On Second Order Symmetric Duality For Multiobjective Optimization Problems

다목적 최적화 문제에 대한 2계 대칭 쌍대성 문제

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요　　　약

본 논문에서는 변수가 분리된 다목적 최적화 문제에 대한 2계 대칭 쌍대문제를 생각한다. 2계 인벡스티 함수 조건 아래서, 약 효율해에 대한 2계 약 쌍대정리, 2계 강 쌍대정리, 2계 역 쌍대정리를 증명한다. 그리고 주 정리들로부터 많은 최적화 문제들에 대한 쌍대정리가 유도 될 수 있음을 보인다.

1. Introduction

Multiobjective programming problems consist of more than one objective function and a constrained set. Their optima are the solution concepts that appear to be the natural extension of the optimization from a single objective to multi objectives, called properly efficient solutions, efficient solutions and weakly efficient solutions. Dantzig, Eisenberg and Cottle [3] first formulated a pair of symmetric dual nonlinear programs in which the dual of the dual equaled the primal one, and established the weak and strong duality for these problems concerning convex and concave functions. Mond [12] presented a slightly different pair of symmetric dual nonlinear programs and obtained more generalized duality results than those of Dantzig, Eisenberg and Cottle [3]. Later on, Mond and Weir [14] gave different pair of symmetric dual nonlinear programs in which could be applied to the pseudo-convexity and pseudo-concavity functions, and established the weak and strong duality of these programs. Recently, Mond [13] introduced the concept of second order convex functions and proved second order duality and symmetric duality results under the assumptions of second order convexity on functions. And Mangasarian [11] formulated a second order dual program for a nonlinear program and established second order duality theorems under assumption that was rather difficult to be verified. Afterwards, Mond and Weir [15] introduced a generalized second order dual and establised the duality results under the second order pseudo-convexity assumption on the dual objective function and second order quasi-convexity assumption on the dual constraints. Independently, Bector and Chandra [1] established second order symmetric and self duality results under pseudo bonvexity and pseudo boncavity assumptions.

As a generalization of differentiable convex function, Hanson [6] introduced the weak convex function, where it was shown that the Kuhn-Tucker conditions were sufficient for optimality of nonlinear programming problems under the condition of weak convex function. The weak convex function was called invex function by Craven [12]. Afterwards, the second order invexity was introduced by Egudo and Hanson [5].

In multiobjective optimization case, Weir and Mond [16] established the symmetric and self duality relations in multiobjective programming. Mond and Weir [15] proved symmetric duality theorems for nonlinear multiobjective programming. In 1996, Kim, Yun and Kuk [8] suggested another second order symmetric and self dual programs in multiobjective nonlinear programming and proved the weak, strong and converse duality theorems under convexity and concavity conditions. Lee and Kim [10] formulated a pair of generalized multiobjective symmetric dual nonlinear programs which were unifying several known symmetric programs and established the duality theorems. Kim, Lee and Lee [9] extended the results to second order cases. Very recently, Jeung [7] proved generalized second order symmetric duality for multiobjective optimization problems on the basis of efficiency and under second order invexity and incavity assumptions when the variables were seperated into two parts. The aim of this paper is to consider Jeung [7]'s results on the basis of weak efficiency and to give generalized second order symmetric duality results which can be applied under second order convexity and concavity assumptions. In this thesis, we formulate a pair of generalized second order mulitobjective symmetric dual nonlinear programs, which can be reduced to several known programs and establish the second order duality relations for our programs on the basis of weak efficiency. Moreover, we will show that many duality results can be deduced from our second order dual ones. This paper consists of four sections. In Section 2, we fix mathematical notations and give definitions for generalized second order invexity. In Section 3, we consider the pair of generalized multiobjective symmetric dual nonlinear programs and eatablish duality results (weak duality, strong duality, converse duality) for the two programs. Finally, in Section 4, we apply our generalized second order symmetric duality results to the usual (first order) multiobjective optimization problems and then we will get several duality results for the problems.

2. Notations and Definitions

In this section, we give notations and definitions which will be used for next sections. The following conventions for vectors in $\mathbb{R}^n$ will be used:

$$x < y \Leftrightarrow x_i < y_i, \ \ i = 1, 2, \cdots, n;$$

$$x \leqq y \Leftrightarrow x_i \leqq y_i, \ \ i = 1, 2, \cdots, n;$$

$$x \leq y \Leftrightarrow x_i \leqq y_i, \ \ i = 1, 2, \cdots, n \text{ but } x \neq y;$$

$$x \nless y \text{ is the negation of } x < y; and$$

$$x \nleq y \text{ is the negation of } x \leq y.$$

Let f be a twice differentiable function from $\mathbb{R}^n \times \mathbb{R}^m$ into $\mathbb{R}^k$ and $N = \{1, 2, \cdots, n\}$, $M = \{1, 2, \cdots, m\}$, $A \subset N$, $I \subset M$, $N \setminus A = B$ and $M \setminus I = J$. Note that A, B, I or J can be empty. We rearrange x, y as $x = (x_A, x_B)$ and $y = (y_I, y_J)$, respectively. $\nabla_x f(x, y)$ denotes $k \times n$ matrix of first partial derivatives. If $\lambda \in \mathbb{R}^k$, then $\lambda^T f$ is a scalar valued function. Let $\nabla_x(\lambda^T f)(x, y)$ and $\nabla_y(\lambda^T f)(x, y)$ denote gradient (column) vectors with respect to x and y, respectively. Subsequently, let $\nabla_{xx}(\lambda^T f)$ and $\nabla_{yy}(\lambda^T f)$ denote respectively the $n \times n$ and $m \times m$ matrices of second partial derivatives.

We consider the following multiobjective programming problem.

$$(MP) \qquad \begin{array}{ll} \text{Minimize} & f(x) \\ \text{subject to} & g(x) \leqq 0, \end{array}$$

where $f : \mathbb{R}^n \to \mathbb{R}^k$, $g : \mathbb{R}^n \to \mathbb{R}^m$ and $X = \{x \in \mathbb{R}^n \mid g(x) \leqq 0\}$ is the feasible set of (MP).

DEFINITION 2.1. *(1) A point $\bar{x} \in X$ is an efficient solution (or Pareto optimal point) of (MP) if there exists no other $x \in X$ such that $f(x) \leq f(\bar{x})$.*

(2) A point $\bar{x} \in X$ is a weakly efficient solution of (MP) if there exists no other $x \in X$ such that $f(x) < f(\bar{x})$.

DEFINITION 2.2. *A twice differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be*

(i) second order convex if for any $x, r, u \in \mathbb{R}^n$,

$$f(x) - f(u) \geqq (x-u)^T \nabla f(u) + (x-u)^T \nabla^2 f(u) r - \frac{1}{2} r^T \nabla^2 f(u) r,$$

(ii) second order concave if for any $x, u, p \in \mathbb{R}^n$,

$$f(x) - f(u) \leqq (x-u)^T \nabla f(u) + (x-u)^T \nabla^2 f(u) p - \frac{1}{2} p^T \nabla^2 f(u) p,$$

DEFINITION 2.3. *A twice differentiable function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$ is said to be*

(i) second order convex ([13]) if for fixed $y \in \mathbb{R}^m$ and for all $x, r, u \in \mathbb{R}^n$,

$$f_i(x,y)-f_i(u,y) \geqq (x-u)^T\nabla_x f_i(u,y)+(x-u)^T\nabla_{xx}f_i(u,y)r-\frac{1}{2}r^T\nabla_{xx}f_i(u,y)r,$$

for each $i = 1, 2, \cdots, k$.

(ii) second order concave if for fixed $x \in \mathbb{R}^n$ and for all $y, p, v \in \mathbb{R}^m$,

$$f_i(x,v)-f_i(x,y) \leqq (v-y)^T\nabla_y f_i(x,y)+(v-y)^T\nabla_{yy}f_i(x,y)p-\frac{1}{2}p^T\nabla_{yy}f_i(x,y)p,$$

for each $i = 1, 2, \cdots, k$.

DEFINITION 2.4. *A twice differentiable function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$ is said to be*

(i) second order invex ([1],[5]) in x with respect to the $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ if for fixed $y \in \mathbb{R}^m$ and for all $x, r, u \in \mathbb{R}^n$,

$$f_i(x,y)-f_i(u,y) \geqq \eta(x,u)^T\nabla_x f_i(u,y)+\eta(x,u)^T\nabla_{xx}f_i(u,y)r-\frac{1}{2}r^T\nabla_{xx}f_i(u,y)r,$$

for each $i = 1, 2, \cdots, k$.

(ii) second order incave in y with respect to the $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ if for fixed $x \in \mathbb{R}^n$ and for all $y, p, v \in \mathbb{R}^m$,

$$f_i(x,v)-f_i(x,y) \leqq \eta(v,y)^T\nabla_y f_i(x,y)+\eta(v,y)^T\nabla_{yy}f_i(x,y)p-\frac{1}{2}p^T\nabla_{yy}f_i(x,y)p,$$

for each $i = 1, 2, \cdots, k.$

3. Generalized Second Order Symmetric Duality

In this section we establish generalized second order symmetric duality theorems. Now we consider the following pair of generalized multiobjective symmetric dual nonlinear programs.

$$
\begin{aligned}
(GSP)\ \text{Minimize}\ \ & f(x,y) - ({y_I}^T \nabla_{y_I} (\lambda^T f)(x,y))e \\
& - ({y_I}^T \nabla_{yy_I} (\lambda^T f)(x,y)p)e - \frac{1}{2}(p^T \nabla_{yy} (\lambda^T f)(x,y)p)e \\
\text{subject to}\ \ & \nabla_y (\lambda^T f)(x,y) + \nabla_{yy} (\lambda^T f)(x,y)p \leqq 0, \\
& y_J^T \nabla_{y_J} (\lambda^T f)(x,y) + y_J^T \nabla_{yy_J} (\lambda^T f)(x,y)p \geqq 0, \\
& \lambda \geq 0,\ \lambda^T e = 1,\ x \geqq 0,
\end{aligned}
$$

$$
\begin{aligned}
(GSD)\ \text{Maximize}\ \ & f(u,v) - (u_A^T \nabla_{x_A} (\lambda^T f)(u,v))e \\
& - ({u_A}^T \nabla_{xx_A} (\lambda^T f)(u,v)r)e - \frac{1}{2}(r^T \nabla_{xx} (\lambda^T f)(u,v)r)e \\
\text{subject to}\ \ & \nabla_x (\lambda^T f)(u,v) + \nabla_{xx} (\lambda^T f)(u,v)r \geqq 0, \\
& u_B^T \nabla_{x_B} (\lambda^T f)(u,v) + u_B^T \nabla_{xx_B} (\lambda^T f)(u,v)r \leqq 0, \\
& \lambda \geq 0,\ \lambda^T e = 1,\ v \geqq 0,
\end{aligned}
$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$, $\lambda \in \mathbb{R}^k$ and $e = (1, \cdots, 1)^T \in \mathbb{R}^k$.

$\nabla_{x_A}(\lambda^T f)(x,y)$, $\nabla_{x_B}(\lambda^T f)(x,y)$, $\nabla_{y_I}(\lambda^T f)$ and $\nabla_{y_J}(\lambda^T f)$ are gradient vectors with respect to x_A, x_B, y_I and y_J, respectively. $\nabla_{xx} f(x,y)$ and $\nabla_{yy} f(x,y)$ are respectively the $n \times n$ and $m \times m$ symmetric Hessian matrices.

Now we establish the symmetric duality theorems for (GSP) and (GSD).

THEOREM 3.1. **(Weak Duality)** *Let (x, y, λ, p) be feasible for (GSP) and (u, v, λ, r) be feasible for (GSD). Assume that $f(\cdot, y)$ is second order invex for fixed y with $\eta_1(x, u) + u \geqq 0$ and $f(x, \cdot)$ is second order incave for fixed x with $\eta_2(v, y) + y \geqq 0$. Then*

$$\begin{aligned} f(x,y) &- ({y_I}^T \nabla_{y_I}(\lambda^T f)(x,y))e - ({y_I}^T \nabla_{yy_I}(\lambda^T f)(x,y)p)e \\ &\qquad - \frac{1}{2}(p^T \nabla_{yy}(\lambda^T f)(x,y)p)e \\ \nleq f(u,v) &- (u_A^T \nabla_{x_A}(\lambda^T f)(u,v))e - ({u_A}^T \nabla_{xx_A}(\lambda^T f)(u,v)r)e \\ &\qquad - \frac{1}{2}(r^T \nabla_{xx}(\lambda^T f)(u,v)r)e. \end{aligned}$$

Proof. Assume that

$$\begin{aligned} f(x,y) &- ({y_I}^T \nabla_{y_I}(\lambda^T f)(x,y))e - ({y_I}^T \nabla_{yy_I}(\lambda^T f)(x,y)p)e \\ &\qquad - \frac{1}{2}(p^T \nabla_{yy}(\lambda^T f)(x,y)p)e \\ < f(u,v) &- (u_A^T \nabla_{x_A}(\lambda^T f)(u,v))e - ({u_A}^T \nabla_{xx_A}(\lambda^T f)(u,v)r)e \\ &\qquad - \frac{1}{2}(r^T \nabla_{xx}(\lambda^T f)(u,v)r)e. \end{aligned}$$

Then since $\lambda \geq 0$, we have

$$(\lambda^T f)(x,y) - {y_I}^T \nabla_{y_I}(\lambda^T f)(x,y) - {y_I}^T \nabla_{yy_I}(\lambda^T f)(x,y)p - \frac{1}{2}p^T \nabla_{yy}(\lambda^T f)(x,y)p$$
$$< (\lambda^T f)(u,v) - u_A^T \nabla_{x_A}(\lambda^T f)(u,v) - {u_A}^T \nabla_{xx_A}(\lambda^T f)(u,v)r$$
$$- \frac{1}{2}r^T \nabla_{xx}(\lambda^T f)(u,v)r. \quad (3.1)$$

By the second order invexity of $f(\cdot, y)$,

$$f(x,v) - f(u,v) \geqq \eta_1(x,u)^T \nabla_x f(u,v) + \eta_1(x,u)^T \nabla_{xx} f(u,v)r$$
$$- \frac{1}{2}r^T \nabla_{xx} f(u,v)r.$$

Since $u_B^T \nabla_{x_B}(\lambda^T f)(u,v) + u_B^T \nabla_{xx_B}(\lambda^T f)(u,v)r \leqq 0$,

$$(\lambda^T f)(x,v) - (\lambda^T f)(u,v) \geqq \eta_1(x,u)^T \nabla_x(\lambda^T f)(u,v) + \eta_1(x,u)^T \nabla_{xx}(\lambda^T f)(u,v)r$$
$$- \frac{1}{2}r^T \nabla_{xx}(\lambda^T f)(u,v)r$$
$$\geqq -u^T \nabla_x(\lambda^T f)(u,v) - u^T \nabla_{xx}(\lambda^T f)(u,v)r - \frac{1}{2}r^T \nabla_{xx}(\lambda^T f)(u,v)r$$
$$= -u_A^T \nabla_{x_A}(\lambda^T f)(u,v) - u_B^T \nabla_{x_B}(\lambda^T f)(u,v) - u_A^T \nabla_{xx_A}(\lambda^T f)(u,v)r$$
$$- u_B^T \nabla_{xx_B}(\lambda^T f)(u,v)r - \frac{1}{2}r^T \nabla_{xx}(\lambda^T f)(u,v)r$$
$$\geqq -u_A^T \nabla_{x_A}(\lambda^T f)(u,v) - u_A^T \nabla_{xx_A}(\lambda^T f)(u,v)r - \frac{1}{2}r^T \nabla_{xx}(\lambda^T f)(u,v)r. \quad (3.2)$$

By the second order incavity of $f(x, \cdot)$,

$$f(x,v) - f(x,y) \leqq \eta_2(v,y)^T \nabla_y f(x,y) + \eta_2(v,y)^T \nabla_{yy} f(x,y)p$$
$$- \frac{1}{2}p^T \nabla_{yy} f(x,y)p.$$

Since $y_J^T \nabla_{y_J}(\lambda^T f)(x,y) + y_J^T \nabla_{yy_J}(\lambda^T f)(x,y)p \geqq 0$,

$$
\begin{aligned}
(\lambda^T f)(x,v) - (\lambda^T f)(x,y) &\leqq \eta_2(v,y)^T \nabla_y(\lambda^T f)(x,y) + \eta_2(v,y)^T \nabla_{yy}(\lambda^T f)(x,y)p \\
&\qquad - \frac{1}{2}p^T \nabla_{yy}(\lambda^T f)(x.y)p \\
&\leqq -y^T \nabla_y(\lambda^T f)(x,y) - y^T \nabla_{yy}(\lambda^T f)(x,y)p - \frac{1}{2}p^T \nabla(\lambda^T f)(x,y)p \\
&= -y_I^T \nabla_{y_I}(\lambda^T f)(x,y) - y_J^T \nabla_{y_J}(\lambda^T f)(x,y) - y_I^T \nabla_{yy_I}(\lambda^T f)(x,y)p \\
&\qquad - y_J^T \nabla_{yy_J}(\lambda^T f)(x,y)p - \frac{1}{2}p^T \nabla_{yy}(\lambda^T f)(x,y)p \\
&\leqq -y_I^T \nabla_{y_I}(\lambda^T f)(x,y) - y_I^T \nabla_{yy_I}(\lambda^T f)(x,y)p - \frac{1}{2}p^T \nabla_{yy}(\lambda^T f)(x,y)p. \qquad (3.3)
\end{aligned}
$$

Substracting (3.3) from (3.2) and rearranging yields

$$
\begin{aligned}
&(\lambda^T f)(u,v) - u_A^T \nabla_{x_A}(\lambda^T f)(u,v) - u_A^T \nabla_{xx_A}(\lambda^T f)(u,v)r - \frac{1}{2}r^T \nabla_{xx}(\lambda^T f)(u,v)r \\
&\leqq (\lambda^T f)(x,y) - y_I^T \nabla_{y_I}(\lambda^T f)(x,y) - y_I^T \nabla_{yy_I}(\lambda^T f)(x,y)p - \frac{1}{2}p^T \nabla_{yy}(\lambda^T f)(x,y)p,
\end{aligned}
$$

which contradicts (3.1). Thus

$$
\begin{aligned}
&f(x,y) - ({y_I}^T \nabla_{y_I}(\lambda^T f)(x,y))e - ({y_I}^T \nabla_{yy_I}(\lambda^T f)(x,y)p)e \\
&\qquad - \frac{1}{2}(p^T \nabla_{yy}(\lambda^T f)(x,y)p)e \\
&\not\leq f(u,v) - (u_A^T \nabla_{x_A}(\lambda^T f)(u,v))e - ({u_A}^T \nabla_{xx_A}(\lambda^T f)(u,v)r)e \\
&\qquad - \frac{1}{2}(r^T \nabla_{xx}(\lambda^T f)(u,v)r)e.
\end{aligned}
$$

COROLLARY 3.1. **(Weak Duality)** *Let (x, y, λ, p) be feasible for (GSP) and*

(u, v, λ, r) be feasible for (GSD). Assume that $f(\cdot, y)$ is second order convex for fixed y and $f(x, \cdot)$ is second order concave for fixed x . Then

$$\begin{aligned}
& f(x, y) - ({y_I}^T \nabla_{y_I} (\lambda^T f)(x, y))e - ({y_I}^T \nabla_{yy_I} (\lambda^T f)(x, y)p)e \\
& \qquad\qquad\qquad - \frac{1}{2}(p^T \nabla_{yy} (\lambda^T f)(x, y)p)e \\
& \not\leq f(u, v) - (u_A^T \nabla_{x_A} (\lambda^T f)(u, v))e - ({u_A}^T \nabla_{xx_A} (\lambda^T f)(u, v)r)e \\
& \qquad\qquad\qquad - \frac{1}{2}(r^T \nabla_{xx} (\lambda^T f)(u, v)r)e.
\end{aligned}$$

PROPOSITION 3.1. **(Fritz John optimality Conditions)** *If $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})$ is a weakly efficient solution of (GSP), then there exists $(\alpha, \beta, \gamma, \rho, \delta)$ in $\mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k$ such that*

$$\begin{aligned}
K \equiv\ & \alpha^T [f - (\bar{y}_I^T \nabla_{y_I} (\bar{\lambda}^T f))e - (\bar{y}_I^T \nabla_{yy_I} (\bar{\lambda}^T f)\bar{p})e - \frac{1}{2}(\bar{p}^T \nabla_{yy} (\bar{\lambda}^T f)\bar{p})e] \\
& + \beta^T [\nabla_y (\bar{\lambda}^T f) + \nabla_{yy} (\bar{\lambda}^T f)\bar{p}] - \gamma[\bar{y}_J^T \nabla_{y_J} (\bar{\lambda}^T f) + \bar{y}_J^T \nabla_{yy_J} (\bar{\lambda}^T f)\bar{p}] \\
& - \rho^T x - \delta^T \bar{\lambda}
\end{aligned}$$

satisfies

$$\nabla_x K \geqq 0,$$

$$\nabla_{y_I} K = 0,$$

$$\nabla_{y_J} K = 0,$$

$$\nabla_p K = 0,$$

$$\nabla_\lambda K = 0,$$

$$\beta^T[\nabla_y(\bar\lambda^T f) + \nabla_{yy}(\bar\lambda^T f)\bar p] = 0,$$

$$\gamma[\bar y_J^T \nabla_{y_J}(\bar\lambda^T f) + \bar y_J^T \nabla_{yy_J}(\bar\lambda^T f)\bar p] = 0,$$

$$\rho^T \bar x = 0,$$

$$\delta^T \bar\lambda = 0,$$

$$(\alpha, \beta, \gamma, \rho, \delta) \geqq 0,$$

$$(\alpha, \beta, \gamma, \rho, \delta) \neq 0.$$

THEOREM 3.2. **(Strong Duality)** *Let f be a three times differentiable function from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^k$. Let $(\bar x, \bar y, \bar\lambda, \bar p)$ be a weakly efficient solution of (GSP). Assume that $f(\cdot, y)$ is second order invex for fixed y with $\eta_1(x,u) + x \geqq 0$ and $f(x,\cdot)$ is second order incave for fixed x with $\eta_2(v,y) + v \geqq 0$. Suppose that*

(i) $\nabla_{yy}\left(\bar\lambda^T f\right)(\bar x, \bar y)$ is non-singular,

(ii) $\nabla_{y_J}(\bar\lambda^T f)(\bar x, \bar y) + \nabla_{yy_J}(\bar\lambda^T f)(\bar x, \bar y)\bar p \neq 0$,

(iii) the set $\{\nabla_{y_J} f_i(\bar x, \bar y)\}_{i=1,\cdots,k}$ is linearly independent and

(iv) the matrix $\frac{\partial}{\partial y_i}(\nabla_{yy}(\bar\lambda^T f)(\bar x, \bar y))$ is positive or negative definite, for some $i \in I$.

Then $(\bar x, \bar y, \bar\lambda, \bar p = 0)$ is a feasible solution of (GSD) and $(\bar x, \bar y, \bar\lambda, \bar p)$ is a weakly efficient solution of (GSD).

Proof. Since $(\bar x, \bar y, \bar\lambda, \bar p)$ is a weakly efficient solution of (GSP), it follows from proposition 3.1 that there exist $\alpha \in \mathbb{R}^k$, $\beta \in \mathbb{R}^m$, $\gamma \in \mathbb{R}$, $\rho \in \mathbb{R}^n$

and $\delta \in \mathbb{R}^k$ such that the following Fritz John conditions are satisfied at $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})$:

$$
\begin{aligned}
&\nabla_x(\alpha^T f) - \nabla_{y_I x}(\lambda^T f)(\alpha^T e)\bar{y}_I - \nabla_x(\nabla_{yy_I}(\lambda^T f)\bar{p}(\alpha^T e)\bar{y}_I) \\
&\quad - \nabla_x(\frac{1}{2}\bar{p}^T \nabla_{yy}(\bar{\lambda}^T f)\bar{p}(\alpha^T e)) + \beta^T \nabla_{yx}(\lambda^T f) + \nabla_x(\beta^T \nabla_{yy}(\bar{\lambda}^T f)\bar{p}) \\
&\quad - \nabla_{y_J x}(\bar{\lambda}^T f)\gamma\bar{y}_J - \nabla_x(\nabla_{yy_J}(\bar{\lambda}^T f)\bar{p}(\gamma\bar{y}_J)) \\
&= \nabla_x\left(\alpha^T f\right) - \left(\nabla_{y_I x}\left(\bar{\lambda}^T f\right) \;\; \nabla_{y_J x}\left(\bar{\lambda}^T f\right)\right)\begin{pmatrix} \left(\alpha^T e\right)\bar{y}_I - \beta_I \\ \gamma\bar{y}_J - \beta_J \end{pmatrix} \\
&\quad - \nabla_x(\nabla_{yy}(\bar{\lambda}^T f)\bar{p})\begin{pmatrix} (\alpha^T e)\bar{y}_I + \frac{1}{2}(\alpha^T e)\bar{p}_I - \beta_I \\ \gamma\bar{y}_J + \frac{1}{2}(\alpha^T e)\bar{p}_J - \beta_J \end{pmatrix} - \rho \\
&\geqq 0
\end{aligned} \tag{3.4}
$$

$$
\begin{aligned}
&\nabla_{y_I}(\alpha^T f) - \nabla_{y_I}(\bar{\lambda}^T f)(\alpha^T e) - \nabla_{y_I y_I}(\bar{\lambda}^T f)(\alpha^T e)\bar{y}_I - \nabla_{yy_I}(\bar{\lambda}^T f)(\alpha^T e)\bar{p} \\
&\quad - \nabla_{yI}(\nabla_{yy_I}(\bar{\lambda}^T f)\bar{p}(\alpha^T e)\bar{y}_I) - \nabla_{y_I}(\frac{1}{2}\bar{p}^T \nabla_{yy}(\bar{\lambda}^T f)(\alpha^T e)\bar{p}) + \beta^T \nabla_{yy_I}(\bar{\lambda}^T f) \\
&\quad + \nabla_{y_I}(\beta^T \nabla_{yy}(\bar{\lambda}^T f)\bar{p}) - \nabla_{y_J y_I}(\bar{\lambda}^T f)\gamma\bar{y}_J - \nabla_{y_I}(\nabla_{yy_J}(\bar{\lambda}^T f)\bar{p}(\gamma\bar{y}_J)) \\
&= -(\nabla_{y_I y_I}(\bar{\lambda}^T f) \;\; \nabla_{y_J y_I}(\bar{\lambda}^T f))\begin{pmatrix} (\alpha^T e)\bar{y}_I - \beta_I + (\alpha^T e)\bar{p}_I \\ \gamma\bar{y}_J - \beta_J + (\alpha^T e)\bar{p}_J \end{pmatrix} \\
&\quad - \nabla_{y_I}\{(\nabla_{yy_I}(\bar{\lambda}^T f)\bar{p} \;\; \nabla_{yy_J}(\bar{\lambda}^T f)\bar{p})\begin{pmatrix} (\alpha^T e)\bar{y}_I + \frac{1}{2}(\alpha^T e)\bar{p}_I - \beta_I \\ \gamma\bar{y}_J + \frac{1}{2}(\alpha^T e)\bar{p}_J - \beta_J \end{pmatrix}\} \\
&= 0
\end{aligned} \tag{3.5}
$$

$$
\begin{aligned}
&\nabla_{y_J}(\alpha^T f) - \nabla_{y_I y_J}(\bar{\lambda}^T f)(\alpha^T e)\bar{y}_I - \nabla_{y_J}(\nabla_{yy_I}(\bar{\lambda}^T f)\bar{p}(\alpha^T e)\bar{y}_I) \\
&\quad - \nabla_{y_J}(\frac{1}{2}(\bar{p}^T\nabla_{yy}(\alpha^T f)\bar{p}(\alpha^T e)) + \beta^T\nabla_{yy_J}(\bar{\lambda}^T f) + \nabla_{y_J}(\beta^T\nabla_{yy}(\bar{\lambda}^T f)\bar{p}) \\
&\quad - \gamma^T\nabla_{y_J}(\bar{\lambda}^T f) - \nabla_{y_J y_J}(\bar{\lambda}^T f)\gamma\bar{y}_J - \gamma^T\nabla_{yy_J}(\bar{\lambda}^T f)\bar{p} \\
&\quad - \nabla_{y_J}(\nabla_{yy_J}(\bar{\lambda}^T f)\bar{p}(\gamma\bar{y}_J)) \\
&= (\alpha - \gamma\bar{\lambda})^T\nabla_{y_J} f - (\nabla_{y_I y_J}(\bar{\lambda}^T f)\ \ \nabla_{y_J y_J}(\bar{\lambda}^T f)) \begin{pmatrix} (\alpha^T e)\bar{y}_I - \beta_I + \gamma\bar{p}_I \\ \gamma\bar{y}_J - \beta_J + \gamma\bar{p}_J \end{pmatrix} \\
&\quad - \nabla_{y_J}\{(\nabla_{yy_I}(\bar{\lambda}^T f)\bar{p}\ \ \nabla_{yy_J}(\bar{\lambda}^T f)\bar{p}) \begin{pmatrix} (\alpha^T e)\bar{y}_I + \frac{1}{2}(\alpha^T e)\bar{p}_I - \beta_I \\ \gamma\bar{y}_J + \frac{1}{2}(\alpha^T e)\bar{p}_J - \beta_J \end{pmatrix}\} \\
&= 0
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
&- \nabla_{yy_I}(\bar{\lambda}^T f)(\alpha^T e)\bar{y}_I - (\nabla_{yy}(\bar{\lambda}^T f)\bar{p})(\alpha^T e) + \beta^T\nabla_{yy}(\bar{\lambda}^T f) - \nabla_{yy_J}(\bar{\lambda}^T f)\gamma\bar{y}_J \\
&= \nabla_{yy}(\bar{\lambda}^T f) \begin{pmatrix} \beta_I - (\alpha^T e)\bar{p}_I - (\alpha^T e)\bar{y}_I \\ \beta_J - (\alpha^T e)\bar{p}_J - \gamma\bar{y}_J \end{pmatrix} \\
&= 0
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
&- \nabla_{y_I} f(\alpha^T e)\bar{y}_I^T - \nabla_{yy_I} f\bar{p}(\alpha^T e)\bar{y}_I - \frac{1}{2}\bar{p}^T\nabla_{yy} f(\alpha^T e)\bar{p} \\
&\quad + \beta^T\nabla_y f + \beta^T\nabla_{yy} f\bar{p} - \nabla_{y_J} f(\gamma\bar{y}_J) - \nabla_{yy_J} f(\gamma\bar{y}_J)\bar{p} - \delta \\
&= (\nabla_{y_I} f\ \ \nabla_{y_J} f) \begin{pmatrix} \beta_I - (\alpha^T e)\bar{y}_I \\ \beta_J - \gamma\bar{y}_J \end{pmatrix} - (\nabla_{yy_I} f\bar{p}\ \ \nabla_{yy_J} f\bar{p}) \begin{pmatrix} (\alpha^T e)\bar{y}_I + \frac{1}{2}(\alpha^T e)\bar{p}_I - \beta_I \\ \gamma\bar{y}_J + \frac{1}{2}(\alpha^T e)\bar{p}_J - \beta_J \end{pmatrix} \\
&\quad - \delta \\
&= 0
\end{aligned}
\tag{3.8}
$$

$$\beta^T(\nabla_y(\bar{\lambda}^T f) + \nabla_{yy}(\bar{\lambda}^T f)\bar{p}) = 0. \tag{3.9}$$

$$\gamma(\bar{y}_J^T \nabla_{y_J}(\bar{\lambda}^T f) + \bar{y}_J^T \nabla_{yy_J}(\bar{\lambda}^T f)\bar{p}) = 0. \tag{3.10}$$

$$\rho^T \bar{x} = 0. \tag{3.11}$$

$$\delta^T \bar{\lambda} = 0. \tag{3.12}$$

$$(\alpha, \beta, \gamma, \rho, \delta) \geqq 0. \tag{3.13}$$

$$(\alpha, \beta, \gamma, \rho, \delta) \neq 0. \tag{3.14}$$

Since $\nabla_{yy}(\bar{\lambda}^T f)$ is non-singular, (3.7) yields

$$\beta_I = (\alpha^T e)(\bar{p}_I + \bar{y}_I) \quad \text{and} \quad \beta_J = (\alpha^T e)\bar{p}_J + \gamma \bar{y}_J \tag{3.15}$$

From (3.5) and (3.15), we have

$$\frac{1}{2}(\alpha^T e)\nabla_{y_I}(\bar{p}^T \nabla_{yy}(\bar{\lambda}^T f)\bar{p}) = 0 \tag{3.16}$$

Suppose that $\alpha = 0$.

From (3.6), $\gamma(\nabla_{y_J}(\bar{\lambda}^T f) + \nabla_{yy_J}(\bar{\lambda}^T f)\bar{p}) = 0$. Since $\nabla_{y_J}(\bar{\lambda}^T f) + \nabla_{yy_J}(\bar{\lambda}^T f)\bar{p} \neq 0$, $\gamma = 0$. From (3.15) and (3.8), we have $\beta = 0$ and $\delta = 0$. From (3.4), we have $0 \geqq \rho$. But since $\rho \geqq 0$, $\rho = 0$

This is a contradiction to (3.14). Hence $\alpha \neq 0$ (i.e. $\alpha \geq 0$). Using the hypothesis that $\frac{\partial}{\partial y_i}(\nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y}))$ is positive or negative definite in (3.16), we get

$$\bar{p} = 0 \tag{3.17}$$

Using (3.4),(3.15) and (3.17), we get

$$\nabla_x(\bar{\alpha}^T f)(\bar{x}, \bar{y}) \geqq 0 \tag{3.18}$$

From (3.6), (3.14) and (3.16), we have $(\alpha - \gamma\bar{\lambda})^T \nabla_{y_J} f = 0$.
Since $\{\nabla_{y_J} f_i(\bar{x}, \bar{y})\}_{i=1,\cdots,k}$ is linearly independent,

$$\alpha = \gamma\bar{\lambda} \tag{3.19}$$

and substituting (3.19) in (3.18) ,we have

$$\nabla_x(\bar{\lambda}^T f)(\bar{x}, \bar{y}) \geqq 0 \tag{3.20}$$

Using $\bar{p} = 0$, and (3.20),

$$\nabla_x(\bar{\lambda}^T f) + \nabla_{xx}(\bar{\lambda}^T f)\bar{p} \geqq 0, \tag{3.21}$$

$$\bar{x_B}^T \nabla_{x_B}(\bar{\lambda}^T f) + \bar{x_B}^T \nabla_{xx_B}(\bar{\lambda}^T f)\bar{p} = 0 \quad \text{and} \tag{3.22}$$

Since $\bar{p} = 0$,

$$\frac{1}{2}(\bar{p}^T \nabla_{xx}(\bar{\lambda}^T f)\bar{p})e = 0. \tag{3.23}$$

From (3.21)and (3.22), $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p} = 0)$ is a feasible for(GSD)

Now multiplying (3.8) by λ and using (3.9),(3.10) and (3.12) gives

$$\bar{y}_I^T \nabla_{y_I}(\bar{\lambda}^T f)(\bar{x},\bar{y}) + \bar{y}_I^T \nabla_{yy_I}(\bar{\lambda}^T f)(\bar{x},\bar{y})\bar{p} + \frac{1}{2}\bar{p}^T \nabla_{yy}(\bar{\lambda}^T f)(\bar{x},\bar{y})\bar{p} = 0.$$

By theorem 3.1 (weak duality), $(\bar{x},\bar{y},\bar{\lambda},\bar{r})$ is a weakly efficient solution for (GSD).

COROLLARY 3.2. **(Strong Duality)** *Let f be a three times differentiable function from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^k$. Let $\left(\bar{x},\bar{y},\bar{\lambda},\bar{p}\right)$ be a weakly efficient solution of (GSP). Assume that $f(\cdot,y)$ is second order covex for fixed y and $f(x,\cdot)$ is second order concave for fixed x . Suppose that*

(i) $\nabla_{yy}\left(\bar{\lambda}^T f\right)(\bar{x},\bar{y})$ is non-singular,

(ii) $\nabla_{y_J}(\bar{\lambda}^T f)(\bar{x},\bar{y}) + \nabla_{yy_J}(\bar{\lambda}^T f)(\bar{x},\bar{y})\bar{p} \;\neq\; 0,$

(iii) the set $\{\nabla_{y_J} f_i(\bar{x},\bar{y})\}_{i=1,\cdots,k}$ is linearly independent and

(iv) the matrix $\frac{\partial}{\partial y_i}(\nabla_{yy}(\bar{\lambda}^T f)(\bar{x},\bar{y}))$ is positive or negative definite, for some $i \in I$.

Then $\left(\bar{x},\bar{y},\bar{\lambda},\bar{p}=0\right)$ is a feasible solution of (GSD) and $(\bar{x},\bar{y},\bar{\lambda},\bar{p})$ is a weakly efficient solution of (GSD).

By the similar method of Theorem 3.2, we can prove the following converse duality theorem.

THEOREM 3.3. **(Converse Duality)** *Let f be a three times differentiable function from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^k$. Let $\left(\bar{u}, \bar{v}, \bar{\lambda}, \bar{r}\right)$ be a weakly efficient solution of (GSD). Assume that $f(\cdot, y)$ is second order invex for fixed y with $\eta_1(x, u) + x \geqq 0$, and $f(x, \cdot)$ is second order incave for fixed x with $\eta_2(v, y) + v \geqq 0$. Suppose that*

(i) $\nabla_{xx}\left(\bar{\lambda}^T f\right)(\bar{u}, \bar{v})$ *is non-singular,*

(ii) $\nabla_{x_B}(\bar{\lambda}^T f)(\bar{u}, \bar{v}) + \nabla_{xx_B}(\bar{\lambda}^T f)(\bar{u}, \bar{v})\bar{r} \neq 0,$

(iii) the set $\{\nabla_{x_B} f_i(\bar{u}, \bar{v})\}_{i=1,\cdots,k}$ *is linearly independent and*

(iv) the matrix $\frac{\partial}{\partial x_i}(\nabla_{xx}(\bar{\lambda}^T f)(\bar{u}, \bar{v}))$ *is positive or negative definite, for some $i \in A$.*

Then $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{r} = 0)$ is a feasible solution of (GSD) and $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{r} = 0)$ is a weakly efficient solution of (GSD) and (GSP).

COROLLARY 3.3. **(Converse Duality)** *Let f be a three times differentiable function from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^k$. Let $\left(\bar{u}, \bar{v}, \bar{\lambda}, \bar{r}\right)$ be a weakly efficient solution of (GSD). Assume that $f(\cdot, y)$ is second order connvex for fixed y and $f(x, \cdot)$ is second order concave for fixed x. Suppose that*

(i) $\nabla_{xx}\left(\bar{\lambda}^T f\right)(\bar{u}, \bar{v})$ *is non-singular,*

(ii) $\nabla_{x_B}(\bar{\lambda}^T f)(\bar{u}, \bar{v}) + \nabla_{xx_B}(\bar{\lambda}^T f)(\bar{u}, \bar{v})\bar{r} \neq 0,$

(iii) the set $\{\nabla_{x_B} f_i(\bar{u}, \bar{v})\}_{i=1,\cdots,k}$ *is linearly independent and*

(iv) the matrix $\frac{\partial}{\partial x_i}(\nabla_{xx}(\bar{\lambda}^T f)(\bar{u}, \bar{v}))$ *is positive or negative definite, for some $i \in A$.*

Then $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{r} = 0)$ is a feasible solution of (GSD) and $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{r} = 0)$ is a weakly efficient solution of (GSD) and (GSP).

4. Applications

In this section, we apply results in Section 3 to the first order multiobjective symmetric dual programs and the usual multiobjective optimization problems. If $I = \emptyset$ and $A = \emptyset$, then our pair of programs (GSP) and (GSD) is reduced to the following multiobjective second order symmetric dual problems (MSP) and (MSD), which are Mond-Weir type ones.

$$
\begin{aligned}
(MSP) \qquad & \text{Minimize } f(x,y) - \frac{1}{2}(p^T \nabla_{yy}(\lambda^T f)(x,y)p)e \\
& \text{subject to } \nabla_y(\lambda^T f)(x,y) + \nabla_{yy}(\lambda^T f)(x,y)p \leqq 0, \\
& y^T \nabla_y(\lambda^T f)(x,y) + y^T \nabla_{yy}(\lambda^T f)(x,y)p \geqq 0, \\
& \lambda \geq 0,\ \lambda^T e = 1,\ x \geqq 0. \\
(MSD) \qquad & \text{Maximize } f(u,v) - \frac{1}{2}(r^T \nabla_{xx}(\lambda^T f)(u,v)r)e \\
& \text{subject to } \nabla_x(\lambda^T f)(u,v) + \nabla_{xx}(\lambda^T f)(u,v)r \geqq 0, \\
& u^T \nabla_x(\lambda^T f)(u,v) + u^T \nabla_{xx}(\lambda^T f)(u,v)r \leqq 0, \\
& \lambda \geq 0,\ \lambda^T e = 1,\ v \geqq 0.
\end{aligned}
$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$, $\lambda \in \mathbb{R}^k$ and $e = (1, \cdots, 1)^T \in \mathbb{R}^k$.

We can easily give the weak, strong and converse duality for (MSP) and (MSD) from Theorems 3.1, 3.2, and 3.3.

THEOREM 4.1. **(Weak Duality)** *Let (x, y, λ, p) be feasible for (GSP) and (u, v, λ, r) be feasible for (GSD). Assume that $f(\cdot, y)$ is second order invex for fixed y with $\eta_1(x,u) + u \geqq 0$ and $f(x, \cdot)$ is second order incave for fixed x with $\eta_2(v, y) + y \geqq 0$. Then*

$$f(x,y) - \frac{1}{2}(p^T \nabla_{yy}(\lambda^T f)(x,y)p)e \not\leq f(u.v) - \frac{1}{2}(r^T \nabla_{xx}(\lambda^T f)(u,v)r)e.$$

THEOREM 4.2. **(Strong Duality)** *Let f be a three times differentiable function from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^k$. Let $\left(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p}\right)$ be a weakly efficient solution of (GSP). Assume that $f(\cdot, y)$ is second order invex for fixed y with $\eta_1(x, u) + x \geqq 0$ and $f(x, \cdot)$ is second order incave for fixed x with $\eta_2(v, y) + v \geqq 0$. Suppose that*

(i) $\nabla_{yy}\left(\bar{\lambda}^T f\right)(\bar{x}, \bar{y})$ is non-singular,

(ii) $\nabla_{y_J}(\bar{\lambda}^T f)(\bar{x}, \bar{y}) + \nabla_{yy_J}(\bar{\lambda}^T f)(\bar{x}, \bar{y})\bar{p} \neq 0.$

(iii) the set $\{\nabla_{y_J} f_i(\bar{x}, \bar{y})\}_{i=1,\cdots,k}$ is linearly independent and

(iv) the matrix $\frac{\partial}{\partial y_i}(\nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y}))$ is positive or negative definite, for some $i \in I$.

Then $\left(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p} = 0\right)$ is a feasible solution of (GSD) and $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})$ is a weakly efficient solution of (GSD) and $f(\bar{x}.\bar{y})$.

THEOREM 4.3. **(Converse Duality)** *Let f be a three times differentiable function from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^k$. Let $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{r})$ be a weakly efficient solution of (GSD). Assume that $f(\cdot, y)$ is second order invex for fixed y with $\eta_1(x, u) + x \geqq 0$, and $f(x, \cdot)$ is second order incave for fixed x with $\eta_2(v, y) + v \geqq 0$. Suppose that*

(i) $\nabla_{xx}\left(\bar{\lambda}^T f\right)(\bar{u}, \bar{v})$ is non-singular,

(ii) $\nabla_{x_B}(\bar{\lambda}^T f)(\bar{u}, \bar{v}) + \nabla_{xx_B}(\bar{\lambda}^T f)(\bar{u}, \bar{v})\bar{r} \neq 0$.

(iii) the set $\{\nabla_{x_B} f_i(\bar{u}, \bar{v})\}_{i=1,\cdots,k}$ is linearly independent and

(iv) the matrix $\frac{\partial}{\partial x_i}(\nabla_{xx}(\bar{\lambda}^T f)(\bar{u}, \bar{v}))$ is positive or negative definite, for some $i \in A$.

Then $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{r} = 0)$ is a feasible solution of (GSD) and $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{r} = 0)$ is a weakly efficient solution of (GSD) and (GSP).

If $I = M$ and $A = N$, then our pair of programs (GSP) and (GSD) is reduced to the following multiobjective second order symmetric dual problems (WSP) and (WSD) which are Wolfe type ones.

$$
\begin{aligned}
(WSP)\ \text{Minimize}\ & f(x, y) - (y^T \nabla_y(\lambda^T f)(x, y))e \\
& - (y^T \nabla_{yy}(\lambda^T f)(x, y)p)e - \frac{1}{2}(p^T \nabla_{yy}(\lambda^T f)(x, y)p)e \\
\text{subject to}\ & \nabla_y(\lambda^T f)(x, y) + \nabla_{yy}(\lambda^T f)(x, y)p \leqq 0, \\
& \lambda \geq 0,\ \lambda^T e = 1,\ x \geqq 0
\end{aligned}
$$

$$
\begin{aligned}
(WSD)\text{Maximize}\ & f(u, v) - (u^T \nabla_x(\lambda^T f)(u, v))e \\
& - (u^T \nabla_{xx}(\lambda^T f)(u, v)r)e - \frac{1}{2}(u^T \nabla_{xx}(\lambda^T f)(u, v)r)e \\
\text{subject to}\ & \nabla_x(\lambda^T f)(u, v) + \nabla_{xx}(\lambda^T f)(u, v)r \geqq 0, \\
& \lambda \geq 0,\ \lambda^T e = 1,\ v \geqq 0
\end{aligned}
$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$, $\lambda \in \mathbb{R}^k$ and $e = (1, \cdots, 1)^T \in \mathbb{R}^k$.

We can easily give the weak, strong and converse duality for (WSP) and (WSD) from Theorems 3.1, 3.2, and 3.3.

THEOREM 4.4. **(Weak Duality)** *Let (x, y, λ, p) be feasible for (GSP) and (u, v, λ, r) be feasible for (GSD). Assume that $f(\cdot, y)$ is second order invex for fixed y with $\eta_1(x, u) + u \geqq 0$ and $f(x, \cdot)$ is second order incave for fixed x with $\eta_2(v, y) + y \geqq 0$. Then*

$$\begin{aligned} f(x,y) - (y^T \nabla_y (\lambda^T f)(x,y))e - (y^T \nabla_{yy} (\lambda^T f)(x,y)p)e \\ - \frac{1}{2}(p^T \nabla_{yy} (\lambda^T f)(x,y)p)e \\ \nleq f(u,v) - (u^T \nabla_x (\lambda^T f)(u,v))e - (u^T \nabla_{xx} (\lambda^T f)(u,v)r)e \\ - \frac{1}{2}(u^T \nabla_{xx} (\lambda^T f)(u,v)r)e. \end{aligned}$$

THEOREM 4.5. **(Strong Duality)** *Let f be a three times differentiable function from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^k$. Let $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})$ be a weakly efficient solution of (GSP). Assume that $f(\cdot, y)$ is second order invex for fixed y with $\eta_1(x, u) + x \geqq 0$ and $f(x, \cdot)$ is second order incave for fixed x with $\eta_2(v, y) + v \geqq 0$. Suppose that*

(i) $\nabla_{yy} \left(\bar{\lambda}^T f\right) (\bar{x}, \bar{y})$ is non-singular,

(ii) $\nabla_{y_J} (\bar{\lambda}^T f)(\bar{x}, \bar{y}) + \nabla_{yy_J} (\bar{\lambda}^T f)(\bar{x}, \bar{y})\bar{p} \neq 0$,

(iii) the set $\{\nabla_{y_J} f_i(\bar{x}, \bar{y})\}_{i=1,\cdots,k}$ is linearly independent and

(iv) the matrix $\frac{\partial}{\partial y_i}(\nabla_{yy}(\bar{\lambda}^T f)(\bar{x},\bar{y}))$ *is positive or negative definite, for some* $i \in I$.

Then $(\bar{x},\bar{y},\bar{\lambda},\bar{p}=0)$ *is a feasible solution of* (GSD) *and* $(\bar{x},\bar{y},\bar{\lambda},\bar{p})$ *is a weakly efficient solution of* (GSD).

THEOREM 4.6. **(Converse Duality)** *Let* f *be a three times differentiable function from* $\mathbb{R}^n \times \mathbb{R}^m$ *to* $\mathbb{R}^k$. *Let* $(\bar{u},\bar{v},\bar{\lambda},\bar{r})$ *be a weakly efficient solution of* (GSD). *Assume that* $f(\cdot,y)$ *is second order invex for fixed* y *with* $\eta_1(x,u)+x \geqq 0$. *and* $f(x,\cdot)$ *is second order incave for fixed* x *with* $\eta_2(v,y)+v \geqq 0$. *Suppose that*

(i) $\nabla_{xx}\left(\bar{\lambda}^T f\right)(\bar{u},\bar{v})$ *is non-singular,*

(ii) $\nabla_{x_B}(\bar{\lambda}^T f)(\bar{u},\bar{v}) + \nabla_{xx_B}(\bar{\lambda}^T f)(\bar{u},\bar{v})\bar{r} \neq 0$,

(iii) the set $\{\nabla_{x_B} f_i(\bar{u},\bar{v})\}_{i=1,\cdots,k}$ *is linearly independent and*

(iv) the matrix $\frac{\partial}{\partial x_i}(\nabla_{xx}(\bar{\lambda}^T f)(\bar{u},\bar{v}))$ *is positive or negative definite, for some* $i \in A$.

Then $(\bar{u},\bar{v},\bar{\lambda},\bar{r}=0)$ *is a feasible solution of* (GSD) *and* $(\bar{u},\bar{v},\bar{\lambda},\bar{r}=0)$ *is a weakly efficient solution of* (GSD) *and* (GSP).

If $p=0$ and $r=0$, then our pair of programs (GSP) and (GSD) is reduced to the following multiobjective first order symmetric dual problems (SP) and (SD).

$$(SP) \qquad \text{Minimize } f(x,y) - ({y_I}^T \nabla_{y_I} (\lambda^T f)(x,y))e$$
$$\text{subject to } \nabla_y (\lambda^T f)(x,y) \leqq 0,$$
$${y_J}^T \nabla_{y_J} (\lambda^T f)(x,y) \geqq 0,$$
$$\lambda \geq 0, \ \lambda^T e = 1, \ x \geqq 0.$$

$$(SD) \qquad \text{Maximize } f(u,v) - ({u_A}^T \nabla_{x_A} (\lambda^T f)(u,v))e$$
$$\text{subject to } \nabla_x (\lambda^T f)(u,v) \geqq 0,$$
$${u_B}^T \nabla_{x_B} (\lambda^T f)(u,v) \leqq 0,$$
$$\lambda \geq 0, \ \lambda^T e = 1, \ v \geqq 0.$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$, $\lambda \in \mathbb{R}^k$ and $e = (1, \cdots, 1)^T \in \mathbb{R}^k$.

We can easily give the weak, strong duality for (SP) and (SD) from Theorems 3.1, 3.2, and 3.3.

THEOREM 4.7. **(Weak Duality)** *Let (x, y, λ, p) be feasible for (GSP) and (u, v, λ, r) be feasible for (GSD). Assume that $f(\cdot, y)$ is second order invex for fixed y with $\eta_1(x, u) + u \geqq 0$ and $f(x, \cdot)$ is second order incave for fixed x with $\eta_2(v, y) + y \geqq 0$. Then*

$$f(x,y) - ({y_I}^T \nabla_{y_I} (\lambda^T f)(x,y))e \not< f(u,v) - (u_A^T \nabla_{x_A} (\lambda^T f)(u,v))e$$

Theorem 4.8. **(Strong Duality)** *Let f be a three times differentiable function from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^k$. Let $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})$ be a weakly efficient solution of (GSP). Assume that $f(\cdot, y)$ is second order invex for fixed y with $\eta_1(x, u) + x \geqq 0$ and $f(x, \cdot)$ is second order incave for fixed x with $\eta_2(v, y) + v \geqq 0$. Suppose that*

(i) $\nabla_{yy}\left(\bar{\lambda}^T f\right)(\bar{x}, \bar{y})$ is non-singular.

(ii) $\nabla_{y_J}(\bar{\lambda}^T f)(\bar{x}, \bar{y}) + \nabla_{yy_J}(\bar{\lambda}^T f)(\bar{x}, \bar{y})\bar{p} \neq 0$.

(iii) the set $\{\nabla_{y_J} f_i(\bar{x}, \bar{y})\}_{i=1,\cdots,k}$ is linearly independent and

(iv) the matrix $\frac{\partial}{\partial y_i}(\nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y}))$ is positive or negative definite, for some $i \in I$.

Then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p} = 0)$ is a feasible solution of (GSD) and $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})$ is a weakly efficient solution of (GSD).

Theorem 4.9. **(Converse Duality)** *Let f be a three times differentiable function from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^k$. Let $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{r})$ be a weakly efficient solution of (GSD). Assume that $f(\cdot, y)$ is second order invex for fixed y with $\eta_1(x, u) + x \geqq 0$. and $f(x, \cdot)$ is second order incave for fixed x with $\eta_2(v, y) + v \geqq 0$. Suppose that*

(i) $\nabla_{xx}\left(\bar{\lambda}^T f\right)(\bar{u}, \bar{v})$ is non-singular.

(ii) $\nabla_{x_B}(\bar{\lambda}^T f)(\bar{u}, \bar{v}) + \nabla_{xx_B}(\bar{\lambda}^T f)(\bar{u}, \bar{v})\bar{r} \neq 0$,

(iii) the set $\{\nabla_{x_B} f_i(\bar{u}, \bar{v})\}_{i=1,\cdots,k}$ is linearly independent and

(iv) the matrix $\frac{\partial}{\partial x_i}(\nabla_{xx}(\bar{\lambda}^T f)(\bar{u}, \bar{v}))$ is positive or negative definite, for some $i \in A$.

Then $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{r} = 0)$ *is a feasible solution of* (GSD) *and* $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{r} = 0)$ *is a weakly efficient solution of* (GSD) *and* (GSP).

Letting $f(x, y) = f(x) + y^T g(x)e$, and $p = r = 0$ where $f : \mathbb{R}^n \to \mathbb{R}^k$, $g : \mathbb{R}^n \to \mathbb{R}^m$, and $J = \emptyset$, (GSP) and (GSD) collapse to the following vector optimization problems.

$$
\begin{aligned}
(VP) \quad & \text{Minimize } f(x) \\
& \text{subject to } g(x) \leqq 0, \\
& \qquad x \geqq 0. \\
(VD) \quad & \text{Maximize } f(u) + v^T g(u)e - (u_A^T(\lambda^T \nabla_{x_A} f(u) + v^T \nabla_{x_A} g(u)))e \\
& \text{subject to } \lambda^T \nabla_x f(u) + y^T \nabla_x g(u)) \geqq 0, \\
& \qquad {u_B}^T(\lambda^T \nabla_{x_B} f(u) + y^T \nabla_{x_B} g(u)) \leqq 0, \\
& \qquad \lambda \geqq 0, \ \lambda^T e = 1, \ v \geqq 0.
\end{aligned}
$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$, $\lambda \in \mathbb{R}^k$ and $e = (1, \cdots, 1)^T \in \mathbb{R}^k$.

THEOREM 4.10. **(Weak Duality)** *Let* x *be feasible for* (VP) *and* (u, v, λ, r) *be feasible for* (VD). *Assume that* $f(\cdot) + y^T g(\cdot)e$ *is second order invex for fixed* y *with* $\eta(x, u) + u \geqq 0$. *Then*

$$f(x) \nleq f(u) + v^T g(u)e - (u_A^T(\lambda^T \nabla_{x_A} f(u) + v^T \nabla_{x_A} g(u)))e$$

Proof. *It is directly deduced from theorem 3.1 (weak dualty) and definition 2.3.*

THEOREM 4.11. **(Strong Duality)** *Suppose that $f(\cdot)+y^Tg(\cdot)e$ is second order invex for fixed y with $\eta(x,u)+u \geqq 0$. If $\bar{x} \in X$ is a weakly efficient solution of (VP) and a constraint qualification holds at $\bar{x}$, then $(\bar{x},\bar{y},\bar{\lambda})$ is a feasible solution of (VD) and $(\bar{x},\bar{y},\bar{\lambda})$ is a weakly efficient solution of (VD).*

Proof. By the usual Kuhn - Tuker Theorem, there exist

$\lambda_i^* \geqq 0$ $(\lambda_1^*,\cdots,\lambda_p^*) \neq 0$, $\mu_j^* \geqq 0$, $j=1,\cdots,m$, $\gamma_k^* \geqq 0$ $(k=1,\cdots,n)$

such that

$\sum_{i=1}^p \lambda_i^* \nabla f_i(\bar{x}) + \sum_{j=1}^m \mu_j^* \nabla g_j(\bar{x}) + \sum_{k=1}^n \gamma_k(0,\cdots,-1,0,0) = 0,$

$\mu_j^* g_j(\bar{x}) = 0,$ $(j=1,\cdots,m)$, $-\gamma_k^* \bar{x_k} = 0$, $(k=1,\cdots,n)$.

Thus there exist $\lambda_i^* \geqq 0$ $(\lambda_1^*,\cdots,\lambda_p^*) \neq 0$, $\mu_j^* \geqq 0.(j=1,\cdots,m)$, $\gamma_k^* \geqq 0$ $(k=1,\cdots,n)$ such that

$$\sum_{i=1}^p \lambda_i^* \nabla f_i(\bar{x}) + \sum_{j=1}^m \mu_j^* \nabla g_j(\bar{x}) = (\gamma_1^*,\gamma_2^*,\cdots,\gamma_n^*)\mu_j^* g_j(\bar{x}) = 0.$$

$$\gamma_k^* \bar{x_k} = 0, \quad (k=1,\cdots,n).$$

So, there exist $\lambda_i^* \geqq 0$, $(\lambda_1^*,\cdots,\lambda_p^*) \neq 0$, $\mu_j^* \geqq 0$, $(j=1,\cdots,m)$, such that

$$\sum_{i=1}^p \lambda_i^* \nabla f_i(\bar{x}) + \sum_{j=1}^m \mu_j^* \nabla g_j(\bar{x}) \geqq 0. \tag{4.1}$$

$\bar{x}^T[\sum_{i=1}^p \lambda_i^* \nabla f_i(\bar{x}) + \sum_{j=1}^m \mu_j^* \nabla g_j(\bar{x})] = 0,$ $\mu_j^* g_j(\bar{x}) = 0.$

Dividing (4.1) with $\sum_{i=1}^p \lambda_i^*$ and letting $\bar{\lambda}_i = \dfrac{\lambda_i^*}{\sum_{i=1}^p \lambda_i^*}$ and $\dfrac{\mu_j^*}{\sum_{i=1}^p \lambda_i^*} = \bar{y}_j$, we have

$$\sum_{i=1}^{p} \bar{\lambda}_i \nabla f_i(\bar{x}) + \sum_{j=1}^{m} \bar{y}_j \nabla g_j(\bar{x}) \geqq 0,$$

$$\bar{x}^T [\sum_{i=1}^{p} \bar{\lambda}_i \nabla f_i(\bar{x}) + \sum_{j=1}^{m} \bar{y}_j \nabla g_j(\bar{x})] = 0,$$

$$\bar{y}_j g_j(\bar{x}) = 0 \ \ (j = 1, \cdots, m).$$

Hence

$$f(\bar{x}) + \bar{y}^T g(\bar{x})e - (\bar{x}_A^T(\bar{\lambda}^T \nabla_{x_A} f(\bar{x}) + \bar{y}^T \nabla_{x_A} g(\bar{x}))e = f(\bar{x}),$$

$$\bar{\lambda}^T \nabla_x f(\bar{x}) + \bar{y}^T \nabla_x g(\bar{x}) \geqq 0$$

$$\text{and} \quad \bar{x}_B^T [\bar{\lambda}^T \nabla_{x_B} f(\bar{x}) + \bar{y}^T \nabla_{x_B} g(\bar{x})] = 0.$$

Thus $(\bar{x}, \bar{y}, \bar{\lambda})$ is a feasible solution of (VD). And $f(\bar{x}) = f(\bar{x}) + \bar{y}^T g(\bar{x})e - (\bar{x}_A^T(\bar{\lambda}^T \nabla_{x_A} f(\bar{x}) + \bar{y}^T \nabla_{x_A} g(\bar{x}))e$. By Theorem 4.10, for any feasible (u, v, λ) for (VD),

$$f(\bar{x}) \not< f(u) + v^T g(u)e - (u_A^T(\lambda^T \nabla_{x_A} f(u) + v^T \nabla_{x_A} g(u)))e.$$

Thus $(\bar{x}, \bar{y}, \bar{\lambda})$ is a weakly efficient solution of (VD).

Letting $f(x, y) = f(x) + y^T g(x)e$, where $f : \mathbb{R}^n \to \mathbb{R}^k$, $g : \mathbb{R}^n \to \mathbb{R}^m$, and $J = \emptyset$, (GSP) and (GSD) becomes the following generalized second order vector optimization problems.

(SVP) Minimize $f(x)$

subject to $g_i(x) \leqq 0, i = 1, \cdots, m$

$x \geqq 0.$

$$(SVD) \text{ Maximize } f(u) + v^T g(u)e - (\sum_{i=1}^{m} \lambda_i u_A^T \nabla_{x_A} f_i(u)$$

$$+ \sum_{i=1}^{m} v_i u_A^T \nabla_{x_A} g_i(u))e - (\sum_{i=1}^{m} \lambda_i u_A^T \nabla_{xx_A} f_i(u) r)e$$

$$- (\sum_{i=1}^{m} v_i u_A^T \nabla_{xx_A} g_i(u) r)e - \frac{1}{2}(r^T \sum_{i=1}^{m} \lambda_i \nabla_{xx} f_i(u) r)e$$

$$- \frac{1}{2}(r^T \sum_{i=1}^{m} v_i \nabla_{xx} g_i(u) r)e$$

$$\text{subject to } \sum_{i=1}^{m} \lambda_i \nabla_x f_i(u) + \sum_{i=1}^{m} v_i \nabla_x g_i(u)$$

$$+ \sum_{i=1}^{m} \lambda_i \nabla_{xx} f_i(u) r + \sum_{i=1}^{m} v_i \nabla_{xx} g_i(u) r \geqq 0,$$

$$\sum_{i=1}^{m} \lambda_i u_B^T \nabla_{x_B}^T f_i(u) + \sum_{i=1}^{m} v_i u_B^T \nabla_x g_i(u)$$

$$+ \sum_{i=1}^{m} \lambda_i u_B^T \nabla_{xx_B} f_i(u) r + \sum_{i=1}^{m} v_i u_B^T \nabla_{xx_B} g_i(u) r \leqq 0,$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$, $\lambda \in \mathbb{R}^k$ and $e = (1, \cdots, 1)^T \in \mathbb{R}^k$.

THEOREM 4.12. **(Weak Duality)** *Let x be feasible for (SVP) and (u, v, λ, r) be feasible for (SVD). Assume that f_i, g_i are second order convex. Then*

$$\begin{aligned} f(x) \;\not\leq\; & f(u) + v^T g(u)e - (\sum_{i=1}^{m} \lambda_i u_A^T \nabla_{x_A} f_i(u) + \sum_{i=1}^{m} v_i u_A^T \nabla_{x_A} g_i(u))e \\ & - (\sum_{i=1}^{m} \lambda_i u_A^T \nabla_{xx_A} f_i(u) r)e - (\sum_{i=1}^{m} v_i u_A^T \nabla_{xx_A} g_i(u) r)e \\ & - \frac{1}{2}(r^T \sum_{i=1}^{m} \lambda_i \nabla_{xx} f_i(u) r)e - \frac{1}{2}(r^T \sum_{i=1}^{m} v_i \nabla_{xx} g_i(u) r)e \end{aligned}$$

Proof. Since $f_i(\cdot)$ and $g_i(\cdot)$ are second order convex $f(\cdot) + v^T g(\cdot)e$ is second order convex for fixed $v \geqq 0$. $f(x) + v^T g(x)$ is automatically second order concave for fixed x with respect to v. So, by Corollary 3.1 we can get weak duality.

THEOREM 4.13. **(Strong Duality)** *Suppose that $f_i(\cdot)$ and $g_i(\cdot)$ are second order convex If $\bar{x} \in X$ is a weakly efficient solution of (SVP) and constraint qualification hold at $\bar{x}$, then $(\bar{x}, \bar{y}, \bar{\lambda})$ is a feasible solution of (SVD) and $(\bar{x}, \bar{y}, \bar{\lambda})$ is a weakly efficient solution of (SVD).*

Proof. By the same argument in the proof of Throrem 4.11, we can get the conclusion.

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