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Metacyclic Groups of Odd Order with Two Prime Divisors

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두 소수 약수를 갖는 홀수 위수의 메터순환군

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요 약

메터순환군(metacyclic group)은 한 순환군(cyclic group)을 또 하나의 순환군에 의하여 확장한 군이다. 군의 확장이론의 가장 기본적인 경우일 뿐만아니라 가해군(soluble)의 일종으로 여러 군론 연구자들의 주목을 받아왔다. 본 논문에서는 두 소수 약수를 갖는 메터순환군의 분류문제를 연구하였다. 분류문제 해결의 첫 단계로서 주어진 닐포턴트(nilpotent)가 아닌 유한 메터순환 {p,q}-군의 실로우(Sylow) q-부분군(subgroup)의 구조를 분석하므로써 소위 표준 군표시(presentation)을 구하였다. 이들 표준 군표현은 7개의 매개변수들로 구성되어 있는데 이 중에서 6개는 그 동형족(isomorphism type)의 불변량(invariants)이지만 나머지 하나는 불변량이 아니다. 따라서 두 번째 단계는 불변량이 아닌 매개변수에 관한 동형문제를 해결하는 것으로 실로우 p-군의 어떤 상군(factor group)의 자기동형사상군(automorphism group)의 작용(action)을 분석하여 이를 해결하였다.

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1 Introduction

A group G is said to be an extension of a group N by a group Q if there exists a normal subgroup K such that $K \cong N$ and $G/K \cong Q$. The case when both N and Q are cyclic is basic and very important. For the case, the group G is called a **metacyclic group**. In other wards, a metacyclic group is a group G that has a cyclic normal subgroup K such that G/K is also cyclic. Given a metacyclic group G, if K is a cyclic normal subgroup of G, then there exists a cyclic subgroup S such that $G = SK$. Therefore each metacyclic group G has a factorization $G = SK$; such a factorization is called a **metacyclic factorization** of G. If $S \cap K = 1$, then the factorization is called **split**. Subgroups and quotient groups of metacyclic groups are also metacyclic. As a special subfamily of soluble groups, metacyclic groups have been received considerable attention by many authors.

Metacyclic groups are usually presented on two generators with three defining relations. In fact each metacyclic group has presentations of the form

$$\langle\, x, y \,:\, x^m = y^s,\ y^n = 1,\ x^{-1}yx = y^r \,\rangle$$

with the arithmetic conditions

$$0 < m, n,\ r^m \equiv 1 \bmod n,\ s(r-1) \equiv 0 \bmod n.$$

Conversely, every group defined by such a presentation is a metacyclic group of order mn. This characterization is well-known and found several times in the literature. It is unsatisfactory that the parameters involved in this **metacyclic presentation** may not be invariants. Basmaji [1] in 1969 attempted to give a partial answer for the isomorphism problem for these presentations. However, the isomorphism problem for general metacyclic presentations still remains unsolved.

Various classifications for metacyclic p-groups were given in 1970s, for example, [7, 8, 9, 2, 6, 11]. In 1992, Sim [13] gave a classification of metacyclic groups of odd order up to isomorphism in terms of presentations, and then in 1998, Hempel [4] extended Sim's result to all metacyclic groups.

The purpose of this thesis is to describe a classification of metacyclic groups of odd orders with two prime divisors. In logical sense, the classification may be considered as a special case of the general classification given by Sim. We here provide a direct and alternative approach to the classification.

The first step towards the solution of our classification is to choose so called a **standard factorization** of a given nonnilpotent metacyclic $\{p, q\}$-group G with p dividing $q-1$. We observe that the Sylow q-subgroup is normal and has the canonical split metacyclic factorization. A special metacyclic factorization of a Sylow p-subgroup gives rise to a standard factorization of G. From a standard factorization, we have some special form of presentations, which we call **standard presentations**. Each standard presentation involves 7 parameters; 6 of the parameters are invariants of the isomorphism type but the other is not. So the isomorphism problem of the groups defined by standard presentations still remains to be solved. Our answer for the isomorphism problem can be outlined as follows:

Given odd prime p, q-group with p dividing $q-1$, let G be a non-nilpotent metacyclic $\{p, q\}$ with a Sylow p-subgroup P and the Sylow q-subgroup Q, and let $X = G/\mathsf{O_p}(G)$, $Y = G/Q$ where $\mathsf{O_p}(G)$ is the largest normal p-subgroup of G. Regarding G as a subgroup of the direct product of X and Y through the canonical embedding of G into $X \times Y$, the group G is a subdirect subgroup of $X \times Y$. The subgroup G of $X \times Y$ determine subgroups X, Y as the images of the natural projections and subgroups $X_0 = G \cap X$ and $Y_0 = G \cap Y$. Then we know that $X/X_0 \cong Y/Y_0 \cong P/\mathsf{O_p}(G)$. Let $SD(G)$ be the set of all subgroup $G(\lambda) = \{(x, y) : \lambda(xX_0) = yY_0\}$ for each isomorphism λ from X/X_0 onto Y/Y_0. Identifying Y by P, Y_0 becomes $\mathsf{O_p}(G)$. We then show that $\mathrm{Aut}(P/\mathsf{O_p}(G))$ acts transitively on the isomorphic types of $SD(G)$ and the induced automorphisms of $\mathsf{N}_{\mathsf{Aut(P)}}(\mathsf{O_p}(G))$ on $P/\mathsf{O_p}(G)$ is the stabilizer of the action. It turns out that this observation yields our answer for the isomorphism problem.

We now set up some notation, which will be used in this thesis. The identity element of a multiplicative group is denoted by 1 and the same notation is also used for the trivial subgroup consisting of the identity element only.

Let G be a group. The Frattini subgroup, namely the intersection of all maximal subgroups of G, is denoted by $\Phi(G)$. The automorphism group of the group G is denoted by $\mathrm{Aut}(G)$. If H is a subgroup of G, then the

normalizer of H in G is denoted by $\mathsf{N}_G(H)$; the **centralizer** of H in G is denoted by $\mathsf{C}_G(H)$.

Let H, N be groups and $\phi : H \longrightarrow \mathrm{Aut}(N)$ a homomorphism. Then the homomorphism defines a **semidirect product** of N by H; we denote the semidirect product by $H \ltimes_\phi N$, or simply by $H \ltimes N$. We may usually regard H and N as subgroups of $H \ltimes N$ via the natural identifications.

Let G be a group, H a subgroup of G and K a normal subgroup of H. Let N be a group of automorphisms of G which normalize (that is, stabilize setwise) both H and K. Every automorphism in N induces an automorphism of H/K in a natural way; the set of all such automorphisms of N will be denoted by $N\downarrow_{H/K}$. For example, if C is a normal subgroup of a group P then $\mathsf{N}_{\mathrm{Aut}(P)}(C)\downarrow_{P/C}$ is the group of automorphisms of P/C induced from all automorphisms of P which normalize C.

Most of notation and terminology not defined in this thesis is standard and can be found in almost all standard books on related areas, for example, see [14, 12].

2 Preliminaries

In this section, we present some basic concepts and facts that will be used frequently in this thesis, while we assume that we are familiar with most of basic concepts and properties on elementary group theory.

Lemma 2.1. G is a group and H and K are normal subgroup of G. If $\theta : G \longrightarrow G/H \times G/K$ is a homomorphism de¯ned by $\theta(g) = (gH, gK)$, then $\ker(\theta) = H \cap K$

Let $G = H \times K$ be the direct product of two finite cyclic groups H and K, let η and κ be the projection onto the factors H and K, respectively. We regard H and K as subgroups of G. Then we have the following lemma, see [14] for the proof.

Lemma 2.2. A subgroup U of G determines four subgroups $U \cap H,\ \eta(U),\ U \cap K,$ and $\kappa(U)$, and the isomorphism φ_U from $\eta(U)/(U \cap H)$ onto $\kappa(U)/(U \cap K)$. Conversely, suppose that we have subgroups $H_1,\ H_2$ of H and subgroups $K_1,\ K_2$ of K, such that $H_2 \leq H_1$ and $K_2 \leq K_1$, and an isomorphism θ from H_1/H_2 onto K_1/K_2. De¯ne the subset U of $H \times K$ as follows:

$$U = \{hk : h \in H_1, k \in K_1, \theta(hH_2) = kK_2\}.$$

Then U is a unique subgroup of G such that $H_1 = \eta(U),\ H_2 = U \cap H,\ K_1 = \kappa(U),\ K_2 = U \cap K,$ and $\theta = \varphi_U$.

De¯nition 2.3. A subgroup U is called subdirect in G if $\eta(U) = H$ and $\kappa(U) = K$.

We need the following lemmas.

Lemma 2.4. If π^0-group A acts on a π-group P, then $P = C_P(A)[P, A]$. In particular, if P is abelian, then $P = C_P(A) \times [P, A]$.

The proof of the above lemma can be found in [3, Theorem 2.3 and Theorem 3.5].

We will close this section with some investigation about the automorphism groups of finite cyclic group.

Let Z_n denote the additive group of integers modulo n for a positive integer n. The set U_n of integers m modulo n which are relatively prime to n forms an abelian group under multiplication modulo n. It is well-known that the automorphism group of a cyclic group of order n can be identified with this multiplicative group U_n.

The structure of U_n is well-known, see [10] for example. We here just state the special case when n is a power of an odd prime number.

Theorem 2.5. If p is an odd prime, then U_{p^α} is the cyclic group of order $(p-1)p^{\alpha-1}$.

We adopt following well-known facts without proofs.

Lemma 2.6. Let p be an odd prime. If $r \equiv 1 \bmod p$, then the multiplicative order of r in U_{p^n} is equal to $p^n/\gcd(p^n, r-1)$.

3 Metacyclic groups

A finite group G is called a **metacyclic group** if G has a cyclic normal subgroup K such that G/K is also cyclic. Then a metacyclic group G has a factorization $G = SK$ such that S is a cyclic subgroup and K is a cyclic normal subgroup of G; such a factorization of a metacyclic group is called a *metacyclic factorization*. Then S and K are called the *supplement* and the *kernel* of the metacyclic factorization, respectively. In particular, if $S \cap K = 1$, the metacyclic factorization is called split. A metacyclic group is split if it has a split metacyclic factorization.

Let C be a fixed subgroup of a metacyclic group G. Then a kernel K is called a C-*maximal kernel* if K is contained in C and K has the greatest order among all the kernels contained in C.

Lemma 3.1. Consider the group

$$G = \langle\, a, b \,:\, a^m = b^s, b^n = 1, a^{-1}ba = b^r \,\rangle,$$

where m, n, r, s are integers and $r, s \leq n$, and $r^m \equiv 1, rs \equiv s \mod n$. Then $K = \langle\, b \,\rangle$ is a normal subgroup of G such that $K \cong \mathbb{Z}_n, G/K \cong \mathbb{Z}_m$. Thus G is a finite metacyclic group of order m, n respectively and moreover every finite metacyclic group has a presentation of this form.

For the proof of the above theorem, see ([5, p64, Theorem 1]).

Lemma 3.2. Let G be a metacyclic group with a metacyclic factorization $G = SK$. Let $S = \langle\, x \,\rangle$, $K = \langle\, y \,\rangle$. Let r be an integer such that $y^x = y^r$. Then $G' = \langle\, y^{r-1} \,\rangle$.

Lemma 3.3. ([13, Lemma 2.2.6]) Let P be a non-cyclic p-group for an odd prime p and K a subgroup.

(i) K is cyclic if and only if K does not contain $\Omega_1(P)$.

(ii) K is normal and P/K is cyclic if and only if K contains P' and K is not contained in $\Phi(P)$

Lemma 3.4. ([13, Lemma 3.1.1]) Let P be a metacyclic p-group, p odd. If K is a kernel of a metacyclic factorization of P, then there exists a cyclic subgroup S of P such that $P = SK$ and $|S| = \exp(P)$.

4 Factorization of metacyclic $\{p,q\}$-groups

Let G be a $\{p,q\}$ group with a Sylow p-subgroup P and a Sylow q-subgroup Q for primes p,q with $p<q$. Let $G = SK$ be a metacyclic factorization. Then $S = S_p S_q$, $K = K_p K_q$ where S_p, K_p are the Sylow p-subgroups of S, K, respectively and S_q, K_q are the Sylow q-subgroups of S, K, respectively. Since K is normal subgroup of G and K_p, K_q are characteristic subgroups of K, K_p, K_q are normal subgroup of G. Thus $K_p \leq \mathsf{O}_p(G)$. Since $\mathsf{O}_p(G)$ commutes Q and $\mathsf{O}_p(G)$ is a subgroup P, $\mathsf{O}_p(G) \leq \mathsf{C}_p(Q)$. Moreover $\mathsf{C}_P(Q)^g = \mathsf{C}_P(Q^g) = \mathsf{C}_P(Q)$ for all g in G, and so $\mathsf{C}_P(Q)$ is a normal p-subgroup of G. So $O_p(G) = \mathsf{C}_P(Q)$. It follows that $K_p \leq \mathsf{C}_P(Q)$ and hence $[K_p, S_q] = 1$. Then we have $G = SK = S_p S_q K_p K_q = (S_p K_p)(S_q K_q)$. Replacing S by conjugate if necessary, we arrange that $S_p \leq P$. Then $G = P \mathrm{n}\, Q$ and we have $S_p = P \cap S$ and in any case $S_q = Q \cap S$. $K_p = P \cap K$. $K_q = Q \cap K$, and thus . Furthermore $[P,Q] = [S_p, K_q]$

and $S_q \leq C_Q(P)$. Suppose $[P, Q] \neq 1$. Since $K_q = C_{K_q}(S_p) \times [S_p, K_q]$ by Lemma 2.4 and K_q is directly indecomposable, $C_{K_q}(S_p) = 1$. So we have $K_q = [P, Q]$ and $K_q \cap C_Q(P) = 1$. It follows from $S_q \leq C_Q(P)$ and Dedekind's Law that $S_q = C_Q(P)$. Therefore $S_q \cap K_q = 1$, i.e. Q is split. Consequently, $Q = C_Q(P)[P, Q]$ is a split metacyclic factorization. It is obvious that $[P, Q] \leq G' \cap Q$. Since $G' \leq K$, we have $G' \cap Q \leq K \cap Q = K_q = [P, Q]$. Therefore $[P, Q] = G' \cap Q$.

Theorem 4.1. Let G be a finite $\{p, q\}$-group for given primes p, q such that $p < q$, P a Sylow p-subgroup and Q a Sylow q-subgroup. Then we have: if G is metacyclic, then

(i) $G = P \ltimes Q$

(ii) there exists a metacyclic factorization $P = S_P K_P$ such that $K_P \leq C_P(Q)$;

(iii) either $[P, Q] = 1$ or $Q = C_Q(P)[P, Q]$ is a split metacyclic factorization;

(iv) $C_P(Q) = O_p(G)$ and $[P, Q] = G' \cap Q$.

Let G be a metacyclic $\{p, q\}$-group for given primes p, q such that $p < q$, P a Sylow p-subgroup and Q the Sylow q-subgroup. For a subgroup C of P, a metacyclic factorization $P = S_P K_P$ is called C-**standard** if $|S_P| = \exp(P)$ and K_P is a C-maximal kernel.

A factorization of $G = S_P K_P S_Q K_Q$ is called **standard** if $P = S_P K_P$ is a $O_p(G)$-standard metacyclic factorization of P and $S_Q = C_Q(P)$,

$K_Q = G' \cap Q$.

5 Presenting metacyclic $\{p, q\}$-groups

Lemma 5.1. Let $P = SK$ be a metacyclic factorization. Define $\alpha, \beta, \gamma, \delta$ by

$$p^{\alpha} = |S : S \cap K|,\ p^{\beta} = |K : S \cap K|,\ p^{\gamma} = |K : P'|,\ p^{\delta} = |S \cap K|.$$

Then P has the presentation

$$\langle\, a, b \,:\, a^{p^{\alpha}} = b^{p^{\beta}},\ b^{p^{\beta+\delta}} = 1,\ b^{x} = b^{1+p^{\gamma}} \,\rangle.$$

Moreover, if $P = SK$ is C-standard for a subgroup C with index p^{i}, then

(i) $\alpha \geq \beta \geq \gamma - \delta = 0$;

(ii) if $\gamma = 0$ *then* $\beta = 0$;

(iii) if $\beta \leq \gamma$ *then* $\alpha - \beta \leq i$.

Proof. Let x, y be generators of S, K, respectively. It follows that

$$x^{p^{\alpha}} = y^{sp^{\beta}},\ y^{p^{\beta+\delta}} = 1,\ y^{x} = y^{r}$$

for some nonnegative integer r and nonnegative integer s relatively prime to p. By Lemma 3.2, $r = 1 + tp^{\gamma}$ for some nonnegative integer t relatively prime to p since $|P'| = p^{\beta+\delta-\gamma}$. By Lemma 2.6, there exists a positive integer t' such that $(1 + tp^{\gamma})^{t'} = 1 + p^{\gamma}$. Let $a := x^{t'}$ and $b := y^{st'}$. It follows that

$$a^{p^{\alpha}} = b^{p^{\beta}},\ b^{p^{\beta+\delta}} = 1,\ b^{a} = b^{1+p^{\gamma}}.$$

So P is a homomorphic image of the group presented by

$$\langle a, b : a^{p^{®}} = b^{p^{¯}},\ b^{p^{¯+±}} = 1,\ b^{x} = b^{1+p^{°}} \rangle.$$

On the other hand, the presentation gives a group of order at most $p^{®+¯+±}$, and so P is isomorphic to the group so defined.

Suppose $P = SK$ is C-standard. It follows form $|S| = \exp(P)$ that $\alpha \geq \beta$. Since $P^0 \leq K$, we also have $\beta \geq \gamma - \delta$. Since $b^{p^{¯}} = (b^{p^{¯}})^{a} = b^{¯(1+p^{°})}$, we have $p^{¯+°} \equiv 0 \bmod b^{¯+±}$, so $\gamma \geq \delta$; this completes the proof (i). To prove (ii), assume $\beta \geq 1$. Then P is not cyclic. Then Frattini subgroup $\Phi(P)$ contains the commutator subgroup P^0, but by Lemma 3.3 it does not contain K, so we have that P^0 is a proper subgroup of K and hence $\gamma \geq 1$. To prove (iii), suppose that $\alpha - \beta > i$ when $\beta \leq \gamma$. The proof will done by producing a contradiction. Since $\beta \leq \gamma$, we have $P^0 \leq S \cap K$ Let X be the subgroup of S with $|S : X| = p^{¯+i}$. Since $\alpha - \beta > i$, we have $S \cap K < X$ so that $P^0 < X$. Let $\theta : XK/X \longrightarrow S/X$ be a injective homomorphism. Define $Y := \{ts : \theta(tX) = sX\}$. Then by Lemma 2.2, Y is a subgroup such that $Y \cap XK = X$, $S \cap Y = X$ and $YK = C$. Since $\Omega_1(P)$ is contained Y and $\Omega_1(P)$ does not contained X, Y does not contain $\Omega_1(P)$. So Y is a cyclic subgroup of P by Lemma 3.3. Also Y contains P^0, so Y is a cyclic normal subgroup of P. Moreover, $|SY| = |S||Y|/|S \cap Y| = |S||Y|/|X| = p^{¯+i}p^{®¡i+±} = p^{®+¯+±} = |P|$ and so P/Y is cyclic. Since $|Y| = p^{®¡i+±} > p^{¯+±} = |K|$, we have $|Y| > |K|$. So there exists a cyclic normal subgroup Y such that P/Y is cyclic, $Y \leq C$ and

$|Y| < |K|$. This contradicts the C-maximality of K. Consequently if $\beta \leq \gamma$ then $\alpha - \beta \leq i$, which completes the proof of (iii). $\square$

Once a classification of metacyclic groups of prime-power order was given (see [13, Theorem 3.1.4]), it is trivial matter to classify the nilpotent metacyclic $\{p,q\}$-groups. So we consider the nonnilpotent case only here.

Let p,q be odd primes with $p|(q-1)$. Let α, β, γ and δ be non-negative integers such that

$$\text{either} \quad (i)\, \alpha \geq \beta \geq \gamma - \delta \geq 0,\ \gamma \geq 1, \quad \text{or} \quad (ii)\, \alpha > \beta = \gamma = \delta = 0.$$

Let u, v and w be non-negative integers such that

$$u \geq v,\ w \geq 1.$$

Let θ be a positive integer such that

(i) $1 < \theta < q^{v+w}$

(ii) $\theta^{p^\alpha} \equiv 1 \mod q^{v+w}$

(iii) $\beta \leq \gamma \Rightarrow \alpha - \beta \leq |\theta \mod q^{v+w}|$.

Define

$$\wp[\alpha, \beta, \gamma, \delta, u, v, w, \theta]$$

to be the presentation with the generators a, b, c, d and u, v, w and the following defining relations;

$$a^{p^\alpha} = b^{p^\beta}, \quad b^{p^{\beta+\delta}} = 1, \quad b^a = b^{1+p^\gamma},$$
$$c^{q^u} = 1, \quad d^{q^{v+w}} = 1, \quad d^c = d^{1+q^w}$$
$$c^a = c, \quad c^b = c, \quad d^a = d^\theta, \quad d^b = d\,.$$

Such a presentation with the above conditions for the parameters is called a standard presentation.

Theorem 5.2. Each standard presentation $\wp[\alpha, \beta, \gamma, \delta, u, v, w, \theta]$ de¯nes a non-nilpotent metacyclic group of order $p^{®+¯+±}q^{u+v+w}$. The parameters $\alpha, \beta, \gamma, \delta, u, v, w$ and $|\theta \mod p^{v+w}|$ in the presentation are invariants.

Conversely, every non-nilpotent metacyclic $\{p, q\}$-group has a standard presentation $\wp[\alpha, \beta, \gamma, \delta, u, v, w, \theta]$ for some nonnegative integers $\alpha, \beta, \gamma, \delta, u, v, w, \theta$ with the above conditions.

Proof. Let G be the group defined by the presentation $\wp[\alpha, \beta, \gamma, \delta, u, v, w, \theta]$. Let

$$P = \langle x, y \,:\, x^{p^®} = y^{p^¯}, y^{p^{¯+±}} = 1, y^x = y^{1+q^°} \rangle$$

and

$$Q = \langle e, f \,:\, e^{q^u} = 1, f^{q^{v+w}} = 1, f^e = f^{1+q^w} \rangle.$$

Then by Lemma 3.1, P, Q are metacyclic group of order $p^{®+¯+±}$, q^{u+v+w} respectively. Define the action of P on Q via $e^x = e^y = e$, $f^x = f^µ$, $f^y = f$ and let $H =: P$ n Q. The map a $\mapsto x$, $b \mapsto y$, $c \mapsto e$, $d \mapsto f$ yields a homomorphism from G onto H. On the other hand, $G/\langle c, d \rangle \cong P$ where $\langle c, d \rangle$ is a homomorphic image of Q. $|P||Q| = |H| \leq |G| = |P||\langle c, d \rangle| \leq |P||Q|$. This implies the map $a \mapsto x$, $b \mapsto y$, $c \mapsto e$, $d \mapsto f$ yields an isomorphism from G onto H. Identifying G with H by this isomorphism, $G = P$ n Q where $P = \langle a, b \rangle, Q = \langle c, d \rangle$. In fact $P = \langle a, b \rangle, Q = \langle c, d \rangle$ and

$\mathsf{O}_p(G) = \langle a^{p^i}, b \rangle$, $\mathsf{C}_Q(P) = \langle c \rangle$, $G' \cap Q = \langle d \rangle$ where p^i is the multiplicative order of θ modulo q^{u+w}. It follows Lemma 3.1 that the standard presentation gives a non-nilpotent metacyclic group of order $p^{\alpha+\beta+\delta}q^{u+v+w}$. Moreover $|P/P'| = p^{\alpha+\gamma}, \exp(P) = p^{\alpha+\delta}$, and $|\mathsf{O}_p(G)| = p^{\alpha+\beta+\delta-i}$. Thus we have invariants $\alpha+\beta+\delta$, $\alpha+\gamma$, $\alpha+\delta$, $\alpha+\beta+\delta-i$. Moreover $\beta+\delta$ is invariant if $\beta \le \gamma$, while α is invariant if $\beta > \gamma$. So α, β, γ, δ and i are invariants of G. Obviously u, v and w are invariants of G.

Let $G = S_P K_P S_Q K_Q$ be a standard factorization of G. Define $\alpha, \beta, \gamma, \delta, u, v, w, i$ by

$$p^{\alpha} = |S_P : S_P \cap K_P|,\ p^{\beta} = |K_P : S_P \cap K_P|,\ p^{\gamma} = |K_P : P'|,\ p^{\delta} = |S_P \cap K_P|,$$

$$p^{i} = |P : \mathsf{C}_P(Q)|,\ q^{u} = |S_Q|,\ q^{v} = |Q'|,\ q^{w} = |K_Q : Q'|.$$

By Lemma 5.1, $\alpha \ge \beta \ge \gamma - \delta \ge 0$, $\gamma \ge 1$ or $\alpha > \beta = \gamma = \delta = 0$. We also know $w \ge 1$, since $K_Q = [P, Q] \ne 1$. By Lemma 5.1, we can choose a, b, c, d such that $\langle a \rangle = S_P, \langle b \rangle = K_P, \langle c \rangle = S_Q, \langle d \rangle = K_Q$, so that the following relation holds:

$$\begin{array}{llll} a^{p^{\alpha}} = b^{p^{\beta}}, & b^{p^{\beta+\delta}} = 1, & b^{a} = b^{1+p^{\gamma}}, & \\ c^{q^{u}} = 1, & d^{q^{v+w}} = 1, & d^{c} = d^{1+q^{w}} & \\ c^{a} = c, & c^{b} = c, & d^{a} = d^{\mu}, & d^{b} = d\ . \end{array}$$

If $v + w = 0$, then G is nilpotent, so that $v + w \ge 1$. If Q is abelian, then $v = 0$ and so $w \ge 1$. Suppose that Q is nonabelian. Then the Frattini subgroup $\Phi(Q)$ contains Q' but by Lemma 3.3, it does not contain K_Q; so we see that Q^0 is a proper subgroup of K_Q, and hence $w \geq 1$. We have seen that $w \geq 1$ in any case. It also follows from $(1+q^w)^{q^u} \equiv 1 \mod q^{v+w}$ that $u \geq v$. Since $\wp[\alpha, \beta, \gamma, \delta, u, v, w, \theta]$ has order $p^{®+¯+±}q^{u+v+w}$, we know that G has the presentation $\wp[\alpha, \beta, \gamma, \delta, u, v, w, \theta]$ for some non-negative integers with the above conditions. $\square$

6 Isomorphism problem of metacyclic $\{p, q\}$-groups

Now we investigate how the presentations in (4.1) depend on the choices of primitive p^i-th roots θ.

Given odd primes p, q with $p < q$, let G be a non-nilpotent metacyclic $\{p, q\}$-group with the presentation with the standard presentation $\wp[\alpha, \beta, \gamma, \delta, u, v, w, \theta]$. Denote $X = G/C_P(Q), Y = G/Q$, where P is a Sylow p-subgroup and Q is the Sylow q-subgroup. We regard G as a subgroup of $X \times Y$ through the canonical embedding of G into $X \times Y$, i.e., there exists an isomorphism from G into $X \times Y$. Denote $X_0 = G \cap X$ and $Y_0 = G \cap Y$. Then $X/X_0 \cong Y/Y_0 \cong P/C_P(Q) \cong Z_{p^i}$, and G is subdirect in $X \times Y$. For each λ in $\Lambda = \mathrm{Iso}(X/X_0, Y/Y_0)$, let $SD(G)$ be the set of all subgroups $G(\lambda) = \{(x, y) : \lambda(xX_0) = yY_0\}$, which are called diagonals. The set Λ is permuted by $\mathrm{Aut}(X/X_0)$ on the right and $\mathrm{Aut}(Y/Y_0)$ on the left. Write A for $N_{\mathrm{Aut}(X)}(X_0)$ and $\bar{A}$ for the induced automorphism of A on

X/X_0, so $\bar{A}$ is a homomorphic image of A; define B and $\bar{B}$ correspondingly.

Suppose that $\lambda, \lambda^* \in \Lambda$ and ϕ is an isomorphism of $G(\lambda)$ onto $G(\lambda^*)$. Then the isomorphism ϕ induces an automorphism μ of X in A, namely, the map $x \mapsto \eta(\phi(g))$, where g is any element of $G(\lambda)$ with $\eta(g) = x$ and η is the natural projection of $X \times Y$ onto X. So ϕ induces an automorphism $\overline{\mu}$ in $\bar{A}$. Similarly ϕ induces an automorphism ν of Y in B, so it induces an automorphism $\overline{\nu}$ in $\bar{B}$. Then we have $\lambda^* = \overline{\nu}\lambda\overline{\mu}^{-1}$. Conversely, if $\lambda^* = \overline{\nu}\lambda\overline{\mu}^{-1}$ for some $\mu \in A$ and $\nu \in B$ then the restriction of $\mu \times \nu$ gives an isomorphism of $G(\lambda)$ onto $G(\lambda^*)$. Consequently, if $\lambda \in \Lambda$ then the set of all diagonals isomorphic to the group $G(\lambda)$ correspond to $\bar{B}\lambda\bar{A}$. We also observe that $\overline{A} = 1$.

Identifying Y with P, Y_0 becomes $O_p(G)$ and so $B = N_{Aut(P)}(O_p(G))$ and $\overline{B} = B\downarrow_{P=O_p(G)}$. Then we can summerize the above investigation as follows:

Theorem 6.1. Let $B = N_{Aut(P)}(O_p(G))$ and $\overline{B} = B\downarrow_{P=O_p(G)}$. Suppose $\lambda, \lambda^* \in \Lambda = \mathrm{Iso}(X/X_0, Y/Y_0)$. Then

$$G(\lambda) \cong G(\lambda^*) \text{ if and only if } \lambda^*\lambda^{-1} \in \overline{B}.$$

Now we associate $\overline{B}$ with the standard presentations. Let $p^{i} = |P : O_p(G)|$. Recall that $\mathrm{Aut}(P/O_p(G)) \cong U_{p^{i}}$, and note that the automorphisms of Y/Y_0 are precisely the map $\lambda_s : yY_0 \mapsto y^sY_0$ with $s \in U_{p^{i}}$.

Lemma 6.2. If $G = G(\lambda)$ has the standard presentation $\wp[\alpha, \beta, \gamma, \delta, u, v, w, \theta]$

and $\lambda^{¤} = \lambda_{s}^{i\,1}\lambda$ then $\wp[\,\alpha,\beta,\gamma,\delta,u,v,w,\theta^{s}\,]$ is a standard presentation of $G(\lambda^{¤})$.

Proof. By the definition, G is generated by elements a,b,c,d satisfying the following relations:

$$a^{p^{®}} = b^{p^{¯}}, \quad b^{p^{¯+±}} = 1, \quad b^{a} = b^{1+p^{°}},$$
$$c^{q^{u}} = 1, \qquad d^{q^{v+w}} = 1, \qquad d^{c} = d^{1+q^{w}}$$
$$c^{a} = c, \qquad c^{b} = c, \qquad d^{a} = d^{\mu}, \qquad d^{b} = d\,.$$

Let η and κ be the natural projections of $X \times Y$ on X and Y, respectively. Then $X = \langle\, \eta(a), \eta(b), \eta(c), \eta(d)\,\rangle = \langle\, e, c, d\,\rangle$ and $Y = \langle\, \kappa(a), \kappa(b), \kappa(c), \kappa(d)\,\rangle = \langle\, f, b\,\rangle$ where $e = \eta(a), f = \kappa(a)$, and $\lambda(eX_0) = fY_0$; so $\lambda^{¤} = \lambda_{s}^{i\,1}\lambda$ yields that $\lambda^{¤}(e^{s}X_0) = fY_0$ and hence $G(\lambda^{¤}) = \langle\, e^{s}f, b, c, d\,\rangle$. In terms of these generators, $G(\lambda^{¤})$ has the presentation $\wp[\,\alpha,\beta,\gamma,\delta,u,v,w,\theta^{s}\,]$. $\square$

Let V be the set $\{s \in U_{p^{i}} : \lambda_{s} \in \overline{B}\}$. Obviously V is the unique subgroup of order $|\overline{B}|$ in $U_{p^{i}}$. Then we have the following consequence.

Theorem 6.3. Let θ_1 and θ_2 be primitive p^{i}th roots of 1 $\bmod q^{v+w}$. Then $\wp[\,\alpha,\beta,\gamma,\delta,u,v,w,\theta_1\,] \cong \wp[\,\alpha,\beta,\gamma,\delta,u,v,w,\theta_2\,]$ if and only if there exists $s \in V$ such that $\theta_2 \equiv \theta_1^{s} \bmod q^{v+w}$.

References

[1] B.G. Basmaji, `On the isomorphisms of two metacyclic groups', Proc. Amer. Math. Soc 22(1969), 175-182

[2] F.R. Beyl, The Classi¯cation of Metacyclic p-groups, and other Application of homological Algebra to Group Theory, PhD Thesis (Cornell University, 1972)

[3] Daniel Gorenstein, Finite Groups, (Harper's Series in Mordern Mathematics. Harper and Row, New York, Evanston, London, 1968)

[4] E.J.C.Hempel, Metacyclic Groups, PhD Thesis (Australian National University, 1998)

[5] D.L.Johnson, Topics in the Theory of Group Presentation, (Cambridge University Press , 1980)

[6] Bruce W. King, `Presentation of metacyclic groups', Bull. Austral. Math. Soc. 8(1973), 103-131

[7] Wolfgang Lindenberg, `Uber die Struktur zerfallender bizyklischer p-Gruppen', J. Reine Angew. Math. 241 (1970a),118-146

[8] Wolfgang Lindenberg, `Uber die Struktur zerfallendernicht-modularer bizyklischer 2-Gruppen', Gesellschaft fÄur Math. und Datenverarbeitung, Bonn 29 (1970b).

[9] Wolfgang Lindenberg, `Struktur and Klassi¯zierung bizyklischer p-Gruppen', Gesellschaft fÄur Math. und Datenverarbeitung, Bonn 40 (1971)

[10] Ian D. Macdonald, The Theory of Group, (Oxford University Press , 1968)

[11] M.F. Newman and Ming-Yao Xu, `Metacyclic Groups of Prime-power Order' Preprint.

[12] Derek J. Robinson, `A Course in the Theory of Groups', Graduate Texts in Math. 80(1982)

[13] Hyo-Seob Sim, Metacyclic Groups of Odd Order, PhD Thesis (Australian National University, 1992)

[14] Michio Suzuki, Group Theory I, (Springer-Verlag , 1982)