On semilinear retarded integrodifferential functional equations

준선형 지연 함수 적분미분방정식에 관하여

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On semilinear retarded integrodifferential functional equations

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by

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요 약

이 논문은 주어진 Hilbert 공간에서 지연항과 적분미분항을 포함한 준선형 미분방정식의 초기치 문제와 주작용소가 비선형인 미분방정식에 대한 해의 정규성 문제를 함수해석적으로 다루고자 한다. 먼저 다음과 같이 주어진 지연항을 포함한 방정식:

$$(1)\quad \begin{cases} \dfrac{du(t,x)}{dt} + A_0(x,D_x)u(t,x) + A_1(x,D_x)u(t-h,x) \\ \qquad + \displaystyle\int_{-h}^{0} a(s)A_2(x,D_x)u(t-s,x)ds \\ \quad = F(t, u(t-h,x), \displaystyle\int_0^t k(t,s,u(s-h,x)ds + f(t,x) \\ u(0,x) = g^0, \quad u(s,x) = g^1(s), \quad x \in \Omega,\ s \in [-h, 0). \end{cases}$$

여기서 Ω 는 유한공간에서 경계가 매끄러운 유계집합이며 $A_i(x, D_x)(i = 0, 1, 2)$는 주어진 공간에서 비유계인 선형작용소이고 타원형 미분연산자이다. 먼저 (1)식을 추상적인 함수미분방정식으로 전환하여 주어진 비선형항의 Lipschitz 연속의 가정과 방정식에 포함된 작용소들의 함수성질을 이용한 부동점정리를 응용하여 해의 존재성과 정규성을 증명하고자 한다.

(주결과) H 와 V 를 Hilbert 공간으로 하고 V 가 조밀한 공간으로서 그의 공액공간을 V^*로 하자. 다음과 같이 각각의 조건:

1) $(g^0, g^1) \in H \times L^2(0, T; V)$
2) 비선형항의 Lipschitz 조건
3) $f \in L^2(0, T; V^*)$
4) 주작용소의 해석적 반군

으로 주어지면 두 방정식의 초기치 문제의 해는 유일하게 존재하며, 아울러

$$x \in L^2(0, T; V) \cap W^{1,1/2}(0, T; V^*) \subset C([0, T]; H).$$

임을 증명하였다.

1. Introduction

This paper is concerned with the existence, uniqueness and norm estimations of solutions for a class of partial functional integrodifferential systems with delay terms:

$$(1.1)\qquad \begin{cases} \dfrac{\partial}{\partial t}u(t,x)+\mathcal{A}(x,D_x)u(t,x)+\mathcal{A}_1(x,D_x)u(t-h,x) \\ \quad +\displaystyle\int_{-h}^{0} a(s)\mathcal{A}_2(x,D_x)u(t+s,x)ds \\ \quad = F(t,u(t-h,x),\displaystyle\int_0^t k(t,s,u(s-h,x))ds)+f(t,x), \\ 0\le t\le T, \quad x\in\Omega. \end{cases}$$

Here, $\Omega\subset\mathcal{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, $\mathcal{A}(x,D_x)$, $\mathcal{A}_\iota(x,D_x)u$, $\iota=1,\ 2$, are second order linear differential operators with real coefficients, and $\mathcal{A}(x,D_x)$ is an elliptic in $\overline{\Omega}$. The function $a(s)$ is a real scalar function on $[-h,0]$, where $h>0$ is a delay time and f is a forcing function. The boundary condition attached to (1.1) is given by Dirichlet boundary condition

$$(1.2)\qquad u|_{\partial\Omega}=0, \quad 0<t\le T$$

and the initial condition is given by

$$(1.3)\qquad u(0,x)=g^0, \quad u(s,x)=g^1(s,x) \quad -h\le s\le 0.$$

Set

$$G(t,u)=F(t,u(t-h),\int_0^t k(t,s,u(s-h))ds).$$

The nonlinear term $G(t,\cdot)$, which is a Lipschitz continuous operator from $L^2(-h,T;V)$ to $L^2(-h,T;H)$, is a semilinear version of the quasilinear one considered in Yong and Pan [1]. Precise assumptions are given in the next section.

The abstract formulations of many partial integrodifferential equations arise in the mathematical description of the dynamical processes with heat flow in material with memory, viscoelasticity, and many physical phenomena (See [2,3]). When $F\equiv 0$ in (1.1), this linear type of equations is studied extensively by [4], Tanabe [5] and Jeong, Nakagiri [6,7].

In order to prove the solvability of the initial value problem (1.1) we establish necessary estimates applying the result of Di Blasio, Kunisch and Sinestrari [4] to (1.1) considered as an equation in a Hilbert space. In this paper, we give preliminaries on linear equations, and then prove the local existence and uniqueness for solution of (1.1)-(1.3) by using the contraction principle. Finally, we establish the norm estimation of solutions by using the regularity for solutions associated with the linear part of the given equations and the global existence of solutions by the step by step method.

2. Preliminaries and local existence

Let H and V be two complex Hilbert spaces such that V is a dense subspace of H. The norm of H(resp. V) is denoted by $|\cdot|$ (resp. $||\cdot||$) and the corresponding scalar product by $(\cdot,\cdot)$($resp.$$((\cdot,\cdot))$). Assume that the injection of V into H is continuous. The antidual of V is denoted by V^*, and the norm of V^* by $||\cdot||^*$. Identifying H with its antidual we may consider that H is embedded in V^*. Hence we have $V \subset H \subset V^*$ densely and continuously.

We realize the operator $\mathcal{A}(x, D_x)$, $\mathcal{A}_\iota(x, D_x)$, $\iota = 1,\ 2$, in Hilbert spaces by

$$A_0 v = -\mathcal{A}(x, D_x)v, \quad A_\iota v = -\mathcal{A}_\iota(x, D_x)v,\ \iota = 1,\ 2, \quad v \in V$$

in the distribution sense. The mixed problem (1.1)-(1.3) can be formulated abstractly as

(SLE)

$$\begin{cases} \dfrac{d}{dt}u(t) = A_0 u(t) + A_1 u(t-h) + \displaystyle\int_{-h}^{0} a(s)A_2 u(t+s)ds \\ \qquad\qquad + F(t, u(t-h), \displaystyle\int_0^t k(t,s,u(s-h))ds) + f(t), \quad 0 \le t \le T \\ u(0) = g^0, \quad u(s) = g^1(s) \quad -h \le s \le 0. \end{cases}$$

Let $b(\cdot,\cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding's inequality

$$\text{(2.1)} \qquad \operatorname{Re} b(v,v) \ge c_0||v||^2 - c_1|v|^2, \quad c_0 > 0, \quad c_1 \ge 0.$$

Let A_0 be the operator associated with the sesquilinear form $-b(\cdot,\cdot)$:

$$(A_0 v_1, v_2) = -b(v_1, v_2), \quad v_1,\ v_2 \in V.$$

A_0 is a bounded linear operator from V to V^*, and its realization in H which is the restriction of A_0 to

$$D(A_0) = \{v \in V; A_0 v \in H\}$$

is also denoted by A_0. Then A_0 generates an analytic semigroup in both of H and V^*.

The operators A_1 and A_2 are bounded linear operators from V to V^* such that their restrictions to $D(A_0)$ are bounded linear operators from $D(A_0)$ equipped with the graph norm of A_0 to H. The function $a(\cdot)$ is assumed to be real valued and belongs to $L^2(-h, 0)$.

First we consider the fundamental results on the following linear functional differential initial value problem:

$$
\text{(LE)} \quad \begin{cases} \dfrac{d}{dt}u(t) = A_0 u(t) + A_1 u(t-h) + \displaystyle\int_{-h}^{0} a(s) A_2 u(t+s) ds + f(t), \\ u(0) = g^0, \quad u(s) = g^1(s) \quad -h \le s \le 0. \end{cases}
$$

By assumption there exists a positive constant M_0 such that

$$
(2.2) \qquad |v| \le M_0 ||v||.
$$

Then, for any $f \in H$ we have

$$
(2.3) \qquad ||f||_* \le M_0 |f|.
$$

It follows from (2.1) that for $u \in V$

$$
\operatorname{Re}((c_1 - A_0)v, v) \ge c_0 ||v||^2.
$$

Hence there exists a constant C_0 such that

$$
(2.4) \qquad ||v|| \le C_0 ||v||_{D(A_0)}^{1/2} |v|^{1/2}
$$

for every $v \in D(A_0)$, where

$$
||v||_{D(A_0)} = (|A_0 v|^2 + |v|^2)^{1/2}
$$

is the graph norm of $D(A_0)$.

If X is a Banach space and $1 < p < \infty$, $L^p(0, T; X)$ is the collection of all strongly measurable functions from $(0, T)$ into X the p-th powers whose of norms are integrable and $W^{m,p}(0, T; X)$ is the set of all functions f whose derivatives $D^\alpha f$ up to degree m in the distribution sense belong to $L^p(0, T; X)$.

By virtue of Theorem 3.3 of [4] we have the following result on the corresponding linear equation of (LE).

Proposition 2.1. *Suppose that the assumptions stated above are satisfied. Then the following properties hold:*
1) Let $X = (D(A_0), H)_{\frac{1}{2},2}$ *where* $(D(A_0), H)_{\frac{1}{2},2}$ *is the real interpolation space between* $D(A_0)$ *and* H *(see [8; section 1.3.3]) and* $G(\cdot, u) \equiv 0$. *For* $(g^0, g^1) \in X \times L^2(-h, 0; D(A_0))$ *and* $f \in L^2(0, T; H)$, $T > 0$, *there exists a unique solution* u *of (LE) belonging to*

$$L^2(-h, T; D(A_0)) \cap W^{1,2}(0, T; H) \subset C([0, T]; X)$$

and satisfying

$$\begin{aligned} ||u||_{L^2(-h,T;D(A_0))\cap W^{1,2}(0,T;H)} \leq C_1(||g^0||_X \\ + ||g^1||_{L^2(-h,0;D(A_0))} + ||f||_{L^2(0,T;H)}), \end{aligned} \tag{2.5}$$

where C_1 *is a constant depending on* T.
2) Let $(g^0, g^1) \in H \times L^2(-h, 0; V)$ *and* $f \in L^2(0, T; V^*)$, $T > 0$. *Then there exists a unique solution* u *of (LE) in case* $G(\cdot, u) \equiv 0$ *belonging to*

$$L^2(-h, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H)$$

and satisfying

$$\begin{aligned} ||u||_{L^2(-h,T;V)\cap W^{1,2}(0,T;V^*)} \leq C_1(|g^0| \\ + ||g^1||_{L^2(-h,0;V)} + ||f||_{L^2(0,T;V^*)}), \end{aligned} \tag{2.6}$$

where C_1 *is a constant depending on* T.

For a given $u \in L^2(0, T; V)$ we extend it to the space $L^2(-h, T; V)$ by setting $u(s) = g^1(s)$ for $s \in (-h, 0)$.

We assume the following hypotheses on the nonlinear mappings F, k in (SLE):
(A1) $F : [0, T] \times L^2(0, T; V) \times H \to H$ is a nonlinear mapping such that for $\phi \in L^2(0, T; V)$ and $x \in H$, $F(t, \phi, x)$ is strongly measurable on $[0, T]$ and there exist positive constants L_0, L_1, L_2 and L_3 such that

$$|F(t, \phi_1, x_1) - F(t, \phi_2, x_2)| \leq L_1||\phi_1 - \phi_2|| + L_2|x_1 - x_2|, \quad t \in [0, T].$$

(A2) Let $\Delta_T = \{(s, t) : 0 \leq s \leq t \leq T\}$. Then $k : \Delta_T \times L^2(0, T; V) \to$

H is a nonlinear mapping such that for $x \in H$, $k(t,s,x)$ is strongly measurable on Δ_T and there exists positive constant L_3 such that

$$|k(t,s,x_1) - k(t,s,x_2)| \leq L_3||x_1 - x_2||, \quad (s,t) \in \Delta_T.$$

(A3) $\quad |F(t,0,0)| \leq L_0, \quad |k(t,s,0)| \leq L_0.$

Remark 2.1. The above operator F is the semilinear case of the nonlinear part of quasilinear equations considered by Yong and Pan [1].

For $u \in L^2(-h,T;V)$, $T > 0$ we set

$$G(t,u) = F(t, u(t-h), \int_0^t k(t,s,u(s-h))ds).$$

Lemma 2.1. *Let $u \in L^2(-h,T;V)$ $T > 0$. Then $G(\cdot,u) \in L^2(0,T;H)$ and*

$$(2.7) \qquad ||G(\cdot,u)||_{L^2(0,T;H)} \leq L_0\sqrt{T} + (L_1 + L_2L_3T/\sqrt{2})||u||_{L^2(-h,T-h;V)}.$$

Moreover if $u_1,\ u_2 \in L^2(-h,T;V)$, then

$$(2.8) \qquad ||G(\cdot,u_1) - G(\cdot,u_2)||_{L^2(0,T;H)} \leq (L_1 + L_2L_3T/\sqrt{2})||u_1 - u_2||_{L^2(-h,T-h;V)}.$$

Proof. For $u \in L^2(-h,T;V)$, since

$$\begin{aligned}\int_0^T |\int_0^t k(t,s,u(s-h))ds|^2 dt &\leq L_3^2 \int_0^T (\int_0^t ||u(s-h)||ds)^2 dt \\ &\leq L_3^2 \int_0^T t \int_0^t ||u(s-h)||^2 ds dt \\ &\leq L_3^2 \frac{T^2}{2} \int_0^T ||u(s-h)||^2 ds,\end{aligned}$$

from (A1) and (A2), it is easily seen that

$$
\begin{aligned}
&||G(\cdot,u)||_{L^2(0,T;H)} = \{\int_0^T |F(t,u(t-h),\int_0^t k(t,s,u(s-h))ds)|^2 dt\}^{1/2}\\
&= \{\int_0^T |F(t,u(t-h),\int_0^t k(t,s,u(s-h))ds) - F(t,0,0) + F(t,0,0)|^2 dt\}^{1/2}\\
&\le \{\int_0^T |F(t,u(t-h),\int_0^t k(t,s,u(s-h))ds) - F(t,0,0)|^2 dt\}^{1/2} + L_0\sqrt{T}\\
&\le L_0\sqrt{T} + L_1||u||_{L^2(-h,T-h;V)} + L_2\{\int_0^T |\int_0^t k(t,s,u(s-h))ds|^2 dt\}^{1/2}.
\end{aligned}
$$

The proof of (2.8) is similar. □

Now we are ready to give the following result on a local solvability of (SLE).

Theorem 2.1. *Suppose that the assumptions (A1), (A2) and (A3) are satisfied. Then for any* $(g^0,g^1) \in H \times L^2(-h,0;V)$ *and* $f \in L^2(0,T;V^*)$, $T>0$, *there exists a time* $T_0 > 0$ *such that the functional differential equation (SLE) admits a unique solution* u *in* $L^2(-h,T_0;V) \cap W^{1,2}(0,T_0;V^*)$.

Proof. Let us fix $T_0 > 0$ so that

$$C_0C_1(L_1 + L_2L_3\frac{T_0}{\sqrt{2}})(\frac{T_0}{\sqrt{2}})^{1/2} < 1, \tag{2.9}$$

where C_0 and C_1 are constants in (2.4) and (2.5) respectively. Let w be the solution of

(2.10)
$$
\begin{aligned}
\frac{d}{dt}w(t) =& A_0w(t) + A_1w(t-h)\\
&+ \int_{-h}^0 a(s)A_2w(t+s)ds + G(t,v) + f(t),
\end{aligned}
$$

$$w(0) = g^0, \quad w(s) = g^1(s), \qquad s\in[-h,0). \tag{2.11}$$

We are going to show that $v \mapsto w$ is strictly contractive from $L^2(0,T_0;V)$ to itself if the condition (2.9) is satisfied. Let w_1, w_2 be the solutions of (2.10), (2.11) with v replaced by $v_1,\ v_2 \in L^2(0,T_0;V)$, respectively. From (2.5) and (2.8) it follows that

$$
\begin{aligned}
&||w_1 - w_2||_{L^2(0,T_0;D(A_0))\cap W^{1,2}(0,T_0;H)} \\
&\le C_1||G(\cdot,v_1) - G(\cdot,v_2)||_{L^2(0,T_0;H)} \\
&\le C_1(L_1 + L_2L_3\frac{T_0}{\sqrt{2}})||v_1 - v_2||_{L^2(0,T_0;V)},
\end{aligned}
$$

and hence in view of (2.4) we have

(2.12)

$$
\begin{aligned}
||w_1 - w_2||_{L^2(0,T_0;V)} &\le C_0||w_1 - w_2||^{\frac{1}{2}}_{L^2(0,T_0;D(A_0))}||w_1 - w_2||^{\frac{1}{2}}_{L^2(0,T_0;H)} \\
&\le C_0||w_1 - w_2||^{\frac{1}{2}}_{L^2(0,T_0;D(A_0))}(\frac{T_0}{\sqrt{2}})^{\frac{1}{2}}||w_1 - w_2||^{\frac{1}{2}}_{W^{1,2}(0,T_0;H)} \\
&\le C_0(\frac{T_0}{\sqrt{2}})^{\frac{1}{2}}||w_1 - w_2||_{L^2(0,T_0;D(A_0))\cap W^{1,2}(0,T_0;H)} \\
&\le C_0C_1(L_1 + L_2L_3\frac{T_0}{\sqrt{2}})(\frac{T_0}{\sqrt{2}})^{1/2}||v_1 - v_2||_{L^2(0,T_0;V)}.
\end{aligned}
$$

Here we used the following inequality

$$
\begin{aligned}
||w_1 - w_2||_{L^2(0,T_0;H)} &= \{\int_0^{T_0} |w_1(t) - w_2(t)|^2 dt\}^{\frac{1}{2}} \\
&= \{\int_0^{T_0} |\int_0^t (\dot{w}_1(\tau) - \dot{w}_2(\tau))d\tau|^2 dt\}^{\frac{1}{2}} \\
&\le \{\int_0^{T_0} t\int_0^t |\dot{w}_1(\tau) - \dot{w}_2(\tau)|^2 d\tau dt\}^{\frac{1}{2}} \\
&\le \frac{T_0}{\sqrt{2}}||w_1 - w_2||_{W^{1,2}(0,T_0;H)}.
\end{aligned}
$$

So by virtue of (2.9) the contraction mapping principle gives that the equation (SLE) has a unique solution in $[-h, T_0]$. □

3. Global existence and behavior of solution

In this section we give a norm estimation of the solution of (SLE) and establish the global existence of solutions with the aid of norm estimations.

Theorem 3.1. *Suppose that the assumptions (A1), (A2) and (A3) are satisfied. Then for any* $(g^0, g^1) \in H \times L^2(-h, 0; V)$ *and* $f \in L^2(0, T; V^*)$, $T > 0$, *the solution* u *of (SLE) exists and is unique in* $L^2(-h, T; V) \cap W^{1,2}(0, T; V^*)$, *and there exists a constant* C_2 *depending on* T *such that*

$$\begin{aligned} ||u||_{L^2(-h,T;V)\cap W^{1,2}(0,T;V^*)} \le C_2(1 + |g^0| \\ + ||g^1||_{L^2(-h,0;V)} + ||f||_{L^2(0,T;V^*)}). \end{aligned} \tag{3.1}$$

Proof. Let $u(\cdot)$ be the solution of (SLE) in the interval $[-h, T_0]$ where T_0 is a constant in (2.9) and $w(\cdot)$ be the solution of the following equation

$$\begin{aligned} \frac{d}{dt}w(t) =& A_0 w(t) + A_1 w(t-h) + \int_{-h}^{0} a(s) A_2 w(t+s) ds + f(t), \\ w(0) =& g^0, \quad w(s) = g^1(s), \qquad -h \le s < 0. \end{aligned}$$

Then in view of (2.5), (2.7)

$$\begin{aligned} &||u - w||_{L^2(0,T_0;D(A_0))\cap W^{1,2}(0,T_0;H)} \le C_1 ||G(\cdot, u)||_{L^2(0,T_0;H)} \\ &\le C_1\{L_0\sqrt{T_0} + (L_1 + L_2 L_3 T_0/\sqrt{2})(||u||_{L^2(0,T_0;V)} \\ &\ + ||g^1||_{L^2(-h,0;V)})\}. \\ &\le C_1\{L_0\sqrt{T_0} + (L_1 + L_2 L_3 T_0/\sqrt{2})(||u - w||_{L^2(0,T_0;V)} \\ &\ + ||w||_{L^2(0,T_0;V)} + ||g^1||_{L^2(-h,0;V)})\}. \end{aligned}$$

Thus, arguing as in the proof of (2.12)

$$\begin{aligned} &||u - w||_{L^2(0,T_0;V)} \\ &\le C_0\left(\frac{T_0}{\sqrt{2}}\right)^{\frac{1}{2}} ||u - w||_{L^2(0,T_0;D(A_0))\cap W^{1,2}(0,T_0;H)} \\ &\le C_0\left(\frac{T_0}{\sqrt{2}}\right)^{\frac{1}{2}} C_1\{L_0\sqrt{T_0} + (L_1 + L_2 L_3 T_0/\sqrt{2})(||u - w||_{L^2(0,T_0;V)} \\ &\ + ||w||_{L^2(0,T_0;V)} + ||g^1||_{L^2(-h,0;V)})\}. \end{aligned}$$

Therefore, we have

$$||u-w||_{L^2(0,T_0;V)} \leq \frac{C_0C_1(T_0/\sqrt{2})^{1/2}}{1-C_0C_1(L_1+L_2L_3T_0/\sqrt{2})(T_0/\sqrt{2})^{1/2}}$$
$$\times \{L_0\sqrt{T}_0+(L_1+L_2L_3T_0/\sqrt{2})(||w||_{L^2(0,T_0;V)}+||g^1||_{L^2(-h,0;V)})\},$$

and hence, with the aid of 2) of Proposition 2.1

(3.2)

$$||u||_{L^2(0,T_0;V)}$$
$$\leq \frac{C_0C_1L_0\sqrt{T_0}(T_0/\sqrt{2})^{1/2}}{1-C_0C_1(L_1+L_2L_3T_0/\sqrt{2})(T_0/\sqrt{2})^{1/2}}$$
$$+\frac{||w||_{L^2(0,T_0;V)}+||g^1||_{L^2(-h,0;V)}}{1-C_0C_1(L_1+L_2L_3T_0/\sqrt{2})(T_0/\sqrt{2})^{1/2}}$$
$$\leq \frac{C_0C_1L_0\sqrt{T_0}(T_0/\sqrt{2})^{1/2}}{1-C_0C_1(L_1+L_2L_3T_0/\sqrt{2})(T_0/\sqrt{2})^{1/2}}$$
$$+\frac{1}{1-C_0C_1(L_1+L_2L_3T_0/\sqrt{2})(T_0/\sqrt{2})^{1/2}}\{C_1(|g^0|+||g^1||_{L^2(-h,0;V)}$$
$$+||f||_{L^2(0,T_0;V^*)})+||g^1||_{L^2(-h,0;V)}\}.$$

On the other hand using (2.6), (2.3), (2.7) we get

(3.3)

$$||u||_{L^2(-h,T_0;V)\cap W^{1,2}(0,T_0;V^*)}$$
$$\leq C_1(|g^0|+||g^1||_{L^2(-h,0;V)}+||G(\cdot,u)+f||_{L^2(0,T_0;V^*)})$$
$$\leq C_1(|g^0|+||g^1||_{L^2(-h,0;V)}+M_0||G(\cdot,u)||_{L^2(0,T_0;H)}+||f||_{L^2(0,T_0;V^*)})$$
$$\leq C_1[|g^0|+||g^1||_{L^2(-h,0;V)}+||f||_{L^2(0,T_0;V^*)}$$
$$+M_0\{L_0\sqrt{T}_0+(L_1+L_2L_3T_0/\sqrt{2})(||u||_{L^2(0,T_0;V)}$$
$$+||g^1||_{L^2(-h,0;V)})\}].$$

Combining (3.2), and (3.3) we obtain

(3.4)
$$||u||_{L^2(-h,T_0;V)\cap W^{1,2}(0,T_0;V^*)} \leq C(1+|g^0|$$
$$+||g^1||_{L^2(-h,0;V)}+||f||_{L^2(0,T_0;V^*)})$$

for some constant C. Since the condition (2.9) is independent of initial values, the solution of (SLE) can be extended to the interval $[-h, nT_0]$ for every natural number n. An analogous estimate to (3.4) holds for the solution in $[-h, nT_0]$, and hence for the initial value $(u(nT_0), u_{nT_0})$ in the interval $[nT_0, (n+1)T_0]$. □

Theorem 3.2. *Suppose that the assumptions (A1), (A2) and (A3) are satisfied. If $(g^0, g^1) \in X \times L^2(-h, 0; D(A_0))$ and $f \in L^2(0, T; H)$, then $u \in L^2(-h, T; D(A_0)) \cap W^{1,2}(0, T; H)$, and the mapping $(g^0, g^1, f) \mapsto u \in L^2(-h, T; D(A_0)) \cap W^{1,2}(0, T; H)$ is continuous.*

Proof. It is easy to show that if $(g^0, g^1) \in X \times L^2(-h, 0; D(A_0))$ and $f \in L^2(0, T; H)$, then from Proposition 2.1 it follows that u belongs to $L^2(-h, T; D(A_0)) \cap W^{1,2}(0, T; H)$. Let $(g_i^0, g_i^1, f_i) \in X \times L^2(-h, 0; D(A_0)) \times L^2(0, T; H)$, and u_i be the solution of (SLE) with (g_i^0, g_i^1, f_i) in place of (g^0, g^1, f) for $i = 1,\ 2$. Then in view of Proposition 2.1 and Lemma 2.1 we have

(3.5)

$$\begin{aligned}
&||u_1 - u_2||_{L^2(-h,T;D(A_0)) \cap W^{1,2}(0,T;H)} \leq C_1 \{ ||g_1^0 - g_2^0||_X \\
&\quad + ||g_1^1 - g_2^1||_{L^2(-h,0;D(A_0))} + ||G(\cdot, u_1) - G(\cdot, u_2)||_{L^2(0,T;H)} \\
&\quad + ||f_1 - f_2||_{L^2(0,T;H)} \} \\
&\quad \leq C_1 \{ ||g_1^0 - g_2^0||_X + ||g_1^1 - g_2^1||_{L^2(-h,0;D(A_0))} + ||f_1 - f_2||_{L^2(0,T;H)} \\
&\quad + (L_1 + L_2 L_3 T/\sqrt{2})(||u_1 - u_2||_{L^2(0,T;V)} + ||g_1^1 - g_2^1||_{L^2(-h,0;V)}) \}.
\end{aligned}$$

Since

$$u_1(t) - u_2(t) = g_1^0 - g_2^0 + \int_0^t (\dot{u}_1(s) - \dot{u}_2(s)) ds,$$

we get

$$||u_1 - u_2||_{L^2(0,T;H)} \leq \sqrt{T} |g_0^1 - g_2^0| + \frac{T}{\sqrt{2}} ||u_1 - u_2||_{W^{1,2}(0,T;H)}.$$

Hence, arguing as in (2.12) we get

(3.6)

$$\begin{aligned}
||u_1 - u_2||_{L^2(0,T;V)} &\leq C_0||u_1 - u_2||^{1/2}_{L^2(0,T;D(A_0))}||u_1 - u_2||^{1/2}_{L^2(0,T;H)} \\
&\leq C_0||u_1 - u_2||^{1/2}_{L^2(0,T;D(A_0))} \\
&\times \{T^{1/4}|g_1^0 - g_2^0|^{1/2} + (\frac{T}{\sqrt{2}})^{1/2}||u_1 - u_2||^{1/2}_{W^{1,2}(0,T;H)}\} \\
&\leq C_0 T^{1/4}|g_1^0 - g_2^0|^{1/2}||u_1 - u_2||^{1/2}_{L^2(0,T;D(A_0))} \\
&+ C_0(\frac{T}{\sqrt{2}})^{1/2}||u_1 - u_2||_{L^2(0,T;D(A_0))\cap W^{1,2}(0,T;H)} \\
&\leq 2^{-7/4}C_0|g_1^0 - g_2^0| \\
&+ 2C_0(\frac{T}{\sqrt{2}})^{1/2}||u_1 - u_2||_{L^2(0,T;D(A_0))\cap W^{1,2}(0,T;H)}.
\end{aligned}$$

Combining (3.5), and (3.6) we obtain

(3.7)

$$\begin{aligned}
||u_1 - u_2||_{L^2(-h,T;D(A_0))\cap W^{1,2}(0,T;H)} &\leq C_1\{||g_1^0 - g_2^0||_X \\
&+ ||g_1^1 - g_2^1||_{L^2(-h,0;D(A_0))} + ||f_1 - f_2||_{L^2(0,T;H)} \\
&+ (L_1 + L_2L_3T/\sqrt{2})||g_1^1 - g_2^1||_{L(-h,0;V))}\} \\
&+ 2^{-7/4}C_0C_1(L_1 + L_2L_3T/\sqrt{2})|g_1^0 - g_2^0| + 2C_0C_1(\frac{T}{\sqrt{2}})^{1/2} \\
&\times (L_1 + L_2L_3T/\sqrt{2})||u_1 - u_2||_{L^2(0,T;D(A_0))\cap W^{1,2}(0,T;H)}.
\end{aligned}$$

Suppose that $(g_n^0, g_n^1, f_n) \to (g^0, g^1, f)$ in $X\times L^2(-h, 0; D(A_0))\times L^2(0, T; H)$, and let u_n and u be the solutions (SLE) with (g_n^0, g_n^1, f_n) and (g^0, g^1, f) respectively. Let $0 < T_1 \leq T$ be such that

$$2C_0C_1(T_1/\sqrt{2})^{1/2}(L_1 + L_2L_3T_1/\sqrt{2}) < 1.$$

Then by virtue of (3.7) with T replaced by T_1 we see that $u_n \to u$ in $L^2(-h, T_1; D(A_0))\cap W^{1,2}(0, T_1; H)$. This implies that $(u_n(T_1), (u_n)_{T_1}) \mapsto (u(T_1), u_{T_1})$ in $X\times L^2(-h, 0; D(A_0))$. Hence the same argument shows that $u_n \to u$ in

$$L^2(T_1, \min\{2T_1, T\}; D(A_0)) \cap W^{1,2}(T_1, \min\{2T_1, T\}; H).$$

Repeating this process we conclude that $u_n \to u$ in $L^2(-h, T; D(A_0)) \cap W^{1,2}(0, T; H)$. □

Theorem 3.3. *For $f \in L^2(0,T;H)$ let u_f be the solution of equation (SLE). Let us assume the natural assumption that the embedding $D(A_0) \subset V$ is compact. Then the mapping $f \mapsto u_f$ is compact from $L^2(0,T;H)$ to $L^2(0,T;V)$.*

Proof. If $f \in L^2(0,T;H)$, then in view of Theorem 3.1

$$
\begin{aligned}
(3.8) \qquad ||u_f||_{L^2(-h,T;V)\cap W^{1,2}(0,T;V^*)} \le C_2(1 + |g^0| \\
+ ||g^1||_{L^2(-h,0;V)} + M_0||f||_{L^2(0,T;H)}).
\end{aligned}
$$

Since $u_f \in L^2(0,T;V)$, $G(\cdot,u_f) \in L^2(0,T;H)$. Consequently $u_f \in L^2(0,T;D(A_0))\cap W^{1,2}(0,T;H)$ and with aid of Proposition 2.1, Lemma 2.1, and (3.8),

(3.9)

$$
\begin{aligned}
&||u_f||_{L^2(0,T;D(A_0))\cap W^{1,2}(0,T;H)} \\
&\le C_1(||g^0||_X + ||g^1||_{L^2(-h,0;D(A_0))} + ||G(\cdot,u_f) + f||_{L^2(0,T;H)}) \\
&\le C_1\{||g^0||_X + ||g^1||_{L^2(-h,0;D(A_0))} + L_0\sqrt{T} \\
&+ (L_1 + L_2L_3T/\sqrt{2})||u||_{L^2(-h,T-h;V)} + ||f||_{L^2(0,T;H)}\} \\
&\le C_1[||g^0||_X + ||g^1||_{L^2(-h,0;D(A_0))} + L_0\sqrt{T} \\
&+ (L_1 + L_2L_3T/\sqrt{2})\{||g^1||_{L^2(-h,0;V)} + C_2(1 + M_0||f||_{L^2(0,T;H)})\} \\
&+ ||f||_{L^2(0,T;H)}].
\end{aligned}
$$

Hence if f is bounded in $L^2(0,T;H)$, then so is u_f in $L^2(0,T;D(A_0))\cap W^{1,2}(0,T;H)$. Since $D(A_0)$ is compactly embedded in V by assumption, the embedding $L^2(0,T;D(A_0)) \cap W^{1,2}(0,T;H) \subset L^2(0,T;V)$ is compact in view of Theorem 2 of J. P. Aubin [9]. □

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