On non-prime 2-bridge θ-curves

(비기약적인 이교 θ-곡선들에 대하여)

Advisor : Prof. Hyun Jong Song

by

Seung Chan Lim

A thesis submitted in partial fulfillment of the requirement
for the degree of

Master of Education

Graduate School of Education
Pukyong National University

August 2005

On non-prime 2-bridge θ-curves

A Dissertation

by

Seung Chan Lim

Approved by :

(Chairman) Young Gheel Baik, Ph. D.

(Member) Hyo Seob Sim, Ph. D.

(Member) Hyun Song Song, Ph. D.

August 31, 2005

Contents

비기약적인 이교 θ-곡선들에 대하여

임 승 찬

부경대학교 교육대학원 수학교육전공

요 약

Goda가 이교 매듭(2-bridge knot)과 같이 이교 θ-곡선(2-bridge θ-curve) 역시 기약(prime)임을 주장하였으나 Motohashi가 비 기약인 이교 θ-곡선이 존재함을 밝히고 이들을 완전히 분류하였다.

본 논문에서는 Song의 결과를 이용하여 강전위적 이교 매듭(strognly invertible 2-bridge knot)들의 equivariant connected sum으로부터 Motohashi의 비 기약인 이교 θ-곡선들을 얻을 수 있음을 보인다.

1 Introduction

Let a be a properly embedded arc in the 3-ball B, namley $a \cap \partial B = \partial a$. Then an arc a is said to be trivial if and only if there exists a disk D embedded in B so that ∂D consists of two arcs a and b with $\partial a = \partial b$ and b on ∂B. Here such a disk is said to be a spanning disk of a trivial arc. More generally, a set $\{a_1, \cdots, a_n\}$ of n-number of arcs properly embedded in the 3-ball B is said to be a n-trivial tangle if and only if each arc $a_i, 1 \leq i \leq n$ has a spanning disk which is disjoint with the others.

Let K be a knot in the three sphere S^3. We say that K admits a n-bridge decomposition if and only if (S^3, K) is decomposed into a union $(B^+, B^+ \cap K) \cup_{S=\partial B^+ = \partial B^-} (B^-, B^- \cap K)$ of two trivial n-string tangles so that $(B^+, B^+ \cap K) \cap (B^-, B^- \cap K) = S \cap K$ consists of $2n$-points.

The 2-sphere S in the above discussion is said to be a n-bridge decomposing sphere of K. It is easy to see that every knot K admits a n-bridge decomposition and the minimal such a number n is said to be a bridge index of K (denote $b(K)$). And, K is said to be a n-bridge knot if and only if $b(K) = n$. Using a smooth embedding of K in S^3, we have a following description of a n-bridge decomposition which is equivalent to the above piece-wise linear description. Let $I = [-1, 1]$ be a closed interval. Without loss of a generality we may assume that any knot K is embedded in $R^2 \times I \subset S^3 = R^2 \times R \cup \{\infty\}$. For the projection map $\pi : R^2 \times I \to I$, K is embedded in $R^2 \times I$ so that each plane $\pi^{-1}(t)$ transversely meets K except finitely many critical points with either local maxima or local minima. We see that K admits a n-bridge decomposition if and only if K is embedded in $R^2 \times I \subset S^3 = R^2 \times R \cup \{\infty\}$ so that each local maximum appears in the above of all local minima where n is the number of all local maxima which is equal to that of all local minima. It is easy to see that any 1-bridge knot is an unknot which bounds a disk in S^3. Fig. 1(a) shows one of well-known 2-bridge knots;

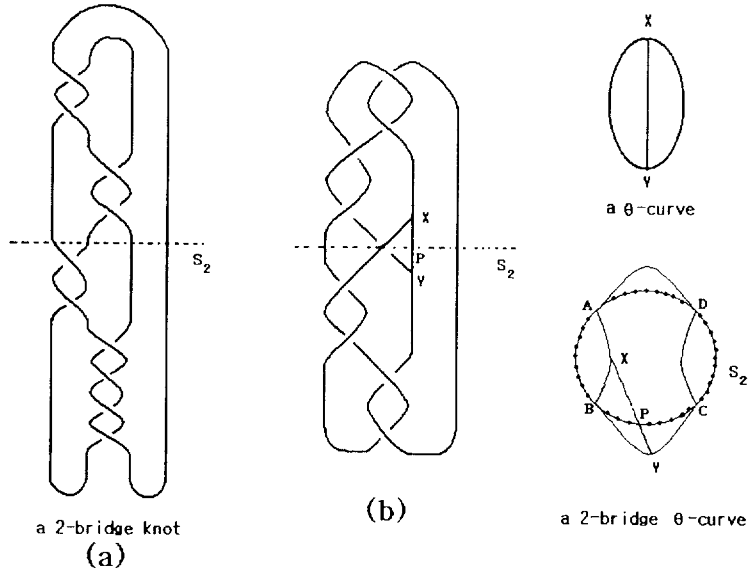
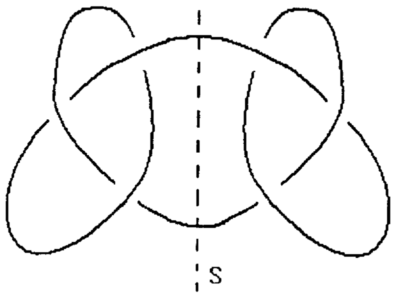


Fig. 1

The connected sum (or composition) of the knots K_1 and K_2, denoted by $K_1 \# K_2$, is an oriented knot obtained from the disjoint union of the manifold pairs $(S^3 - Int(B_i^3), K_i - Int(B_i^1))(i = 1, 2)$ by pasting their boundaries along an orientation-reversing homeomorphism ψ : $(\partial B_2^3, \partial B_2^1) \to (\partial B_1^3, \partial B_1^1)$ as illustrated in Fig. 2. In this definition the 2-sphere $S = \partial B_1^3 = \partial B_2^3$ is said to be a prime decomposing sphere if and only if both K_1 and K_2 are non-trivial knots.

A knot K is called prime if and only if it does not admit a prime decomposing sphere, namely suppose we have a decomposition $K = K_1 \# K_2$, then one of K_1 and K_2 must be a trivial knot. It is well known that any 2-bridge knot is prime whereas there are non-prime 3-bridge knots. For instance, by using Schubert theorem, namely $b(K_1 \# K_2) = b(K_1) + b(K_2) - 1$, we see that a connected sum of two 2-bridge knots is a 3-bridge non-prime knot.



S : a prime decomposing sphere

Fig. 2

For a circle $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, take a graph $\theta = S^1 \cup \overline{NS}$ with two vertices and three edges where $\overline{NS}$ is the edge joining $N = (0, 1)$ and $S = (0, -1)$. Then an embedding $i(\theta)$ of θ in S^3 is said to be a θ-curve. Thus a θ-curve can be thought of as a natural generalization of a knot. We may introduce a concept of a bridge decomposition and that of prime decomposition as we did on a knot. In [6] and [11] they showed that it plays a crucial role in investigating a tunnel number one knot. For more details, see section 2.3. Goda[3] claimed that a θ-curve with a 2-bridge decomposition is prime but Motohashi[7] provided counterexamples to the Goda's claim. Moreover she showed that they are a complete set of non-prime 2-bridge θ-curves. The purpose of this paper is to show that the Motohashi's non-prime 2-bridge θ-curves arise from equivariant prime decompositions of 2-bridge knots.

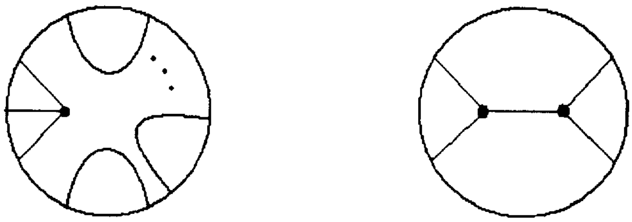
2 Preliminaries

2.1 2-bridge θ-curves

Now we introduce a concept of a *θ-curve* which naturally arises from study of either a strongly invertible knot or a tunnel number one knot. A *θ-curve* is a graph in S^3 which consists of two vertices and three edges joining the two vertices. A *labeling* of a θ-curve is a total ordering on the set of the edges and a choice of one of the vertices. The ith edge in the ordering is denoted by e_i, and the vertex of our choice is denoted by v_1 and the other v_2. All our θ-curves will be labeled. Two θ-curves are *equivalent* if there exists an orientation preserving self-homeomorphism of S^3 taking one to the other which respects the labelings. A θ-curve Γ is *trivial* if there exists a 2-sphere in S^3 which contains Γ. For a concept of bridge decomposition of a *θ-curve*, we slightly extend the well known term "tangle" for a knot to that for a *θ-curve* as follows;

The pair (B, t) is called a *tangle* if B is a 3-ball and each component of t is a graph properly embedded in B. A tangle (B, t) is *trivial* if there is a union Δ of mutually disjoint disks Δ_i properly embedded in B such that Δ contains t and each Δ_i contains just one component of t. We call Δ *trace disks* of the tangle. A trivial tangle (B, t) is *k-bridge* if t is a union of one star of 3-degree and $k - 1$ arcs, see Fig. 3(a).

Let Γ be a θ-curve and T a 2-sphere in S^3. We call T a *bridge decomposing sphere* of Γ if $(B_i, B_i \cap \Gamma)$ is a k-bridge tangle for $i = 1, 2$, where B_i is a 3-ball in S^3 bounded by T which contains v_i. Then we say that Γ has a *k-bridge decomposition* and denote it by $(S^3, \Gamma) = (B_1, B_1 \cap \Gamma) \cup (B_2, B_2 \cap \Gamma)$. The *bridge number* $b(\Gamma)$ of Γ is the smallest integer k for which Γ has a k-bridge decomposition. If $b(\Gamma) = k$, then Γ is said to be *k-bridge*. Note that any θ-curve Γ has a k-bridge decomposition for some integer k, see



(a) (b)

Fig. 3

[3, Proposition 2.1], and that $b(\Gamma) = 1$ if and only if Γ is trivial, see [3, Proposition 2.2].

Let Γ be a θ-curve and S a 2-sphere in S^3. We call S a *decomposing sphere* of Γ if S does not contain a vertex of Γ and S meets each edge of Γ transversely at exactly one point as illustrated in Fig. 1(b). Let B_i be a 3-ball in S^3 bounded by S which contains v_i. We construct a new θ-curve Γ_1 (respectively Γ_2) from Γ by replacing $(B_2, B_2 \cap \Gamma)$ (respectively $(B_1, B_1 \cap \Gamma)$) by a 1-bridge tangle. The labeling of Γ_i is defined to be the one induced from Γ. Note that Γ_1 (respectively Γ_2) is equivalent to a θ-curve which is obtained from Γ by contracting B_2 (respectively B_1) to v_2 (respectively v_1). Then we say that Γ is decomposed into Γ_1 and Γ_2, and denote it by $\Gamma = \Gamma_1 \# \Gamma_2$. A decomposition $\Gamma = \Gamma_1 \# \Gamma_2$ by a decomposing sphere S is *efficient* if each Γ_i is nontrivial, and then we say that S is *efficient*. A θ-curve Γ is *prime* if Γ is nontrivial and does not have an efficient decomposition.

A θ-curve Γ is *(i, j)-rational* (or *rational*) if Γ is nontrivial and there exists a 2-sphere which bounds two 3-balls B_1 and B_2 in S^3 such that $(B_1, B_1 \cap \Gamma)$ is a trivial tangle as in Fig. 3(b) and $(B_2, B_2 \cap \Gamma)$ is a trivial tangle with two arcs contained in $e_i \cup e_j$. Note that a (i, j)-rational θ-curve Γ contains a nontrivial knot $e_i \cup e_j$. For, if $e_i \cup e_j$ is trivial, then Γ is trivial, see [4].

The following theorem is proved by Motohashi[7].

Theorem 2.1.1([7, Theorem1.1]) A θ-curve Γ is non-prime and 2-bridge if and only if Γ is decomposed into a (i,j)-rational θ-curve and a (j,k)-rational θ-curve, where $\{i,j,k\} = \{1,2,3\}$.

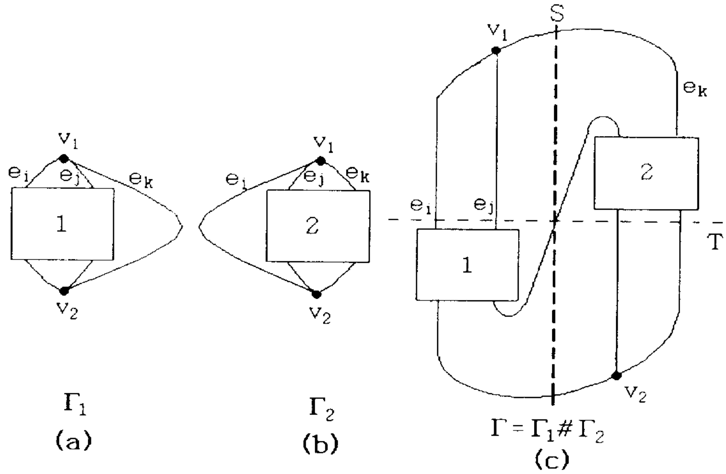


Fig. 4

2.2 Equivariant prime decompositions of 2-bridge θ-curves

For given two strongly invertible knots $(K_i, h_i)(i = 1,2)$, we define their equivariant connected sum $(K_1,h_1)\#(K_2,h_2)$ as follows. Let z_i be a point of $\mathrm{Fix}(h_i) \cap K_i$ and B_i be an equivariant regular neighbourhood of z_i for each $i = 1,2$, and let f be an orientation-reversing equivariant homeomorphism from $\partial(B_1, B_1 \cap K_1)$ to $\partial(B_2, B_2 \cap K_2)$. Then the manifold pair

$$\{(S^3,K_1)-(Int(B_1),Int(B_1)\cap K_1)\}\cup_f\{(S^3,K_2)-(Int(B_2),Int(B_2)\cap K_2)\}$$

is homeomorphic to $(S^3, K_1\#K_2)$, and the involutions h_1 and h_2 naturally determine an inverting involution h of $(S^3, K_1\#K_2)$. We want to define

$(K_1, h_1)\#(K_2, h_2)$ to be $(K_1\#K_2, h)$. But there remains the following ambiguities in this "definition".

(1) The choice of the point $z_i \in \mathrm{Fix}(h_i) \cap K_i \cong S^0$ for each $i = 1, 2$

(2) The choice of the equivariant homeomorphism f.

To remove these ambiguities, we attach the following additional informations to each strongly invertible knot (K, h).

(i) An orientation of $\mathrm{Fix}(h)$.

(ii) A "base point" ∞ of $\mathrm{Fix}(h)$, which lies in one of the components of $\mathrm{Fix}(h) - K$.

We call these additional informations a *direction* of (K, h), and a strongly invertible knot with a direction is said to be *directed.* We can now define the equivariant connected sum of two directed strongly invertible knots $(K_i, h_i)(i = 1, 2)$. This indication clearly resolves the ambiguity (1). Concerning the ambiguity (2), it specifies the restriction of f to $\partial B_1 \cap \mathrm{Fix}(h_1)$. Let g be another orientation-reversing equivariant homeomorphism from $\partial(B_1, B_1 \cap K_1)$ to $\partial(B_2, B_2 \cap K_2)$ of which restriction to $\partial B_1 \cap \mathrm{Fix}(h_1)$ is equal to that of f. Then g is equivariantly isotopic to either f or $(h_2|_{\partial B_2}) \circ f$ rel. $\partial B_1 \cap K_1$. [Proof. The equivariant homeomorphism $g \circ f^{-1}$ on $\partial(B_2, B_2 \cap K_2)$ induces a homeomorphism ψ on $\partial B_2 / h_2$ which is identity on the subset $P = \{\partial B_2 \cap (\mathrm{Fix}(h_2) \cup K_2)\}/h_2$. Note that P consists of three points. Then, by Theorem 4.5 of [1], ψ is isotopic to the identity map rel. P. This isotopy lifts to an equivariant isotopy between $g \circ f^{-1}$ and either the identity map or $h_2|_{\partial B_2}$.] Moreover, f and $(h_2|_{\partial B_2}) \circ f$ determine the equivalent strongly invertible knots. Thus the equivariant connected sum is well-defined for directed strongly invertible knots.

Definition 2.2.1

(1) (K, h) is said to be *trivial*, if K is a trivial knot and h is the standard inverting involution.

(2) (K, h) is said to be *prime*, if it is not trivial, and is not equivalent to a sum of two nontrivial strongly invertible knots.

(3) For an oriented knot $k = (S^3, k)$, $D(k)$ denotes the strongly invertible knot $(k\# - k, h)$, where $-k$ denotes the knot $(S^3, -k)$ and h is the inverting involution which interchanges the factors k and $-k$.

(4) For a finite sequence $\{(K_i, h_i)|1 \leq i \leq n\}$ of directed strongly invertible knots, $\#_{i=1}^{n}(K_i, h_i)$ denotes the directed strongly invertible knot $(((K_1, h_1)\#(K_2, h_2))\#(K_3, h_3))\# \cdots)\#(K_n, h_n)$.

The set $\tilde{\mathcal{S}}$ of all directed strongly invertible knots together with the operation $\#$ forms a non-commutative semi-group. It is easily seen that $D(k)$ (with any direction) belongs to the center of the semi-group. (Furthermore, by the unique decomposition theorem stated below, the center consists only of $D(k)$'s.) We have the followings.

Lemma 2.2.2 (1) (K, h) is trivial, iff K is trivial.
(2) (K, h) is prime, iff K is prime or $(K, h) \cong D(k)$ for some prime knot k.

Theorem 2.2.3 (1) Any nontrivial, directed, strongly invertible knot (K, h) has an equivariant prime decomposition. Any prime decomposition is equivalent to a decomposition $(K, h) \cong \{\#_{i=1}^{r}(K_i, h_i)\}\#\{\#_{j=1}^{s}D(k_j)\}$ where $K_i (1 \leq i \leq r)$ and $k_j (1 \leq j \leq s)$ are prime knots.
(2) Let $\{\#_{i=1}^{r}(K_i, h_i)\}\#\{\#_{j=1}^{s}D(k_j)\}$ and $\{\#_{i=1}^{r'}(K'_i, h'_i)\}\#\{\#_{j=1}^{s'}D(k'_j)\}$ be prime decompositions of a directed strongly invertible knot. Then the following hold.

(a) $r = r'$ and $(K_i, h_i) \cong (K'_i, h'_i)$ for each $i (1 \le i \le r)$.

(b) $s = s'$ and after a permutation $D(k_j) \cong D(k'_j)$ for each $j (1 \le j \le s)$.

Here, $\cong$ denotes the equivalence as directed strongly invertible knots.

To prove Lemma 2.2.2(2) and Theorem 2.2.3, we need the following: For a strongly invertible knot (K, h), let $\theta(K, h) = p(\mathrm{Fix}(h)) \cup p(K)$, where p is the projection $S^3 \to S^3/h \cong S^3$. We call it the *θ-curve* associated with (K, h). $\theta(K, h)$ is said to be *prime*, if it is "nontrivial", and if every 2-sphere intersecting $\theta(K, h)$ transversely at three points bounds a 3-ball B such that $(B, B \cap \theta(K, h))$ is homeomorphic to the cone over $\partial(B, B \cap \theta(K, h))$. $\theta(K, h)$ is said to be *irreducible*, if it is prime, and does not contain a local knot. Note that the 2-fold branched cover $\Sigma(K)$ of K is homeomorphic to the $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ branched cover of $\theta(K, h)$. Let τ be the covering transformation of $\Sigma(K)$, and $\tilde{h}$ be a lift of h to $\Sigma(K)$. Then using known results, we obtain the following equivalences. For more details, see [9].

(i) K is trivial. $\leftrightarrows$ $\Sigma(K) \cong S^3$.
$\leftrightarrows$ $\theta(K, h)$ is trivial.

(ii) K is prime. $\leftrightarrows$ $\Sigma(K)$ has no essential 2-sphere.
$\leftrightarrows$ $\theta(K, h)$ is irreducible.

(iii) (K, h) is prime. $\leftrightarrows$ $\Sigma(K)$ does not have $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ invariant essential 2-sphere.
$\leftrightarrows$ $\theta(K, h)$ is prime.

(iv) (K, h) is prime, but K is not prime.
$\leftrightarrows$ $\Sigma(K)$ contains an essential 2-sphere S, such that $\tau(S) = S$ and $\tilde{h}(S) \cap S = \emptyset$.
$\leftrightarrows$ $\theta(K, h)$ is prime, but not irreducible.
$\leftrightarrows$ $(K, h) \cong D(k)$ for some prime knot k.

In particular, we obtain Lemma 2.2.2(2). The first half of Theorem 2.2.3 follows from Lemma 2.2.2 and the existence of the prime decomposition of a knot. The above observations say that we have only to show the uniqueness of the "prime decomposition" of a θ-curve to prove the latter half of Theorem 2.2.3. But this can be done by a standard cut and paste method; so, we omit it. We note that the prime decomposition of a θ-curve mentioned here may be considered as the prime decomposition of it as an orbifold [A θ-curve is viewed as the branch line of a $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ branched cover.], and the existence and the uniqueness of the prime decomposition of a "pseudo-good" orbifold are claimed by Bonahon-Siebenmann.

2.3 Dihedral branched coverings of 2-bridge θ-curves

A knot K in S^3 is said to be *strongly invertible* if there is an involution h (called a *strong inversion*) of the pair (S^3, K) such that $\mathrm{Fix}(h)$ is a circle intersecting K in two points. Considering the double covering projection $p : S^3 \to S^3/h\,(\cong S^3)$ branched over a trivial knot $p(\mathrm{Fix}(h))$, we have a θ-curve $\theta(K, h) \equiv p(\mathrm{Fix}(h) \cup K)$ induced by the pair (K, h).

Let K be a knot with a n-bridge decomposition $(S^3, K, S) = (B_1, t_1) \cup (B_2, t_2)$, where $S = \partial B_1 \cap \partial B_2$ denotes the associated bridge decomposing sphere. Then a strong inversion h of a knot with a n-bridge decomposition (S^3, K, S) is said to be *bridge-preserving* (respectively *bridge-exchanging*) if and only if $h(B_i, t_i) = (B_i, t_i)$ for each $i = 1$ and 2 (respectively $h(B_1, t_1) = (B_2, t_2)$ and vise versa).

Recall from section 2.1 that a θ-curve is said to admit a 2-bridge decomposition, if and only if (S^3, θ) is a union of (B_1, t_1, a_1) and (B_2, t_2, a_2) along their boundary $S_2 = \partial B_1 = \partial B_2$, where (B_i, t_i) (respectively a_i) is a 2-strand trivial tangle (respectively a trivial arc in (B_i, t_i)) for $i = 1, 2$ as illustrated in Fig. 5(a), and it is said to admit a 3-bridge decomposition, if and only if (S^3, θ) is a union of (B_1, t_1, a) and $(B_2, t_2, \emptyset)$ along their boundary $S_3 = \partial B_1 = \partial B_2$, where (B_i, t_i) is a 3-strand trivial tangle for $i = 1, 2$ and a is a trivial arc in (B_1, t_1) as illustrated in Fig. 5(b).

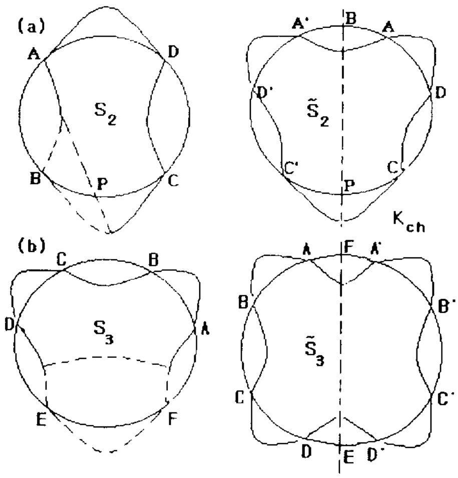


Fig. 5

In the above definition S_g is said to be a *bridge decomposing sphere* and a θ-curve admitting bridge decomposition sphere S_g is denoted by (θ, S_g).

Remark. With replacement of the 3-strand trivial tangle by a 2-strand trivial tangle in the definition of (θ, S_3), we have a rational θ-curve as studied by Harikae[4].

In the sequel, we assume that $g = 2$ or 3 otherwise it is stated explicitly. The following lemma immediately follows from the definition of (θ, S_g).

Lemma 2.3.1 A θ-curve with a bridge decomposition (θ, S_g) induces those of its three constituent knot $C_i = \theta - \mathrm{Int}(e_i)$ such that
(i) for $g = 2$, C_1 has a 1-bridge decomposition and C_2, C_3 have 2-bridge decompositions,
(ii) for $g = 3$, C_1, C_2 and C_3 have 1, 2 and 3-bridge decomposition, respectively.

Lemma 2.3.2 For each pair (θ, S) of a θ-curve and its g-bridge decomposing sphere $S(\equiv S_g)$, we have a triple $(K, \tilde{S}, h)$ of a knot K, its $(g+1)$-bridge decomposing sphere $\tilde{S}$ and a bridge preserving strong inversion h.

Conversely a bridge preserving strong inversion of a knot with a $(g+1)$-bridge decomposition induces a θ-curve with a g-bridge decomposition.

Proof. Consider the double covering projection $\pi : \tilde{S}^3 = \tilde{B}_1 \cup \tilde{B}_2 \to S^3 = B_1 \cup B_2$ branched over a 1-bridge constituent knot C_1 where each $\tilde{B}_i$ is the 3-ball covering B_i. Since C_1 is a trivial knot, so is $\pi^{-1}(C_1)$ in the covering 3-sphere $\tilde{S}^3$. Then $\pi^{-1}(e_1)$, the lifting of the edge $e_1 = \overline{\theta - C_1}$ is a knot in $\tilde{S}^3$ with a bridge decomposition $(\tilde{S}^3, \pi^{-1}(e_1)) = (\tilde{B}_1, \pi^{-1}(e_1 \cap B_1)) \cup (\tilde{B}_2, \pi^{-1}(e_1 \cap B_2))$ where $\pi^{-1}(e_1 \cap B_i)$ consists of $g+1$ trivial arcs for each $i = 1, 2$ as illustrated in Fig. 5. Moreover $\pi^{-1}(C_1)$ forms the fixed circle of a bridge preserving strong inversion for a pair $(K_{ch} \equiv \pi^{-1}(e_1),\ \tilde{S} \equiv \pi^{-1}(S))$.

By tracing the above argument backwards, we have the converse. $\square$

We call the knot K in Lemma 2.3.2 *the characteristic knot* of (θ, S_g) and denote it by K_{ch}.

Remarks. There are strong inversions of 3-bridge knots which may not be bridge preserving. For instance, a 3-bridge knot 9_{40} has a strong inversion h such that (θ, h) has a constituent knot $8_{21} = M(1; (2,1), (3,2), (3,2))$. Thus h cannot be bridge preserving by Lemmas 2.3.1 and 2.3.2. On the other hand, each of 3-bridge knots 10_{155} and 10_{157} with the antipodal (i.e., 2-freely periodic) symmetry has two strong inversions such that one is bridge preserving and the other is bridge exchanging.

A pair of θ-curves with bridge decompositions (θ_i, S_i), $i = 1$ and 2, is said to be homeomorphic, if and only if there exists an orientation preserving homeomorphism $\Psi : S^3 \to S^3$ such that $\Psi(\theta_1, S_1) = (\theta_2, S_2)$. On the other hand a pair of bridge preserving strong inversions h_i of K_i with a bridge decomposing sphere S_i, $i = 1$ and 2, is said to be homeomorphic, if and only if there exists an orientation preserving homeomorphism $\Pi : S^3 \to S^3$ such that $\Pi(K_1, S_1) = (K_2, S_2)$ and $h_2 = \Pi \circ h_1 \circ \Pi^{-1}$. Under these concepts of equivalences, we can easily see that each θ-curve with a bridge decomposition (θ, S) uniquely corresponds to a triple $(K_{ch}, \tilde{S}, h)$ up to their homeomorphic types.

The $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-branched covering of (θ, S_g)

Denote the dihedral group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ by D_2. It is well known that for any θ-curve in S^3, we have the D_2 covering projection $\pi_{D_2} : M \to S^3$ branched over θ which is induced by a monodromy map from the fundamental group of θ to D_2.

If a θ-curve admits a bridge decomposing sphere S_g, then we shall see that the branch set upstairs $\pi_{D_2}^{-1}(\theta)$ can be realized by fixed point circles of three (orientation preserving) involutions of M which preserve each handlebody in a Heegaard decomposition of M with genus g. Hence restriction of π_{D_2} on the associated Heegaard surface F_g induces the covering projection $\pi_{D_2}|_{F_g} : F_g \to S_g$ branched over $\theta \cap S_g$.

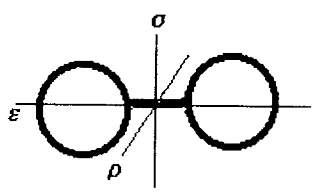


Fig. 6

Let S^1_ϵ, S^1_σ and S^1_ρ be a triple of circles in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ each pair of which meets orthogonally at $S^1_\epsilon \cap S^1_\sigma \cap S^1_\rho$. Then π-rotation with respect to S^1_ϵ, S^1_σ and S^1_ρ induce involutions ϵ, σ and ρ of S^3 respectively such that $\rho = \epsilon \circ \sigma = \sigma \circ \epsilon$. Let $D_2 = \langle \epsilon, \sigma : \epsilon \circ \sigma = \sigma \circ \epsilon \rangle$. Then S^3 has D_2 symmetry with a pair of global fixed points $S^1_\epsilon \cap S^1_\sigma \cap S^1_\rho$.

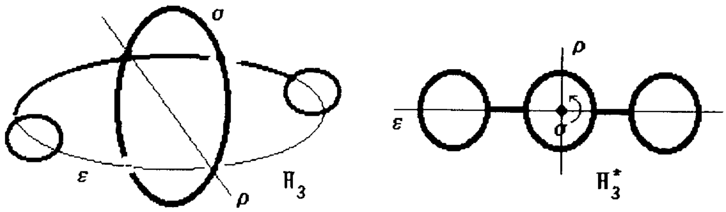


Fig. 7

Now, we consider a handlebody H_g of genus g standardly imbedded in S^3 so that the D_2-action of S^3 can be restricted on H_g. Here we assume that one of involutions in D_2, say ϵ, is always taken as the standard involution of H_g so that $S^1_\epsilon \cap H_g$ may consist of $(g+1)$-arcs.

Then we have a set $\mathcal{M}_g$ of $(g+1)$-meridian discs of H_g with the following action of a non-trivial involution $I \in D_2$ on each meridian disc $D \in \mathcal{M}_g$:

(1) if $\mathrm{Fix}(I) \cap D = \emptyset$, then $I(D)$ is another meridian disc $E \in \mathcal{M}_g$;

(2) if $\mathrm{Fix}(I) \cap D$ is a single point (and hence a global fixed point of D_2), then $I(D) = D$ and I preserves the orientation of D;

(3) if $\mathrm{Fix}(I) \cap D$ is an arc, then $I(D) = D$ and I reverses the orientation of D.

We call $\mathcal{M}_g$ a system of D_2-equivariant meridian discs of H_g. Let $\mathcal{M}_g^*$ be a system of D_2-equivariant meridian discs of $H_g^* = S^3 - Int(H_g)$. Since a non-trivial involution $I \in D_2$, $I \neq \epsilon$, has the action of type (1) on each meridian disc in $\mathcal{M}_g$ or $\mathcal{M}_g^*$ which does not contain any global fixed point of D_2, we have:

Case $g = 2$. Each global fixed point of D_2 should lie on each handlebody H_2 and H_2^*, respectively. For a meridian disc D_f in $\mathcal{M}_2$ (respectively D_{f^*} in $\mathcal{M}_2^*$) containing a global fixed point f (respectively f^*), two involutions which have the action of type (2) on D_f and D_{f^*} must be the same. We take such an involution as σ. Then σ transposes the two meridian discs of $\mathcal{M}_2 - \{D_f\}$ and those of $\mathcal{M}_2^* - \{D_{f^*}\}$ as illustrated in Fig. 6.

Case $g = 3$. Both global fixed points f_1, f_2 of D_2 should lie on one of the two handlebodies, say H_3. And, two involutions which have the action of type (2) on D_{f_1} and D_{f_2} must be the same. We take such an involution as σ. Then S^1_σ forms a core of H_3 transversely meeting the meridian discs D_{f_1} and D_{f_2}, and σ transposes the two meridian discs of $\mathcal{M}_3 - \{D_{f_1}, D_{f_2}\}$. On the other hand, σ acts freely on H_3^* and pairwise transposes two meridian discs of $\mathcal{M}_3^*$ as illustrated in Fig. 7.

Since an orientation-preserving involution of S^3 is conjugate to an orthogonal transformation, we see that a D_2-symmetry of H_g with its standard involution in D_2 is uniquely determined.

If we can choose a gluing homeomorphism ψ of the two handlebodies H_g and H_g^* so that it may be compatible with ϵ and σ, i.e., $\epsilon \circ \psi = \psi \circ \epsilon$ and $\sigma \circ \psi = \psi \circ \sigma$, then we have a 3-manifold with a Heegaard decomposition $M_g = H_g \cup_\psi H_g^*$ on which the dihedral group D_2 acts so that it may preserve each handlebody. We call such a Heegaard decomposition of a 3-manifold *D_2-symmetric.*

Further we assume that the gluing homeomorphism ψ is chosen so that M may be a $\mathbb{Z}_2$-homology 3-sphere, which is necessary for M to be the double branched covering of a knot K in S^3 or the D_2-branched covering of a θ-curve in S^3. Then by classification of a D_2 action on a $\mathbb{Z}_2$-homology 3-sphere, it is guaranteed that the fixed point sets of all three involutions of M form three circles intersecting in exactly two points.

If we denote the fixed point set of each involution $I \in \{\epsilon, \sigma, \rho\}$ of M by $\text{Fix}(I)$ and the union of them by $\text{Fix}(\mathcal{I})$, then we have the D_2-covering projection $\pi_{D_2} : M \to M/D_2 (\cong S^3)$ branched over a θ-curve $\pi_{D_2}(\text{Fix}(\mathcal{I}))$ with a bridge decomposing sphere $\pi_{D_2}(F_g)$ where F_g is a Heegaard surface associated with the Heegaard decomposition of M. And, for each $I \in \{\epsilon, \sigma, \rho\}$ we have the double covering projection $\pi_I : M \to M/I$ branched over a knot $K_I = \pi_I(\text{Fix}(I))$ in M/I whose Heegaard decomposition of genus g^*, $M/I = H_g/I \cup_{\tilde{\psi}} H_g^*/I$, naturally induces a (g^*, b)-decomposition of K_I in M/I where $\tilde{\psi} = \pi_I \circ \psi \circ (\pi_I)^{-1}$.

Details of such decomposition of K_I is given in the following proposition which can be easily read off given the D_2-action on the handlebodies.

Proposition 2.3.3 Let M be a $\mathbb{Z}_2$-homology 3-sphere admitting a D_2-symmetric Heegaard decomposition of genus g. Then we have:

Case $g = 2$.

(i) K_ϵ is a knot in S^3 with a 3-bridge decomposition and with a bridge preserving strong inversion h_ϵ such that $\text{Fix}(h_\epsilon) = \pi_\epsilon(\text{Fix}(\sigma) \cup \text{Fix}(\rho))$.

(ii) K_σ (respectively K_ρ) is a (1,1)-knot in a lens space M/σ (respectively M/ρ). And $\pi_\sigma(\text{Fix}(\epsilon) \cup \text{Fix}(\rho))$ (respectively $\pi_\rho(\text{Fix}(\epsilon) \cup \text{Fix}(\sigma))$) form the fixed point set of the standard involution of the lens space M/σ (respectively M/ρ) intersecting each unknotted string once in the (1,1)-decomposition of K_σ (respectively K_ρ).

Case $g = 3$.

(i) K_ϵ is a knot in S^3 with a 4-bridge decomposition and with a bridge preserving strong inversion h_ϵ such that $\text{Fix}(h_\epsilon) = \pi_\epsilon(\text{Fix}(\sigma) \cup \text{Fix}(\rho))$.

(ii) K_σ admits a (2,0)-decomposition in M/σ;

$$(M/\sigma, K_\sigma) = (H_3/\sigma, K_\sigma) \cup_{\psi_\sigma} (H_3^*/\sigma, \emptyset).$$

Further, $\pi_\sigma(\text{Fix}(\epsilon) \cup \text{Fix}(\rho))$ form the fixed point set of the standard involution of M/σ intersecting K_σ twice.

(iii) K_ρ admits a (1,2)-decomposition in a lens space M/ρ. And $\pi_\rho(\text{Fix}(\epsilon) \cup \text{Fix}(\sigma))$ form the fixed point set of the standard involution of the lens space M/ρ intersecting each of two unknotted strings once on one side of a solid torus of the (1,2)-decomposition of K_ρ.

Remark. All lens spaces (including S^3) in Proposition 2.3.3 must be of odd type, i.e., $L(p, q)$, $p \equiv 1 \pmod 2$ because they are the double branched coverings of constituent knots of the θ-curve with 2-bridge decompositions.

Conversely we have:

Theorem 2.3.4([11, Theorem 4]) Let (K, S_{g+1}, h) be a triple of knot K with a $(g+1)$-bridge decomposing sphere S_{g+1} and a bridge-preserving strong inversion h. Then the double branched covering space of (S^3, K) admits a D_2-symmetric Heegaard decomposition of genus g.

Proof. Taking a gluing homeomorphism ψ of the two handlebodies H_g and H_g^* provided by the $(g+1)$-bridge decomposition of K through the method in [2], we have the double covering projection $\pi : M = H_g \cup_\psi H_g^* \to S^3$ branched over K. Thus we have a set $\mathcal{M}_g$ (respectively $\mathcal{M}_g^*$) of $(g+1)$-meridian discs of H_g (respectively H_g^*) such that they may doubly cover the spanning discs of $(g+1)$-trivial arcs in the bridge decomposition of K. And, we have an involution ϵ of M with $\pi^{-1}(K)$, the lifting of K as the fixed circle. Since h is a bridge-preserving strong inversion of K, there are a pair of involutions $\tilde{h}_1, \tilde{h}_2$ of M such that $h \circ \pi = \pi \circ \tilde{h}_i$ $(i = 1, 2)$, $\pi^{-1}(\mathrm{Fix}(h)) = \mathrm{Fix}(\tilde{h}_1) \cup \mathrm{Fix}(\tilde{h}_2)$ and $\pi^{-1}(K \cap \mathrm{Fix}(h)) = \mathrm{Fix}(\tilde{h}_1) \cap \mathrm{Fix}(\tilde{h}_2)$.

In the case of $g = 3$, both points of $\pi^{-1}(K \cap \mathrm{Fix}(h))$ lie on one side of the two handlebodies, say H_3. Then one of the two circles, say $\mathrm{Fix}(\tilde{h}_1)$, in $\pi^{-1}(\mathrm{Fix}(h))$ transversely meets a pair of meridian discs in $\mathcal{M}_3$ which are determined by the two spanning discs of the trivial arcs containing $K \cap \mathrm{Fix}(h)$. Thus, $\mathrm{Fix}(\tilde{h}_1)$ forms a core of H_3 and $\tilde{h}_1$ is equivalent to σ. In the case of $g = 2$, $\pi^{-1}(K \cap \mathrm{Fix}(h))$ consists of a pair of points $\{p, p^*\}$ such that $p \in H_2$ and $p^* \in H_2^*$, respectively. Then, one of the two circles, say $\mathrm{Fix}(\tilde{h}_1)$, in $\pi^{-1}(\mathrm{Fix}(h))$ transversely meets a meridian disc in $\mathcal{M}_2$ (respectively $\mathcal{M}_2^*$) which is determined by the spanning disc of the trivial arc containing p (respectively p^*) in the given bridge decomposition of K. Thus $\tilde{h}_1$ is equivalent to σ. □

By Lemma 2.3.2 and Theorem 2.3.4, we have:

Corollary 2.3.5 Let (θ, S_g) be a θ-curve with a bridge decomposing sphere S_g. Then the D_2-branched covering of (θ, S_g) admits a D_2-symmetric Heegaard decomposition of genus g such that the associated Heegaard surface covers S_g.

By considering the D_2-branched covering of (θ, S_g), we have a refinement of the Morimoto–Sakuma–Yokota's method of studying tunnel 1 knots.

Theorem 2.3.6([6, Theorem 1.2 (1) and (2)]) A knot K in S^3 is a $(1,1)$-knot (respectively a tunnel-1 knot). if and only if there exists a strong inversion h of K such that
(i) θ-curve $\theta(K, h)$ admits a 2 (respectively 3)-bridge decomposing sphere S_2 (respectively S_3) and
(ii) $p(\mathrm{Fix}(h))$ forms a trivial constituent knot of $(\theta(K, h), S_2)$ (respectively $(\theta(K, h), S_3)$) with a 2-bridge (respectively 3-bridge) decomposition where p is the projection $S^3 \to S^3/h$.

3 Main Results

This section contains a main result of this paper which shows that non-prime 2-bridge θ-curves in Motohashi's theorem are induced by strong inversions of 2-bridge knots. For more precise statement of our result, we briefly recall the following classification theorem of the strong inversions of 2-bridge knots; Let $b(p, q)$ be the 2-bridge knot of type (p, q), $1 < |q| < p$, namely for the unique continued fraction expansion

$$p/q = b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{b_4 + \cdots + \cfrac{1}{b_n}}}}$$

where b_i, $1 \le i \le n$ and n are non-zero even integers(see [9]), we have a following 2-bridge presentation of $b(p,q)$ which naturally reveals its strong inversion σ;

Since $b(p,q)$ and $b(p,q')$ with $q \cdot q' \equiv 1 \pmod p$ represents the equivalent 2-bridge knot, we have another strong inversion of $b(p,q)$ through the continued fraction expansion of $b(p,q^{-1})$ if $q^2 \not\equiv 1 \pmod p$. But it is more convenient to express both strong inversions of $b(p,q)$ through so called a trisymmetric projection diagram of $b(p,q)$ as illustrated in Fig. 8(a). On the other hand, if $q^2 \equiv 1 \pmod p$, then there is another strong inversion ρ whose axis, namely the fixed point circle $\mathrm{Fix}(\rho)$ lies on the 2-bridge decomposing sphere of $b(p,q)$ as illustrated in Fig. 8(b) ; The strong inversion of type h_i is said to be bridge-preserving whereas that of g is said to be bridge-exchanging.

Now we recall the following classification theorem for strong inversions of 2-bridge knots, which is refined for forthcoming application.

Theorem 3.1([9, Proposition 3.6]) (1) If $q^2 \not\equiv 1 \pmod p$, then $b(p,q)$ has two non-isotopic strong inversions h_1 and h_2 such that $\theta(K,h_1)$ (resp. $\theta(K,h_2)$) is a rational θ-curve of type $p/2q$ (resp. $p/2q^{-1}$) .
(2) If $q^2 \equiv 1 \pmod p$, then $b(p,q)$ has two non-isotopic strong inversions h_3 and g such that $\theta(K,h_3)$ is a rational θ-curve of type $p/2q$ whereas $\theta(K,g)$ is not a rational θ-curve.
(3) h_i are bridge preserving whereas g is bridge exchanging and $\mathrm{Fix}(g)$ lies on the bridge decomposing sphere of $b(p,q)$.

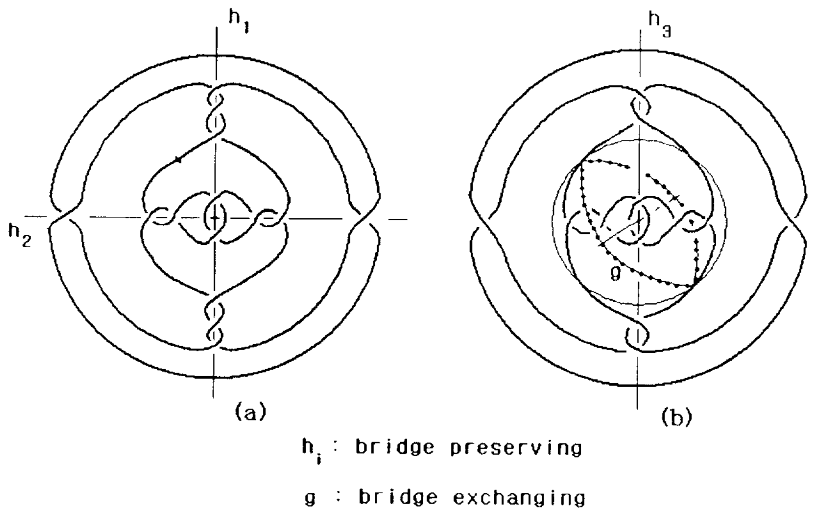

h1
h2
(a)
h3
g
(b)
hi : bridge preserving
g : bridge exchanging

Fig. 8

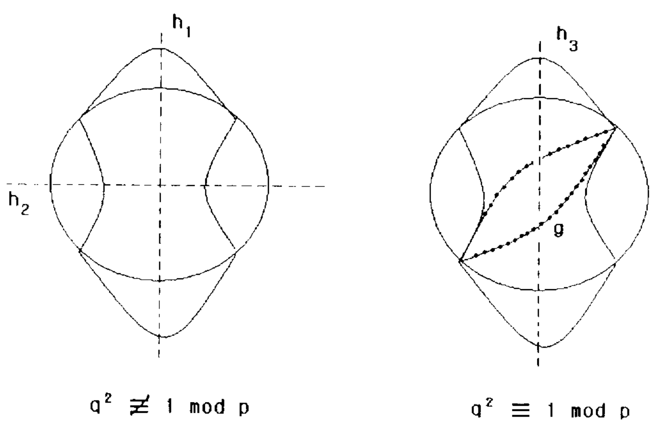

h1
h2
h3
g
q² ≢ 1 mod p
q² ≡ 1 mod p

Fig. 9

Remark. Harikae[4] called $\mathrm{Fix}(\rho) \cup \pi_\rho(K)$ a pseudo-rational θ-curve.

Although there is only one way of composing two knots (up to its knot type), this is not true of the equivariant composition of two strongly invertible knots as illustrated in Fig. 10.;

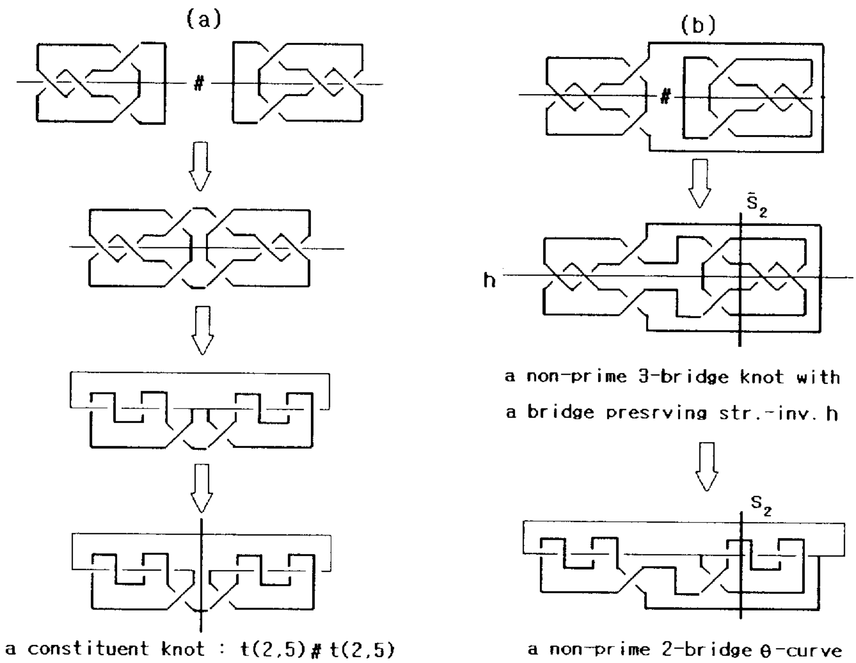


Fig. 10

In Fig. 10(a) we have an equivariant connected sum of two copies of $(b(5,2), \sigma)$ the associated θ-curve $\theta_1 \equiv \theta(b(5,2)\#b(5,2), h_1)$ of which has $b(5,1)\#b(5,1)$ as one of its constituent knots whereas in Fig. 10(b) we have another equivariant connected sum of two copies of $(b(5,2), \sigma)$ the associated θ-curve $\theta_2 \equiv \theta(b(5,2)\#b(5,2), h_2)$ of which has two copies of $(b(5,1)$ as its constituent knots. Note that the strong inversion h_2 of $b(5,2)\#b(5,2)$ is bridge preserving and hence by Lemma 2.3.2, θ_2 admits a 2-bridge decomposing sphere whereas θ_1 cannot admit a 2-bridge decomposing sphere by Theorem 2.3.6. Indeed we claim that all of the Motohashi's non-prime

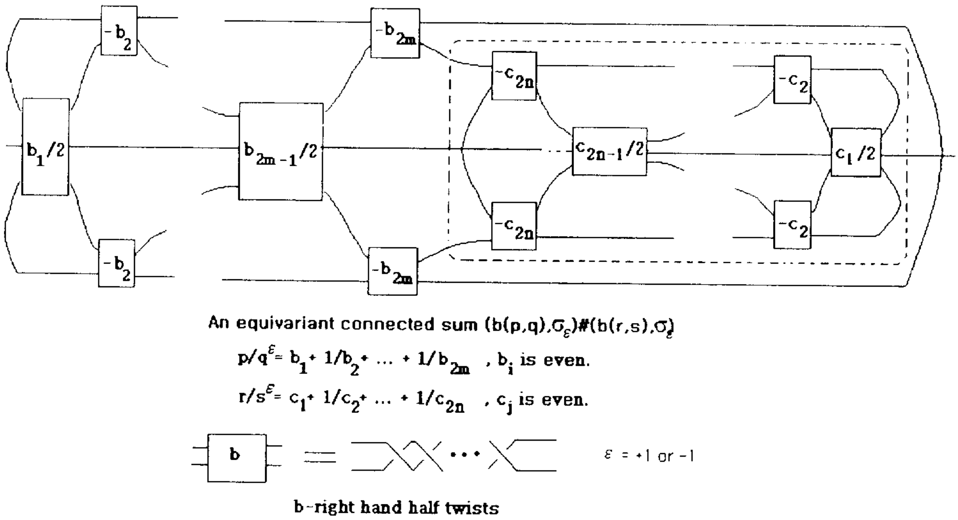
2-bridge θ-curves are obtained by taking equivariant connected sums of 2-bridge knots $(b(p,q)\#b(r,s),h)$ so that the strong inversion h may be bridge-preserving. This example is generallized as follows;

Theorem 3.2 An equivariant connected sum $(b(p,q),\sigma_i)\#(b(r,s),\sigma_j) = (b(p,q)\#b(r,s),h)$ with a bridge-preserving strong inversion h induces a non-prime 2-bridge θ-curve of type $(p/2q^i, r/2s^j)$ where $i,j \in \{+1,-1\}$.

Proof. From Fig. 11, we have an equivariant connected sum $(b(p,q),\sigma_i)\#(b(r,s),\sigma_j) = (b(p,q)\#b(r,s),h)$ with a strong inversion h which preserves a 3-bridge decomposing sphere of $b(p,q)\#b(r,s)$. By Lemma 2.3.2 and Theorem 2.2.3 $\theta(h)$, the θ-curve induced by h is non-prime and admits a 2-bridge decomposing sphere. Moreover, it is easy to see that the two non-trivial constituent knots of $\theta(h)$ determine type of the Motohashi's non-prime θ-curve as claimed in the theorem. □

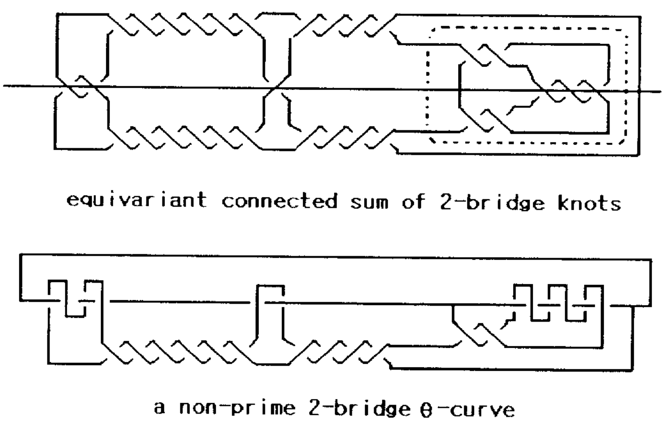
Combining Theorem 3.2 and Motohashi's theorem(Theorem 2.1.1), we have;

Corollary 3.3 Let $b(p_1.q_1)$ and $b(p_2,q_2)$ be 2-bridge knots with strong inversions h_1 and h_2 respectively. Then an equivariant connected sum $(b(p_1,q_1),h_1)\#(b(p_2,q_2),h_2) = (b(p_1,q_1)\#b(p_2,q_2),h)$ yields a 3-bridge non-prime knot with a bridge preserving strong inversion h if and only if both strong inversions h_1, h_2 are bridge preserving.



$-b_2$
$b_1/2$
$-b_2$
$b_{2m-1}/2$
$-b_{2m}$
$-b_{2m}$
$-c_{2n}$
$c_{2n-1}/2$
$-c_{2n}$
$-c_2$
$c_1/2$
$-c_2$
An equivariant connected sum $(b(p,q),\sigma_\varepsilon)\#(b(r,s),\sigma_\varepsilon)$
$p/q^{\varepsilon} = b_1 + 1/b_2 + \ldots + 1/b_{2m}$, b_i is even.
$r/s^{\varepsilon} = c_1 + 1/c_2 + \ldots + 1/c_{2n}$, c_j is even.
b
ε = +1 or -1
b-right hand half twists

Fig. 11



equivariant connected sum of 2-bridge knots
a non-prime 2-bridge θ-curve

Fig. 12

References

[1] J.S. Birman, Braid, links, and mapping class groups, Ann. Math. Studies, 82, Princeton Univ. Press, 1974

[2] J.S. Birman, H.M. Hilden, Heegaard splittings and branched coverings of S^3, Trans. Amer. Math. Soc. 213 (1975) 315-352

[3] H. Goda, Bridge index for theta curves in the 3-sphere, Topology Appl. 79 (1997) 177-196.

[4] T. Harikae, On rational and pseudo-rational θ-curves in the 3-sphere, Kobe J. Math. 7 (1990) 125-138.

[5] K. Kodama, M. Sakuma, Symmetry groups of prime knots up to 10 crossings, in: A. Kawauchi (Ed.), Knots '90. 323-340.

[6] K. Morimoto, M. Sakuma and Y. Yokota, Identifying tunnel number one knots, J. Math. Soc. Japan 48 (1996) 667-688.

[7] T. Motohashi, 2-bridge θ-curves in S^3, Topology Appl. 108 (2000) 267-276.

[8] D. Rolfsen, Knots and links, Math. Lecture Ser. 7, Publish or Perish Inc., Berkeley, 1976.

[9] M. Sakuma, On strongly invertible knots, Algebraic and Topological Theories, edited by M. Nagata et al., Kinokuniya, Tokyo, 1985, 176-196.

[10] H. Schubert, Knoten mit zwei Brüken, Math. Z., 65 (1956), 133-170.

[11] H.J. Song, Morimoto-Sakuma-Yokota's geometric approach to tunnel number one knots, Topology Appl. 127 (2003) 375-392.

[12] K. Taniyama, Cobordism of theta curves in S^3, Math. proc. Camb. Phil. Soc. 113 (97)(1993) 97-106.

[13] K. Wolcott, The knotting of theta curves and other graphs in S^3, in: Geometry and Topology, Marcel Dekker, New York, 1987, pp. 325-346.