On Primitive/Seifert-fibered construction of twisted torus knots K(p,q,p+q,n)

꼬인 토러스 매듭 K(p,q,p+q,n)에 대한 프리미티브/사이퍼 속 만들기

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요 약

최근 Miyazaki-Motegi가 토러스 knot T=T(p,q)을 lingking number n 인 unkont U 에 대해 n-회 full twisting 시켜 얻은 knot K=K(p,q,p+q,n)이 Dean이 소개한 small Seifert-fibered surgery를 갖는 소위 Primitive/Seifert-fibered knot임을 밝힌 바 있다. 이들의 증명 방법은 K와 관련된 disk D 상의 index p, |q|인 exceptional fiber를 갖는 Seifert-fibered space 의 위상적 구조를 이용한 고급적인 것이다.
본 연구자는 K를 standard double torus상의 knot으로 표현하여 Dean 이 이 종류의 knot들의 Primitive/Seifert-fibered 구조를 밝힐 때 사용한 초보적인 대수적 증명 방법을 소개하고자 한다.

On primitive/Seifert-fibered construction of twisted torus knots $K(p, q, p+q, n)$

1 INTRODUCTION

After Max Dehn's discovery of an ingenious methcd of representing a 3-manifold, so called *Dehn surgery*, a great deal of efforts have been devoted to understand it. Thanks to Thurston's positive solution of geometrization conjecture on Haken 3-manifolds, any (prime) knot in S^3 are classified as the following three types;
(H) a knot K whose exterior admitting a complete hyperbolic strucure
(T) a knot K whose exterior admitting a Seifert-fibered structure
(S) a knot K whose exterior admitting an essential torus
A knot in S^3 of type (H), (T) and (S) is said to be *hyperbolic*, *torus* and *satellite* respectively. Moreover, he showed that hyperbolic knots ,which are most abundant in the above three categories of knots in S^3, admit only finitely many non-hyperboic Dehn surgeries (which are referred as *exceptional surgeries*). And, his famous geometrization conjecture implies that those non-hyperboic Dehn surgeries are of the following three types;
(R:reducible) a 3-manifold admitting an essential sphere S
(T:toroidal) a 3-manifold admitting an essential torus T
(SSF: small Seifert-fibered) a 3-manifold admitting a Seifert-fibered structure with a base space S^2, the 2-sphere and at most three exceptional fibers $\{f_1, f_2, f_3\}$. Furthermore, it is conjectured that any hyperbolic knot has no surgery of type R which is referred as *the cabling conjecture.* Although reducible or toroidal surgeries have been extensively investigated with ,for instance, the famous Gordon-Luecke's combinatorial tool (see [16]), relatively little was known about SSF surgeries until Gabai came up with so called *1-bridge braid knots* or more generally, Berge came up with so called *doubly primitive knots* for cyclic surgeries-SSF

surgeries with two exceptional fibers. Then, Dean generallized Berge's concept of doubly primitivity to primitive/Seifert-fibered property. For definitions of these terms, see section 3. These tools show a nice interplay between topology and algebra(a comninatorial group theory). For instance, a tunnel number 1 knot in S^3 corresponds to a primitive word in $F(s,t)$, a free group of rank 2. Moreover, most of known examples of SSF surgeries are interpreted as Dean's primitive/Seifert-fibered construction ([10]), although very recently ([19]) and Song([24]) showed that there are SSF surgeries which do not arise from such construction.

Miyazaki-Motegi introduced a family of knots $K(p,q,p+q,n), 2 \leq |p| \leq q\ n \in \mathbb{Z}$ obtained by n-full twisting of torus knots $t(p,q)$ with respect to unknots u with $|lk(t(p,q),u)| = p+q$. Recently they showed that these knots admit Dean's primitive/Seifert-fibered property([21]). Their elegant proof is based on another result that $K(p,q,p+q,0) \cup u(pq,\infty)$ is $D^2(|p|,q)$, a Seifert-fibered space with a base space D^2, the 2-disk and two exceptional fibers of index $|p|, q$. The purpose of this thesis is to provide more rudimentary proof for primitive/Seifert-fibered property of $K(p,q,p+q,n)$ by following Dean's algebraic approach. This thesis are organized as follows;

Section 1. Introduction.

Section 2. Dehn surgery of a knot K in S^3.

-In this section we deal with some known general aspects of Dehn surgery.

Section 3. Primitive/Seifert-fibered constructions.

-In this section we mainly reproduce Dean's work on primitive/Seifert-fibered constructions, though in section 3.2 we provide a little details of necessary algebraic theorems regarding primitive worlds of $F(s,t)$ and generating systems of $< x,y|X^m y^n >$.

Section 4. primitive/Seifert-fibered constructions of $K(p,q,p+q,n)$

-This section is the heart of the thesis. In section 4.1 we describe $K(p,q,p+q,n)$ as double torus knots. In section 4.2 we reproduce Miyazaki-Motegi's proof for primitive/Seifert-fibered property of $K(p,q,p+q,n)$ for comparison with our rudimentary proof. Section 4.3 contains the statement of our main theorem and its proof.

2 Dehn surgery of a knot K in S^3

The surgery operation is a two-stage construction. First a tubular neighbourhood of a knot is removed from the ambient 3-manifoid (drilling), and then it is reattached (filling). Let us consider the second stage more closely. Take T to be a toral boundary component of an orientable 3-manifold M. Given any homeomorphism $f : \partial\ (S^1 \times D^2) \to T$, from the identification space $M(T; f) = (s^1 \times D^2) \cup_f M$ obtained by identifying the points of $\partial(S^1 \times D^2)$ with their images by f. We refer to $M(T : f)$ as a (Dehn) filling of M along T. A Dehn surgery on a knot K is then a filling of M_K along $\partial\ N(K)$.

In this section we introduce the notion of a slope on a torus, the basic parameter of the filling operation.

2.1 Fillings and slopes

Fix a 3-manifold M and a torus $T \subset \partial M$. By the nature of its construction, a filling $M(T; F)$ depends only on the isotopy class of the attaching homeomorphism $f : \partial(S^1 \times D^2) \to T)$. In fact the dependence on f is much weaker, for if $C_0 = \{pt\} \times \partial D^2 \subset \partial(S^1 \times D^2)$, then $M(T; f)$ dependsonly on the isotopy class of the curve $f(C_0)$ in T. To see this, let $D_0 = \{pt\} \times D^2 \subset S^1 \times D^2$ and observe that $S^1 \times D^2$ splits into two pieces A and B, where A is a closed tubular neighbourhood of D_0 and B is the 3-ball $S^1 \times D^2 \setminus A$. Now think of $M(T; f)$ as being built in two stages. In the first we form $A \cup M$, which amounts to attaching a 2-handle to M along a tubular neighbourhood of $f(C_0) \subset T$. Now a manifold obtained from such an attachment depends only on the isotopy class in T of the attaching 1-sphere, which in our case is $f(C_0)$. In the second stage we from $M(T; f) = B \cup (A \cup M)$ by attaching the 3-ball B to $A \cup M$ along its 2-sphere boundary. As any homeomorphism of a 2-sphere extends over the 3-ball

(e.g., by coning), $M(T;f)$ is completely determined by $A \cup M$, and hence by the isotopy class of $f(C_0)$ in T. Figure 1 depicts a possible $f(C_0)$ in the case where M is the exterior of the righthanded trefoil knot. This analysis indicates the importance of understanding the isotopy classes of simple closed curves on a torus. In fact these classes are particularly well-behaved, as the following lemma indicates. Proofs of its assertions may be found in Chapter 2. C of Rolfsen's text [27].

LEMMA 2.1.1 Let T be a torus.

(i) A separating simple closed curve on T bounds a 2-disk in T.

(ii) For any essential, simple closed curve C on T, there is a dual simple closed curve C' which intersects C exactly once and transversely.

(iii) Disjoint essential, simple closed curves on T are parallel, that is they cobound an annulus embedded in T.

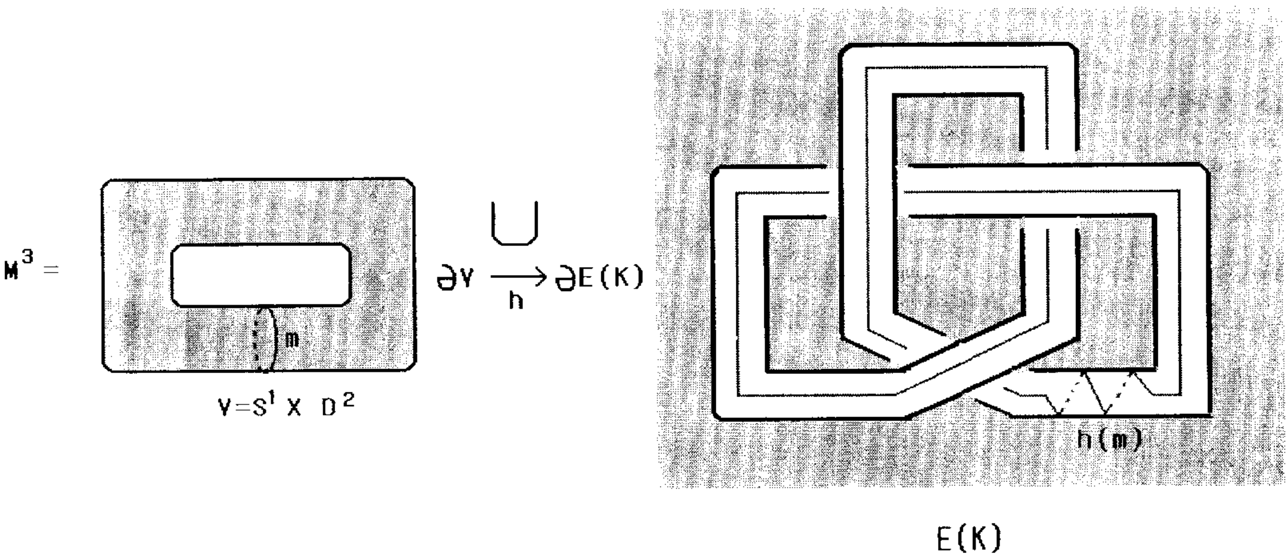


Fig. 1

(iv) Two oriented essential simple closed curves on T are isotopic if and only if the 1-cycles they define are homologous.

(v) A homology class in $H_1(T)$ is represented by the fundamental class of an oriented, eaaential, simple closed curve if and only if it is primitive, that is if and only if it is an element of some basis of $H_1(T)$.

(vi) Given any two essential, simple closed curves C,C' on T, there is an orientation preserving homeomorphism $f : T \to T$ such that $f(C) = C'$.

DEFINITION 2.1.2 A slope on a torus T is the isotopy class of an essential, unoriented, simple closed curve on T. The set of slopes on T will be denoted by $Slope(T)$. Two slopes r_1, r_2 on T are called dual if they have representative curves which intersect exactly once and transversely. Finally if K is a knot in a 3-manifold W, then a slope of K is any slope on $\partial N(K)$.

We summarize the discussion above in the next proposition.

PROPOSITION 2.1.3 A Dehn fulling of M along a torus $T \subseteq \partial M$ is determined, up to orientation preserving homeomorphism, by a slope on T. Furthermore, any slope on T arises as the slope of a Dehn filling of M. Denote by $M(T;r)$ any Dehn filling of M along T corresponding to a gluing homeomrphism f for which $f(C_0)$ represents the slope r. When $T = \partial M$ we shall abbreviate this notation to $M(r)$. There is one distinguished slope determined by any knot. A meridian for a knot $K \subset W$ is any essential, simple closed curve on $\partial N(K)$ which is homologically trivial in $N(K)$ (such curves actually bound 2-disks in $N(K)$). Meridians are well-defined up to isotopy (cf. Lemma 2.1.1) and so determine a slope μ_k of K, called the meridional slope of K. The trivial Dehn surgery on a knot $K \subset W$ is the surgery corresponding to the meridional slope. Evidently $M_k(\mu_k) \cong W$, for in this case we can equate $S^1 \times D^2$ with $N(K)$ and then choose the gluing map so that its effect in attaching $S^1 \times D^2$ to M_k is just to return $N(K)$ to W. There is another distinguished slope for knots in the 3-sphere, or more generally for null-homologous knots in an arbitrary orientable 3-manifold W. A null-homology of a knot $K \subset W$ can be realized by a compact, connected, orientable, smooth subsurface of W whose boundary is K. Such a surface, called a Seifert surface of K, may be isotoped to intersect $N(K)$ in an annulus whose boundary consists of K and an essential, simple closed curve lying on $\partial N(K)$. The latter curve, called a longitude of K, is characterized up to isotopy on $\partial N(K)$

by the fact that it is essential on $\partial N(K)$ while homologically trivial in the exterior of K (cf. Lemma 2.1.1). The longitudinal slope of K, denoted by λk, is the slope of any longitude of K.It is evident that μ_k and λ_k are dual slope (cf. Definition 2.1.2).

2.2 Parameterizing slopes

It follows from Lemma 2.1.1(iv),(v) that the set of slopes on a torus T corresponds bijectively to the set of $\pm$ pairs of primitive classes in $H_1(T)$. Explicitly, if we choose any representative for a slope r and orient it, the fundamental homology class of this oriented circle determines a primitive homology class $\alpha \in H_1(T)$. Changing orientation changes the sign of α. We shall call $\pm\alpha$ the homology classes carried by r. These observations lead to a parameterization of the set of slopes.

PROPOSITION 2.2.1 Let T be a torus and set $P^1(Q) = Q \cup \{\frac{1}{0}\}$. Each choice of ordered basis for $H_1(T)$ determines a bijection between the set of slopes $Slope(T)$ on T and $P^1(Q)$. If K is a null-homologous knot in the interior of an oriented 3-manifold and $T = \partial N(K)$, then the correspondence can be made canonical.

Proof. Fix an ordered basis $\{\alpha, \beta\}$ for $H_1(T)$. Any slope r on T determines a pair of relatively prime integers p, q such that the homology classes in $H_1(T)$ carried by r are $\pm(p\alpha + q\beta)$. This correspondence gives rise to the desired bijection $Slope(T) \leftrightarrow P^1(Q)$ via $r \leftrightarrow \pm(p\alpha + q\beta) \leftrightarrow p/q$. Suppose now that K is a null-homologous knot in the interior of an oriented 3-manifold W and note that the induced orientation on $N(K) \subset W$ determines an orientation on T. Choose classes $\alpha(\mu k)\alpha(\lambda k) \in H_1(T)$ carried by μ_k, λ_k. Since μ_k and λ_k are dual slopes, the algebraic intersection $\alpha(\mu k)\cdot\alpha(\lambda k)$ is either $\{+1\}$ or $\{-1\}$. If we require that this intersection be $+1$, then up to a simultaneous change of sign, $\{\alpha(\mu_k)\alpha(\lambda_k)\}$ is a well-defined ordered basis of $H_1(T)$. Thus the correspondence $r \leftrightarrow \pm(p\alpha(\mu_k) + q\alpha(\lambda_k)) \leftrightarrow p/q$ becomes canonical.

We shall always assume that S^3 is given its usual orientation based on the right-hand rule. Thus slopes of knots in the 3-sphere are canonically identified with $P^1(Q)$.

DEFINITION 2.2.2 Let K be a knot in S^3. For a slope r of K corresponding ti the fraction $p/q \in P^1(Q)$, $M_k(r)$ will also be denoted by $M_k(p/q)$. An integral slope of K is a slope corresponding to an integer, and an integral surgery on K is a surgery whose slope is integral.

Figure 1 depicts a representative curve for the 9/2 slope of the right-handed trefoil knot.

EXAMPLE 2.2.3 Let $K \subset S^3$ be the trivial knot. We determine the manifolds $M_k(p/q)$, $p \geqq 0$. There is a canonical identification $M_k \equiv S^1 \times D^2$ for which $S^1 \times \{1\} \subset \partial M_k$ represents μ_k and $\{ 1\} \times \partial D^2 \subset \partial M_k$ represents λ_k. Hence $M_k(p/q)$ is the union $S^1 \times D^2 \cup_f S^1 \times D^2$ where the meridian $\{1\} \times \partial D^2$ of the first $S^1 \times D^2$ is identified vis f with a curve on the boundary of the second $S^1 \times D^2$ which is homologous to the sum of p copies of $S^1 \times \{1\}$ and q copies of $\{1\} \times \partial D^2$. Thus if $L(p, q)$ denotes the (p, q) lens space

$$M_K(p/q) \cong \begin{cases} S^1 \times S^2 & \text{if} \quad p = 0, \\ S^3 & \text{if} \quad p = 1, \\ L(p,q) & \text{otherwise.} \end{cases}$$

3 Primitive/Seifert-fibered Knots

In this section we will describe a way to construct knots in S^3 that have a Dehn surgery that is a Seifert-fibered space with base S^2 and three or fewer critical fibers. The construction is a generalization of Berge's construction of knots with lens space Dehn surgeries [2] . From the dafinition of the construction, it is not clear that any nontrivial non-Berge examples arise, hence we will describe a simple family of nontrivial examples that arise from the construction. We begin with the definitions of some relevant concepts. We will consider only orientable 3-manifolds throughout.

3.1 Knots in separating surfaces, the surface slope, and 2-handle addition

We begin by defining the notion of 2-handle addition for a 3-manifold with boundary.

Definition 3.1.1 Let γ be a simple closed curve in the boundary of 3-manifold M, and let A be a regular neighborhood of γ in ∂M . Then $M \cup_A (D^2 \times I)$ is the result of 2-handle addition along γ, where A and $\partial D^2 \times I$ are identified.

Next we define the surface slope for a knot contained in a surface in a 3-manifold, and show how Dehn surgery at this slope is related to 2-handle addition when the surface is separating.

Notation $M(K, \gamma)$ denotes the manifold obtained from Dehn surgery on K with slope γ. $N(K)$ is a regular neighborhood of a knot K.

Definition 3.1.2 If K is a knot embedded in a surface F in a 3-manifold then the isotopy class in $\partial N(K)$ of the curve(s) in $\partial N(K) \cap F$ is called *the surface slope of* K with respect to F.

Lemma 3.1.3 Let K be a knot contained in a separating surface F in a 3-manifold M, i.e. $M = V \cup_F V'$, and let γ be the surface slope of K with respect to F. Then $M(K, \gamma) \cong W \cup_{\widetilde{F}} W'$ where $W(resp. W')$ is obtained from $V(resp. V')$ by attaching a 2-handle along K, and $\widetilde{F} = (F - N(K)) \cup (D^2 \times \{0, 1\})$.

Proof Let A (*resp.* A') be the annulus $\partial(N(K)) \cap V$ (*resp.* $\partial(N(K)) \cap V'$), and let c_1 and c_2 be their (shared) boundary curves. Denote the Dehn surgery solid torus by U, and let $h : \partial U \to \partial(S^3 - N(K))$ be the attaching map for the Dehn surgery.

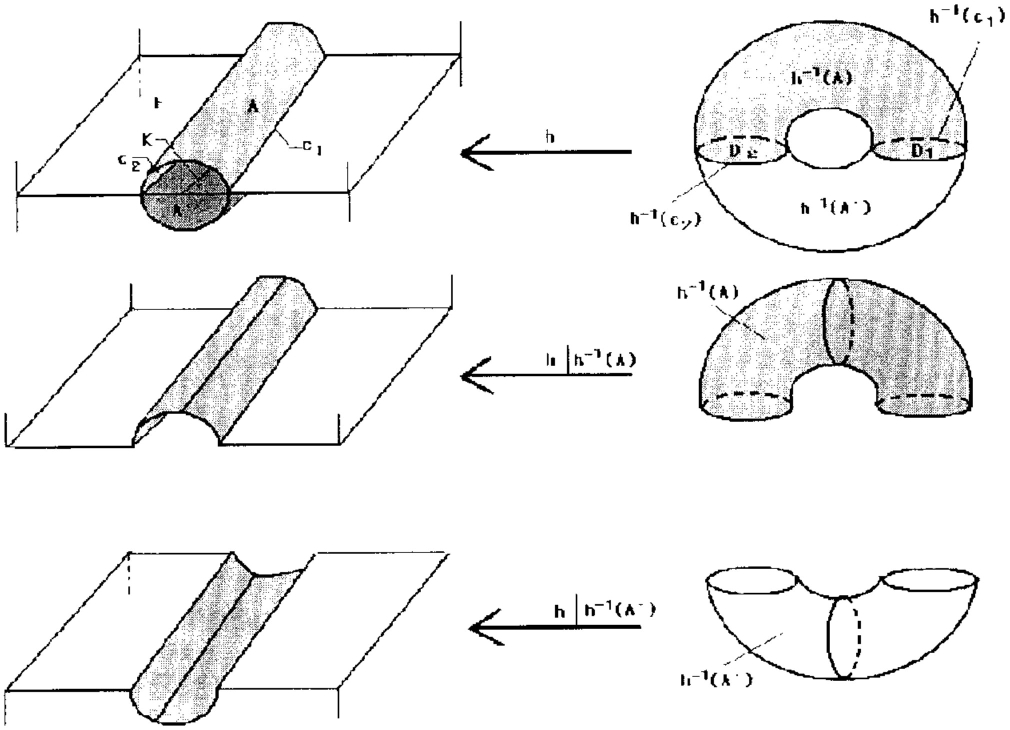


Figure 2: Surface slope Dehn surgery

Since we are considering surface slope Dehn surgery, the curves $h^{-1}(c_1)$ and $h^{-1}(c_2)$ bound disks D_1 and D_2 in U. We may cut the Dehn surgered manifold along $(F-N(K))\cup D_1 \cup D_2$. The resulting pieces (see Figure 2) are homeomorphic to

$$W = V \cup_k \quad 2 - handle$$

and

$$W' = V' \cup_k \quad 2 - handle.$$

3.2 Special elements in a free group and curves on handlebodies

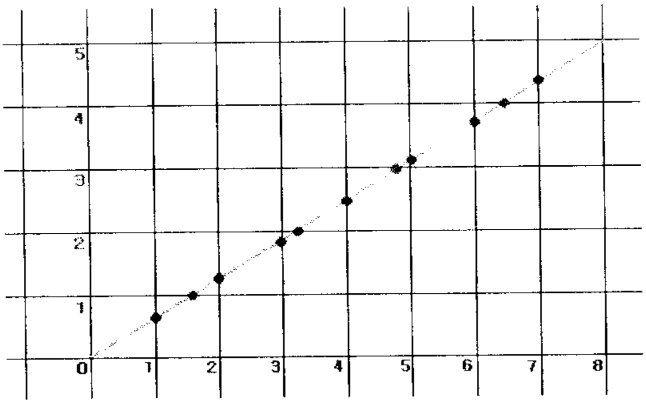
Let G_a, b denots the group $\left\langle x, y \mid x^a y^b \right\rangle$. When a and b are coprime this group is the fundamental group of the (a, b) torus knot. More generally, G_a, b is the fundamental group of a Seifert-fibered space over the disk with two critical fibers of multiplicity a and b. Recall that a basis for a free group is set of elements that freely generates the group.

Definition 3.2.1 An element in a free group is *primitive* if it is part of a basis.

Let $F(s,t)$ denote the free group of rank 2 with the free generators s and t, $A(s,t)$ the abelianized group, and $\psi : F(s,t) \to A(s,t)$ the canonical abelianizing homomorphism. An element $w_1 \in F(s,t)$ is called a primitive if there exists a $w_2 \in F(s,t)$ such that w_1, w_2 generate $F(s,t)$; w_1 and w_2 are called associated primitives. From a nice result of Nielsen [26] it follows easily, that two primitives w, w' of $F(s,t)$ are conjugate if $\psi(w) = \psi(w')$, i,e, the primitives are characterized (up to conjugation) by the abelianized element. In this paper we give a simple procedure for finding the unique (up to a conjugation) primitive corresponding to the abelianized expression $s^m t^n$. Further, we give a geometric interpretation, which is fundamental for our proof, and describe some symmetry properties of primitives. The characterization of primitives arose naturally in the study of certain methods of generating simple closed curves on the genus 2 surface by the frist author [28]. The geometric interpretation is due to Engmann and the second author.

Let Z^2 be the points in $\mathbb{R}^2$ with integral coordinates. Each point (m,n) in Z^2 corresponds to an element $s^m t^n$ of $A(s,t)$. Let Ξ be the set of all lines in R^2 that are parallel to one of the coordinate axes of R^2 and pass through a point of Z^2. A directed line segment in R^2 not containing a point of Z^2 determines a word in $F(s,t)$ obtained by traveling along the segment and writing s whenever we cross a vertical line of Ξ from left to right and writing t when we cross the horizontal lines of Ξ from below. Write $\{s^{-1}\}$ or $\{t^{-1}\}$ if the crossings appear in the opposite direction.

Example



$W_{8,5} = (st)stsststssts{\sim}\, s^2tsts^2tsts^2t$

Fig. 3

Definition of the Osborne-Ziechang's primitive word $W_{m,n}(s,t)$ 3.2.2 Assume that $m, n \geqq 0$ and $(m,n) = 1$. Then the open segment from $(0,0)$ to (m,n) contains no point from Z^2; hence, it defines a word(Ablesung) $V'_{m,n}(s,t)$. If $m, n > 0$ then we define $V_{m,n}$ by

$$W_{m,n}(s,t) = stV'_{m,n}(s,t).$$

In addition, $W_{0,1}(s,t) = t$, $V_{1,0}(s,t) = s$ and $W_{-m,n}(s,t) = W_{m,n}(s^{-1},t)$, $W_{m,-n}(s,t) = W_{m,n}(s,t^{-1})$.

Theorem 3.2.3([29]) A set $\{W_{m,n}(s,t) | m, n \in \mathbb{Z}.(m,n) = 1\}$ forms all primitives of $F(s,t)$ up to cojugation.

REMARK. Gonzalez-Acuna and Ramirez 3.2.4 ([14]) pointed out that the last relation in the above definition should be modified to $W_{m,-n}(s,t) = W^{-1}_{-m,-n}(s,t)$ in order for the above theorem to hold for m, n, p, and q containing negative integers.

Definition 3.2.5 An element w in the free group on x and y is (a,b) Seifert-fiberd if $\langle x, y \mid w \rangle \cong G_{a,b}$ for integers a and b both non-zero.

Note that a primitive element in $\langle x, y \rangle$ is Seifert-fibered.

Theorem3.2.6([34], [8])

(a) Any pair of generators of a group $< s, t \mid s^a t^b >$, $a, b \geq 2$ ($for\ gcd(a,b) = 1$ *a torus knot group*) is Nielsen equivalent to exactly one of the following pairs :

$$\{(s^\alpha, t^\beta) : 1 \leq 2 \leq \beta a, 1 \leq 2\beta \leq \alpha b,$$

$$gcd(a, \alpha) = gcd(b, \beta) = gcd(\alpha, \beta) = 1\}.$$

For all these pairs there are presentations with one or two defining relators. If one of the numbers α, β equals 1 then one relator suffices, [34].

(b) The last condition is necessary, [8]. Hence, there are only finitely many Nielsen equivalence classes of one-relator presentations for the torus knot groups, although there are infinitely many Nielsen equivalence classes of generating pairs.

Combining the above two theorems , we have following description of Seifert-fibered words up to Nielsen equivalence which often helps us to recognize Seifert-fibered words.

Theorem3.2.7([34], [8]) The Nielsen equivalent classes of generating sets that define one-relator presentation of the groups $G_{m,n} =< x, y | x^m y^n >$ correspond to the generators in the following presentations:

$$< x, y | W_{m,a}(x, y^n) >$$

$$< x, y | W_{b,n}(x^m, y) >$$

where $(a, m) = (b, n) = 1, 0 < 2a < m$ and $0 < 2b < n$.

Let γ be a simple closed curve contained in the boundary of a genus two handlebody H. Since γ represents an element (defined up to conjugacy) of $\pi_1(H)$, which is a free group of rank two, we say that γ is *primitive with respect to H* if it represents a primitive element in $\pi_1(H)$. We define Seifert-fibered simple closed curves on the boundary of a genus two handlebody similarly.

The following lemma establishes the link between Seifert-fibered curves on a genus two handlebody and Seifert-fibered spaces.

Lemma 3.2.8 Let γ be a curve in the boundary of a genus two handlebody H that is (a, b) Seifert-fidered with respect to H. Then the manifold M obtained by adding a 2-handle to H along γ is a Seifert-fibered space over D^2 with two critical fibers of multiplicities a and b. In paricular,

$$M \cong D^2 \times S^1 \Leftrightarrow a \text{ or } b \text{ equals } 1 \Leftrightarrow \gamma \text{ is primitive.}$$

Proof The fundamental group of M is $G_{a,b}$, which has a non-trivial center. M is irreducible and Haken, hence by [31], is a Seifert-fibered manifold. It is known that a Seifert-fibered manifold with such a fundamental group is a Seifert-fibered manifold space with base space a disk and critical fibers of multiplicity a and b. By considering when $G_{a,b}$ is isomorphic to Z, the last part follows.

3.3 Primitive/Seifert-fibered knots

Putting the these definitions and lemmas together, we describe a property of a knot that ensures that the knot will have a Dehn surgery that is a small Seifert-fibered space or a connected sum of two lens spaces.

Definition 3.3.1 Let K be a knot contained in a genus two Heegaard surface F for S^3, that is, $S^3 - H \cup_F H'$, where H and H' are genus two handlebodies. Then K is *primitive/Seifert-fibered with respect to* F if it is primitive with respect to H and Seifert-fibered with respect to H'.

Proposition 3.3.2 If a knot K in S^3 is primitive/Seifert-fibered with respect to a genus two Heegaard surface, then Dehn surgery at the surface slope is either a small Seifert-fibered space or a connected sum of two lens spaces.

Proof By Lemma 3.1.3 and Lemma 3.2.8 the Dehn surgered manifold is the union along a torus of a Seifert-fibered space over the disk with at most two critical fibers and a solid torus. Thus surface slope Dehn surgery on a primitive/Seifert-fibered knot results in a manifold that is a Dehn filling of a Seifert-fibered space over the disk with two critical fibers.

Since any non-meridinal simple closed curve on the boundary of a solid torus can be extended to a Seifert fibration of the solid torus, only one Dehn filling on such a Seifert-fibered manifold may fail to be Seifert-fibered. This occurs exactly when the slope is an ordinary fiber. For any other filling, a new critical fiber appears with multiplicity equal to the algebraic intersection number of the slope with the ordinary fiber. So for any Dehn filling but one, the resulting manifold is a Seifert-fibered space over the sphere with at most three critical fibers, i.e. a small Seifert-fibered space.

A Seifert-fibered space over the disk with two fibers is the union of two solid tori glued along an annulus. When each fiber is non-trivial, this annulus intersects a meridian of each solid torus algebraically more than once. A curve parallel to the annulus is an ordinary fiber. When a solid torus is attached with slope equal to the ordinary fiber, the resulting manifold can be cut apart into two solid tori, each with a 2-handle attached along a curve which intersects the meridian more than once algebraically. Thus each piece is a punctured lens space, so the manifold is a connected sum of two lens spaces.

Boileau, Rost, and Zieschang have classified those curves γ on the boundary of an abstract genus two handlebody H that are Seifert-fibered [4]. An embedding of such a pair (H, γ) into S^3 such that H is unknotted and γ is primitive with respect to $S^3 - H$ would give a P/SF knot. However, it would be defficult to consider all possible unknotted embeddings of such pairs, and to determine which are primitive on the "outside" handlebody.

Remarks

• According to cabling conjecture, only cabled knots have reducible Dehn surgeries [15]. In particular, the conjecture implies that a hyperbolic primitive/Seifert-fibered knot would always have a small Seifert-fibered surgery.

• One could define a knot to be primitive/Seifert fibered in any 3-manifold of Heegaard genus less than or equal to two and the proposition would hold.

• When the knot is primitive/primitive *(doubly primitive)*, a lens space results from the surface slope Dehn surgery (this is Berge's construction mentioned above).

• Surface slope Dehn surgery on a doubly Seifert-fibered knot results in the union along a torus of two Seifert manifolds over the disk, each with two critical fibers. Such a manifold is either a graph manifold or a Seifert manifold with base S^2 and four critical fibers. No example is known of a hyperbolic knot with a Dehn surgery of the latter type. However, there are satellite knots with such Dehn surgeries [22].

Note that any primitive/Seifert-fibered knot has tunnel number 1. In fact, any knot that is primitive with respect to one side of a genus two Heegaard surface has tunnel number 1. This is ture since, by [33], there is a homeomorphism of the handlebody after which the knot K appears as in Figure 4. If one pushes K

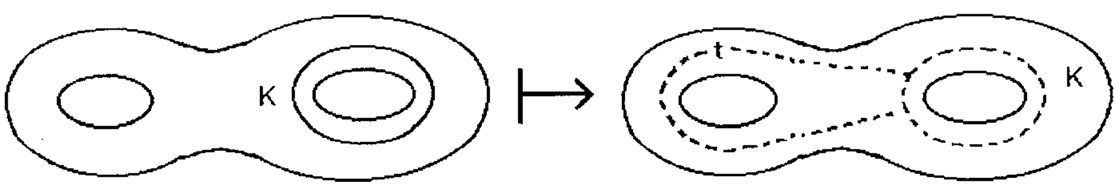


Figure 4 : Primitive on one side implies tunnel number one

slightly into the handlebody, removes a regular neighborhood of K, and then removes an appropriate tunnel t, then what remains in the handlebody is the product of a surface and an interval (see Figure 4). Thus the complement in S^3 is a handlebody, so the knot has tunnel number 1.

3.4 Known examples of Primitive/Seifert-fibered knots

It is not clear from the definition of a primitive/Seifert-fibered knot whether any non-trivial examples exist. We do not consider torus knots that arise from the construction to be interesting since Dehn surgery on torus knots is completely understood. Berge's work shows that, in fact, there are a plethora of interesting knots that are doubly primitive, many of which are known to be hyperbolic. In particular, they include *Gabai's 1-bridge braid knots* as illustrated in figure for a pretzel knot $p(-2,3,7)$.

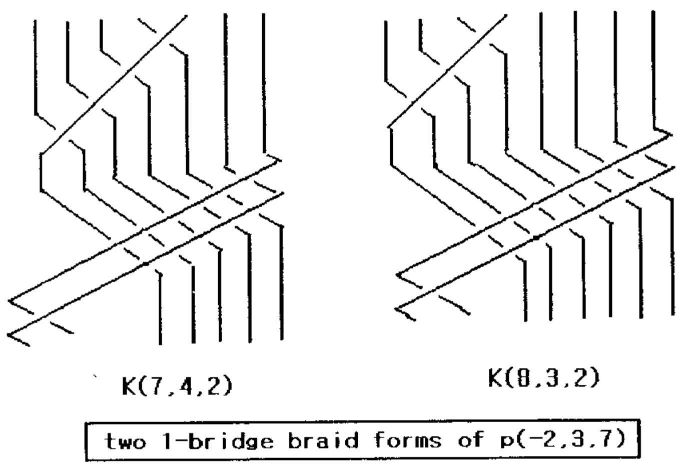


Fig. 5-Example 1. **twist 2-bridge knots**

Let b_n be a twist 2-bridge knots obtained by n-full twisting of one component of the right-handed Whitehead link with respect to the other. Brittenhem-Wu([6]) showed that the Seifert fibered manifold $b_n(2)$ (resp. $b_n(3)$) can be obtained by 2-surgery of f_3 (resp. 3- surgery of f_2), the exceptional fiber of index 3 (resp. 2) in Dehn surgery of the right-handed trefoil, $t(3,2)(-1/n+2)$ (resp. $t(2,3)(-1/n+3)$). And, Dean([10]) has shown that $b_n(2)$ (resp. $b_n(3)$) can be represented by a primitive/Seifert-fibered construction with $(2,4n+1)$ (resp. $(3,3n+1)$) as a type of exceptional fibers corresponding to Seifert-fibered part.

DEFINITION 3.4.1 Let p, q be a pair of integers such that $2 \le |p| < q$ and $(p, q) = 1$. For a positive integer $1 \le r \le |p| + q$, We take an unknot u in the exterior of a torus $t(p, q)$ so that r- parallel strands of $t(p, q)$ may transversally intersect a spanning disk of u and $|lk(t(p, q), u)| = r$. Then, A knot obtained by $n-$ full twisting of $t(p, q)$ with respect to u is denoted by $K(p, q, r, n)$ and said to be *a twisted torus knot.*

Example 2. Dean has detected many twisted torus knots admitting primitive/Seifert-fibered structures under the conditions; $2 \le p < q$, $r < q$ and $n = \pm 1$.

For examples, he has shown that

Theorem 3.4.2([10])

1. $K(p, q, 2p - q, 1)$ with $(q + 1)/2 < p < q$ is $(2, 2q - p)-$ Seifert-fibered.
2. $K(p, q, q - kp, 1)$ with $1 < p < q/2$ and $2 \le k \le (q - 2)/p$ is $(k, q - kp)-$ Seifert-fibered.

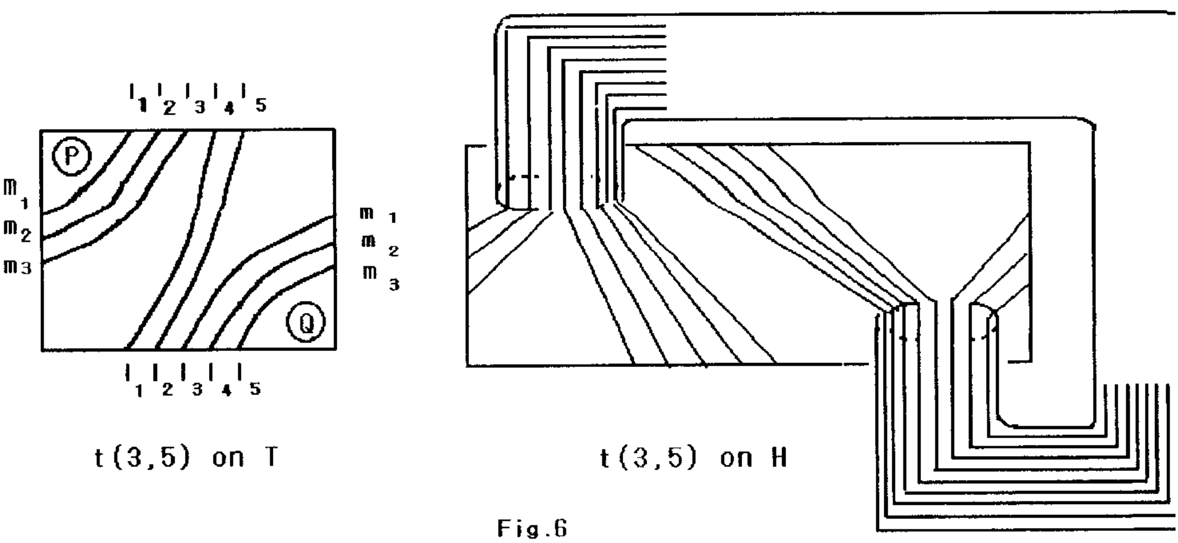
4 Primitive/Seifert-fibered constructions of $K(p, q, p + q, n)$

4.1 embeddings of $K(p, q, p + q, n)$ into a genus-2 Heegaard surface of S^3

In this thesis, we shall investigate primitive/ Seifert-fibered property of twist torus knots $K(p, q, p + q, n)$, $2 \le |p| < q$ and $(p, q) = 1$. In a recent work of Miyazaki and Motegi([21]), they have shown that $K(p, q, p + q, n)$ admits a primitive/ Seifert-fibered construction with (q, n) Seifert-fibered part and the third exceptional fiber of index $|p|$. For comparison with our elementary method, their proof will be reproduced in a next section. In this section, we shall show how to describe $K(p, q, p + q, n)$ as a double torus knot for investigation of its primitive/ Seifert-fibered property.

CASE1. $0 < p$

Consider the standard plane model of a 2-torus $T = S^1 \times S^1$. A pair of vertical edges, denote the left edge (resp. the right edge) by m^- (resp. m^+), represent the meridean $m = S^1 \times 1$ of T after identification of them ($m = m^+ = m^-$). Likewise, a pair of horizontal edges, denote the top edge (resp. the bottom edge) by l^+ (resp. l^-), represent the longitude $= 1 \times S^1$ of T after identification of them ($l = l^+ = l^-$). Along m (resp. l) , we take p- (q-) number of equally spaced points $\{m_1, m_2, \cdots, m_p\}$ from the top to the bottom (resp. $\{l_1, l_2, \cdots, l_q\}$ from the left to the right. For each $1 \leq i \leq p$, join m_i on m^- with l_i on l^+ by an arc α_i on T and m_i on m^+ with l_{p-i+1} on l^- by an arc β_i on T. Finally, for each $1 \leq j \leq q-p$, join l_i on l^- with l_{p+j} on l^+ by an arc γ_j on T. Further, we assume that the arcs α_i, β_i, and γ_j are chosen so that they may be mutually disjoint. Then, a simple closed curve $\alpha_i \cup \beta_i \cup \gamma_j$ on T form a torus knot $t(p,q)$ as depicted in figure 6 . Now, we choose a pair of points $\{P, Q\}$ on T so that they may be disjoint from $t(p,q)$ and an arc joining them may transversally intersects $t(p,q)$ at $p+q$-number of points. Attaching an orientable handle on T after removing the small regular neighbourhood of P and Q, we have an embedding of $T(p,q)$ on H, a genus-2 Heegaard surface of S^3 such that $t(p,q)$ transversally meets the meridian of the attached handle, say it n, at $p+q$-number of points as illustrated in figure 6 for $t(3,5)$.



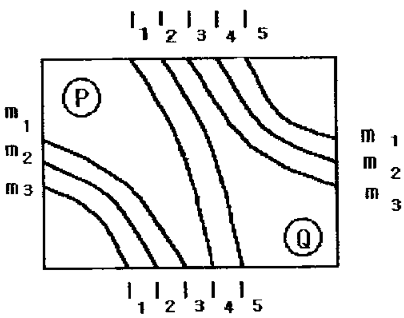
t(3,5) on T

t(3,5) on H

Fig.6

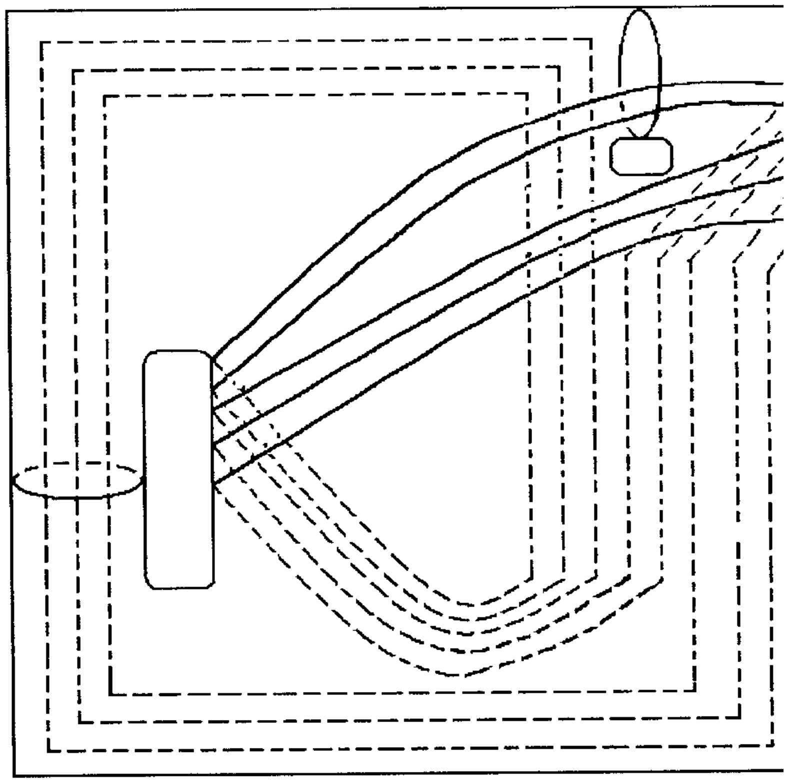
CASE2. $p < 0$

Note that $t(p,q)$ is the mirror image of $t(-p,q)$. In this case we can embedd $t(p,q)$ on T so that the arc joining P and Q may transversally intersect $t(p,q)$ at $p+q = (q-|p|)$-number of points by taking diagonally opposed embeddings of the arcs of case 2 as illustrated in figure for $t(-3,5)$. Figure 7-(b) represents an explicit embedding of $t(-3,5)$ to H, a genus 2 Heegaard surface of S^3.



t(−3,5) on T

Fig.7-(a)



t(−3,5) on H

Fig.7-(b)

Fig.7

Now, taking n-full twisting of $t(p,q)$ on H with respect to the meridian of the attached handle, we get the desired twisted torus knot $K(p,q,p+q,n)$ embedded to H as illustrated in figure 8 for $K(3,5,8,1)$.

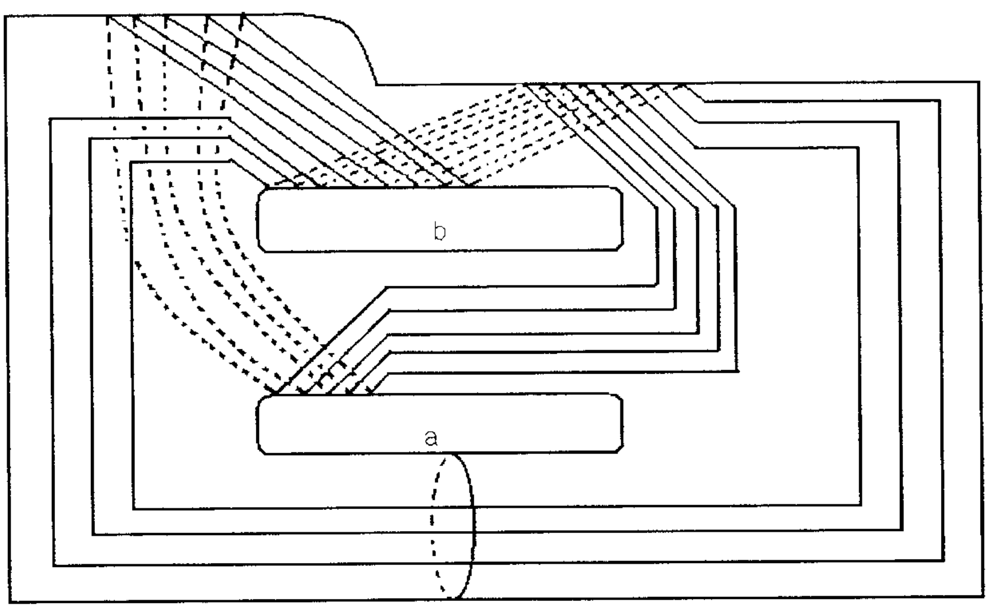


Fig.8

4.2 Miyazaki-Motegi's proof for primitive/seifert-fibered property of $K(p, q, p+q, n)$

Let V_1 be a standardly embedded solid torus in S^3 and V_2 the complementary solid torus $S^3 - int\ V_1$. Let A be an annulus on ∂V_1 which winds around p times meridionally and q times longitudinally ($q > |p| \geq 2$), and set $A' = \partial V_1 - int\ A$. Now we take a trivial knot τ in S^3 as depicted in Fig.9, and put $\tau_i = \tau \cap V_i$ for $i = 1, 2$. Take a tubular neighbourhood $N(\tau)$ of τ such that $N(\tau) \cap A = \emptyset$. Let $V = S^3 - int\ N(\tau)$, an unknotted solid torus. Then the core curve $C_{p,q}$ of A is a knot in V. It should be noted that a meridian of $N(\tau)$ is a longitude of V and $wind_V(C_{p,q}) = lk(\tau, C_{p,q}) = p + q$. Furthermore, we can observed that the minimal geometric intersection number of $C_{p,q}$ with a meridian disk of V also equals $p + q$.

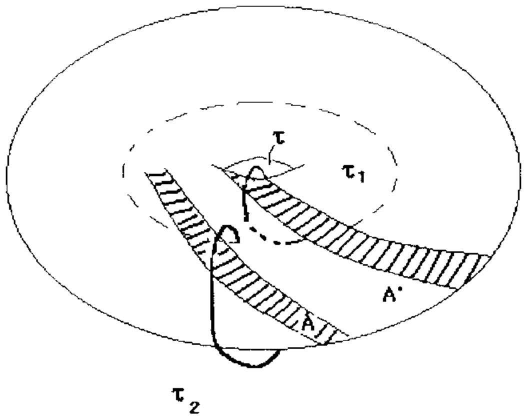


Fig.9

Lemma 4.2.1 The surgered manifold $V(C_{p,q}; pq)$ is a seifert fibered manifold over the disk with two exceptional fibers of induces $|p|$ and q. Furthermore, a longitude of V is a regular fiber of $V(C_{p,q}; pq)$. Proof. Choose a tubular n.b.d. $N(C_{p,q})$ of $C_{p,q}$ so that $N(C_{p,q} \cap \partial V_1 = A$. We identify $V(C_{p,q}; pq)$ with the union of two manifolds, which turn out to be soild tori. Firstly, note that $V - int\ N(C_{p,q}) \cong (V_1 - int\ N(\tau_1)) \cup (V_2 - int\ N(\tau_2)) - M$, say, where $V_i - int\ N(\tau_i)$ are pasted along $A' - int\ N(\tau)$, an annulus with two holes; then the component of ∂M corresponding to $\partial N(C_{p,q})$ is the union of two copies of A. Since a component of $\partial A (\subset \partial N(C_{p,q}))$ has the slope pq in terms of a meridian-longitude pair of $C_{p,q}$, in $V(C_{p,q}; pq) = (V - int\ N(C_{p,q}) \cup (S^1 \times D^2)$ the components of ∂A bound two disjoint meridian disks of the attached solid torus $S^1 \times D^2$. The disks decompose $S^1 \times D^2$ into two 3-balls $h_i^2, i = 1, 2$, each of which is attached to $V_i - int\ N(\tau_i)$ along a copy of A as a 2-handle. Hence, we can regard $V(C_{p,q}; pq)$ as the natural union of two manifolds $U_1 = (V_1 - int\ N(\tau_1)) \cup_A h_1^2$ and $U_2 = (V_2 - int\ N(\tau_2)) \cup_A h_2^2$. Since each τ_i is an unknotted arc in V_i, $V_i - int\ N(\tau_i)$ is a handlebody of genus 2. Furthermore, U_i is a sloid torus. This can be explained for U_1 as follows. Firstly,"expanding" $N(\tau_1 \cup A')$ by an ambient isotopy of V_1, we can see $V_1 - int\ N(\tau_1 \cup A') \cong N(A) \cup h^1$, where h_1 is a 1-handle depicted in Fig 10-(b). Clearly $V_1 - int\ N(\tau) \cong V_1 - int\ N(\tau_1 \cup A')$, hence U_1 is homeomorphic to $N(A) \cup_A h_1^2 \cup h^1$, which is a solid torus. A meridian disk of U_1 can be viewed as in Fig 10 . We remark that a meridian of $N(\tau_i)$ intersects that of U_1 algebarically q times ($q \geq 2$).

Let us see how the solid tori U_1 and U_2 are glued together. Set $T = \partial N(\tau_1) - \partial V_1$. Then T is an annulus on ∂U_1, and $U_1 \cap U_2$ is the complementary annulus $\partial U_1 - int\ T$. Since a meridian of $N(\tau_1)$ intersects that of U_1 q times, the annulus $U_1 \cap U_2$ also winds ∂U_1 around q times longitudinally. The same argument works for U_2, and shows that $U_1 \cap U_2$ winds around ∂U_2 p times logitudinally. Therefore, $V(C_{p,q}; pq) \cong U_1 \cup U_2$ is a Seifert fibered manifold over the disk with two exceptional fibers of indices $|p|$ and q. from the construction, a longitude of V(= a meridian of $N(\tau)$ is a regular fiber of $V(C_{p,q}; pq)$.

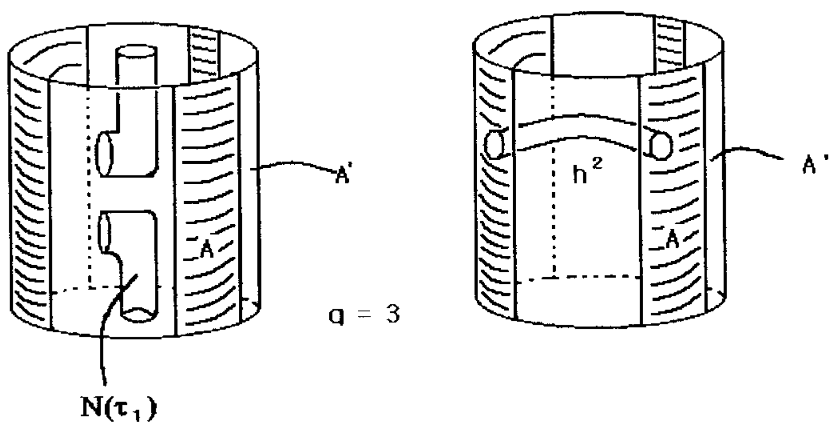


Fig. 10-a

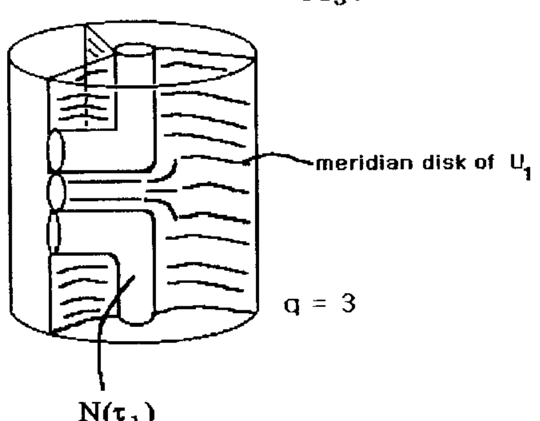


Fig. 10-b

proposition 4.2.2([21]) $K(p,q,p+q,n)$ is *primitive/Seifert-fibered* with surface slope $pq+(p+q)^2n$.

Proof.

Set $V_1 = W$ and $V_2 = S^3 - \text{int}V_1$; V_i are solid tori. By the choice of τ, $H_1 = V_1 - \text{int}N(\tau)$ and $H_2 = V_2 - \text{int}N(\tau)$ are genus 2 handlebodies, and $K_{p,q}$ lies on $\partial V_i - \text{int}N(\tau)$. Let A_i be the annulus $H_i \cap \partial N(\tau)$, whose core is a meridian of τ. Let V be the solid torus glued to $S^3 - \text{int}N(\tau)$ to construct $(\tau; -1/n)$. Then, H_1 and V are pasted so that $A_1(\subset \partial H_1)$ is identified with an annulus in ∂V whose core is the $(1,n)$-cable of V. Since a core of A_1 is primitive with respect to H_1, $H_1' = H_1 \cup V$ is also a handlebody of genus 2. We have $(\tau; -1/n) = H_1' \cup H_2$, a Heegaard decomposition of genus 2. The twisted torus knot $K(p,q,p+q,n)$ lies on the Heegaard surface $\partial H_1'$ as the image of $K_{p,q} \subset \partial H_1$. For simplicity, we denote the simple loop $K_{p,q} = K(p,q,p+q,n)$ by K. As shown in Lamma 4.2.1, $H_1[K]$ is a fibered solid torus over the base orbifold $D^2(q)$ with a core of A_1 a fiber, and $H_2[K]$ is a fibered solid torus over $D^2(|p|)$ with a core of A_2 a fiber. It follows that $H_1'[K] = H_1[K] \cup V$ is a Seifert fibered manifold over $D^2(q,|n|)$. Therefore $K(p,q,p+q,n)$ is primitive with respect to H_2 and Seifert-fibered with respect to H_1'. Since a meridian of τ is a regular fiber in $H_1'[K]$ and $H_2[K]$, surgery on K along the surface slope of $K \subset H_1'$ produces a Seifert fibered manifold with base orbifold $S^2(|p|,q,|n|)$, as stated in Proposition 4.2.2. The surface slope of $K_{p,q} = K(p,q,p+q,0)$ in ∂H_1 is pq, and the image of the slope after $-1/n$-surgery on τ is the surface slope of $K(p,q,p+q,n) \subset \partial H_1'$. The linking number of $K_{p,q}$ and τ is $p+q$, so that the surface slope of $K(p,q,p+q,n)$ is $pq+(p+q)^2n$. This completes the proof.

4.3 Main Results

Now we take a view of a torus $T = \mathbb{R}^2/\mathbb{Z}^2$ as the quotient group of $\mathbb{R}^2$ by the planar lattice group $\mathbb{Z}^2 = \{(m,n)|m,n \in \mathbb{Z}\}$ under the ususual addition operation of vectors. Then, T can be described as

$$T = I^2/(s,0) \sim (s,1) \text{ and } (0,t) \sim (1,t), s,t \in I = [0,1]$$

by taking the unit square I^2 as the fundamental domain of the translation action of $\mathbb{Z}^2$ on $\mathbb{R}^2$. Let $\pi : \mathbb{R}^2 \to T$ be the associated universal covering projection For each lattice point $(m,n) \in \mathbb{Z}^2$ with $(m,n) = 1$, let $\overline{(m,n)}$ be the straight line segment joining $(0,0)$ and (m,n). Then, it is easy to see that $\pi(\overline{(m,n)})$ $((m,n) = 1)$ is a torus curve of type (n,m) where we take $m \equiv \pi(\overline{(1,0)})$ and $l \equiv \pi(\overline{(0,1)})$ as the meridian and the longitude of T respectively. Let $Z_{p,q} \in F(m,l)$ be a word defined by the intersection patten of a torus curve $t(p,q)$ with the meridian m and the longitude l ; we write m (resp. l) for each intersection point of $t(p,q)$ and m (resp. l). . Recalling the intersection patten of $\overline{(m,n)}$ and the two axes of $\mathbb{R}^2$ with the occurence patten of words s,t in the Osborne-Ziechang's primitive word, we have; **Lemma 4.3.1** For a torus curve $t(q,p)$ on $T = I^2/(s,0) \sim (s,1)$ *and* $(0,t) \sim (1,t), s,t \in I = [0,1]$ with the meridian $m \equiv \pi(\overline{(1,0)})$ and the longitude $m \equiv \pi(\overline{(1,0)})$, $Z_{p,q}(s,t)$ is the Osborne-Zieschang's primitive word $W_{q,p}$.

Example.

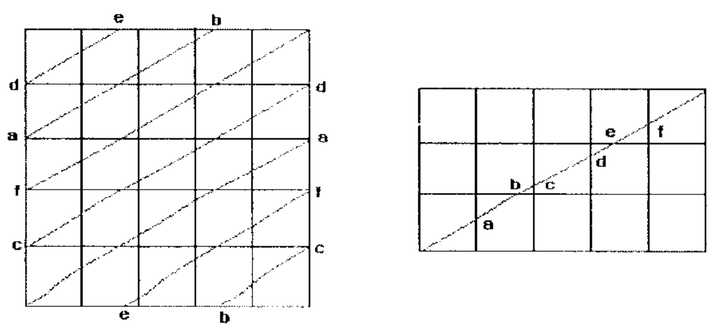


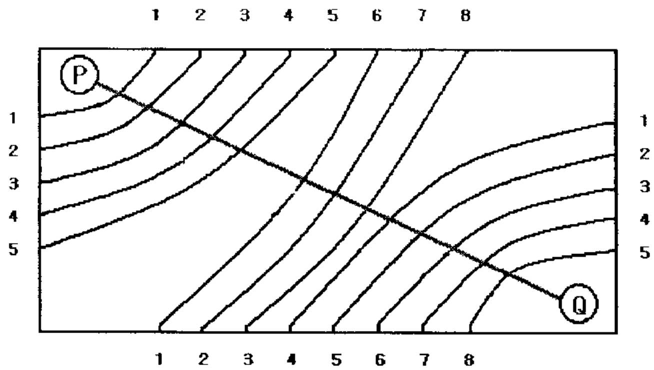
Fig. 11

Theorem 4.3.2 $K \equiv K(p,q,p+q,n)$ admits an embedding to a genus 2

Heegaard surface H of S^3 such that
(1)$[K]_P \in \pi_1(W_1, *)$ is a primitive word $W_{p+q,p}(x, y)$, and
(2)$[K]_{SF} \in \pi_1(W_2, *)$ is a Seifert-fibered word $W_{p+q,q}(a^n, b)$
(3) the Seifert fibered word is Nielsen equivalent to is $A^n B^q$.
,where W_i are genus 2 handlebodies in the Heegaard decomposition, $S^3 = W_1 \cup_H W_2$ and $W_{m,n}$ is a Osborne-Zieschang's primitive word.

Proof. Note that H in figure 6 (or equivalently figure) determines a genus 2 handlebody W_1 with a pair of meridian disks corresponding to the meridian of T and that of the attached handle. On the other hand, H determines another genus 2 handlebody $W_2 = S^3 - Int(W_1)$ with a pair of meridian disks corresponding to the longitude of T and that of the attached handle as deficted in figure . Then, Two free generators a, b (resp. x, y) of $\pi_1(W_2, *) = F(a, b)$ (resp. $\pi_1(W_1, *) = F(x, y)$ are represented by the meridian (resp. longitude) of the attached handle and the meridian (resp. the longitude) of T .
Since K is uniformly oriented in one direction, we may take the free generators a, b (resp. x, y) of $\pi_1(W_2, *) = F(a, b)$ (resp. $\pi_1(W_1, *) = F(x, y)$) so that $[K] \in \pi_1(W_1, *)$ (resp. $[K] \in \pi_1(W_2, *)$) may always have positive exponents a, b (resp. x, y).

For computation of the Seifert-fibered word $[K]_{SF} \in \pi_1(W_2, *)$, it is convenient to consider the embedding of $t(p, q)$ on T and the diagonal arc d joining P and Q as illustrated in figure 12 for $K(5, 8, 13, n)$.



K(5,8,13,n)

$a^{2n}b\ a^{2n}b\ a^{n}\ b\ a^{2n}b\ a^{2n}b\ a^{n}\ b\ a^{2n}b\ a^{n}\ b$

Fig.12

Note that whenever $t(p,q)$ meets d , K travels along the attached handle while transversally intersecting with the longitude of the attached handle $n-$ times, which contributes word a^n, and $p+q$ is the total number of occurences of a^n in $[K]_{SF}$. On the other hand, K transversally intersects with the longitude of T $q-$ times and hence q is the total number of occurences of b in $[K]_{SF}$.
Then by lemma 4.3.1 $Z_{p.p+q}(a,b)$ is a Osborne-Zieschang's primitive word $W_{p+q.p}$. On the other hand $[K]_{SF} \in \pi_1(W_2,*)$ can be obtained by substituting a^n into a in the word $Z_{p.p+q}(a,b)$, where the word $Z_{m.l}(a,b)$ is obtained by considering a torus curve $t(p,q)$ on T cut by the diagonal edge d. Thus we have $[K] = W_{p+q.q}(a^n,b)$. By the similar argument, we see that $[K]_P \in \pi_1(W_1,*)$ is a primitive word $W_{p+q.p}(x,y)$. Finally claim (3) in the theorem is obtained by applying theorem 3.2.7 to the fact that $W_{p+q.q}(a^n,b)$ is Nielsen equivalent to $W_{q-p.q}(a^n,b)$(resp. $W_{|p|.q}(a^n,b)$) if $q < 2|p|$ (resp. $q > 2|p|$)

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