(1 - \lambda)\delta(\epsilon)$ for $0 \leq \lambda \leq 1$.*

Our Theorem 3.1 carries over Theorem 2 of Huang [6] to a more general Ishikawa type scheme and a non-Lipschitzian self mapping.

THEOREM 3.1. *Let X be a uniformly convex Banach space and let C be a nonempty closed convex subset of X. Suppose that $S, T : C \to C$ are both completely and uniformly continuous. Assume also that $S, T : C \to C$ are mappings of strongly ANT with $F(S) \cap F(T) \neq \emptyset$. Put*

$$c_n = \sup_{x,y \in C} (\|S^n x - S^n y\| - \|x - y\|) \vee \sup_{x,y \in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0,$$

so that $\sum_{n=1}^{\infty} c_n < \infty$. Then for any x_1 in C, the sequence $\{x_n\}$ defined by (3), where, in addition to (i) and (ii) in (2), $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that $0 < a \leq \alpha_n \leq 1$, $0 < b_1 \leq \beta_n \leq 1 - b_2 < 1$ for all $n \geq 1$ and some $a, b_1, b_2 \in (0,1)$, $\limsup_{n \to \infty} \beta'_n \leq b < 1$ for some $b \in (0,1)$, and $\lim {\alpha_n}' = 0$, converges strongly to some common fixed point of S and T.

Proof. For a common fixed $z \in F(S) \cap F(T)$, since $\{u_n\}$ and $\{v_n\}$ are bounded in C, let

$$M := \sup_{n \geq 1} \|u_n - z\| \vee \sup_{n \geq 1} \|v_n - z\| < \infty.$$

From

$$
\begin{aligned}
\|T^n y_n - z\| &\le \|y_n - z\| + c_n \\
&= \|\alpha'_n S^n x_n + \beta'_n T^n x_n + \gamma'_n v_n - z\| + c_n \\
&\le \alpha'_n \|S^n x_n - z\| + \beta'_n \|T^n x_n - z\| + \gamma'_n \|v_n - z\| + c_n \\
&\le \alpha'_n \{\|x_n - z\| + c_n\} + \beta'_n \{\|x_n - z\| + c_n\} + \gamma'_n \|v_n - z\| + c_n \\
&\le (1 - \gamma'_n)\|x_n - z\| + 2c_n + \gamma'_n \|v_n - z\|,
\end{aligned}
$$

we have

$$
\begin{aligned}
\|x_{n+1} - z\| &= \|\alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n - z\| \\
&\le \alpha_n \|x_n - z\| + \beta_n \|T^n y_n - z\| + \gamma_n \|u_n - z\| \\
&\le \alpha_n \|x_n - z\| + \beta_n \{(1 - \gamma'_n)\|x_n - z\| + 2c_n + \gamma'_n \|v_n - z\|\} \\
&\quad + \gamma_n \|u_n - z\| \\
&\le (1 - (\gamma_n + \beta_n \gamma'_n))\|x_n - z\| + 2c_n + \gamma_n M + \gamma'_n M \\
&\le \|x_n - z\| + 2c_n + (\gamma_n + \gamma'_n)M.
\end{aligned}
$$

Since all assumptions of Lemma 3.1 with $a_n := \|x_n - z\|$ and $b_n := 2c_n + (\gamma_n + \gamma'_n)M$ are fulfilled, $\lim_{n\to\infty} \|x_n - z\| (\equiv r)$ exists. Without loss of generality, we assume $r > 0$. Note that $d_n := \max\{\gamma'_n, \gamma_n/a\} \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} d_n < \infty$. Since $\|T^n y_n - z\| \le \|x_n - z\| + 2c_n + d_n M$ and

$\|\frac{\alpha_n S^n x_n}{\alpha_n+\gamma_n} + \frac{\gamma_n u_n}{\alpha_n+\gamma_n} - z\| \leq \|x_n - z\| + 3c_n + d_n M$, it follows from Lemma 3.2 that

$$\begin{aligned}
&\|x_{n+1} - z\| \\
=&\|\alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n - z\| \\
=&\Big\|\beta_n(T^n y_n - z) + (1-\beta_n)\Big(\frac{\alpha_n x_n}{\alpha_n+\gamma_n} + \frac{\gamma_n u_n}{\alpha_n+\gamma_n} - z\Big)\Big\| \\
\leq&(\|x_n - z\| + 2c_n + d_n M)\Big[1 - 2\beta_n(1-\beta_n)\cdot \\
&\quad \delta\Big(\frac{1}{\alpha_n+\gamma_n}\cdot\frac{\|\alpha_n(T^n y_n - x_n) + \gamma_n(T^n y_n - u_n)\|}{\|x_n - z\| + 2c_n + d_n M}\Big)\Big].
\end{aligned}$$

Thus

$$\begin{aligned}
&2\beta_n(1-\beta_n)(\|x_n - z\| + 2c_n + d_n M)\cdot \\
&\delta\Big(\frac{1}{\alpha_n+\gamma_n}\cdot\frac{\|\alpha_n(T^n y_n - x_n) + \gamma_n(T^n y_n - u_n)\|}{\|x_n - z\| + 2c_n + d_n M}\Big)\Big] \\
\leq&\|x_n - z\| - \|x_{n+1} - z\| + 2c_n + d_n M.
\end{aligned}$$

Summing all terms in both sides, dividing by 2 and applying $0 < b_1 < \beta_n \leq 1 - b_2 < 1$, we get

$$\begin{aligned}
&b_1 b_2 \sum_{n=1}^{\infty}(\|x_n - z\| + 2c_n + d_n M)\cdot \\
&\delta_X\Big(\frac{1}{\alpha_n+\gamma_n}\cdot\frac{\|\alpha_n(T^n y_n - x_n) + \gamma_n(T^n y_n - u_n)\|}{\|x_n - z\| + 2c_n + d_n M}\Big) < \infty.
\end{aligned}$$

Since $\sup_{n\geq 1}\|T^n y_n - u_n\| < \infty$, and δ_X is strictly increasing and continuous, this implies $\lim_{n\to\infty}\|T^n y_n - x_n\| = 0$. Note next that since

$$\begin{aligned}
\|x_n - S^n x_n\| &\leq \|x_n - z\| + \|S^n x_n - z\| \\
&\leq 2\|x_n - z\| + c_n,
\end{aligned}$$

$\lim \|x_n - z\| = r$ exists and $c_n \to 0$, the set $\{\|x_n - S^n x_n\| : n \in \mathbb{N}\}$ must be bounded, that is, say $M' = \sup_n \|x_n - S^n x_n\| < \infty$. On the other hand, since

$$\begin{aligned}
&\|T^n x_n - x_n\| \leq \|T^n x_n - T^n y_n\| + \|T^n y_n - x_n\| \\
\leq &\|x_n - y_n\| + c_n + \|T^n y_n - x_n\| \\
= &\|x_n - (\alpha_n' S^n x_n + \beta_n' T^n x_n + \gamma_n' v_n)\| + c_n + \|T^n y_n - x_n\| \\
\leq &\alpha_n' \|x_n - S^n x_n\| + \beta_n' \|x_n - T^n x_n\| + \gamma_n' \|x_n - v_n\| + c_n + \|T^n y_n - x_n\| \\
\leq &\alpha_n' M' + \beta_n' \|x_n - T^n x_n\| + \gamma_n' \|x_n - v_n\| + c_n + \|T^n y_n - x_n\|,
\end{aligned}$$

and so

$$\begin{aligned}
&(1 - \beta_n')\|T^n x_n - x_n\| \\
\leq &\alpha_n' M' + \gamma_n' \|x_n - v_n\| + c_n + \|T^n y_n - x_n\| \qquad (5)\\
\leq &\alpha_n' M' + \gamma_n' M'' + c_n + \|T^n y_n - x_n\|,
\end{aligned}$$

where $M'' = \sup_{n \geq 1} \|x_n - v_n\| < \infty$. Since $\limsup_{n\to\infty} \beta_n' \leq b < 1$, we have $\liminf_{n\to\infty}(1 - \beta_n') \geq 1 - b > 0$. Combined this with $\alpha_n', \gamma_n', c_n, \|T^n y_n - x_n\| \to 0$, it easily follows from (5) that

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \qquad (6)$$

Finally remark that

$$\begin{aligned}
x_{n+1} - x_n &= (\alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n) - x_n \\
&= \beta_n (T^n y_n - x_n) + \gamma_n (u_n - x_n)
\end{aligned}$$

and $\|u_n - x_n\| \le \|u_n - z\| + \|z - x_n\| \le M + \sup_n \|x_n - z\| < \infty$, this implies $\|x_{n+1} - x_n\| \to 0$. Since

$$\begin{aligned}\|x_n - Tx_n\| \le & \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| \\ & + \|T^{n+1}x_{n+1} - T^{n+1}x_n\| + \|T^{n+1}x_n - Tx_n\|\end{aligned}$$

and by the uniform continuity of T and (6), we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{7}$$

Since C is closed convex (hence weakly closed) in a uniformly convex (hence reflexive) space X, and moreover $\{x_n\}$ is bounded in C, there exist a subsequence $\{x_m\}$ of the sequence $\{x_n\}$ and a point $w \in C$ such that $x_m \rightharpoonup w$. Since T is completely continuous, $Tx_m \to Tw$. Applying for the lower semicontinuity of the norm $\|\cdot\|$, we easily get

$$\|w - Tw\| \le \liminf_{m\to\infty} \|x_m - Tx_m\| = \lim_{m\to\infty} \|x_m - Tx_m\| = 0,$$

that is, w is a fixed point of T. Therefore, $Tx_m \to w$. Combined this with (7), we get

$$\|x_m - w\| \le \|x_m - Tx_m\| + \|Tx_m - w\| \to 0,$$

and so $x_m \to w$. On the other hand, since

$$\begin{aligned}\|x_m - S^m x_m\| &\le \|x_m - w\| + \|w - S^m x_m\| \\ &\le 2\|x_m - w\| + c_m \to 0,\end{aligned}$$

and similarly $\|x_m - S^{m+1}x_m\| \to 0$, we obtain that

$$(8) \qquad \|x_m - Sx_m\| \leq \|x_m - S^{m+1}x_m\| + \|S^{m+1}x_m - Sx_m\| \to 0.$$

By complete continuity of S, $Sx_m \to Sw$. The lower simicontinuity of the norm and (8) as before imply w is a fixed point of S. Therefore, $w \in F(T) \cap F(S)$. For this common fixed point w of S and T, by repeating the first part proof applying for Lemma 1, we can similarly obtain that $\lim \|x_n - w\|$ exists. Since the subsequence $\{x_m\}$ of the sequence $\{x_n\}$ converges to w, we have $x_n \to w$ and the proof is complete. □

REMARK 3.1. Note that if S and T are commutative, i.e., $ST = TS$, then $F(S) \cap F(T) \neq \emptyset$. Indeed, from the Schauder fixed point theorem [16] $F(S) \neq \emptyset$. Also, the restriction of T to $F(S)$ has a fixed point z from the Schauder fixed point theorem again, which yields $z \in F(S) \cap F(T)$.

Since every asymptotically nonexpansive mapping is uniformly continuous, we immediately improves Theorem 2 due to Huang [6] to a more general Ishikawa type scheme (3) instead of (1).

COROLLARY 3.1. *Let X be uniformly convex, $\emptyset \neq C \subset X$ closed bounded and convex. Let S, T be two completely continuous asymptotically nonexpansive self-mappings of C with $\{L_n(S)\}$, $\{L_n(T)\}$satisfying $L_n(S)$, $L_n(T) \geq 1$,*

$$\sum_{n=1}^{\infty}(L_n(S) - 1),\ \sum_{n=1}^{\infty}(L_n(T) - 1) < \infty.$$

Then for any x_1 in C, the sequence $\{x_n\}$ defined by (3), where, in addition to (i) and (ii), $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that $0 < a \le \alpha_n \le 1$, $0 < b_1 \le \beta_n \le 1 - b_2 < 1$ for all $n \ge 1$ and some $a, b_1, b_2 \in (0,1)$, $\limsup_{n\to\infty} \beta_n' \le b < 1$ for some $b \in (0,1)$ and $\lim \alpha_n' = 0$, converges strongly to some common fixed point of S and T.

Proof. Note that

$$\sum_{n=1}^{\infty} c_n \le \sum_{n=1}^{\infty} (L_n(S) - 1)\mathrm{diam}(C) + \sum_{n=1}^{\infty} (L_n(T) - 1)\mathrm{diam}(C) < \infty,$$

where $\mathrm{diam}(C) = \sup_{x,y\in C} \|x - y\| < \infty$. The conclusion now follows easily from Theorem 3.1.

As a direct consequence, taking $\beta_n' = 0$ and $\gamma_n' = 0$ for $n \in \mathbb{N}$ in Theorem 3.1, we have the following result, which carries over Theorem 1 due to Huang [6] to a more general Mann type scheme and a non-Lipschitzian self mapping.

THEOREM 3.2. *Let X be a uniformly convex Banach space and let C be a nonempty closed convex subset of X. Suppose that $S, T : C \to C$ are both completely and uniformly continuous. Assume also that $S, T : C \to C$ are mappings of strongly ANT with $F(S) \cap F(T) \neq \emptyset$. Put*

$$c_n = \sup_{x,y\in C} (\|S^n x - S^n y\| - \|x - y\|) \vee \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0,$$

so that $\sum_{n=1}^{\infty} c_n < \infty$. Then for any x_1 in C, the sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + \beta_n T^n S^n x_n + \gamma_n u_n,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *in* $[0, 1]$ *are chosen so that* $0 < a \leq \alpha_n \leq 1$, $0 < b_1 \leq \beta_n \leq 1 - b_2 < 1$ *for all* $n \geq 1$ *and some* $a, b_1, b_2 \in (0, 1)$, *and* $\{u_n\}_{n=1}^{\infty}$ *is a bounded sequence in* C *such that* (i) $\alpha_n + \beta_n + \gamma_n = 1$, *for all* $n \geq 1$, (ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, *converges strongly to some common fixed point of* S *and* T.

As a direct consequence of Theorem 3.2 with $S = I$, we improves Theorem 1 due to Huang [6] to a more general Mann type scheme (3) instead of (1).

COROLLARY 3.2. *Let* X *be uniformly convex,* $\emptyset \neq C \subset X$ *closed bounded and convex. Let* T *be a completely continuous asymptotically nonexpansive self map of* C *with* $\{k_n\}$ *satisfying* $k_n \geq 1$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. *Then for any* x_1 *in* C, *the sequence* $\{x_n\}$ *defined by*

$$x_{n+1} = \alpha_n x_n + \beta_n T^n x_n + \gamma_n u_n,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *in* $[0, 1]$ *are chosen so that* $0 < a \leq \alpha_n \leq 1$, $0 < b_1 \leq \beta_n \leq 1 - b_2 < 1$ *for all* $n \geq 1$ *and some* $a, b_1, b_2 \in (0, 1)$, *and* $\{u_n\}_{n=1}^{\infty}$ *is a sequence in* C *such that* (i) $\alpha_n + \beta_n + \gamma_n = 1$, *for all* $n \geq 1$, (ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, *converges strongly to some fixed point of* T.

The following example shows that Theorem 3.1 and 3.2 generalize the results due to Huang [6].

EXAMPLE. Let $E = \mathbb{R}$ and $C = [-\frac{1}{\pi}, \frac{1}{\pi}]$ and let $|k| < 1$. For each $x \in C$, we define

$$T(x) = \begin{cases} kx \sin \frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

It is well-known in [10] that $T^n x \to 0$ uniformly but $T : C \to C$ is not a Lipschitz function. Here we show that T is asymptotically nonexpansive in the intermediate sense. Indeed, for each fixed $n \in \mathbb{N}$, define

$$f_n(x, y) = \|T^n x - T^n y\| - \|x - y\|$$

for all $x, y \in C$. Then $f_n : C \times C \to \mathbb{R}$ is continuous and since $C \times C$ is compact it achieves its maximum at $(x_n, y_n) \in C \times C$, that is,

$$c_n = \sup_{x,y \in C} f_n(x, y) \vee 0 = f_n(x_n, y_n) \vee 0 = \|T^n x_n - T^n y_n\| - \|x_n - y_n\| \vee 0.$$

Since $T^n x \to 0$ uniformly on C, it immediately follows that

$$\limsup_{n \to \infty} c_n = \limsup_{n \to \infty} \{\|T^n x_n - T^n y_n\| - \|x_n - y_n\| \vee 0\} = 0.$$

Since $T : C \to C$ is obviously continuous, it easily follows that it is uniformly continuous and completely continuous (cf., Joshi and Bose [8]).

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