On Optimal Solution Sets of Convex Quadratic Optimization Problems

볼룩 이차형 최적화 문제의 최적 해집합에 대한 연구

Adivisor : Prof. Gue Myung Lee

by

Bo Kyung Choi

A thesis submitted in partial fulfillment of the requirements
for the degree of

Master of Science

in the Department of Applied Mathematics, Graduate School,
Pukyong National University

August 2005

최보경의 이학석사 학위논문을 인준함.

2005년 8월 31일

주 심 이학박사 정 진 문

위 원 이학박사 이 규 명

위 원 이학박사 심 효 섭

On Optimal Solution Sets of Convex Quadratic Optimization Problems

A dissertation

by

Bo Kyung Choi

Approved by:

(Chairman) Jin Mun Jeong

(Member) Gue Myung Lee

(Member) Hyo Seob Sim

August 31, 2005

CONTENTS

볼록 이차형 최적화 문제의 최적 해집합에 대한 연구

최 보 경

부 경 대 학 교 대 학 원 응 용 수 학 과

요 약

본 논문에서는 이브 정리(Eaves Theorem)를 이용하여 볼록 2차형 최적화 문제의 최적해의 존재성을 밝히고, 라그랑지 승수(Lagrange multiplier)를 이용하여 볼록 2차형 최적화 문제의 최적해 집합의 특성화와 유계조건을 직접적으로 증명한다. 그리고 이러한 결과들을 적용하여 선형최적화 문제의 최적해 집합과 선형상보성 문제의 해집합에 대한 특성화와 유계조건을 얻는다. 본 논문에서 얻은 결과들의 의미와 중요성을 보이기 위해 볼록 2차형 최적화 문제, 선형 최적화 문제와 선형 상보성 문제에 대한 예들을 준다.

1 Introduction and Preliminaries

Quadratic optimization problems consist of quadratic objective functions and affine constraint functions. The main research topics for the problems are existence criteria of optimal solutions, properties of optimal solution sets, stability and sensitivity properties and numerical methodologies for finding optimality solutions. Affine variational inequality problems and linear complementarity problems are closely related to the problems. We say the quadratic optimization problem as a convex quadratic optimization problem when the quadratic objective function is defined with a symmetric positive semidefinite matrix.

On 1956, Frank and Wolfe [10] gave a very famous existence theorem for a quadratic optimization problems, which says that if a quadratic function is bounded from below on a nonempty polyhedral set, then the problem of minimizing the function on the set must have an optimal solution, and informed that the problem of minimizing a polynomial function of degree greater than 2 on a nonempty polyhedral set may not have optimal solutions even in the case that the function is bounded from below on the set. Blum and Oettli [4] presented direct proof of the existence result of Frank and Wolfe. Eaves [9] established existence criteria for quadratic optimization problem, and Lee, Tam and Yen [15] proved the criteria by using arguments of Blum and Oettli [4].

Recently, Mangasarian [17] initially presented simple and elegant characterizations of the optimal solution set of a convex minimization problem over a convex set when one optimal solution is known, which are very useful for understanding of the behaviour of optimal solution methods when the problem has multiple optimal solutions. In recent years, many authors [5, 8, 13, 14, 11, 12, 18] extended the Mangasarian's results to several optimization problems. Jeyakumar, Lee and Dinh [13] gave simple Lagrange based characterizations of solution set of a convex minimization problem with a geomeric constraint set and cone ineqaulity constraints by using Lagrange multipliers.

It is the purpose of this thesis to induce an existence theorem for optimal solutions of a convex quadratic optimization problem from Eaves theorem, to give a direct proof for Lagrange based characterizations and a boundedness condition of the optimal solution set of the problem when its one optimal solution is given, and to apply such results to a linear optimization problem and a linear complementarity problem. Moreover, we give examples illustrating our Lagrange based characterizations for a convex quadratic optimization problem, a linear optimization problem and a linear complementarity problem.

This thesis is organized as follows; In Section 1, we give definitions and preliminary results which will be used in next sections. In Section 2, we introduce Frank-Wolfe Theorem and Eaves Theorem, which are fundamental for existence of optimal solutions of a quadratic optimization problem. From Eaves Theorem, we induce an existence theorem for optimal solutions of a convex quadratic optimization problem. In Section 3, we directly prove Lagrange based characterizations and a boundedness condition of optimal solution set of a convex quadratic optimization problem when its one optimal solution is given, and give an example which illustrates the characterization result. In Section 4, we apply our results in Sections 2 and 3 to a linear optimization problem and a linear complementarity problem, and give examples ilustrating results in Section 4.

Now we give definitions and preliminary results which will be used in next sections. For two vectors $x = (x_1, \cdots x_n)$, $y = (y_1, \cdots y_n) \in \mathbb{R}^n$, the inequality $x \geqq y$ means $x_i \geqq y_i$ for all $i = 1, \cdots, n$.

Definition 1.1 *Let D be an $n \times n$ matrix. Then D is said to be positive semidefinite if for any $x \in \mathbb{R}^n$, $x^T Dx \geqq 0$.*

Remark 1.1 *If D is an $n \times n$ symmetric positive semidefinite matrix and $c \in \mathbb{R}^n$, then $f(x) := \frac{1}{2}x^T Dx + c^T x$ is convex on $\mathbb{R}^n$.*

Definition 1.2 [19] *Let S be a closed convex set in $\mathbb{R}^n$. The recession cone of S is defined as the following set ;*

$$S^\infty := \{v \in \mathbb{R}^n \mid x + \alpha v \in S, \ \forall x \in S, \ \forall \alpha \geqq 0\}.$$

Now we give examples of recession cones.

Example 1.1 *(1) Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$, C an $s \times n$ matrix and $d \in \mathbb{R}^s$ and let $S := \{x \in \mathbb{R}^n \mid Ax \geqq b, \ Cx = d\}$. Then $S^\infty := \{v \in \mathbb{R}^n \mid Av \geqq 0, \ Cv = 0\}$.*
(2) Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$ and let $S := \{x \in \mathbb{R}^n \mid Ax = b, \ x \geqq 0\}$. Then $S^\infty := \{v \in \mathbb{R}^n \mid Av = 0, \ v \geqq 0\}$.
(3) Let D be an $n \times n$ matrix and $c \in \mathbb{R}^n$ and let $S := \{x \in \mathbb{R}^n \mid Dx + c \geqq 0, \ x \geqq 0\}$. Then $S^\infty := \{v \in \mathbb{R}^n \mid Dv \geqq 0, \ v \geqq 0\}$.

Theorem 1.1 [19] *Let S be a closed convex set in $\mathbb{R}^n$. Then S is bounded if and only if $S^\infty = \{0\}$.*

Definition 1.3 *Let S be a set in $\mathbb{R}^n$. Then S is called a polyhedral set if it is the intersection of finite number of closed half-spaces, that is,*

$$S = \{x \in \mathbb{R}^n \mid {a_i}^T x \geqq b_i, \ \ i = 1, \cdots, m\}$$

where $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ for $i = 1, \cdots, m$.

Example 1.2 *All the sets S in Example 1.1 are polyhedral.*

Remark 1.2 [19] *Let S be a nonempty set in $\mathbb{R}^n$. Then S is polyhedral if and only if there exist finite elements in $\mathbb{R}^n$, $x_1, \cdots, x_k$, $d_1, \cdots, d_l$ such that*

$$\begin{aligned} S = \{\textstyle\sum_{i=1}^k \lambda_i x_i + \sum_{j=1}^l \mu_j d_j \ \mid \ & \lambda_i \geqq 0, \ \ i = 1, \cdots, k, \ \ \textstyle\sum_{i=1}^k \lambda_i = 1, \\ & \mu_j \geqq 0, \ \ j = 1, \cdots, l\} \end{aligned}$$

Definition 1.4 *Let S be a nonempty closed convex set in $\mathbb{R}^n$.*

(1) A vector $x \in S$ is called an extreme point of S if $x = \lambda x_1 + (1 - \lambda) x_2$ with $x_1, \ x_2 \in S$ and $\lambda \in (0, 1)$ implies that $x = x_1 = x_2$.

(2) $d \in S^\infty \backslash \{0\}$ is said to be an extreme direction if $d = \lambda_1 d_1 + \lambda_2 d_2$, $d_1, \ d_2 \in S^\infty \setminus \{0\}$ and $\lambda_1 > 0, \ \lambda_2 > 0$ implies that $d_1 = \alpha d_2$ for some $\alpha > 0$.

Theorem 1.2 [2] *Let $\Delta = \{x \in \mathbb{R}^n \mid Ax = b, \ \ x \geqq 0\}$ where A is an $m \times n$ matrix and $b \in \mathbb{R}^m$ and assume that $\Delta \neq \emptyset$ and rank$A = m$.*

(1) A point x is an extreme point (a basic feasible solution) of Δ if and only if A can be decomposed into (B, N) such that

$$x = \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}$$

where B is an $m \times m$ invertible matrix satisfying $B^{-1}b \geqq 0$.

(2) A nonzero vector $\bar{d}$ is an extreme direction of Δ if and only if A can be decomposed into (B, N) such that $B^{-1}a_j \leqq 0$ for some column a_j of N and $\bar{d}$ is a positive multiple of $d = \begin{pmatrix} -B^{-1}a_j \\ e_j \end{pmatrix}$ where e_j is an $n - m$ vector of zeros except for a 1 in position j.

Theorem 1.3 [1]**(Representation Theorem)** *Let $\Delta = \{x \in \mathbb{R}^n \mid Ax = b,\ x \geqq 0\}$ be a nonempty set. Then the set of extreme points is not empty and has a finite number of points, say $x_1, \cdots, x_k$. Furthermore, the set of extreme direction is empty if and only if Δ is bounded. If Δ is not bounded, then the set of extreme directions is nonempty and has a finite number of vectors, say $d_1, \cdots, d_l$. Furthermore,*

$$\Delta = \{\textstyle\sum_{i=1}^k \lambda_i x_i + \sum_{j=1}^l \mu_j d_j \mid \lambda_i \geqq 0,\ i = 1, \cdots, k,\ \sum_{i=1}^k \lambda_i = 1,\ \mu_j \geqq 0,\ j = 1, \cdots, l\}$$

2 Existence Theorems

In the following, we consider the following quadratic optimization problem;

$$
\begin{array}{lll}
\text{(QP)} & \text{Minimize} & f(x) := \frac{1}{2}x^T Dx + c^T x \\
& \text{subject to} & x \in \Delta := \{x \in \mathbb{R}^n \mid Ax \geqq b,\ Cx = d\},
\end{array}
$$

where D is an $n \times n$ symmetric matrix, $c \in \mathbb{R}^n$, A is an $m \times n$ matrix, $b \in \mathbb{R}^m$, C is an $s \times n$ matrix and $d \in \mathbb{R}^s$.

Now we introduce Frank-Wolfe Theorem and Eaves Theorem, which are fundamental for existence of optimal solutions of (QP).

Theorem 2.1 [10] **[Frank-Wolfe Theorem]** *If* $\bar{\theta} := \inf\{f(x) \mid x \in \Delta\}$ *is finite, that is,* $f(x)$ *is bounded below on* Δ*, then (QP) has an optimal solution.*

Theorem 2.2 [9] **[Eaves Theorem]** *(QP) has an optimal solution if and only if the following hold;*

(i) $\Delta \neq \emptyset$*;*

(ii) If $Av \geqq 0$ *and* $Cv = 0$*, then* $v^T Dv \geqq 0$*;*

(iii) If $Av \geqq 0$*,* $Cv = 0$*,* $v^T Dv = 0$*,* $Ax \geqq b$ *and* $Cx = d$ *then* $(Dx+c)^T v \geqq 0$.

Now we induce an existence theorem for a convex (QP) from Theorem 2.2.

Theorem 2.3 *If D is an $n \times n$ symmetric positive semidefinite matrix, then the following are equivalent;*

(i) (QP) has an optimal solution.

(ii) $\Delta \neq \emptyset$ and if $Av \geqq 0$, $Cv = 0$, $v^T Dv = 0$, $Ax \geqq b$ and $Cx = d$, then $(Dx + c)^T v \geqq 0$.

(iii) $\Delta \neq \emptyset$ and if $Av \geqq 0$, $Cv = 0$ and $Dv = 0$, then $c^T v \geqq 0$.

Proof. By Theorem 2.2, (ii) implies (i). Now we will prove that (iii) implies (ii). Let $v \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$ be such that $Av \geqq 0$, $Cv = 0$, $v^T Dv = 0$, $Ax \geqq b$ and $Cx = d$. Since $(y^T Dv)^2 \leqq (y^T Dy)(v^T Dv)$ for any $y,\ v \in \mathbb{R}^n$, then $y^T Dv = 0$, for any $y \in \mathbb{R}^n$ and hence $Dv = 0$. By assumption (iii), $c^T v \geqq 0$. So, $(Dx + c)^T v = x^T Dv + c^T v = c^T v \geqq 0$. Hence (ii) holds.

Now we will prove that (i) implies (iii). Since (QP) has a solution, $\Delta \neq \emptyset$. Let $v \in \mathbb{R}^n$ be such that $Av \geqq 0$, $Cv = 0$ and $Dv = 0$, and let x_0 be a solution of (QP). Then for any $t > 0$, $A(x_0 + tv) = Ax_0 + tAv \geqq b$ and $C(x_0 + tv) = Cx_0 + tCv = d$. Hence for any $t > 0$, $x_0 + tv \in \Delta$. Since x_0 is a solution of (QP), we have, for any $t > 0$,

$$\frac{1}{2}(x_0 + tv)^T D(x_0 + tv) + c^T(x_0 + tv) \geqq \frac{1}{2}x_0^T Dx_0 + c^T x_0.$$

Thus for any $t > 0$, $\frac{1}{2}t^2 v^T Dv + tx_0^T Dv + tc^T v \geqq 0$. Since $Dv = 0$, $c^T v \geqq 0$. Hence (i) holds. □

Remark 2.1 *The condition (iii) in Theorem 2.3 can be found in* [3], *but our proof is quitely different from one in* [3].

For the completeness, we give a proof for a necessary and sufficient optimality theorem (Lagrange multiplier theorem) for a convex (QP), which can be found in [15].

Theorem 2.4 *Let D be an $n \times n$ symmetric positive semidefinite matrix. Then $x_0 \in \mathbb{R}^n$ be an optimal solution of (QP) if and only if there exist $\lambda \in \mathbb{R}^m$ and $\mu \in \mathbb{R}^s$ satisfying the following Kuhn-Tucher system:*

$$\begin{aligned} Dx_0 + c - A^T\lambda - C^T\mu &= 0, \\ \lambda^T(Ax_0 - b) &= 0, \\ \lambda &\geqq 0, \\ Ax_0 &\geqq b, \\ Cx_0 &= d. \end{aligned} \tag{2.1}$$

Proof. Let $x_0 \in \mathbb{R}^n$ be an optimal solution of (QP). Let $I_1 := \{i \mid A_i x_0 = b_i\}$ and $I_2 := \{i \mid A_i x_0 > b_i\}$, where A_i is the i-th row vector of A. Let v be any point such that $A_i v \geqq 0$ for any $i \in I_1$ and $C_j v = 0$, $j = 1, \cdots, s$. C_j is the j-th row vector of C. Then there exists $\delta > 0$ such that for any $t \in (0, \delta)$, and any $i \in I_2$, $A_i(x_0 + tv) \geqq b_i$. Moreover, for any $t \in (0, \delta)$, for any $i \in I_1$, $A_i(x_0 + tv) = A_i x_0 + tA_i v \geqq b_i$ and $C(x_0 + tv) = Cx_0 + tCv = d$. Hence for any $t \in (0, \delta)$, $x_0 + tv \in \Delta$. Since x_0 is an optimal solution of (QP), for any $t \in (0, \delta)$

$$\frac{1}{2}(x_0 + tv)^T D(x_0 + tv) + c^T(x_0 + tv) \geqq \frac{1}{2}x_0^T Dx_0 + c^T x_0$$

and hence $\frac{1}{2}t^2v^TDv+t(Dx_0+c)^Tv \geqq 0$. So, for any $t \in (0,\delta)$, $\frac{1}{2}tv^TDv+(Dx_0+c)^Tv \geqq 0$. Letting $t \to 0^+$ we have $(Dx_0+c)^Tv \geqq 0$. Thus $(Dx_0+c)^Tv \geqq 0$ for any $v \in \{v \in \mathbb{R}^n \mid A_iv \geqq 0,\ i \in I_1$ and $C_jv = 0,\ j = 1,\cdots,s\}$. By Farkas Theorem in [16], there exist $\lambda_i \in \mathbb{R},\ i \in I_1,\ \mu_j \in \mathbb{R},\ j = 1,\cdots,m$ such that

$$Dx_0 + c - \sum_{i\in I_1} \lambda_i A_i - \sum_{j=1}^{m} \mu_i C_j = 0$$

and $\lambda_i \geqq 0,\ i \in I_1$. Letting $\lambda_i = 0$ for any $i \in I_2$ then we have

$$\begin{aligned}
&Dx_0 + c - A^T\lambda - C^T\mu = 0,\\
&\lambda^T(Ax_0 - b) = 0,\\
&\text{and } \ \lambda \geqq 0.
\end{aligned}$$

Conversely, that (2.1) holds, For any $x \in \Delta$,

$$\begin{aligned}
&\tfrac{1}{2}x^TDx + c^Tx - \tfrac{1}{2}x_0^TDx_0 - c^Tx_0\\
= \ &\tfrac{1}{2}(x-x_0)^TD(x-x_0) - x_0^TDx_0 + x_0^TDx + c^Tx - c^Tx_0\\
\geqq \ &(Dx_0+c)^T(x-x_0) \quad \text{(since } D \text{ is positive semidefinite)}\\
= \ &(A^T\lambda + C^T\mu)^T(x-x_0) \quad \text{(by (2.1))}\\
= \ &\lambda^TA(x-x_0) + \mu^TC(x-x_0)\\
= \ &\lambda^TAx - \lambda^TAx_0 + \mu^TCx - \mu^TCx_0\\
\geqq \ &\lambda^Tb - \lambda^Tb \quad \text{(by (2.1))}\\
= \ &0.
\end{aligned}$$

Thus for any $x \in \Delta$,

$$\frac{1}{2}x^TDx + c^Tx \geqq \frac{1}{2}x_0^TDx_0 + c^Tx_0$$

and hence x_0 is an optimal solution of (QP). □

3 Characterization of Solution Sets of Convex Quadratic Optimization Problem

In this section, we directly prove Lagrange based characterizations and a boundedness condition of optimal solution sets of a convex quadratic optimization problem when its one solution is given, and give an example which illustrates the characterizations.

Let S be the optimal solution set (the set of all optimal solutions) of (QP) in Section 2 and assume that $S \neq \emptyset$. Let $a \in S$. By Theorem 2.4, there exists $(\lambda, \mu) \in \mathbb{R}^n \times \mathbb{R}^s$ such that

$$\begin{aligned} &Da + c = A^T\lambda + C^T\mu, \quad \lambda^T(Aa - b) = 0 \\ &\text{and} \quad \lambda \geqq 0. \end{aligned} \tag{3.1}$$

We call (λ, μ) the Lagrange multipliers corresponding to a. Let $I = \{1, \cdots, m\}$, $\tilde{I}_1(a) = \{i \mid A_i a = b_i, \ \lambda_i > 0\}$.

Theorem 3.1 *Let D be an $n \times n$ symmetric positive semidefinite matrix. Then we have,*

$$\begin{aligned} (i)\ S &= \{x \in \mathbb{R}^n \mid Ax \geqq b, \ Cx = d, \ \lambda^T(Ax - b) = 0, \ Dx = Da\} \\ &= \{x \in \mathbb{R}^n \mid Ax \geqq b, \ Cx = d, \ \lambda^T(Ax - b) = 0, \\ &\qquad\qquad Dx + c = A^T\lambda + C^T\mu\}. \end{aligned}$$

(ii) $S \;=\{x\in\mathbb{R}^n \;|A_ix=b_i,\;\; \forall i\in\tilde{I}_1(a),$

$$A_ix\geq b_i,\;\; \forall i\in I\setminus\tilde{I}_1(a),\;\; Cx=d,\;\; Dx=Da\}.$$

(iii) S is polyhedral.

(iv) S is bounded if and only if

$$\{v\in\mathbb{R}^n \mid A_iv=0,\;\; \forall i\in\tilde{I}_1(a),\;\; A_iv\geqq 0,\;\; \forall i\in I\setminus\tilde{I}_1(a),\;\; Cv=0,\;\; Dv=0\}$$
$$=\{0\}.$$

Proof. Now we firstly prove that S is convex. Let $x_1,\;\; x_2\in S$ and $\alpha\in[0,1]$. Since Δ is convex, $\alpha x_1+(1-\alpha)x_2\in\Delta$. By Remark 1.1, we have,

$$\begin{aligned}
&\tfrac{1}{2}(\alpha x_1+(1-\alpha)x_2)^TD(\alpha x_1+(1-\alpha)x_2)+c^T(\alpha x_1+(1-\alpha)x_2)\\
\leqq\;& \alpha[\tfrac{1}{2}x_1^TDx_1+c^Tx_1]+(1-\alpha)[\tfrac{1}{2}x_2^TDx_2+c^Tx_2]\\
=\;& \tfrac{1}{2}a^TDa+c^Ta.
\end{aligned}$$

Thus $\alpha x_1+(1-\alpha)x_2\in S$ and hence S is convex.

(i) Let $x\in S$. Then $Ax\geqq b$, $Cx=d$ and $a+t(x-a)\in S$ for any $t\in(0,1]$ since S is convex. Moreover, for any $t\in(0,1]$,

$$\begin{aligned}
0\;&=\tfrac{1}{2}(a+t(x-a))^TD(a+t(x-a))+c^T(a+t(x-a))-\tfrac{1}{2}a^TDa-c^Ta\\
&=\tfrac{1}{2}t^2(x-a)^TD(x-a)+t(x-a)^TDa+tc^T(x-a).
\end{aligned}$$

Hence for any $t\in(0,1]$,

$$\frac{1}{2}t(x-a)^TD(x-a)+(Da+c)^T(x-a)=0.$$

Letting $t \to 0^+$, we have

$$(Da + c)^T(x - a) = 0. \tag{3.2}$$

By (3.1) and (3.2),

$$\begin{aligned} 0 &= (A^T\lambda + C^T\mu)^T(x - a) \\ &= \lambda^T Ax - \lambda^T Aa + \mu^T Cx - \mu^T Ca \\ &= \lambda^T(Ax - b). \end{aligned}$$

For any $y \in \mathbb{R}^n$, we have,

$$\begin{aligned} & \tfrac{1}{2}y^T Dy + c^T y - \tfrac{1}{2}a^T Da - c^T a - (Da + c)^T(y - a) \\ = \; & \tfrac{1}{2}y^T Dy + \tfrac{1}{2}a^T Da - a^T Dy \\ = \; & \tfrac{1}{2}(y - a)^T D(y - a) \\ \geqq \; & 0 \quad \text{(since } D \text{ is positive semidefinite).} \end{aligned}$$

Hence for any $y \in \mathbb{R}^n$,

$$\frac{1}{2}y^T Dy + c^T y - \frac{1}{2}a^T Da - c^T a \geqq (Da + c)^T(y - a).$$

Moreover for any $y \in \mathbb{R}^n$,

$$\begin{aligned} & \tfrac{1}{2}y^T Dy + c^T y - \tfrac{1}{2}x^T Dx - c^T x \\ = \; & \tfrac{1}{2}y^T Dy + c^T y - \tfrac{1}{2}a^T Da - c^T a \\ \geqq \; & (Da + c)^T(y - a) \\ = \; & (Da + c)^T(y - x) + (Da + c)^T(x - a) \\ = \; & (Da + c)^T(y - x) \quad \text{(by (3.2))}. \end{aligned}$$

Furthermore for any $y \in \mathbb{R}^n$,

$$\frac{1}{2}(x+ty)^T D(x+ty) + c^T(x+ty) - \frac{1}{2}x^T Dx - c^T x \geqq t(Da+c)^T y.$$

Thus for any $t > 0$ and $y \in \mathbb{R}^n$,

$$\lim_{t \to 0+} \frac{\frac{1}{2}(x+ty)^T D(x+ty) + c^T(x+ty) - \frac{1}{2}x^T Dx - c^T x}{t}$$
$$\geqq (Da+c)^T y.$$

Since the left-hand side in the above inequality, becomes $(Dx+c)^T y$, $(Dx+c)^T y \geqq (Da+c)^T y$ for any $y \in \mathbb{R}^n$, i.e., $(Dx-Da)^T y \geqq 0$ for any $y \in \mathbb{R}^n$. Thus $Dx = Da$. Consequently, $S \subset \{x \in \mathbb{R}^n \mid Ax \geqq b,\ Cx = d,\ \lambda^T(Ax-b) = 0,\ Dx = Da\}$.

Let x be a point such that $Ax \geqq b$, $Cx = d$, $\lambda^T(Ax-b) = 0$ and $Dx = Da$. Then it follows from (3.1) that $Dx + c = A^T\lambda + C^T\mu$. Thus

$$\begin{aligned} & \{x \in \mathbb{R}^n \mid Ax \geqq b,\ Cx = d,\ \lambda^T(Ax-b) = 0,\ Dx = Da\} \\ \subset\ & \{x \in \mathbb{R}^n \mid Ax \geqq b,\ Cx = d,\ \lambda^T(Ax-b) = 0,\ Dx + c = A^T\lambda + C^T\mu\}. \end{aligned}$$

Moreover, by Theorem 2.4,

$$\{x \in \mathbb{R}^n \mid Ax \geqq b,\ Cx = d,\ \lambda^T(Ax-b) = 0,\ Dx + c = A^T\lambda + C^T\mu\} \subset S.$$

Therefore,

$$\begin{aligned} S =\ & \{x \in \mathbb{R}^n \mid Ax \geqq b,\ Cx = d,\ \lambda^T(Ax-b) = 0,\ Dx = Da\} \\ =\ & \{x \in \mathbb{R}^n \mid Ax \geqq b,\ Cx = d,\ \lambda^T(Ax-b) = 0,\ Dx + c = A^T\lambda + C^T\mu\}. \end{aligned}$$

(ii) $Ax \geqq b$ and $\lambda^T(Ax - b) = 0$ if and only if $A_i x \geqq b_i$ for any $i \in I \setminus \tilde{I}_1(a)$ and $A_i x = b_i$ for any $i \in \tilde{I}_1(a)$. Thus from (i),

$$S = \{x \in \mathbb{R}^n \mid \quad A_i x = b_i,\ \forall i \in \tilde{I}_1(a),$$
$$A_i x \geqq b_i,\ \forall i \in I \setminus \tilde{I}_1(a),\ Cx = d,\ Dx = Da\}.$$

(iii) By (ii), S is polyhedral.

(iv) It follows from (ii) and Example 1.1 that $S^\infty = \{v \in \mathbb{R}^n \mid A_i v = 0,\ \forall i \in \tilde{I}_1(a),\ A_i v \geqq 0,\ \forall i \in I \setminus \tilde{I}_1(a),\ Cv = 0,\ Dv = 0\}$.
So, by Theorem 1.1, S is bounded. □

Remark 3.1 *Theorem 3.1 is a slight extension of Corollaries 2.3 and 2.4 in* [13]*, which are induced from Lagrange based characterizations of solution set of cone-constrained convex program (see Theorem 2.2 and Corollary 2.2 in* [13]*). Here we give a direct proof for Theorem 3.1 only with matrices D, A and C (without using subgradient or gradient of convex functions).*

Now we give example illustrating Theorem 3.1.

Example 3.1 *Consider the following convex quadratic optimization problem:*

$$\text{(QP)} \quad \text{Minimize} \quad f(x) := \frac{1}{2}(x_1,\ x_2)\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (1,\ 0)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$= x_1^2 + x_1$$
$$\text{subject to} \quad x \in \Delta := \{x = (x_1,\ x_2)^T \mid x_1 \geqq 0,\ x_2 \geqq 0\}.$$

Let S be the optimal solution set of (QP). We can easily check that $S = \{(0, x_2)^T \mid x_2 \geqq 0\}$.

Now we identify the solution set by Theorem 3.1. Let $a = (0,0)^T$. *Then* $a \in S$. *We consider the Kuhn-Tucker system corresponding to a in (3.1) :*

$$\begin{aligned}&(1,0)^T - \lambda_1(1,0)^T - \lambda_2(0,1)^T = (0,0)^T,\\ &\lambda_1 \geqq 0, \quad \lambda_2 \geqq 0.\end{aligned}$$

Then $\lambda_1 = 1, \ \lambda_2 = 0$ *and hence* $\tilde{I}_1(a) = \{1\}$. *So, by (ii) of Theorem 3.1, we have,*

$$\begin{aligned} S = \ & \{x \in \mathbb{R}^2 \mid x_1 = 0, \ x_2 \geqq 0, \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \} \\ = \ & \{x \in \mathbb{R}^2 \mid x_1 = 0, \ x_2 \geqq 0\}. \end{aligned}$$

4 Applications

Now we apply our results in Sections 2 and 3 to a linear optimization problem and a linear complementarity problem, and give examples illustrating obtained results.

4.1 Linear Optimization Problem

Consider the following linear optimization problem:

$$\text{(LP)} \qquad \begin{aligned} &\text{Minimize} \quad f(x) := c^T x \\ &\text{subject to} \quad x \in \Delta := \{x \in \mathbb{R}^n \mid Ax = b, \ x \geqq 0\}, \end{aligned}$$

where $c \in \mathbb{R}^n$, A is an $m \times n$ matrix and $b \in \mathbb{R}^m$.

By Theorem 2.3, we can obtain the following corollory.

Corollary 4.1 *The following hold:*

(LP) has an optimal solution if and only if $\Delta = \{x \in \mathbb{R}^n \mid Ax = b,\ x \geqq 0\} \neq \emptyset$, *and if* $Av = 0$ *and* $v \geqq 0$, *then* $c^T v \geqq 0$.

Let S be the optimal solution set of (LP) and assume that $S \neq \emptyset$. Let $a \in S$. Then by Theorem 2.4, there exsits $(\lambda, \mu) \in \mathbb{R}^m \times \mathbb{R}^n$ such that

$$\begin{aligned} &c = A^T\lambda + \mu, \\ &\mu^T a = 0, \\ &\mu \geqq 0. \end{aligned} \tag{4.1}$$

We call this (λ, μ) the Lagrangean multiplier for (LP) corresponding to a. Let $I = \{1, \cdots, n\}$, $\tilde{I}_1(a) = \{i \in I \mid a_i = 0,\ \mu_i > 0\}$.

Applying Theorem 3.1 to (LP), we get the following corollary:

Corollary 4.2 *The following hold:*

(i) $$\begin{aligned} S &= \{x \in \mathbb{R}^n \mid x \geqq 0,\ Ax = b,\ \mu^T x = 0\} \\ &= \{x \in \mathbb{R}^n \mid x \geqq 0,\ Ax = b,\ \mu^T x = 0,\ c = A^T\lambda + \mu\}. \end{aligned}$$

(ii) $S = \{x \in \mathbb{R}^n \mid x_i = 0,\ \forall i \in \tilde{I}_1(a),\ x_i \geqq 0,\ \forall i \in I \setminus \tilde{I}_1(a),\ Ax = b\}$.

(iii) S is polyhedral.

(iv) S is bounded if and only if

$$\{x \in \mathbb{R}^n \mid x_i = 0,\ \forall \in \tilde{I}_1(a),\ x_i \geqq 0,\ \forall i \in I \setminus \tilde{I}_1(a),\ Ax = 0\} = \{0\}.$$

Now we give examples showing the meaning of Theorem 4.1

To illustrate Corollary 4.1, we give an example of which the optimal solution set is bounded.

Example 4.1 *We consider the following linear optimization problem:*

$$(LP) \quad \textit{Minimize} \quad f(x) := x_1$$
$$\textit{subject to} \quad x \in \Delta := \{x \in \mathbb{R}^4 \mid x_1 \geqq 0,\ x_2 \geqq 0,\ x_3 \geqq 0,\ x_4 \geqq 0,$$
$$-x_1 + x_2 + x_3 = 6,\ x_1 - x_2 + x_4 = 5\}.$$

Then $\Delta = \{(x_1,\ x_2,\ x_1 - x_2 + 6,\ -x_1 + x_2 + 5)^T \mid x_1 \geqq 0,\ ,x_2 \geqq 0,\ x_1 - x_2 \geqq -6,\ ,-x_1 + x_2 \geqq -5\} = \{x_1(1,0,1,-1)^T + x_2(0,1,-1,1)^T \mid x_1 \geqq 0,\ x_2 \geqq 0,\ 6 + x_1 - x_2 \geqq 0,\ 5 - x_1 + x_2 \geqq 0\} + (0,0,6,5)^T$*. Thus* $f(\Delta) = \{x_1 \mid x_1 \geqq 0,\ x_2 \geqq 0,\ 6 + x_1 - x_2 \geqq 0,\ 5 - x_1 + x_2 \geqq 0\} = [0,5]$. *So, the optimal value is 0, and hence the optimal solution set of (LP) is* $S = \{x_2(0,1,-1,1)^T + (0,0,6,5)^T \mid 0 \leq x_2 \leq 6\}$.

Now we identify the optimal solution set S *by Corollory 4.2. Let* $a = (0,0,6,5)^T \in S$*. We consider the Kuhn-Tucker system corresponding to* a *as in (4.1):*

$$(1,0,0,0)^T + \lambda_1(-1,1,1,0)^T + \lambda_2(1,-1,0,1)^T - (\mu_1,\mu_2,\mu_3,\mu_4)^T = (0,0,0,0)^T,$$
$$6\mu_3 + 5\mu_4 = 0,$$
$$\mu_1 \geqq 0,\ \ \mu_2 \geqq 0,\ \ \mu_3 \geqq 0,\ \ \mu_4 \geqq 0,$$
$$\lambda_1,\ \ \lambda_2 \in \mathbb{R},$$

that is,

$$
\begin{aligned}
&1 + \lambda_1 - \lambda_2 - \mu_1 = 0,\\
&-\lambda_1 + \lambda_2 - \mu_2 = 0,\\
&-\lambda_1 - \mu_3 = 0,\\
&-\lambda_2 - \mu_4 = 0,\\
&\mu_1 \geqq 0, \quad \mu_2 \geqq 0,\\
&\mu_3 = 0, \quad \mu_4 = 0,\\
&\lambda_1, \quad \lambda_2 \in \mathbb{R}.
\end{aligned}
$$

Hence $\lambda_1,\ \lambda_2 = 0$*,* $\mu_1 = 1$*,* $\mu_2 = 0$*,* $\mu_3 = 0$*,* $\mu_4 = 0$ *and* $\tilde{I}(0,0,6,5) = \{1\}$*. Thus, by Corollay 4.2,*

$$
\begin{aligned}
S &= \{(x_1,\ x_2,\ x_3,\ x_4) \in \mathbb{R}^4 \mid x_1 = 0,\ x_2 \geqq 0,\ x_3 \geqq 0,\ x_4 \geqq 0,\\
&\qquad\qquad -x_1 + x_2 + x_3 = 6,\ x_1 - x_2 + x_4 = 5\}\\
&= \{(0,\ x_2,\ 6 - x_2,\ 5 + x_2) \mid x_2 \geqq 0,\ 6 - x_2 \geqq 0,\ 5 + x_2 \geqq 0\}\\
&= \{(0,\ x_2,\ 6 - x_2,\ 5 + x_2) \mid 0 \leqq x_2 \leqq 6\}\\
&= \{x_2(0,\ 1,\ -1,\ 1)^T + (0,\ 0,\ 6,\ 5)^T \mid 0 \leqq x_2 \leqq 6\}.
\end{aligned}
$$

To illustrate Corollary 4.1, we give an example of which the optimal solution set is unbounded.

Example 4.2 *Consider the following linear optimization problem:*

$$
\begin{aligned}
&\textit{Minimize} && f(x) := x_1 - x_2\\
&\textit{subject to} && x \in \Delta := \{x \in \mathbb{R}^4 \mid x_1 \geqq 0,\ x_2 \geqq 0,\ x_3 \geqq 0,\ x_4 \geqq 0,\\
& && \qquad x_1 - 2x_2 + x_3 = 4,\ -x_1 + x_2 + x_4 = 3\},
\end{aligned}
$$

Now we find extreme points and extreme direction of Δ by using Theorem 1.2.

Let $A = \begin{pmatrix} 1 & -2 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{pmatrix}$ *and* $b = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

Extreme points : *To find extreme points of Δ, we find 2×2 submatrices B's of A with linearly independent row vectors of A as follows;*

$$\begin{pmatrix} -1 & -2 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now we check whether $B^{-1}b$ is nonnegative or not as follows;

$$\begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} -10 \\ -7 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 7 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}.$$

$$\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \end{pmatrix}.$$

$$\begin{pmatrix} -2 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 1 & 0 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}.$$

Thus, by Theorem 1.2, extreme points of Δ *are*

$(4,\ 0,\ 0,\ 7)^T$, $(0,\ 3,\ 10,\ 0)^T$ *and* $(0,\ 0,\ 4,\ 3)^T$.

Extreme directions : *To find extreme directions of* Δ, *with the above* 2×2 *submatrices* B'*s of* A *and* a_j'*s, which is a row vector of* A *(but which is not a row vector of* B*), we check whether* $B^{-1}a_j$ *is nonpositive or not as follows;*

$$\begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

$$\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

$$\begin{pmatrix} -2 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 1 & 0 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$\begin{pmatrix} -2 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 1 & 0 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Now from the above we get vector $\begin{pmatrix} -B^{-1}a_j \\ e_j \end{pmatrix}$ *where* e_j *is an* $n-m$ *vector of zeros except for a 1 in position* j *as follows;*

$(1,\ 1,\ 1,\ 0),\ (2,\ 1,\ 0,\ 1),\ (1,\ \frac{1}{2},\ 0,\ \frac{1}{2})$.

Thus, by Theorem 1.2, extreme directions of Δ *are* $(1,\ 1,\ 1,\ 0)^T,\ (2,\ 1,\ 0,\ 1)^T$.

So, by Theorem 1.3, $\Delta = \{\lambda_1(0,\ 0,\ 4,\ 3)^T + \lambda_2(4,\ 0,\ 0,\ 7)^T + \lambda_3(0,\ 3,\ 10,\ 0)^T + \mu_1(1,\ 1,\ 1,\ 0)^T + \mu_2(2,\ 1,\ 0,\ 1)^T \mid \lambda_1 \geqq 0,\ \lambda_2 \geqq 0,\ \lambda_3 \geqq 0,\ \lambda_1 + \lambda_2 + \lambda_3 = 1,\ \mu_1 \geqq 0,\ \mu_2 \geqq 0\}$.

$$\begin{aligned} f(\Delta) &= \{0 \cdot \lambda_1 + 4\lambda_2 - 3\lambda_3 + \ 0 \cdot \mu_1 + 1 \cdot \mu_2 \mid \lambda_1 \geqq 0,\ \lambda_2 \geqq 0,\ \lambda_3 \geqq 0, \\ &\qquad \lambda_1 + \lambda_2 + \lambda_3 = 1,\ \mu_1 \geqq 0,\ \mu_2 \geqq 0\} \\ &= \{4\lambda_2 - 3\lambda_3 + \mu_2 \mid \lambda_2 \geqq 0,\ \lambda_3 \geqq 0,\ \lambda_2 + \lambda_3 \leqq 1,\ \mu_2 \geqq 0\} \\ &= [-3, +\infty). \end{aligned}$$

Note that $f(4,\ 0,\ 0,\ 7) = 4,\ f(0,\ 3,\ 10,\ 0) = -3,$ *and* $f(0,\ 0,\ 4,\ 3) = 0$.

So, we see that the optimal solution set is

$$S = \{(0,\ 3,\ 10,\ 0)^T + t(1,\ 1,\ 1,\ 0)^T \mid t \geqq 0\}.$$

Then $(0,\ 3,\ 10,\ 0)^T \in S$ *and there exists* $(\lambda,\ \mu) \in \mathbb{R}^2 \times \mathbb{R}^4$ *such that*

$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \lambda_1 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \mu_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \mu_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \mu_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$3\mu_2 + 10\mu_3 = 0,$$

$$\mu_1 \geqq 0,\ \ \mu_2 \geqq 0,\ \ \mu_3 \geqq 0,\ \ \mu_4 \geqq 0,$$

that is,

$$\begin{aligned}
&1 = \lambda_1 - \lambda_2 + \mu_1, \\
&-1 = -2\lambda_1 + \lambda_2 + \mu_2, \\
&0 = \lambda_1 + \mu_3, \\
&0 = \lambda_2 + \mu_4, \\
&\mu_2 = \mu_3 = 0, \\
&\mu_1 \geqq 0,\ \ \mu_4 \geqq 0, \\
&\lambda_1,\ \ \lambda_2 \in \mathbb{R},
\end{aligned}$$

that is,

$$\begin{aligned}
&\lambda_1 = 0,\ \ \lambda_2 = -1, \\
&\mu_1 = \mu_2 = \mu_3 = 0,\ \ \mu_4 = 1.
\end{aligned}$$

Thus $\tilde{I}_1(0,\ 3,\ 10,\ 0) = \{4\}$.

$$\begin{aligned} S = & \ \{x \in \mathbb{R}^4 \mid x_4 = 0,\ x_1 \geqq 0,\ x_2 \geqq 0,\ x_3 \geqq 0,\ x_1 - 2x_2 + x_3 = 4, \\ & \qquad -x_1 + x_2 + x_4 = 3\} \\ = & \ \{(x_1,\ 3 + x_1,\ x_1 + 10,\ 0)^T \mid x_1 \geqq 0\} \\ = & \ \{x_1(1,\ 1,\ 1,\ 0)^T + (0,\ 3,\ 10,\ 0)^T \mid x_1 \geqq 0\}. \end{aligned}$$

Let $x_1 = t$. *Then*

$$S = \{(0,\ 3,\ 10,\ 0)^T + t(1,\ 1,\ 1,\ 0)^T \mid t \geqq 0\}.$$

4.2 Linear Complementarity Problem

Consider the following linear complementarity problem;

(LCP) Find $\bar{x} \in \mathbb{R}^n$ such that $\bar{x} \geqq 0,\ D\bar{x} + c \geqq 0,\ \bar{x}^T(D\bar{x} + c) = 0,$

where D is an $n \times n$ matrix and $c \in \mathbb{R}^n$.

Notice that if D is an $n \times n$ positive semidefinite matrix, then $D + D^T$ is symmetric and positive semidefinite and that $\frac{1}{2}x^T(D + D^T)x = x^T Dx$ for any $x \in \mathbb{R}^n$.

For the completeness, we give the proof of the following theorem, which can be found in [7], [20]

Theorem 4.1 *Suppose that D is an $n \times n$ positive semidefinite matrix. Then $\bar{x}$ is a solution of (LCP) if and only if $\bar{x}$ is an optimal solution of the following quadratic optimization problem;*

$$(QP)_1 \quad \textit{Minimize} \quad \frac{1}{2}x^T(D+D^T)x + c^Tx$$

$$\textit{subject to} \quad x \in \Delta_1 := \{x \in \mathbb{R}^n \mid Dx + c \geqq 0, \;\; x \geqq 0\}$$

Proof. Necessity: Let $\bar{x}$ be a solution of (LCP). Then $\bar{x} \geqq 0$, $D\bar{x}+c \geqq 0$ and $\bar{x}^T(D\bar{x}+c) = 0$. For any $x \in \Delta_1$,

$$\begin{aligned} & \tfrac{1}{2}x^T(D+D^T)x + c^Tx \\ = \; & x^TDx + c^Tx \\ = \; & x^T(Dx+c) \\ \geqq \; & 0 \\ = \; & \bar{x}^T(D\bar{x}+c) \\ = \; & \tfrac{1}{2}\bar{x}^T(D+D^T)\bar{x} + c^T\bar{x}. \end{aligned}$$

Hence $\bar{x}$ is an optimal solution of $(\mathrm{QP})_1$.

Sufficiency: Suppose that $\bar{x}$ is an optimal solution of $(\mathrm{QP})_1$. By Theorem 2.4, there exists $(\lambda, \mu) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$(D + D^T)\bar{x} + c = D^T\lambda + \mu, \tag{4.1}$$

$$\lambda^T(D\bar{x} + c) = 0, \tag{4.2}$$

$$\mu^T\bar{x} = 0, \tag{4.3}$$

$$\lambda \geqq 0, \quad \mu \geqq 0, \tag{4.4}$$

$$D\bar{x} + c \geqq 0, \tag{4.5}$$

$$\bar{x} \geqq 0. \tag{4.6}$$

From (4.5) and (4.6),

$$\bar{x}^T(D\bar{x} + c) \geqq 0. \tag{4.7}$$

From (4.1) and (4.3),

$$\bar{x}^T((D + D^T)\bar{x} + c - D^T\lambda) = 0,$$

that is,

$$\bar{x}^T(D\bar{x} + c) + \bar{x}^T D^T\bar{x} - \bar{x}^T D^T\lambda = 0. \tag{4.8}$$

From (4.7) and (4.8),

$$\bar{x}^T D^T(\bar{x} - \lambda) \leqq 0. \tag{4.9}$$

From (4.1) and (4.4),

$$\lambda^T(D\bar{x} + c) + \lambda^T D^T(\bar{x} - \lambda) \geqq 0.$$

Thus, by (4.2),

$$\lambda^T D^T(\bar{x} - \lambda) \geqq 0. \tag{4.10}$$

From (4.9) and (4.10),

$$(\bar{x}-\lambda)^T D^T(\bar{x}-\lambda) \leqq 0.$$

Since D is positive semidefinite,

$$(\bar{x}-\lambda)^T D(\bar{x}-\lambda) = 0. \tag{4.11}$$

From (4.9), (4.10) and (4.11),

$$0 \geqq (\bar{x}-\lambda)^T D\bar{x} = (\bar{x}-\lambda)^T D\lambda \geqq 0.$$

Hence $(\bar{x}-\lambda)^T D\bar{x} = 0$. So, from (4.8),

$$x^T(D\bar{x}+c) = 0. \tag{4.12}$$

Thus (4.5), (4.6) and (4.12) imply that $\bar{x}$ is a solution of (LCP). □

Applying Theorem 2.3 to $(\mathrm{QP})_1$, we obtain the following corollary:

Corollary 4.3 *The following conditions are equivalent;*

(i) (LCP) has a solution;

(ii) $\Delta \neq \emptyset$.

Proof. It is clear that (i) implies (ii). Suppose that (ii) holds. Then there exists $x_0 \geqq 0$ such that $Dx_0 + c \geqq 0$. Assume that $Dv \geqq 0$, $v \geqq 0$ and $(D+D^T)v = 0$. Since $Dv + D^T v = 0$ and $Dv \geqq 0$, tnen $D^T v \leqq 0$. Since $v \geqq 0$ and $x_0^T D^T + c^T \geqq 0$, then $c^T v \geqq -x_0^T D^T v \geqq 0$. So, if $Dv \geqq 0$, $v \geqq 0$ and $(D+D^T)v = 0$, then $c^T v \geqq 0$. Thus, it follows from Theorem 2.3 that $(\mathrm{QP})_1$ has an optimal solution. So,by Theorem 4.1 (LCP) has a solution. □

Remark 4.1 *The condition (ii) in Corollary 4.2 can be found in* [6, 7]. *But our proof is different from one in* [6, 7].

Let S be the set of solution of (LCP) and assume that $S \neq \emptyset$. Let $a \in S$. Then, by Theorems 2.4 and 4.1, there exists $(\lambda, \mu) \in \mathbb{R}^n \times \mathbb{R}^n$ be such that

$$\begin{aligned} &(D + D^T)a + c = D^T\lambda + \mu, \\ &\lambda^T(Da + c) = 0, \\ &a^T\mu = 0, \\ &\lambda \geqq 0, \quad \mu \geqq 0. \end{aligned} \tag{4.13}$$

Let D_i be the i-th row vector of D, c_i the i-th component of c, $I = \{1, \cdots, n\}$, $\tilde{I}_1(a) = \{i \in I \mid D_i a + c_i = 0, \ \lambda_i > 0\}$ and $\tilde{J}_1(a) = \{j \in I \mid a_j = 0, \ \mu_j > 0\}$.

Applying Theorem 3.1 to $(\mathrm{QP})_1$, we can obtain the following corollary.

Corollary 4.4 *The following hold;*

(i) $$S = \{x \in \mathbb{R}^n \mid Dx + c \geqq 0,\ \ x \geqq 0,\ \ \lambda^T(Dx + c) = 0,\ \ \mu^T x = 0$$
$$(D + D^T)x = (D + D^T)a\}$$
$$= \{x \in \mathbb{R}^n \mid Dx + c \geqq 0,\ \ x \geqq 0,\ \ \lambda^T(Dx + c) = 0,\ \ \mu^T x = 0$$
$$(D + D^T)x + c = D^T\lambda + \mu\}.$$

(ii) $$S = \{x \in \mathbb{R}^n \mid D_i x + c_i = 0,\ \ \forall i \in \tilde{I}_1(a),\ \ x_j = 0,\ \ \forall j \in \tilde{J}_1(a),$$
$$D_i x + c_i \geqq 0,\ \ \forall i \in I \setminus \tilde{I}_1(a),\ \ x_j \geqq 0,\ \ \forall j \in I \setminus \tilde{J}_1(a)$$
$$(D + D^T)x = (D + D^T)a\}.$$

(iii) S *is polyhedral.*

(iv) S *is bounded if and only if*

$$\{v \in \mathbb{R}^n \mid D_i v = 0,\ \ \forall i \in \tilde{I}_1(a),\ \ v_i = 0,\ \ \forall j \in \tilde{J}_1(a),$$
$$D_i v \geqq 0,\ \ \forall i \in I \setminus \tilde{I}_1(a),\ \ v_j \geqq 0,\ \ \forall j \in I \setminus \tilde{J}_1(a),\ \ (D + D^T)v = 0\}$$
$$= \{0\}.$$

Now we give an example illustrating Corollary 4.4:

Example 4.3 *Let* $D = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ *and* $c = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. *Consider the following linear complementarity problem with the above* D *and* c*:*

(LCP) *Find* $\bar{x} \in \mathbb{R}^3$ *such that*

$$\bar{x} \geqq 0,\ \ D\bar{x} + c \geqq 0,\ \ \bar{x}^T(D\bar{x} + c) = 0.$$

Notice that D is not symmetric, but positive semidefinite. Let S be the solution set of (LCP). Then, clearly, $a = (0,\ 0,\ 0)^T \in S$. Now we consider the Lagrange multipliers for a in (4.13).

$$\lambda_1(1,\ 0, -1)^T + \lambda_2(0,\ 0,\ 0)^T + \lambda_3(1,\ 0,\ 0)^T + (\mu_1,\ \mu_2,\ \mu_3)^T = (0,\ 0,\ 0)^T,$$
$$\lambda_1 \geqq 0,\ \ \lambda_2 \geqq 0,\ \ \lambda_3 \geqq 0,$$
$$\mu_1 \geqq 0,\ \ \mu_2 \geqq 0,\ \ \mu_3 \geqq 0.$$

Then $\lambda_1 = \lambda_3 = \mu_1 = \mu_2 = \mu_3 = 0$ and $\lambda_2 = 1$ satisfies the above equations and inequalities, and hence $\tilde{I}_1(a) = \{1\}$ and $\tilde{J}_1(a) = \emptyset$. Thus, by Corollary 4.4, we have,

$$\begin{aligned} S &= \{x \in \mathbb{R}^3 \mid D_1x + c_1 = 0,\ \ D_2x + c_2 \geqq 0,\ \ D_3x + c_3 \geqq 0, \\ &\qquad\qquad x \geqq 0,\ \ (D + D^T)x = 0\} \\ &= \{(x_1,\ x_2,\ x_3) \in \mathbb{R}^3 \mid x_1 - x_3 = 0,\ x_1 \geqq 0,\ x_2 \geqq 0,\ x_3 \geqq 0,\ 2x_1 = 0\} \\ &= \{(0,\ x_2,\ 0) \in \mathbb{R}^3 \mid x_2 \geqq 0\}. \end{aligned}$$

References

[1] M. S. Bazaraa and John J. Jarvis, Linear Programming and Network Flows, John Wiley and Sons, Inc., 1977.

[2] M. S. Bazaraa, H. D. Sherali and C. M. Shetty, Nonlinear Programming: Theory and Algorithms, Second Edition, John Wiley and Sons, Inc, 1993.

[3] D. P. Bertsekas, Convex Analysis and Optimization, Athena Scientific, Belmont, Massachusetts, 2003.

[4] E. Blum and W. Oettli, Direct proof of the existence theorem for quadratic programs, *Operations Research,* **20** (1972), 165-167.

[5] J. V. Burke and M. Ferris, Characterization of solution sets of convex programs, *Operations Research Letters,* **10** (1991), 57-60.

[6] R. W. Cottle, Note on a fundamental theorem in quadratic programming, *SIAM Jouranl on Applied Mathematics,* **12** (1964), 663-665.

[7] R. W. Cottle, J. S. Pang and R. E. Stone, The Linear Complementarity Problem, Academic Press, Inc., 1992.

[8] N. Dinh, V. Jeyakumar and G. M. Lee, Lagrange multiplier characterizations of solution sets of constrained pseudolinear optimization problems, *submitted.*

[9] B. C. Eaves, On quadratic programming, *Management Science*, **17** (1971), 698-711.

[10] M. Frank and P. Wolfe, An algorithm for quadratic programming, *Naval Research Logistics Quarterly*, **3** (1956), 95-110.

[11] V. Jeyakumar and X. Q. Yang, Convex composite multi-objective nonsmooth programming, *Mathematical Programming* **59** (1993), 325-343.

[12] V. Jeyakumar and X. Q. Yang, On characterizing the solution sets of pseudolinear programs, *Journal of Optimization Theory and Applications,* **87** (1995), 747-755.

[13] V. Jeyakumar, G. M. Lee and N. Dinh, Lagrange multiplier conditions characterizing optimal solution sets of cone-constrained convex programs, *Journal of Optimization Theory and Applications,* **123** (2004), 83-103.

[14] V. Jeyakumar, G. M. Lee and N. Dinh, Characterization of solution sets of convex vector minimization problems, *submitted.*

[15] G. M. Lee, N. N. Tam and N. D. Yen, Quadratic Programming and Affine Variational Inequalities. A Qualitative Study, Springer Science+Business Media, Inc., 2005.

[16] O. L. Magasarian, Nonlinear Programming, McGraw-Hill, New York, 1969.

[17] O. L. Mangasarian, A simple characterization of solution sets of convex programs, *Operations Research Letters,* **7** (1988), 21-26.

[18] J. P. Penot, Characterization of solution sets of quasiconvex programs, *Journal of Optimization Theory and Applications,* **117** (2003), 627-636.

[19] R. T. Rockafellar, Convex Analysis, Princeton University Press, 1972.

[20] S. J. Wright, Primal-Dual Interior-Point Methods, Society for Industrial and Applied Mathematics, Philadelphia, 1997.