On Vector Equilibrium Problems with Multifunctions

다가함수를 가지는 벡터 균형점 문제에 관한 연구

Adivisor : Gue Myung Lee

by

Min Ja Eu

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주 심 이학박사 이 병 수

부 심 이학박사 김 도 상

위 원 이학박사 홍 정 희

위 원 이학박사 이 규 명

위 원 이학박사 신 준 용

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A dissertation

by

In Ja Bu

Approved by:

(Chairman) Byung Soo Lee

(Member) Do Sang Kim

(Member) Jeong Hee Hong

(Member) Gue Myung Lee

(Member) Jun Yong Shin

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부 인 자

부경대학교 대학원 응용수학과

요 약

본 논문에서는 비컴팩트(noncompact) 집합상에서 정의되는 다가함수를 가지는 두 개의 벡터 균형점 문제 $(VEP)_1$, $(VEP)_2$ 를 다룬다. 먼저 점근 추(asymptotic cone)를 사용하여 $(VEP)_1$ 에 대한 해의 존재성과 해집합의 컴팩트성(compactness)을 보이고 이 결과를 관련된 랜덤 벡터 균형점 문제(random vector equilibrium problem)로 확장하였다. 그리고 컴팩트한 볼록 집합들의 증가 수열을 사용하여 $(VEP)_2$ 에 대한 해의 존재성을 보이고, 이 결과를 비협조 벡터 내쉬 균형점 문제 (noncooperative Nash vector equilibrium problem)에 적용하고 관련된 랜덤 벡터 균형점 문제로 확장하였다. 나아가서, 비컴팩트 다면체 제약집합과 양의 반정부호 (또는 단조)행렬로 정의되는 아핀 벡터 변분 부등식(affine vector variational inequality) 의 해집합의 유계성 (boundedness)과 연결성 (connectedness)을 조사하고 그 결과를 다목적 선형 분수 최적화문제와 다목적 볼록 이차형 최적화문제에 적용하였다.

Chapter 1
Introduction and Preliminaries

In the fifties and sixties, as Giannessi told in the preface of his editing book entitled "Vector Variational Inequalities and Vector Equilibria ([36])", many real problems in physics, mechanics, fluid-dynamics, structural engineering and economics had shown the need of new mathematical models for studying the equilibrium of systems. In sixties, such requirements had led to the formulations of variational inequality, which was first introduced in the context of partial differential equations by Stampacchia, and complementarity problem, which can be regarded as a special case of variational inequality. Until now, the variational inequality and complementarity problems have played important roles in the formulation and treatment of equilibrium, in particular, in operation research, transportation and economics. In 1994, Blum and Oettli [14] coined the terminology "Equilibrium Problem" for giving a unified formulation for optimization problem, variational inequality, Nash equilibrium in noncooperative games and other problems related to equilibrium. Main theorems in nonlinear analysis [9, 10, 15, 16, 24, 25, 26], for example, Brouwer fixed point theorem, Broweder fixed point theorem, Kakutani fixed point theorem, Ky Fan's minimax inequality and Knaster-Kuratowski-Mazurkiewicz principle (Fan-KKM theorem), have supported strong mathematical tools for analyzing such equilibrium problems. So many authors have generalized such main theorems and then applied them to several kinds of equilibrium problems.

Most of decision making situations require a simultaneous consideration of more than two objectives which are in conflict or trade-off. For example, in the panel design problem, it would be reasonable to minimize weight and maximize strength simultaneously. Such requirements had led to multiobjective (vector) optimization problem, which was initiated by Pareto [69]. In 1980, Giannessi [34] first introduced vector variational inequality for studying vector optimization problem. Since then, many authors studied several kinds of vector variational inequalities [1, 7, 17, 18, 28, 45, 46, 47, 49, 48, 54, 56, 60, 61, 70, 73, 81, 80, 83, 84] and have shown that vector variational inequality can be efficient tools for studying vector optimization problems [7, 35, 42, 53, 50, 52, 57, 72, 81, 82, 83]. Many authors have formulated and studied vector equilibrium problems [2, 3, 4, 5, 6, 8, 13, 20, 21, 23, 27, 29, 30, 31, 38, 40, 41, 51, 55, 62, 63, 67, 68, 74] which are vector versions of the equilibrium problem and which contain several kinds of vector variational inequalities and vector optimization problems as special cases.

Now we will introduce two vector equilibrium problems with multifunctions considered in this dissertation.

Let X be a convex and closed subset of $\mathbb{R}^n$ and $Y = \mathbb{R}^m$. Let $F : X \times X \to 2^Y$ be a multifunction and let $C : X \to 2^Y$ be a multifunction such that $C(x)$ is a non-empty convex cone in Y with $intC(x) \neq \emptyset$ and $C(x) \neq Y$ for all $x \in X$. In this dissertation, we will consider the following two vector equilibrium problems with multifunctions :

$(VEP)_1$ Find $\bar{x} \in X$ such that

$$F(\bar{x}, x) \subset Y \setminus (-intC(\bar{x})) \text{ for any } x \in X.$$

$(VEP)_2$ Find $\bar{x} \in X$ such that

$$F(\bar{x}, x) \cap [Y \setminus (-intC(\bar{x}))] \neq \emptyset \text{ for any } x \in X.$$

It is clear that each solution of $(VEP)_1$ is also a solution of $(VEP)_2$. When $C(x)$ is a constant convex cone for any x $\in X$, the above problem $(VEP)_1$ is reduced to the one studied in [74]. Both problems include several kinds of vector equilibrium problems and vector variational inequalities as special cases.

If F is singe-valued, then both problems $(VEP)_1$ and $(VEP)_2$ collapse to the following vector equilibrium problem which was studied in [3, 23, 38, 54];

$$\text{Find } \bar{x} \in X \text{ such that } \phi(\bar{x}, x) \in Y \setminus (-intC(\bar{x})) \text{ for any } x \in X,$$

where $\phi : X \times X \to Y$ is a function.

Setting $\phi(x, y) = T(x)(y - x)$, where $T : X \to L(\mathbb{R}^n, \mathbb{R}^m)$ is a function, we get the following vector variational inequality which was investigated in [18, 84];

$$\text{Find } \bar{x} \in X \text{ such that } T(\bar{x})(x - \bar{x}) \in Y \setminus (-intC(\bar{x})) \text{ for any } x \in X.$$

Setting $\phi(x, y) = f(y) - f(x)$, where $f : X \to Y$ is a function, we obtain the following vector optimization problem which was studied in [43], and it becomes usual vector optimization problem when $C(x)$ is a nonnegative orthant $\mathbb{R}^n_+$;

$$\text{Find } \bar{x} \in X \text{ such that } f(y) - f(\bar{x}) \in Y \setminus (-intC(\bar{x})) \text{ for any } x \in X.$$

On the other hand, Qun [70] used an increasing sequence of nonempty compact convex sets to get the existence theorems for a vector variational inequality defined on a noncompact set. Auslender and Teboulle [11] studied the compactness conditions of solution sets of scalar optimization problems and scalar variational inequalities by using asymptotic cones, which is a generalization of the recession cone in [71]. Very recently, Fabián Flores-Bazán and Fernando Flores-Bazán [22, 23] have tried to extend the ideas of Auslender and Teboulle [11] to vector optimization problems and vector equilibrium problem for vector valued functions. Many authors [40, 41, 43, 55] have studied random vector variational inequalities, which are random versions of vector variational inequalities. In particular, Kalmoum [41] gave the existence theorems for random vector equilibrium problem for vector valued functions. Recently, the connectedness properties for solution sets of several kinds of vector variational inequalities for vector valued function have been studied for understanding the behavior of their solution sets ([19, 52, 58, 82, 83]). In particular, the connectedness of solution sets for affine vector variational inequalities with 2×2 monotone matrices was investigated in [58].

The main purposes of this dissertation are as follows;

(1) Using the asymptotic cone of the solution set of $(VEP)_1$, we give conditions under which the solution set is nonempty and compact, and then extend them to a random vector equilibrium problem with multifunctions.

(2) Using an increasing sequence of nonempty compact convex sets, we establish existence theorems for $(VEP)_2$ in noncompact settings.

(3) We investigate the boundedness and connectedness of solution sets of affine vector variational inequalities with noncompact polyhedral constraint sets and positive semidefinite (or monotone) matrices.

Now we give some definitions and preliminary results which will be used in the next chapters:

Definition 1.1. Let X be a convex and closed subset of $\mathbb{R}^n$ and $Y = \mathbb{R}^m$. Let $F : X \to 2^Y$ be a multifunction and C is a convex cone in Y with $C \neq Y$.

(1) ([76]) F is said to be upper (lower, respectively) C-convex on X if for any $x_1, x_2 \in X,\ t \in [0,1]$,

$$tF(x_1) + (1-t)F(x_2) \subset F(tx_1 + (1-t)x_2) + C$$

$$(F(tx_1 + (1-t)x_2) \subset tF(x_1) + (1-t)F(x_2) - C, \text{ respectively})$$

holds. If F is both upper C-convex on X and lower C-convex on X, we say that F is C-convex on X.

(2) F is said to be upper (lower, respectively) C-lower semicontinuous at $\bar{x} \in X$ if for any open set V in $\mathbb{R}^m$ with $F(\bar{x}) \cap V \neq \emptyset$, there exists a

neighborhood $N(\bar{x})$ of $\bar{x}$ such that for any $x \in N(\bar{x}) \cap X$

$$F(x) \cap (V + C) \neq \emptyset$$

$$(F(x) \cap (V - C) \neq \emptyset, \text{ respectively}).$$

If F is upper (lower, respectively) C-lower semicontinuous at every $x \in X$, then F is said to upper (lower, respectively) C-lower semicontinuous on X.

If F is both upper C-lower semicontinuous on X and lower C-lower semicontinuous on X, then F is said to be C-lower semicontinuous on X.

Example 1.1. Define a multifunction $F : \mathbb{R} \to 2^{\mathbb{R}}$ by $F(x) = [x^2, \infty)$ for any $x \in \mathbb{R}$. Then F is $\mathbb{R}_+$-convex, where $\mathbb{R}_+ = [0, \infty)$.

Given any closed set K in $\mathbb{R}^n$, we define the asymptotic cone of K as the closed set

$$K^\infty = \{x \in \mathbb{R}^n \mid \exists t_n \downarrow 0 \text{ and } x_n \in K \text{ such that } t_n x_n \to x\}.$$

In addition, if K is convex, it is known that for any given $x_0 \in K$,

$$K^\infty = \{x \in \mathbb{R}^n \mid x_0 + tx \in K \quad \text{for all} \quad t > 0\}.$$

Moreover, K is bounded if and only if $K^\infty = \{0\}$.

We give some basic properties of asymptotic cones which will be used in Chapter 2.

Proposition 1.1 [11]. For closed sets K_1, K_2 in $\mathbb{R}^n$, the following holds:

(1) $K_1 \subset K_2$ implies $(K_1)^\infty \subset (K_2)^\infty$;

(2) Let $\{K_i\}_{i\in I}$ be any family of nonempty sets, then $(\bigcap_{i\in I} K_i)^\infty \subset \bigcap_{i\in I} (K_i)^\infty$. If, in addition, $\bigcap_{i\in I} K_i \neq \emptyset$ and each set K_i is closed and convex, then we obtain an equality in the previous inclusion.

Now we recall the definitions of the KKM multifunction and the KKM-Fan Theorem ([25]) needed for the proofs of our existence theorems.

Definition 1.2. Let X be a vector space and K be a nonempty subset of X. Then a multifunction $G : K \rightarrow 2^X$ is called a KKM multifunction if for each finite subset $\{x_1, \cdots, x_n\}$ of K, $co\{x_1, \cdots, x_n\} \subset \bigcup_{i=1}^n G(x_i)$.

Theorem 1.1 (KKM-Fan Theorem). Let X be a Hausdorff topological vector space, K be a nonempty subset of X and $G : K \rightarrow 2^X$ be a KKM multifunction. If all the sets $G(x)$ are closed in X and if one is compact, then $\bigcap_{x\in K} G(x) \neq \emptyset$.

Definition 1.3 ([78]). Let X be a convex and closed subset of $\mathbb{R}^n$ and $Y = \mathbb{R}^m$. Let $F : X \rightarrow 2^Y$ be a multifunction and C be a non-empty convex cone in Y with $C \neq Y$. Then F is said to be natural quasi C- convex on X if for any $x_1, x_2 \in X$ and $t \in [0, 1]$, there exists $\mu \in [0, 1]$ such that

$$F(tx_1 + (1 - t)x_2) \subset \mu F(x_1) + (1 - \mu)F(x_2) - C.$$

It is clear that if F is lower C- convex on X then F is natural quasi C-convex on X. But the converse may not hold.

Definition 1.4. Let X be a convex and closed subset of $\mathbb{R}^n$ and $Y = \mathbb{R}^m$. Let $F : X \to 2^Y$ be a multifunction. Then the multifunction F is said to be closed if the graph of F, $Gr(F) := \{(x, y) \in X \times Y : y \in F(x)\}$, is closed in $X \times Y$.

Definition 1.5. Let X be a convex and closed subset of $\mathbb{R}^n$ and $Y = \mathbb{R}^m$. Let $F : X \to 2^Y$ be a multifunction.

(1) F is said to be upper semicontinuous (shortly, u.s.c.) at $\bar{x} \in X$ if for any open subset $\mathcal{O}$ of Y with $F(\bar{x}) \subset \mathcal{O}$, there exists a neighborhood $N(\bar{x})$ of $\bar{x}$ such that

$$\text{for all } x \in N(\bar{x}) \cap X, \;\; F(x) \subset \mathcal{O}.$$

We say that F is u.s.c. on X if F is u.s.c. at every point $x \in X$.

(2) F called lower semicontinuous (shortly l.s.c.) at $\bar{x} \in X$ if for any open subset $\mathcal{V}$ of Y with $F(\bar{x}) \cap \mathcal{V} \neq \emptyset$, there exists a neighborhood $N(\bar{x})$ of $\bar{x}$ such that

$$\text{for all } x \in N(\bar{x}) \cap X, \;\; F(x) \cap \mathcal{V} \neq \emptyset.$$

We say that F is l.s.c. on X if F is l.s.c. at every point $x \in X$.

Lemma 1.1 ([9, 77]). Let X be a convex and closed subset of $\mathbb{R}^n$ and $Y = \mathbb{R}^m$. Let $F : X \to 2^Y$ be a multifunction.

(1) If X is compact, and F is u.s.c. on X and compact valued, then $F(X)$ is compact.

(2) If F is u.s.c. on X and compact valued, then F is closed.

(3) F is l.s.c. at $x \in X$ if and only if for any $y \in F(x)$ and any sequence $\{x_n\}$ converging to x, there exists a sequence $\{y_n\}$, such that $y_n \in F(x_n)$, converging to y.

(4) If the multifunction F is closed and Y is compact, then F is upper semicontinuous.

This dissertation are organized as follows;

In Chapter 2, the vector equilibrium problem $(VEP)_1$ with multifunctions and the following Minty type vector equilibrium problem $(MVEP)_1$ will be considered.

$$(MVEP)_1 \qquad \text{Find } \bar{x} \in X \text{ such that}$$
$$F(x, \bar{x}) \subset Y \setminus intC(x) \text{ for any } x \in X.$$

Using the asymptotic cone of the solution set of $(VEP)_1$, we will give conditions that the solution set is nonempty and compact, and then extend it to random vector equilibrium problem with multifunctions. Our approaches follow ideas and methods in ([23, 40, 41]), in which vector equilibrium problems with vector valued functions are considered.

In Chapter 3, the vector equilibrium problem $(VEP)_2$ with multifunctions and the following Minty type vector equilibrium problem $(MVEP)_2$ will be considered.

$$(MVEP)_2 \qquad \text{Find } \bar{x} \in X \text{ such that}$$
$$-F(x, \bar{x}) \cap [Y \setminus (-intC(\bar{x}))] \neq \emptyset \text{ for any } x \in X.$$

Using an increasing sequence of nonempty compact convex sets, we establish existence theorems for $(VEP)_2$ in noncompact settings, and then apply our results to a noncooperative vector Nash equilibrium problem. Moreover we will obtain existence theorems for a random vector equilibrium problem with vector valued functions in compact settings.

In Chapter 4, we investigate the boundedness and connectedness of solution sets of affine vector variational inequalities with noncompact polyhedral constraint sets and positive semidefinite (or monotone) matrices, and then apply the boundedness and connectedness results to a multiobjective linear fractional optimization problem and a multiobjective convex linear-quadratic optimization problem. Moreover some numerical examples clarifying and illustrating the results for affine vector variational inequalities will be given.

Chapter 2
Vector Equilibrium Problem $(\mathbf{VEP})_1$

In this chapter, the vector equilibrium problem $(VEP)_1$ and its Minty type vector equilibrium problem $(MVEP)_1$, which are introduced in Chapter 1, are considered. Using the asymptotic cone of the solution set of $(VEP)_1$ and its related set R_0, we give conditions under which the solution set is nonempty and compact. Our approaches follow ideas of Fabián Flores-Bazán and Fernando Flores-Bazán ([23]), in which vector equilibrium problems with vector valued functions are considered.

2.1. Existence Theorems

We will start with assumptions which will be used for next theorems.

(H_0) Let $C : X \to 2^Y$ be a multifunction such that $C(x)$ be a nonempty convex cone in Y with $intC(x) \neq \emptyset$ and $C(x) \neq Y$ for all $x \in X$.

(H_1) Let $F : X \times X \to 2^Y$ be a multifunction satisfying the following conditions;

(A_1) For all $x \in X$, $F(x,x) \subset [C(x) \cap (-C(x))]$.

(A_2) For all $x, y \in X$,

$$F(x,y) \subset Y \setminus (-intC(x)) \text{ implies } F(y,x) \subset Y \setminus intC(y).$$

(A_3) For all $x \in X$, $y \mapsto F(x,y)$ is $C(x)$-convex and upper $C(x)$-lower semicontinuous on X.

(A_4) For all $x, y \in X$, the set $\{\xi \in [x, y] : F(\xi, y) \subset Y \setminus -intC(\xi)\}$ is closed. Here $[x, y]$ stands for the closed line segment joining x and y.

We give existence theorems for $(VEP)_1$ and $(MVEP)_1$ in compact settings. We will denote the solution sets of $(VEP)_1$ and $(MVEP)_1$ by E_w and E'_w, respectively.

Theorem 2.1.1. Let X be a convex and compact subset of $\mathbb{R}^n$ and $Y = \mathbb{R}^m$. Let $C : X \to 2^Y$ be a multifunction satisfying (H_0) and let $F : X \times X \to 2^Y$ be a multifunction satisfying (H_1). Then E_w is nonempty and closed, and $E_w = E'_w$.

Proof. Define a multifunction $G : X \to 2^Y$ by for any $y \in X$,

$$G(y) = \{x \in X : F(y, x) \subset Y \setminus intC(y)\}.$$

By (A_1), for any $y \in X$, $F(y, y) \subset -C(y) \subset Y \setminus intC(y)$ and hence $G(y) \neq \emptyset$. Let $\{x_n\}$ be a sequence in $G(y)$ converging to some $x \in X$. Then

$$F(y, x_n) \subset Y \setminus intC(y). \tag{2.1.1}$$

Since X is closed, $x \in X$. Suppose to the contrary that $F(y, x) \cap intC(y) \neq \emptyset$. Since $F(y, \cdot)$ is upper $C(y)$-lower semicontinuous at x, there exists a neighborhood V of x such that for any $x' \in V$, $F(y, x') \cap (intC(y) + C(y)) \neq \emptyset$. Thus for any $x' \in V$, $F(y, x') \cap intC(y) \neq \emptyset$. So, for n sufficiently large,

$$F(y, x_n) \cap intC(y) \neq \emptyset,$$

which contradicts (2.1.1). Hence $F(y,x) \subset Y \setminus intC(y)$, that is, $x \in G(y)$. Thus for any $y \in X$, $G(y)$ is closed.

Assume to the contrary that G is not a KKM multifunction on X. Then there exist a finite subset $\{y_1, y_2, \cdots, y_k\}$ of X and $\alpha_i \geqq 0,\ i = 1, \cdots, k$ such that

$$\sum_{i=1}^{k} \alpha_i = 1 \text{ and } y := \sum_{i=1}^{k} \alpha_i y_i \notin \bigcup_{i=1}^{k} G(y_i).$$

Thus we have $y \notin G(y_i),\ i = 1, \cdots, k$, i.e., $F(y_i, y) \cap intC(y_i) \neq \emptyset,\ i = 1, \cdots, k$. By (A_2), $F(y, y_i) \cap (-intC(y)) \neq \emptyset,\ i = 1, \cdots, k$. Since $F(y, \cdot)$ is upper $C(y)$-convex, $B := \{\xi \in X : F(y, \xi) \cap (-intC(y)) \neq \emptyset\}$ is convex, and hence $F(y,y) \cap (-intC(y)) \neq \emptyset$, which contradicts (A_1) since $F(y,y) \subset C(y) \subset Y \setminus (-intC(y))$. Hence G is a KKM multifunction. So, by KKM-Fan Theorem, there exists $\bar{x} \in X$ such that $\bar{x} \in \bigcap_{y \in X} G(y)$. Since $E'_w = \bigcap_{y \in X} G(y)$, E'_w is non-empty and closed. Let $\bar{x} \in E'_w$. Then

$$F(x, \bar{x}) \subset Y \setminus intC(x) \text{ for any } x \in X.$$

Let $x \in X$ be fixed. Consider $x_t := tx + (1-t)\bar{x}$, for $t \in (0,1)$. Clearly $x_t \in X$. Since $F(x_t, \cdot)$ is upper $C(x_t)$-convex,

$$tF(x_t, x) + (1-t)F(x_t, \bar{x}) \subset F(x_t, x_t) + C(x_t).$$

Since $F(x_t, x_t) \subset C(x_t)$, $F(x_t, \bar{x}) \subset Y \setminus intC(x_t)$ and $C(x_t) + Y \setminus (-intC(x_t)) \subset Y \setminus (-intC(x_t))$, we have

$$F(x_t, x) \subset Y \setminus (-intC(x_t)).$$

By (A_4), $F(\bar{x}, x) \subset Y \setminus (-intC(\bar{x}))$. Thus $\bar{x} \in E_w$. Hence $E'_w \subset E_w$. By (A_2), $E_w \subset E'_w$. Thus $E_w = E'_w$. □

Remark 2.1.1. Looking carefully at the proof of Theorem 2.1.1 one can realize that the same result holds under the following assumptions (A'_1) and (A'_2) instead of (A_1) and (A_2) in (H_1).

(A'_1) For all $x \in X$, $F(x, x) \subset [W(x) \cap (-W(x))]$,
where $W(x) = Y \setminus (-intC(x))$.

(A'_2) For all $x, y \in X$, $F(x, y) \subset Y \setminus (-intC(x))$
implies $F(y, x) \subset -C(y)$.

Certainly assumption (A'_1) is weaker than (A_1), whereas (A'_2) is stronger than (A_2).

Let us consider the following problem of finding

$$\bar{x} \in X \text{ such that } F(\bar{x}, y) \subset C(\bar{x}) \text{ for all } y \in X.$$

We denote its solution set by E_c.

Corollary 2.1.1. Let X be a convex, closed and bounded subset of $\mathbb{R}^n$ and $Y = \mathbb{R}^m$. Let $C : X \to 2^Y$ be a multifunction satisfying (H_0) and $F : X \times X \to 2^Y$ be a multifunction such that assumptions (A_2) and (A_4) of (H_1) are verified with $C(x)$ instead of $Y \setminus (-intC(x))$. Assume, in addition, that $C(x) \cup (-C(x)) = Y$ for all $x \in X$. Then E_c is a nonempty closed set, i.e., there exists $\bar{x} \in X$ such that $F(\bar{x}, y) \subset C(\bar{x})$ for all $y \in X$.

We consider two sets R_0 and R_1 defined as follows.

$$R_0 := \bigcap_{y \in X} \{v \in X^\infty \ : \ F(y, z + \lambda v) \subset Y \setminus intC(y)$$

$$\text{for all } \lambda > 0 \text{ and for all } \ z \in X \text{ with } F(y,z) \subset -C(y)\}$$

$$R_1 := \bigcap_{y \in X} \{v \in X^\infty \ : \ F(y, y + \lambda v) \subset Y \setminus intC(y) \ \text{ for all } \lambda > 0\}.$$

Then clearly $R_0 \subset R_1$.

Proposition 2.1.1. Let X be a convex and closed subset of $\mathbb{R}^n$ and $Y = \mathbb{R}^m$. Assume that $C : X \to 2^Y$ is a multifunction satisfying (H_0) and $F : X \times X \to 2^Y$ be a multifunction such that $F(x, \cdot)$ is upper $C(x)$-convex and satisfies $F(x,x) \subset C(x)$ for all $x \in X$. Then

$$R_1 \subset \bigcap_{y \in X} \{v \in X^\infty \ : \ F(y + \lambda v, y) \subset Y \setminus (-intC(y + \lambda v)) \ \text{ for all } \lambda > 0\}.$$

Proof. Let $v \in \bigcap_{y \in X} \{v \in X^\infty \ : \ F(y, y+\lambda v) \subset Y \setminus intC(y) \ \text{ for all } \lambda > 0\}$.

Then for any $y \in X$ and $\lambda > 0$, the upper $C(y + \lambda v)$-convexity of $F(y + \lambda v, \cdot)$ implies

$$\frac{1}{2}F(y + \lambda v, y + \lambda v + \lambda v) + \frac{1}{2}F(y + \lambda v, y) \subset F(y + \lambda v, y + \lambda v) + C(y + \lambda v).$$

Then

$$\begin{aligned} \tfrac{1}{2}F(y + \lambda v, y) \ &\subset C(y + \lambda v) + C(y + \lambda v) + Y \setminus -intC(y + \lambda v) \\ &\subset Y \setminus -intC(y + \lambda v). \end{aligned}$$

Thus $F(y+\lambda v, y) \subset Y \setminus -intC(y+\lambda v)$. Since $y \in X$ and $\lambda > 0$ are arbitrary, we conclude the proof. □

Proposition 2.1.2. Let X be a convex and closed subset of $\mathbb{R}^n$ and $Y = \mathbb{R}^m$. Let $C : X \to 2^Y$ be a multifunction satisfies (H_0). Assume the multifunction $F : X \times X \to 2^Y$ satisfies (H_1). Then

$$\begin{aligned} R_1 &\subset \bigcap_{y \in X} \{x \in X : F(x,y) \subset Y \setminus -intC(x)\}^\infty \\ &\subset \bigcap_{y \in X} \{x \in X : F(y,x) \subset Y \setminus intC(y)\}^\infty. \end{aligned}$$

Proof. Let $v \in X^\infty$ such that $F(y, y+\lambda v) \subset Y \setminus intC(y)$, for all $\lambda > 0$ and for all $y \in X$. By Proposition 2.1.1,

$$F(y+\lambda v, y) \subset Y \setminus -intC(y+\lambda v) \quad \text{for all} \quad \lambda > 0.$$

For any fixed $y \in X$, set $x_k = y + kv \in X$, $k \in \mathbb{N}$. Then $F(x_k, y) \subset Y \setminus -intC(x_k)$. By choosing $t_k = \frac{1}{k}$, we have $t_k x_k = \frac{y}{k} + v \to v$ as $k \to \infty$, i.e., $v \in \{x \in X : F(x,y) \subset Y \setminus -intC(x)\}^\infty$. Since y was arbitrary, $R_1 \subset \bigcap_{y \in X} \{x \in X : F(x,y) \subset Y \setminus -intC(x)\}^\infty$. Hence by (A_2), we have

$$\bigcap_{y \in X} \{x \in X : F(x,y) \subset Y \setminus -intC(x)\}^\infty \subset \bigcap_{y \in X} \{x \in X : F(y,x) \subset Y \setminus intC(y)\}^\infty.$$

□

In order to consider existence theorems and compactness condition of solution set in noncompact settings, we consider the asymptotic cone $(E_w)^\infty$ of the solution set E_w and its related set R_0.

Now we give relationships between $(E_w)^\infty$ and R_0.

Proposition 2.1.3. Let X be a convex and closed subset of $\mathbb{R}^n$ and $Y = \mathbb{R}^m$. Let $C : X \to 2^Y$ be a multifunction satisfying (H_0) and $F : X \times X \to 2^Y$ be a multifunction satisfying (A_2) and (A_3). Then $(E_w)^\infty \subset R_0$. If, in addition, (A_4) holds and there exists $x^* \in X$ such that

$$F(y, x^*) \subset -C(y) \text{ for all } y \in X,$$

then $(E_w)^\infty = R_0$.

Proof. Let $v \in (E_w)^\infty$. Then there exist $t_n \downarrow 0$ and $v_n \in E_w$ such that $t_n v_n \to v$. Then we have

$$F(v_n, y) \subset Y \setminus (-intC(v_n)) \text{ for all } n \in \mathbb{N}.$$

By (A_2), $F(y, v_n) \subset Y \setminus intC(y)$ for all $n \in \mathbb{N}$. Take any $z \in X$ such that $F(y, z) \subset -C(y)$. Let $\lambda > 0$ be fixed. For n sufficiently large, by the lower $C(y)$-convexity of $F(y, \cdot)$,

$$\begin{aligned} F(y, (1-\lambda t_n)z + \lambda t_n v_n) &\subset (1-\lambda t_n)F(y, z) + \lambda t_n F(y, v_n) - C(y) \\ &\subset -C(y) + [Y \setminus intC(y)] - C(y) \\ &\subset Y \setminus intC(y). \end{aligned}$$

Since $(1-\lambda t_n)z+\lambda t_n v_n \to z+\lambda v$ and $F(y,\cdot)$ is upper $C(y)$-lower semicontinuous, $F(y,z+\lambda v)\subset Y\setminus intC(y)$. Thus $v\in R_0$. Hence $(E_w)^\infty\subset R_0$.

Assume that there exists $x^*\in X$ such that $F(y,x^*)\subset -C(y)$ for all $y\in X$. Let $v\in R_0$. Then for all $y\in X$ and all $\lambda>0$, $F(y,x^*+\lambda v)\subset Y\setminus intC(y)$. By the same argument as in the proof of Theorem 2.1.1, we have, for all $y\in X$ and all $\lambda>0$,

$$F(x^*+\lambda v,y)\subset Y\setminus(-intC(x^*+\lambda v)).$$

Thus for all $\lambda>0$, $x^*+\lambda v\in E_w$. Hence $v\in(E_w)^\infty$. Consequently, $(E_w)^\infty=R_0$. □

We consider the following set.

$$R_0' := \bigcap_{y\in X}\{v\in X^\infty \;:\; F(y,z+\lambda v)\subset Y\setminus intC(y) \quad \text{for all } \lambda>0 \quad \text{and for all } z\in X \text{ such that } F(y,z)\subset Y\setminus intC(y)\}.$$

Obviously $R_0'\subset R_0\subset R_1$.

Theorem 2.1.2. Let X be a convex and closed subset of $\mathbb{R}^n$ and $Y=\mathbb{R}^m$. Let $C: X\to 2^Y$ be a multifunction satisfying (H_0). Assume the multifunction $F: X\times X\to 2^Y$ satisfies hypothesis (H_1) with (A_2') instead of (A_2). Then

$$\begin{aligned}(E_w)^\infty\subset R_0'\subset R_0\subset R_1 &\subset \bigcap_{y\in X}\{x\in X: F(x,y)\subset Y\setminus -intC(x)\}^\infty\\ &\subset \bigcap_{y\in X}\{x\in X: F(y,x)\subset -C(y)\}^\infty.\end{aligned}$$

If, in addition, $E_w \neq \emptyset$ then $R'_0 \subset (E_w)^\infty$. As a consequence $R'_0 = (E_w)^\infty$.

Corollary 2.1.2. Let X be a convex and closed subset of $\mathbb{R}^n$ and $Y = \mathbb{R}^m$. Let $C : X \to 2^Y$ be a multifunction satisfying (H_0). Let $F : X \times X \to 2^Y$ be a multifunction such that assumptions (A_2) and (A_4) are verified with $C(x)$ instead of $Y \setminus (-intC(x))$. Assume that $C(x) \cup -C(x) = Y$ for all $x \in X$. If $E_c \neq \emptyset$ then

$$(E_c)^\infty = \bigcap_{y \in X} \{v \in X^\infty \ : \ F(y, z + \lambda v) \subset -C(y) \text{ for all } \lambda > 0,$$

$$\text{for all } \ z \in X \text{ such that } \ F(y, z) \subset -C(y)\}.$$

We are now in a position to establish our first main existence theorem.

Theorem 2.1.3. Let X be a convex and closed subset of $\mathbb{R}^n$ and $Y = \mathbb{R}^m$. Let $C : X \to 2^Y$ be a multifunction satisfying (H_0) and $F : X \times X \to 2^Y$ be a multifunction satisfying (H_1). Suppose, in addition, that

(1) for each $x \in X$, $y \mapsto F(x, y)$ is lower $C(x)$-convex.

(2) for every sequence $\{x_n\}$ in X such that $\|x_n\| \to \infty$, $\frac{x_n}{\|x_n\|} \to v$ and $v \in R_0$ and for any $y \in X$, there exists $n_y \in \mathbb{N}$ such that $F(x_n, y) \subset Y \setminus (-intC(x_n))$ for all $n \geq n_y$ and there exists $u \in X$ such that $\|u\| < \|x_n\|$ and $F(x_n, u) \subset -C(x_n)$ for $n \in \mathbb{N}$ sufficiently large.

Then there exists $\bar{x} \in X$ which is a solution of the following vector equilibrium problem defined by F and X:

$(VEP)_1$ Find $\bar{x} \in X$ such that

$$F(\bar{x}, x) \subset Y \setminus (-intC(\bar{x})) \text{ for any } x \in X.$$

Proof. For every $n \in \mathbb{N}$, set

$$X_n := \{x \in X : \|x\| \leq n\}.$$

Then we may assume $X_n \neq \emptyset$ for any $n \in \mathbb{N}$. Also, X_n is a nonempty convex and compact subset of $\mathbb{R}^n$. So, by Theorem 2.1.1, we can find $x_n \in X_n$ which is a solution of the following vector equilibrium problem $(VEP)_{1,n}$ defined by F and X_n:

$$(VEP)_{1,n} \qquad \text{Find } \bar{x} \in X_n \text{ such that}$$
$$F(\bar{x}, y) \subset Y \setminus (-intC(\bar{x})) \text{ for any } y \in X_n.$$

If $\|x_n\| < n$ for some $n \in \mathbb{N}$, then x_n is a solution of $(VEP)_1$. Indeed, suppose to the contrary that x_n is not a solution of $(VEP)_1$. Then there exists $x \in X \setminus X_n$ such that

$$F(x_n, x) \cap (-intC(x_n)) \neq \emptyset.$$

Thus there exists $a_n \in F(x_n, x)$ such that $a_n \in -intC(x_n)$. Since $\|x_n\| < n$, and X is convex, there exists $\lambda \in (0, 1)$ such that

$$\lambda x_n + (1 - \lambda)x \in X_n.$$

Since $F(x_n, \cdot)$ is upper $C(x_n)$-convex, we have

$$\lambda F(x_n, x_n) + (1 - \lambda)F(x_n, x) \subset F(x_n, \lambda x_n + (1 - \lambda)x) + C(x_n).$$

Since $F(x_n, x_n) \subset -C(x_n)$, there exists $c_n \in F(x_n, \lambda x_n + (1 - \lambda)x)$ such that

$$\begin{aligned} -c_n &\in -(1 - \lambda)a_n + C(x_n) \\ &\subset intC(x_n). \end{aligned}$$

Thus $F(x_n, \lambda x_n + (1-\lambda)x) \cap (-intC(x_n)) \neq \emptyset$, which contradicts the choice of x_n.

Now consider the case that $\|x_n\| = n$ for all $n \in \mathbb{N}$. We may assume that $\frac{x_n}{\|x_n\|} \to v \ \ (v \neq 0)$. So, $v \in X^\infty$. Let $y \in X$ be fixed. Then it is clear that $F(x_n, y) \subset Y \setminus (-intC(x_n))$ for n sufficiently large. By assumption (A_2), we have

$$F(y, x_n) \subset Y \setminus intC(y)$$

for n sufficiently large. Take $z \in X$ such that

$$F(y, z) \subset -C(y).$$

By the lower $C(y)$-convexity of $F(y, \cdot)$, for any $\lambda > 0$ and n sufficiently large,

$$\begin{aligned} F\left(y, (1-\frac{\lambda}{\|x_n\|})z + \frac{\lambda}{\|x_n\|}x_n\right) &\subset (1-\frac{\lambda}{\|x_n\|})F(y,z) + \frac{\lambda}{\|x_n\|}F(y,x_n) - C(y) \\ &\subset -C(y) + [Y \setminus intC(y)] - C(y) \\ &\subset Y \setminus intC(y). \qquad (2.1.2) \end{aligned}$$

Assume to the contrary that

$$F(y, z + \lambda v) \cap intC(y) \neq \emptyset.$$

Since $F(y, \cdot)$ is upper $C(x)$-lower semicontinuous and for any $\lambda > 0$,

$$(1-\frac{\lambda}{\|x_n\|})z + \frac{\lambda}{\|x_n\|}x_n \to z + \lambda v,$$

$$F\left(y, (1-\frac{\lambda}{\|x_n\|})z + \frac{\lambda}{\|x_n\|}x_n\right) \cap intC(y) \neq \emptyset$$

for n sufficiently large. This contradicts (2.1.2). Thus $F(y, z + \lambda v) \subset Y \setminus intC(y)$ and hence $v \in R_0$.

By assumption (2), there exists $u \in X$ such that $\|u\| < \|x_n\|$ and $F(x_n, u) \subset -C(x_n)$ for n sufficiently large. We claim that for n sufficiently large, x_n is also a solution of $(VEP)_1$. If not, then there exists $y \in X$, $\|y\| > n$ such that $F(x_n, y) \cap (-intC(x_n)) \neq \emptyset$. Since $\|u\| < \|x_n\|$, we can find $\alpha \in (0, 1)$ such that

$$\alpha u + (1 - \alpha)y \in X_n.$$

By the upper $C(x_n)$-convexity of $F(x_n, \cdot)$,

$$\alpha F(x_n, u) + (1 - \alpha)F(x_n, y) \subset F(x_n, \alpha u + (1 - \alpha)y) + C(x_n).$$

Since $F(x_n, u) \subset -C(x_n)$ and $F(x_n, y) \cap (-intC(x_n)) \neq \emptyset$, we can easily check that

$$F(x_n, \alpha u + (1 - \alpha)y) \cap (-intC(x_n)) \neq \emptyset,$$

which contradicts the choice of x_n. Consequently, for n sufficiently large, x_n is a solution of $(VEP)_1$. □

Theorem 2.1.4. Let X be a convex and closed subset of $\mathbb{R}^n$ and $Y = \mathbb{R}^m$. Let $C : X \to 2^Y$ be a multifunction satisfying (H_0) and $F : X \times X \to 2^Y$ be a multifunction satisfying (H_1) with (A_2') instead of (A_2). Then the problem $(VEP)_1$ has a non-empty closed solution set if and only if the following property $(2)'$ is satisfied,

$(2)'$ for every sequence $\{x_n\}$ in X such that $\|x_n\| \to \infty$, $\frac{x_n}{\|x_n\|} \to v$ and $v \in R'_0$ and for any $y \in X$, it exists $n_y \in \mathbb{N}$ such that $F(x_n, y) \subset Y \setminus (-intC(x_n))$ for all $n \geqq n_y$, there exists $u \in X$ such that $\|u\| < \|x_n\|$ and $F(x_n, u) \subset -C(x_n)$ for all $n \in \mathbb{N}$ sufficiently large.

Proof. The "if" part is similar to the proof of the Theorem 2.1.3. The proof of "only if" part is obtained as follows:

Take any sequence $\{x_n\}$ in X such that $\|x_n\| \to \infty$ and any solution $\bar{x}$ of $(VEP)_1$.

Then by assumption (A'_2), condition $(2)'$ is satisfied by setting $u = \bar{x}$ and choosing x_n with n sufficiently large such that $\|\bar{x}\| < \|x_n\|$. □

2.2. Compactness of Solution Sets

Now we give conditions assuring the nonemptiness and compactness of the solution set E_w.

Theorem 2.2.1. Let X be a convex and closed subset of $\mathbb{R}^n$ and $Y = \mathbb{R}^m$. Let $C : X \to 2^Y$ be a multifunction satisfying (H_0) and $F : X \times X \to 2^Y$ be a multifunction satisfying (H_1). Assume that $R_0 = \{0\}$. Then E_w is nonempty and compact. If, in addition, there exists $x^* \in X$ such that

$$F(y, x^*) \subset -C(y) \text{ for all } y \in X,$$

then E_w is nonempty and compact if and only if $R_0 = \{0\}$.

Proof. For every $n \in \mathbb{N}$, set $X_n := \{x \in X : \|x\| \leq n\}$. Then we may suppose that $X_n \neq \emptyset$ for all $n \in \mathbb{N}$. Also, X_n is a nonempty convex and compact subset of $\mathbb{R}^n$. By Theorem 2.1.1, for all $n \in \mathbb{N}$, there exists $x_n \in X_n$ such that

$$F(x_n, y) \subset Y \setminus (-intC(x_n)) \text{ for any } y \in X_n.$$

Suppose that $\{x_n\}$ is not bounded. Then, up to a subsequence, $\|x_n\| \to \infty$ and $\frac{x_n}{\|x_n\|} \to v$ for some $v \in X$. Then $v \in X^\infty$ and $v \neq 0$. Let $y \in X$ be fixed. Then it is clear that $F(x_n, y) \subset Y \setminus (-intC(x_n))$ for n sufficiently large. By assumption (A_2), we have

$$F(y, x_n) \subset Y \setminus intC(y)$$

for n sufficiently large. Take $z \in X$ such that

$$F(y, z) \subset -C(y).$$

By the lower $C(y)$-convexity of $F(y, \cdot)$, for any $\lambda > 0$ and n sufficiently large,

$$\begin{aligned} F\left(y, (1 - \frac{\lambda}{\|x_n\|})z + \frac{\lambda}{\|x_n\|}x_n\right) &\subset (1 - \frac{\lambda}{\|x_n\|})F(y, z) + \frac{\lambda}{\|x_n\|}F(y, x_n) - C(y) \\ &\subset -C(y) + [Y \setminus intC(y)] - C(y) \\ &\subset Y \setminus intC(y). \end{aligned} \quad (2.2.1)$$

Assume to the contrary that

$$F(y, z + \lambda v) \cap intC(y) \neq \emptyset.$$

Since $F(y, \cdot)$ is upper $C(x)$-lower semicontinuous and for any $\lambda > 0$,

$$(1 - \frac{\lambda}{\|x_n\|})z + \frac{\lambda}{\|x_n\|}x_n \longrightarrow z + \lambda v,$$

$$F\left(y, (1 - \frac{\lambda}{\|x_n\|})z + \frac{\lambda}{\|x_n\|}x_n\right) \cap intC(y) \neq \emptyset$$

for n sufficiently large. This contradicts (2.2.1). Thus $F(y, z + \lambda v) \subset Y \setminus intC(y)$ and hence $v \in R_0$. However, it contradicts the assumption that $R_0 = \{0\}$. Hence $\{x_n\}$ is bounded. Therefore, up to a subsequence, $x_n \longrightarrow \bar{x}$ for some $\bar{x} \in X$. Let x be fixed in X. Then it follows that for n sufficiently large, $F(x_n, x) \subset Y \setminus (-intC(x_n))$, and hence, by (A_2), $F(x, x_n) \subset Y \setminus intC(x)$. Since $F(x, \cdot)$ is upper $C(x)$-lower semicontinuous, $F(x, \bar{x}) \subset Y \setminus intC(x)$. We argue exactly same as in the proof of Theorem 2.1.1 to obtain that $F(\bar{x}, x) \subset Y \setminus (-intC(\bar{x}))$ for any $x \in X$. Hence $\bar{x} \in E_w$. Thus E_w is nonempty. By Proposition 2.1.3, $(E_w)^\infty \subset R_0$.

So, by assumption, $(E_w)^\infty = \{0\}$ and hence E_w is bounded. Now we will prove that E_w is closed. Let $\{z_n\}$ be a sequence in E_w converging to some $z \in X$. Then for all $n \in \mathbb{N}$ and for all $y \in X$, $F(z_n, y) \subset Y \setminus (-intC(z_n))$. Thus, by (A_2), for all $n \in \mathbb{N}$ and for all $y \in X$, $F(y, z_n) \subset Y \setminus intC(y)$. Since F is upper $C(y)$-lower semicontinuous, $F(y, z) \subset Y \setminus intC(y)$ for all $y \in X$. By same argument as in the proof of Theorem 2.1.1, $F(z, y) \subset Y \setminus (-intC(z))$. Thus $z \in E_w$ and hence E_w is closed. Consequently, E_w is nonempty and compact.

Conversely, assume that E_w is nonempty and compact, and that there exists $x^* \in X$ such that $F(y, x^*) \subset -C(y)$ for all $y \in X$. Let $v \in R_0$. Then for all $y \in X$ and for all $\lambda > 0$, $F(y, x^* + \lambda v) \subset Y \setminus intC(y)$. By same argument as in the proof of Theorem 2.1.1, we have, for all $y \in X$ and for all $\lambda > 0$,

$$F(x^* + \lambda v, y) \subset Y \setminus (-intC(x^* + \lambda v)).$$

Thus for all $\lambda > 0$, $x^* + \lambda v \in E_w$. Since E_w is bounded, $v = 0$. Hence $R_0 = \{0\}$. □

2.3. Random Vector Equilibrium Problem $(\mathbf{RVEP})_1$

In this section, we will extend the first part of Theorem 2.2.1 to a random vector equilibrium problem with multifunctions. Our approach follows ideas of Kalmoun ([40, 41]), in which vector equilibrium problems with single-valued functions are considered.

Let Ω be a set and $(\Omega, \mathcal{A})$ be a measurable space, where $\mathcal{A}$ is the σ-algebra of subsets of Ω. Let E be a topological space and let $\mathcal{B}(E)$ be the σ-algebra of all Borel sets of E. Let $\mathcal{A}\otimes\mathcal{B}(E)$ be the σ-algebra generated by all subsets of the form of $A \times B$, where $A \in \mathcal{A}$ and $B \in \mathcal{B}(E)$.

Definition 2.3.1. Let $(\Omega, \mathcal{A})$ be a measurable space and Y be a topological space. Let $F : \Omega \to 2^Y$ be a multifunction. Then F is said to be measurable if the inverse image of each open set in Y is a measurable set in Ω, that is, for every open subset $\mathcal{O}$ of Y, we have

$$F^{-1}(\mathcal{O}) := \{\omega \in \Omega : F(\omega) \cap \mathcal{O} \neq \emptyset\} \in \mathcal{A}.$$

Definition 2.3.2 ([10]). Let $(\Omega, \mathcal{A})$ be a measurable space and Y be a complete separable metric space. Consider a multifunction $F : \Omega \to 2^Y$.

(1) F is said to have a measurable selection f if there exists a measurable function $f : \Omega \to Y$ such that $f(\omega) \in F(\omega)$ for all $\omega \in \Omega$.

(2) F is said to have a Castaing representation if there is a countable family of measurable selections (f_i) such that $(f_i(\omega))_{i\geq 1}$ is dense in $F(\omega)$, i.e., $F(\omega) = \overline{\bigcup_{i\geq 1} f_i(\omega)}$, for each $\omega \in \Omega$.

Lemma 2.3.1 ([10]). Assume that $(\Omega, \mathcal{A})$ is a complete measurable space and Y is a complete separable metric space. If $F : \Omega \longrightarrow 2^Y$ is a multifunction such that $Gr(F) \in \mathcal{A} \otimes \mathcal{B}(X)$, then F has a Castaing representation.

Let $(\Omega, \mathcal{A}, \mu)$ be a complete σ-finite measurable space and X be a nonempty convex and closed subset of $\mathbb{R}^n$. Let $Y = \mathbb{R}^m$. Let $C : \Omega \times X \longrightarrow 2^Y$ be a multifunction such that for any $(\omega, x) \in \Omega \times X$, $C(\omega, x)$ is a convex cone in Y with $intC(\omega, x) \neq \emptyset$ and $C(\omega, x) \neq Y$. Let $F : \Omega \times X \times X \longrightarrow 2^Y$ be a multifunction.

Now we consider the following random vector equilibrium problem $(RVEP)_1$.

$(RVEP)_1$ Find a function $\gamma : \Omega \longrightarrow X$ such that

$$F(\omega, \gamma(\omega), y) \subset Y \setminus (-intC(\omega, \gamma(\omega))) \text{ for any } (\omega, y) \in \Omega \times X.$$

As in [41], for each $\omega \in \Omega$, $\gamma(\omega)$ is called a deterministic solution of $(RVEP)_1$ and the function γ is said to be a random solution of $(RVEP)_1$ when it is measurable.

We obtain the random version of the first part of Theorem 2.2.1 as follows:

Theorem 2.3.1. Suppose that a multifunction $F : \Omega \times X \times X \longrightarrow 2^Y$ satisfies the following conditions:

(i) For any $y \in X$, $\{(\omega, x) \in \Omega \times X | F(\omega, x, y) \subset Y \setminus (-intC(\omega, x))\} \in \mathcal{A} \otimes \mathcal{B}\,(X)$.

(ii) For all $\omega \in \Omega$, $x \in X$, $F(\omega, x, x) \subset [C(\omega, x) \cap (-C(\omega, x))]$.

(iii) For all $\omega \in \Omega$ and $x,\ y \in X$, $F(\omega, x, y) \subset Y \setminus (-intC(\omega, x))$ implies $F(\omega, y, x) \subset Y \setminus intC(\omega, y)$.

(iv) For all $\omega \in \Omega$ and $x \in X$, $y \mapsto F(\omega, x, y)$ is $C(\omega, x)$-convex and $C(\omega, x)$-lower semicontinuous on X.

(v) For all $\omega \in \Omega$ and $x,\ y \in X$, the set $\{\xi \in [x, y] : F(\omega, \xi, y) \subset Y \setminus (-intC(\omega, \xi))\}$ is closed.

(vi) For each $\omega \in \Omega$, $R_0^\omega := \bigcap_{y \in X} \{v \in X^\infty : F(\omega, y, z+\lambda v) \subset Y \setminus intC(\omega, y)$ for all $\lambda > 0$, for all $z \in X$ with $F(\omega, y, z) \subset -C(y)\} = \{0\}$.

Then there exists a countable family of measurable functions $\gamma_i : \Omega \to X$ $(i \geq 1)$ such that

(1) $F(\omega, \gamma_i(\omega), y) \subset Y \setminus (-intC(\omega, \gamma_i(\omega)))$ for any $(\omega, y) \in \Omega \times X$.

(2) $\overline{\bigcup_{i \geq 1} \gamma_i(\omega)} = \{x \in X : F(\omega, x, y) \subset Y \setminus (-intC(\omega, x))$ for all $y \in X\}$ for any $\omega \in \Omega$.

(3) $\overline{\bigcup_{i \geq 1} \gamma_i(\omega)}$ is compact for any $\omega \in \Omega$.

Proof. By Theorem 2.2.1, for each $\omega \in \Omega$, there exists $x_\omega \in X$ such that

$$F(\omega, x_\omega, y) \subset Y \setminus (-intC(\omega, x_\omega)) \text{ for any } y \in X.$$

Since X is separable, there exists y_n in X such that

$$\overline{\{y_1, y_2, \cdots\}} = X.$$

Define a multifunction $S : \Omega \to X$ by for any $\omega \in \Omega$,

$$S(\omega) = \bigcap_{y \in X} \{x \in X | F(\omega, x, y) \subset Y \setminus (-intC(\omega, x))\}.$$

Then it follows from Theorem 2.2.1 that for each $\omega \in \Omega$, $S(\omega)$ is nonempty and compact. Now we will prove that

$$\bigcap_{n=1}^{\infty}\{x \in X : F(\omega, x, y_n) \subset Y \setminus (-intC(\omega, x))\} \subset S(\omega).$$

Indeed, suppose to the contrary that

$$x \in \bigcap_{n=1}^{\infty}\{x \in X : F(\omega, x, y_n) \subset Y \setminus (-intC(\omega, x))\} \tag{2.3.1}$$

but $x \notin S(\omega)$. Then there exists $y \in X$ such that

$$F(\omega, x, y) \cap (-intC(\omega, x)) \neq \emptyset.$$

Moreover, for each n there exists a subsequence $\{y_{n_k}\} \subset \{y_n\}$ such that $y_{n_k} \to y$. Since $F(\omega, x, \cdot)$ is lower $C(\omega, x)$-lower semicontinuous,

$$F(\omega, x, y_{n_k}) \cap (-intC(\omega, x)) \neq \emptyset$$

for k sufficiently large. This contradicts (2.3.1). Thus $Gr(S) = \bigcup_{n=1}^{\infty}\{(\omega, x) \in \Omega \times X : F(\omega, x, y_n) \subset Y \setminus (-intC(\omega, x))\}$. By assumption (i), $Gr(S) \in \mathcal{A} \otimes \mathcal{B}(X)$. By Lemma 2.3.1, S has a Castaing representation, i.e., there exists a countable family of measurable selections $(\gamma_i)_{i \geq 1}$ of S such that for any $\omega \in \Omega$, $S(\omega) = \overline{\bigcup_{i \geq 1} \gamma_i(\omega)}$. Hence the conclusion of Theorem 2.3.1 hold. $\square$

Chapter 3
Vector Equilibrium Problem $(\mathbf{VEP})_2$

In this chapter, the vector equilibrium problem $(VEP)_2$ and its Minty type equilibrium problem $(MVEP)_2$, which are introduced in Chapter 1, are considered. Using an increasing sequence of nonempty compact convex sets, we establish existence theorems for $(VEP)_2$ in noncompact settings and then apply our results to a noncoorperative vector Nash equilibrium problem, which is a vector versions of well-known Nash equilibrium problem ([66]). Moreover we obtain existence theorem for a random vector equilibrium problem with vector valued functions in compact settings.

3.1. Existence Theorems

Now we give an existence theorem for the vector equilibrium problem $(VEP)_2$.

Theorem 3.1.1. Let X be a convex and closed subset of $\mathbb{R}^n$ and $Y = \mathbb{R}^m$. Let $F : X \times X \to 2^Y$ be a multifunction and $C : X \to 2^Y$ be a multifunction such that $C(x)$ is a non-empty convex cone in Y with $intC(x) \neq \emptyset$ and $C(x) \neq Y$ for all $x \in X$. Define a multifunction $W : X \to 2^Y$ by for any $x \in X$, $W(x) := Y \setminus (-intC(x))$, and suppose that $X = \bigcup_{n=1}^{\infty} X_n$, where $\{X_n\}_{n=1}^{\infty}$ is an increasing sequence of nonempty compact convex subsets of X, and $Gr(W)$ is closed in $X \times Y$. Assume that the following conditions are satisfied :

(i) For each $x \in X$, $y \mapsto F(x, y)$ is natural quasi C-convex;

(ii) For each $y \in X$, $x \mapsto F(x,y)$ is u.s.c. and nonempty compact valued ;

(iii) For each $x \in X$, $F(x,x) \subset C(x)$.

Then for each n, there exists a solution $x \in X_n$ of the following vector equilibrium problem:

$$(VEP)_{2,n} \qquad \text{Find } \bar{x} \in X_n \text{ such that for any } x \in X_n,$$

$$F(\bar{x},x) \cap [Y \setminus (-intC(\bar{x}))] \neq \emptyset.$$

Moreover, in addition, suppose that

(iv) $F(\cdot, y)$ is l.s.c. ;

(v) x_{i_n} is a solution of $(VEP)_{2,i_n}$ for each n where $\{i_n\}$ is a sequence in $\mathbb{N}$ and the sequence $\{x_{i_n}\}$ is eventually contained in X_p for some $p \in \mathbb{N}$ (i.e., there exists $k \in \mathbb{N}$ such that for any $n \geqq k$, $x_{i_n} \in X_p$).

Then every cluster point of $\{x_{i_n}\}$ is a solution of $(VEP)_2$.

Proof. Let n be any fixed natural number. Let us consider a multifunction $G : X_n \to 2^X$ defined by for any $y \in X_n$,

$$G(y) := \{x \in X_n : F(x,y) \cap [Y \setminus (-intC(x))] \neq \emptyset\}.$$

Then we have the following;

(1) for each $y \in X_n$, $G(y) \neq \emptyset$.

If not, for any $x \in X_n, F(x,y) \cap [Y \setminus (-intC(x))] = \emptyset$ and hence $F(y,y) \subset (-intC(y))$. By assumption (iii), $F(y,y) \subset C(y) \cap (-intC(y))$, which is a contradiction since $C(y) \cap (-intC(y)) = \emptyset$. Indeed, suppose that $C(y) \cap (-intC(y)) \neq \emptyset$. Then there exists $v \in C(y) \cap (-intC(y))$. So, $0 = -v + v \in$

$intC(y) + C(y) = intC(y)$. This implies $C(y) = Y$ because $intC(y)$ is an absorbing set in Y, which contradicts the assumption that $C(y) \neq Y$.

(2) G is a KKM multifunction on X_n.

Assume to the contrary that G is not a KKM multifunction. Then there exists a finite subset $\{x_1, \cdots, x_n\}$ of X_n and $\alpha_1, \cdots, \alpha_n \geq 0$ such that $\sum_{i=1}^n \alpha_i = 1$ and $x := \sum_{i=1}^n \alpha_i x_i \notin \bigcup_{i=1}^n G(x_i)$. So, we have $x \notin G(x_i), i = 1, \cdots, n$, that is,

$$F(x, x_i) \cap [Y \setminus (-intC(x))] = \emptyset$$

i.e., $F(x, x_i) \subset -intC(x),\ i = 1, \cdots, n.$

Let $U = \{y \in X_n : F(x, y) \subset -intC(x)\}$, and let $z_1, z_2 \in U$ and $\alpha \in [0, 1]$. Then we have

$$F(x, z_i) \subset -intC(x),\ i = 1, 2. \tag{3.1.1}$$

By condition (i), there exists $\mu \in [0, 1]$, we have

$$F(x, \alpha z_1 + (1 - \alpha)z_2) \subset \mu F(x, z_1) + (1 - \mu)F(x, z_2) - C. \tag{3.1.2}$$

From (3.1.1) and (3.1.2), we have

$$\begin{aligned} F(x, \alpha z_1 + (1 - \alpha)z_2) &\subset -intC(x) - intC(x) - C(x) \\ &\subset -intC(x). \end{aligned}$$

Hence U is a convex subset of X_n, and hence

$$x := \sum_{i=1}^n \alpha_i x_i \in U.$$

So, $F(x,x) \subset -intC(x)$. However by condition (iii),

$$F(x,x) \subset C(x) \cap (-intC(x)) = \emptyset,$$

which is a contradiction. Hence G is a KKM multifunction.

(3) for each $y \in X_n$, $G(y)$ is closed in X.

Let $\{x_m\}$ be a sequence in $G(y)$ such that x_m converges to $x_* \in S$. Since X_n is compact, $x_* \in X_n$, and $F(x_m, y) \cap [Y \setminus (-intC(x_m))] \neq \emptyset$. Then there exists $z_m \in F(x_m, y)$ such that $z_m \in W(x_m)$. Since $F(\cdot, y)$ is u.s.c. and nonempty compact valued and $F(X_n, y)$ is compact in Y. So, there exists $\{z_{m_k}\} \subset \{z_m\}$ such that z_{m_k} converges to z_*, for some $z_* \in F(X_n, y)$. Moreover, since $F(\cdot, y)$ is closed, $z_{m_k} \in F(x_{m_k}, y)$ converge to $z_* \in F(x_*, y)$. Therefore $G(y)$ is closed in X.

From (1)-(3), by the KKM-Fan Theorem,

$$\bigcap_{y \in X_n} G(y) \neq \emptyset.$$

So, there exists $\bar{x} \in X_n$ such that for any $y \in X_n$, $\bar{x} \in G(y)$. Hence for any $x \in X_n$,

$$F(\bar{x}, x) \cap [Y \setminus (-intC(\bar{x}))] \neq \emptyset.$$

Let $\{x_{i_n}\}$ be a sequence such that $\{x_{i_n}\}$ is a solution of $(VEP)_{i_n}$. By condition (v) there exists a natural number $k \in \mathbb{N}$ such that $x_{i_n} \in X_p$ for all $n \geq k$ for some $k, p \in \mathbb{N}$. Since X_p is compact, we may assume that the sequence $\{x_{i_n}\}$ converges to some $\bar{x} \in X_p$. Let $x \in X$ be fixed. Since $X = \bigcup_{n=1}^{\infty} X_n$ and X_n is increasing, there exists $i_0 \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ with

$i_n \geqq i_0$, $x \in X_{i_n}$. Since x_{i_n} is a solution of $(VEP)_{i_n}$, for any $n \in \mathbb{N}$ with $i_n \geqq i_0$,

$$F(x_{i_n}, x) \subset W(x_{i_n}).$$

Let z be any point in $F(\bar{x}, x)$. By condition (iv), there exists $z_{i_n} \in F(x_{i_n}, x)$ such that z_{i_n} converges to z. Since $z_{i_n} \in W(x_{i_n})$ and W is closed, $z \in W(\bar{x})$. Hence $F(\bar{x}, x) \subset W(\bar{x})$.

Consequently, for any $x \in X$, $F(\bar{x}, x) \cap [Y \setminus (-intC(\bar{x})] \neq \emptyset$. □

Now we give an existence result for a Minty type vector equilibrium problem:

Theorem 3.1.2. Let X be a convex and closed subset of $\mathbb{R}^n$ and $Y = \mathbb{R}^m$. Let $F : X \times X \to 2^Y$ be a multifunction and $C : X \to 2^Y$ be a multifunction such that $C(x)$ is a non-empty convex cone in Y with $intC(x) \neq \emptyset$ and $C(x) \neq Y$ for all $x \in X$. Define a multifunction $W : X \to 2^Y$ by for any $x \in X$, $W(x) := Y \setminus (-intC(x))$ and suppose that $X = \bigcup_{n=1}^{\infty} X_n$, where $\{X_n\}_{n=1}^{\infty}$ is an increasing sequence of nonempty compact convex subsets of X, and $Gr(W)$ is closed in $X \times Y$. Assume that the following conditions are satisfied:

(i) F is C-pseudo-monotone, that is, for any $x, y \in X$, $F(x, y) \not\subset -intC(x)$ implies $-F(y, x) \not\subset -intC(x)$;

(ii) for each $x \in X$, $y \mapsto F(x, y)$ is natural quasi C-convex;

(iii) for each $x \in X$, $y \mapsto F(x, y)$ is u.s.c. and compact valued;

(iv) for each $x \in X$, $F(x, x) \subset C(x)$.

Then for each n, there exists a solution $x \in X_n$ of the following Minty type vector equilibrium problem:

$$(MVEP)_{2,n} \qquad \text{Find } \bar{x} \in X_n \text{ such that for any } x \in X_n,$$
$$-F(x, \bar{x}) \cap [Y \setminus (-intC(\bar{x}))] \neq \emptyset.$$

Moreover, in addition, suppose that

(v) $F(x, \cdot)$ is l.s.c.;

(vi) if x_{i_n} is a solution of $(MVEP)_{2,i_n}$ for each n where $\{i_n\}$ is a sequence in $\mathbb{N}$ and the sequence $\{x_{i_n}\}$ is eventually contained in X_p for some $p \in \mathbb{N}$.

Then every cluster point of $\{x_{i_n}\}$ is a solution of the following Minty type vector equilibrium problem :

$$(MVEP)_{2,i_n} \qquad \text{Find } \bar{x} \in X \text{ such that for any } x \in X$$
$$-F(x, \bar{x}) \cap [Y \setminus (-intC(\bar{x}))] \neq \emptyset.$$

Proof. Let n be any fixed natural number. Let us consider multifunctions S and $V : X_n \to 2^X$ defined by for any $y \in X_n$,

$$S(y) := \{x \in X_n : -F(y, x) \cap [Y \setminus (-intC(x))] \neq \emptyset\},$$
$$V(y) := \{x \in X_n : F(x, y) \cap [Y \setminus (-intC(x))] \neq \emptyset\}.$$

Then we have the following;

(1) For each $y \in X_n$, $S(y)$ is nonempty and S is a KKM multifunction.

Let $x \in V(y)$. Then $F(x, y) \cap [Y \setminus (-intC(x))] \neq \emptyset$. By the C-pseudomonotonicity of F,

$$-F(y, x) \cap [Y \setminus (-intC(x))] \neq \emptyset.$$

Thus $x \in S(y)$. Hence for any $y \in X_n$, $V(y) \subset S(y)$. Since $V(y)$ is nonempty, $S(y)$ is nonempty. By the same argument in the proof of Theorem 3.1.1, the multifunction V is a KKM multifunction. Therefore S is a KKM multifunction on X_n.

(2) for each $y \in X_n$, $S(y)$ is closed in X.

Let $\{x_m\}$ be a sequence in $S(y)$ such that x_m converges to $x_* \in X_n$. Then $x_* \in X_n$ and $-F(y, x_m) \cap [Y \setminus (-intC(x_m))] \neq \emptyset$. Then there exists $z_m \in -F(y, x_m)$ such that $z_m \in W(x_m)$. Since $-F(y, \cdot)$ is u.s.c. and $-F(y, X_n)$ is compact in Y. So, there exists a subsequence $\{z_{m_k}\} \subset \{z_m\}$ such that z_{m_k} converges to z_*, for some $z_* \in -F(y, X_n)$. Moreover, since $-F(y, \cdot)$ is closed, $z_{m_k} \in -F(y, x_{m_k}) \to z_* \in -F(y, x_*)$. Therefore $S(y)$ is closed in X. Since X_n is compact, $S(y)$ is compact in X_n. So, by the KKM-Fan Theorem,

$$\bigcap_{y \in X_n} S(y) \neq \emptyset.$$

So, there exists $\bar{x} \in X_n$ such that for any $x \in X_n, \bar{x} \in S(x)$. Hence for any $x \in X_n$,

$$-F(x, \bar{x}) \cap [Y \setminus (-intC(\bar{x}))] \neq \emptyset.$$

By the same argument in the proof of Theorem 3.1.1, we can obtain the last part of the conclusion. □

Remark 3.1.1. The last assumption (vi) for the sequence in Theorems 3.1.1 and 3.1.2 are related to the "escaping sequence" defined in [15, 70].

Now we reduce Theorems 3.1.1 and 3.1.2 to the case of vector valued functions.

Corollary 3.1.1. Let X, Y, C, W, and X_n be as in Theorem 3.1.1. Let $f : X \times X \to Y$ be a vector-valued function. Assume that the following conditions are satisfied:

(i) for each $x \in X$, $y \mapsto f(x, y)$ is natural quasi C-convex;

(ii) for each $y \in X$, $x \mapsto f(x, y)$ is continuous;

(iii) for each $x \in X$, $f(x, x) \in C(x)$.

Then, for each n, there exists a solution $x \in X$ of the following vector equilibrium problem:

$$(VEP)_n \qquad \text{Find } \bar{x} \in X_n \text{ such that for any } x \in X_n,$$
$$f(\bar{x}, x) \notin -intC(\bar{x}).$$

Moreover, in addition, suppose that

(iv) if x_{i_n} is a solution of $(VEP)_{i_n}$ for each n, where $\{i_n\}$ is a sequence in $\mathbb{N}$ and the sequence $\{x_{i_n}\}$ is eventually contained in X_p for some $p \in \mathbb{N}$.

Then every cluster point of $\{x_{i_n}\}$ is a solution of the following vector equilibrium problem:

$$(VEP) \qquad \text{Find } \bar{x} \in X \text{ such that for any } x \in X \quad f(\bar{x}, x) \notin -intC(\bar{x}).$$

Corollary 3.1.2. Let X, Y, C, W, and X_n be as in Theorem 3.1.1. Let $f : X \times X \to Y$ be a vector-valued function. Assume that the following conditions are satisfied:

(i) f is C-pseudo-monotone, that is, for any $x, y \in X$, $f(x,y) \notin -intC(x)$ implies $-f(y,x) \notin -intC(x)$;

(ii) for each $x \in X$, $y \mapsto f(x,y)$ is natural quasi C-convex;

(iii) for each $x \in X$, $y \mapsto f(x,y)$ is continuous and natural quasi C-convex;

(iv) for each $x \in X$, $f(x,x) \in C(x)$.

Then, for each n, there exists a solution $x \in X$ of the following Minty type vector equilibrium problem:

$(MVEP)_n$ Find $\bar{x} \in X_n$ such that for any $x \in X_n$, $f(x,\bar{x}) \notin -intC(\bar{x})$.

Moreover, if x_{i_n} is a solution of $(MVEP)_{i_n}$ for each n, where $\{i_n\}$ is a sequence in $\mathbb{N}$ and the sequence $\{x_{i_n}\}$ is eventually contained in X_p for some $p \in \mathbb{N}$ then every cluster point of $\{x_{i_n}\}$ is a solution of the following Minty type vector equilibrium problem:

$(MVEP)$ Find $\bar{x} \in X$ such that for any $x \in X$ $-f(x,\bar{x}) \notin -intC(\bar{x})$.

Corollary 3.1.3. Let X, Y, C, W, and X_n be the same as in Theorem 3.1.1. We denote by $L(X,Y)$ the space of all continuous linear mapping from X to Y. Let $T : X \to L(X,Y)$ be a function. Assume that, for each $y \in X$, $x \mapsto \langle T(x), y-x \rangle$ is continuous, where $\langle\ ,\ \rangle$ denotes the inner product on X. Then, for each n, there exists a solution $x \in X$ of the following vector variational inequality:

$(VVI)_n$ Find $\bar{x} \in X_n$ such that for any $x \in X_n$, $\langle T(\bar{x}), x-\bar{x} \rangle \notin -intC(\bar{x})$.

Moreover, if x_{i_n} is a solution of $(VVI)_{i_n}$ for each n, where $\{i_n\}$ is a sequence in $\mathbb{N}$ and the sequence $\{x_{i_n}\}$ is eventually contained in X_p for some $p \in \mathbb{N}$ then every cluster point of $\{x_{i_n}\}$ is a solution of the following vector variational inequality:

$$(VVI) \qquad \text{Find } \bar{x} \in X \text{ such that for any } x \in X, \ \langle T(\bar{x}), x - \bar{x} \rangle \notin (-intC(\bar{x})).$$

Proof. Putting $f(x, y) = \langle T(x), y - x \rangle$ in Corollary 3.1.1, we get the result. Indeed, it is straightforward to check the conditions (i), (iii) of Corollary 3.1.1 except the continuity of $x \mapsto \langle T(x), y - x \rangle$, for all $x \in X_n$. □

Example 3.1.1. Let $X = Y = \mathbb{R}$ and $C(x) = \mathbb{R}^2_+$ for any $x \in X$. Let $f_1(x, y) = x(y - x)$, $f_2(x, y) = -x(y - x)$ and $f(x, y) = (f_1(x, y), f_2(x, y))$. Take $X_n = [-n, n]$, $n = 1, 2, \cdots$. Then $X = \bigcup_{n=1}^{\infty} X_n$ and $\{X_n\}_{n=1}^{\infty}$ is an increasing sequence of nonempty and compact convex subset of X, for all $n \in \mathbb{N}$. Then we can easily check that the following:

(i) for any $x \in X$, $y \mapsto f(x, y)$ is $\mathbb{R}^2_+$-convex;

(ii) for any $y \in X$, $x \mapsto f(x, y)$ is continuous;

(iii) for any $x \in X$, $f(x, x) = (0, 0) \in C(x)$.

Therefore by Corollary 3.1.1, for each n, there is a solution of:

$$(VEP)_n \qquad \text{Find } \bar{x} \in X_n \text{ such that for any } x \in X_n, \ f(\bar{x}, x) \notin -intC(\bar{x}).$$

Actually every point of X_n is a solution of $(VEP)_n$. Let $x_* \in X$ be fixed. Then there exists $p \in \mathbb{N}$ such that $x_* \in X_p$. Also, we can find a sequence

$\{x_n\}$ in X such that $\{x_n\}$ is eventually contained in X_p and $x_n \to x_*$. By Corollary 3.1.1, x_* is a solution of:

$$(VEP) \qquad \text{Find } \bar{x} \in X \text{ such that for any } x \in X,\ f(\bar{x}, x) \notin -int\mathbb{R}^2_+.$$

So, every point of X is a solution of (VEP).

3.2. Applications to Noncooperative Nash Vector Equilibrium Problem

Let $X := \prod_{i=1}^{n} X^i$ be a nonempty closed convex subset of a product space $E := \prod_{i=1}^{n} \mathbb{R}^{n_i}$, $Y = \mathbb{R}^m$ and $G_i : X \to Y$ be vector valued functions, $i = 1, \cdots, n$. Let for any $x = (x^1, \cdots, x^n) \in X$, for $i \in \{1, 2, \cdots, n\}$, $x^{\hat{i}} = (x^1, \cdots, x^{i-1}, x^{i+1}, \cdots, x^n) \in \prod_{j \neq i} X^j$ and $x = (x^i, x^{\hat{i}}) \in X^i \times \prod_{j \neq i} X^j$.

Definition 3.2.1. Let $C : X \to 2^Y$ be a multifunction such that for each $x \in X$, $C(x)$ is a convex cone in Y with $intC(x) \neq \emptyset$ and $C(x) \neq Y$. Then we say that $\bar{x} \in X$ is a noncooperative vector equilibrium if for each $i \in \{1, \cdots, n\}$, we have

$$G_i(y^i, \bar{x}^{\hat{i}}) - G_i(\bar{x}^i, \bar{x}^{\hat{i}}) \notin -intC(\bar{x}) \text{ for any } y^i \in X^i.$$

Theorem 3.2.1. Let $C : X \to 2^Y$ be a multifunction such that for any $x \in X$, $C(x)$ be a convex cone in Y with $intC(x) \neq \emptyset$, $C(x) \neq Y$ and $F : X \times X \to Y$ be a multifunction. Define a multifunction $W : X \to 2^Y$ by for any $x \in X$, $W(x) := Y \setminus (-intC(x))$, and suppose that $X = \bigcup_{n=1}^{\infty} X_n$, where $\{X_n\}_{n=1}^{\infty}$ is an increasing sequence of nonempty compact convex subsets of X, and $Gr(W)$ is closed in $X \times Y$.

Assume that the following conditions are satisfied:

(i) for each $i \in \{1, \cdots, n\}$, X^i is closed and convex;

(ii) for each $i \in \{1, \cdots, n\}$, the function $y^i \mapsto G_i(y^i, x^{\hat{i}})$ is C-convex;

(iii) for each $i \in \{1, \cdots, n\}$, the function G_i is continuous.

Then for each n, there exists a solution $x \in X_n$ of the following noncooperative vector equilibrium problem $(NVEP)_n$;

$$(NVEP)_n \qquad \text{Find } \ \bar{x} \in X_n \text{ such that for each } i \in \{1, \cdots, n\},$$
$$G_i(y^i, \bar{x}^{\hat{i}}) - G_i(\bar{x}^i, \bar{x}^{\hat{i}}) \notin -intC \text{ , for any } y^i \in X^i.$$

Moreover, if x_{i_n} is a solution of $(NVEP)^{i_n}$ for each n, where $\{i_n\}$ is a sequence in $\mathbb{N}$ and the sequence $\{x_{i_n}\}$ is eventually contained in X_p for some $p \in \mathbb{N}$, then every cluster point of $\{x_{i_n}\}$ is a solution of $(NVEP)$:

Proof. Define a function $f : X \times X \to Y$ by for any $x, y \in X_n$,

$$f(x, y) = \sum_{i=1}^{n} [G_i(y^i, x^{\hat{i}}) - G_i(x^i, x^{\hat{i}})].$$

Then for each $x \in X$, $f(x, x) = 0 \in C(x)$. By the condition (i), X is closed and convex. By the condition (ii), the function $y \mapsto f(x, y)$ is C-convex and hence natural quasi C-convex. By the condition (iii), the function $x \mapsto f(x, y)$ is continuous.

By Remark 3.1.1 and Corollary 3.1.1, there exists $\bar{x} = (\bar{x}^i, \bar{x}^{\hat{i}}) \in X_n$ such that for any $y \in X_n$,

$$f(\bar{x}, y) \notin -intC(\bar{x}). \tag{3.2.1}$$

For each $i \in \{1, \cdots, n\}$ and any $y^i \in X_n^i$, let us take $y = (y^i, \bar{x}^{\hat{i}})$. Then from (3.2.1),

$$f(\bar{x}, y) = G_i(\bar{y}, \bar{x}^{\hat{i}}) - G_i(\bar{x}^i, \bar{x}^{\hat{i}}) \notin -intC(\bar{x}).$$

Hence we have, for each $i \in \{1, 2, \cdots, n\}$,

$$G_i(\bar{y}, \bar{x}^{\hat{i}}) - G_i(\bar{x}^i, \bar{x}^{\hat{i}}) \notin -intC(\bar{x}) \text{ for any } y^i \in X_n^i,$$

that is, $\bar{x}$ is a solution of $(NVEP)_n$. By the same argument in the proof of Theorem 3.1.1, we can obtain the last part of the conclusion. □

3.3. Random Vector Equilibrium Problem $(\mathbf{RVEP})_2$

In this section, we will extend Theorem 3.1.1 to a random vector equilibrium problem with multifunctions. Our approach follows ideas Kalmoun [40, 41], in which vector equilibrium problems with single-valued functions are considered. We apply our results to random vector optimization problems, random vector variational inequality problems and random vector approximate problems.

Let $(\Omega, \mathcal{A})$ be a measurable space where $\mathcal{A}$ is a σ-algebra of subsets of Ω. Let E be a topological space and let $\mathcal{B}(E)$ be the σ-algebra of all Borel sets of E. Let $\mathcal{A} \otimes \mathcal{B}(E)$ be σ-algebra generated by all subsets of the form of $A \times B$, where $A \in \mathcal{A}$ and $B \in \mathcal{B}(E)$.

Let $(\Omega, \mathcal{A}, \mu)$ be a complete σ-finite measurable space and X be a compact and convex subset of $\mathbb{R}^n$. Let $Y = \mathbb{R}^m$. Let $C : \Omega \times X \to 2^Y$ be a multifunction such that for any $(\omega, x) \in \Omega \times X$, $C(\omega, x)$ is a convex cone in Y with $intC(\omega, x) \neq \emptyset$ and $C(\omega, x) \neq Y$ and let P be a convex cone such that for any $(\omega, x) \in \Omega \times X$, $P \subset C(\omega, x)$. Let $F : \Omega \times X \times X \to 2^Y$ be a multifunction.

Now we consider the following random vector equilibrium problem $(RVEP)_2$:

$(RVEP)_2$ Find a function $\gamma : \Omega \to X$ such that

$$F(\omega, \gamma(\omega), y) \cap [Y \setminus (-intC(\omega, \gamma(\omega)))] \neq \emptyset, \text{ for all } (\omega, y) \in \Omega \times X.$$

As defined in [41], for each $\omega \in \Omega$, $\gamma(\omega)$ is called a deterministic solution of $(RVEP)_2$ and the function γ is said to be a random solution of $(RVEP)_2$ when it is measurable.

We obtain the random version of the first part of Theorem 3.1.1 as follows:

Theorem 3.3.1. Let $F : \Omega \times X \times X \to 2^Y$ be a multifunction. Assume that the following conditions are satisfied:

(i) for each $\omega \in \Omega$, $W(\omega, \cdot)$ is closed;

(ii) $Gr(W) \in \mathcal{A} \otimes \mathcal{B}\ (X) \otimes \mathcal{B}\ (Y)$;

(iii) for each $(\omega, x) \in \Omega \times X$, $y \mapsto F(\omega, x, y)$ is natural quasi P-convex and u.s.c. and nonempty compact valued;

(iv) for any $y \in X$ and $C \in \mathcal{B}\ (Y)$,

$$\{(\omega, x) \in \Omega \times X : F(\omega, x, y) \cap C \neq \emptyset\} \in \mathcal{A} \otimes \mathcal{B}(X);$$

(v) for each $(\omega, y) \in \Omega \times X$, $x \mapsto F(\omega, x, y)$ is u.s.c. and compact valued;

and

(vi) for each $(\omega, x) \in \Omega \times X$, $F(\omega, x, x) \in C(\omega, x)$.

Then there exists a countable family of measurable functions $\gamma_i : \Omega \to X$ ($i \geq 1$) such that

(1) $F(\omega, \gamma_i(\omega), y) \cap [Y \setminus -intC(\omega, \gamma_i(\omega))] \neq \emptyset$ for any $(\omega, y) \in \Omega \times X$.

(2) $\overline{\bigcup_{i \geq 1} \gamma_i(\omega)} = \{x \in X : F(\omega, x, y) \cap [Y \setminus -intC(\omega, x)] \neq \emptyset \text{ for all } y \in X\}$

for any $\omega \in \Omega$;

(3) $\overline{\bigcup_{i\geq 1}\gamma_i(\omega)}$ is compact, for any $\omega \in \Omega$.

Proof. By Theorem 3.1.1, for each $\omega \in \Omega$, there exists $x_\omega \in X$ such that

$$F(\omega, x_\omega, y) \cap [Y \setminus (-intC(\omega, x_\omega))] \neq \emptyset \text{ for any } y \in X.$$

Since X is separable, there exists a sequence $\{y_n\}$ in X such that

$$\overline{\{y_1, y_2, \cdots\}} = X.$$

Define a multifunction $S : \Omega \to X$ by for any $\omega \in \Omega$,

$$S(\omega) = \bigcap_{y\in X}\{x \in X | F(\omega, x, y) \cap [Y \setminus (-intC(\omega, x))] \neq \emptyset\}.$$

Then it follows from Theorem 3.1.1 that for each $\omega \in \Omega$, $S(\omega)$ is nonempty and compact. Now we will prove that

$$\bigcap_{n=1}^{\infty}\{x \in X : F(\omega, x, y_n) \cap [Y \setminus (-intC(\omega, x))] \neq \emptyset\} \subset S(\omega).$$

Indeed, suppose to the contrary that

$$x \in \bigcap_{n=1}^{\infty}\{x \in X : F(\omega, x, y_n) \cap [Y \setminus (-intC(\omega, x))] \neq \emptyset\} \qquad (3.3.1)$$

but $x \notin S(\omega)$. Then there exists $y \in X$ such that

$$F(\omega, x, y) \cap [Y \setminus (-intC(\omega, x))] = \emptyset.$$

Moreover, for each n there exists a subsequence $\{y_{n_k}\} \subset \{y_n\}$ such that $y_{n_k} \to y$. Since $F(\omega, x, \cdot)$ is upper semicontinuous,

$$F(\omega, x, y_{n_k}) \cap (-intC(\omega, x)) \neq \emptyset$$

for k sufficiently large. This contradicts (3.3.1). Thus $Gr(S) = \bigcup_{n=1}^{\infty}\{(\omega, x) \in \Omega \times X : F(\omega, x, y_n) \cap [Y \setminus (-intC(\omega, x))] \neq \emptyset\}$. By assumption (iv), $Gr(S) \in \mathcal{A} \otimes \mathcal{B}(X)$. By Lemma 2.3.1, S has a Castaing representation, i.e., there exists a countable family of measurable selections $(\gamma_i)_{i \geq 1}$ of S such that for any $\omega \in \Omega$, $S(\omega) = \overline{\bigcup_{i \geq 1} \gamma_i(\omega)}$. Hence the conclusions of Theorem 3.3.1 hold. $\square$

Chapter 4
Affine Vector Variational Inequality

In this chapter, we discuss the boundedness and connectedness of solution sets for affine vector variational inequalities with noncompact polyhedral constraint sets and positive semidefinite (or monotone) matrices. We give numerical examples clarifying and illustrating the boundedness and connectedness results. Moreover, we show that the boundedness and connectedness result can be applied to the multiobjective linear fractional optimization problems and the multiobjective convex linear-quadratic optimization problems.

4.1. Boundedness and Connectedness of Solution Sets

Now we formulate affine vector variational inequalities.
Let $\Lambda = \{\xi = (\xi_1, \cdots, \xi_p) \in \mathbb{R}^p \mid \xi_i \geqq 0,\ i = 1, \cdots, p,\ \sum_{i=1}^p \xi_i = 1\}$, and $\overset{o}{\Lambda} = \{\xi = (\xi_1, \cdots, \xi_p) \in \mathbb{R}^p \mid \xi_i > 0,\ i = 1, \cdots, p,\ \sum_{i=1}^p \xi_i = 1\}$. Let $\triangle = \{x \in \mathbb{R}^n \mid Ax \geqq b\}$, where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ and $0^+\triangle = \{x \in \mathbb{R}^n \mid Ax \geqq 0\}$, i.e., $0^+\triangle$ denotes the recession cone of $\triangle$. Let $\langle \cdot, \cdot \rangle$ denote the inner product on $\mathbb{R}^n$. Assume that $\triangle \neq \emptyset$. Let $M_i \in \mathbb{R}^{n \times n}$ and $q_i \in \mathbb{R}^n$, $i = 1, \cdots, p$.
Consider the following affine vector variational inequalities:

$$(VVI) \qquad \text{Find } \bar{x} \in \triangle \text{ such that}$$

$$(\langle M_1\bar{x} + q_1,\ x - \bar{x}\rangle, \cdots, \langle M_p\bar{x} + q_p,\ x - \bar{x}\rangle) \notin -\mathbb{R}^p_+ \setminus \{0\} \quad \text{for all } x \in \triangle.$$

$(VVI)^w$ Find $\bar{x} \in \Delta$ such that

$$(\langle M_1\bar{x} + q_1,\ x - \bar{x}\rangle, \cdots, \langle M_p\bar{x} + q_p,\ x - \bar{x}\rangle) \notin -int\mathbb{R}^p_+ \quad \text{for all } x \in \Delta.$$

where $\mathbb{R}^p_+ = \{x := (x_1, \cdots, x_p) \in \mathbb{R}^p \mid x_i \geqq 0,\ i = 1, \cdots, p\}$ and $int\mathbb{R}^p_+$ is the interior of $\mathbb{R}^p_+$.

Consider their related scalar variational inequality: let $\xi = (\xi_1, \cdots, \xi_p) \in \Lambda$.

$(VI)_\xi$ Find $\bar{x} \in \Delta$ such that

$$\langle \sum_{i=1}^p \xi_i M_i\bar{x} + \sum_{i=1}^p \xi_i q_i,\ x - \bar{x}\rangle \geqq 0 \quad \text{for all } x \in \Delta.$$

We denote the solution sets of (VVI), $(VVI)^w$ and $(VI)_\xi$ by $sol(VVI)$, $sol(VVI)^w$ and $sol(VI)_\xi$, respectively. It is clear that $sol(VVI) \subset sol(VVI)^w$. Consider an affine variational inequality:

(VI) Find $\bar{x} \in \Delta$ such that $\langle M\bar{x} + q,\ x - \bar{x}\rangle \geqq 0$ for all $x \in \Delta$,

where $M \in \mathbb{R}^{n\times n},\ q \in \mathbb{R}^n$.

Proposition 4.1.1 ([59]). $\langle v, Mv\rangle > 0$ for all $v \in 0^+\Delta \setminus \{0\}$ if and only if there exists $x^0 \in \Delta$ such that

$$\frac{\langle My - Mx^0,\ y - x^0\rangle}{\| y - x^0 \|} \to +\infty \quad \text{as} \quad \| y \| \to +\infty, \quad y \in \Delta. \tag{4.1.1}$$

The condition (4.1.1) is called a coercivity condition for (VI) (see [44]). By Corollary 4.3 (p. 14, [44]), we have

Proposition 4.1.2. If $\langle v, Mv\rangle > 0$ for all $v \in 0^+\Delta\setminus\{0\}$, then $sol(VI) \neq \emptyset$.

Proposition 4.1.3 ([37], p. 432). Let M be monotone on $\triangle$, i.e., $\langle x-y,\ M(x-y)\rangle \geqq 0$ for all $x, y \in \triangle$. Then $sol(VI) \neq \emptyset$ if and only if there exists $\bar{x} \in \triangle$ such that $\langle M\bar{x}+q,\ v\rangle \geqq 0$ for all $v \in 0^{+}\triangle$.

Let X, Y be two topological spaces and $G : X \to 2^{Y}$ be a multifunction.

Definition 4.1.1. The space X is said to be connected if there do not exist nonempty open subsets $V_i \subset X,\ i = 1, 2$, such that

$$V_1 \cap V_2 = \emptyset \quad \text{and} \quad V_1 \cup V_2 = X.$$

Definition 4.1.2. The multifunction G is said to be upper semicontinuous (shortly u.s.c.) if for every $a \in X$ and every open set $\Omega \subset Y$ satisfying $G(a) \subset \Omega$, there exists a neighborhood U of a such that $G(a') \subset \Omega$ for all $a' \in U$.

Lemma 4.1.1 ([79], Theorem 3.1). Assume that X is connected. If for every $x \in X$, the set $G(x)$ is nonempty and connected, and G is upper semicontinuous, then the set $G(X) := \bigcup_{x \in X} G(x)$ is connected.

From Theorem 2.1 in [60], we can obtain the following proposition:

Proposition 4.1.4. $sol(VVI) = \bigcup_{\xi \in \mathring{\Lambda}} sol(VI)_{\xi} \subset sol(VVI)^{w} = \bigcup_{\xi \in \Lambda} sol(VI)_{\xi}$.

Theorem 4.1.1. Let $M_i,\ i = 1, \cdots, p$ be monotone on $\triangle$. If $\langle v, M_i v\rangle > 0$ for every $i = 1, \cdots, p$ and for every $v \in 0^{+}\triangle \setminus \{0\}$, then $sol(VVI)$ and $sol(VVI)^{w}$ are nonempty, bounded and connected. In particular, $sol(VVI)^{w}$ is compact.

Proof. Let $\xi \in \Lambda$. By assumption, we have

$$\left\langle v, \sum_{i=1}^{p} \xi_i M_i v \right\rangle > 0 \text{ for all } v \in 0^+\triangle \setminus \{0\}.$$

So, by Proposition 4.1.2, $sol(VI)_\xi \neq \emptyset$. Since $\sum_{i=1}^{p} \xi_i M_i$ is monotone on $\triangle$, by Minty's lemma ([65]), $sol(VI)_\xi$ is connected.

Now we will prove that $sol(VVI)^w$ is bounded. Suppose to the contrary that $sol(VVI)^w$ is not bounded. Then there exists $x^k \in sol(VVI)^w$ such that $\| x^k \| \to +\infty$ as $k \to +\infty$. By Proposition 4.1.4, there exists $\xi^k \in \Lambda$ such that

$$x^k \in sol(VI)_{\xi^k}. \tag{4.1.2}$$

We may assume that $\frac{x^k}{\|x^k\|} \to \bar{v}$, $\| \bar{v} \| = 1$ and $\xi^k \to \bar{\xi} \in \Lambda$.

By (4.1.2),

$$\left\langle \sum_{i=1}^{p} \xi_i^k M_i x^k + \sum_{i=1}^{p} \xi_i^k q_i,\ x - x^k \right\rangle \geq 0 \text{ for all } x \in \triangle \tag{4.1.3}$$

$$\text{and} \quad Ax^k \geq b. \tag{4.1.4}$$

Dividing (4.1.3) by $\| x^k \|^2$ and (4.1.4) by $\| x^k \|$, and letting $k \to \infty$, we have

$$\left\langle \sum_{i=1}^{p} \bar{\xi}_i M_i \bar{v},\ \bar{v} \right\rangle \leq 0 \text{ and } \bar{v} \in 0^+\triangle \setminus \{0\},$$

which contradicts the assumption. Hence $sol(VVI)^w$ is bounded. Since $sol(VVI)$ is closed, $sol(VVI)^w$ is compact. Since $sol(VI)_\xi \neq \emptyset$ for all $\xi \in \Lambda$, by Proposition 4.1.4, $sol(VVI)$ is nonempty. Since $sol(VVI) \subset sol(VVI)^w$, $sol(VVI)$ is bounded.

Now we will prove that $sol(VVI)$ and $sol(VVI)^w$ are connected. Define a multifunction $H : \Lambda \to 2^K$ by for all $\xi \in \Lambda$, $H(\xi) = sol(VI)_\xi$. Then we can easily check that H is a closed multifunction. Since $sol(VVI)^w$ is compact, it follows from Lemma 1.1 that H is upper semicontinuous. Since $\overset{o}{\Lambda}$ and Λ are connected, it follows from Lemma 4.1.1 that $H(\overset{o}{\Lambda})$ and $H(\Lambda)$ are connected. By Proposition 4.1.4, $sol(VVI)$ and $sol(VVI)^w$ are connected. □

When $\triangle$ is compact, $0^+\triangle = \{0\}$, so we can obtain the following corollary from Theorem 4.1.1;

Corollary 4.1.1. Let M_i, $i = 1, \cdots, p$ be monotone on $\triangle$. If $\triangle$ is compact, then $sol(VVI)$ and $sol(VVI)^w$ are nonempty, bounded and connected.

Now we introduce the following well-known fact [12].

Lemma 4.1.2. Let $M \in \mathbb{R}^{n \times n}$ be positive semidefinite. If $\bar{v}^T M \bar{v} = 0$, for any $\bar{v} \in \mathbb{R}^n$, then $(M + M^T)\bar{v} = 0$, where T denotes the transposition.

From Lemma 4.1.2 and Proposition 4.1.3, we can obtain the following theorem.

Theorem 4.1.2. Let M_i $i = 1, \cdots, p$ be positive semidefinite $n \times n$ matrices (and hence, monotone on $\triangle$). If for any $v \in 0^+\triangle \setminus \{0\}$, there exists $x \in \triangle$ such that

$$\langle M_i x + q_i,\ v \rangle > 0 \quad \text{for all } i \in \{1, \cdots, p\}, \tag{4.1.5}$$

then $sol(VVI)$ and $sol(VVI)^w$ are nonempty, bounded and connected.

Proof. Let $\xi = (\xi_1, \cdots, \xi_p)$ be any point in Λ. From (4.1.5), for all $v \in 0^+\triangle \setminus \{0\}$, there exists $x \in \triangle$ such that

$$\left\langle \sum_{i=1}^{p} \xi_i M_i x + \sum_{i=1}^{p} \xi_i q_i,\ v \right\rangle > 0. \tag{4.1.6}$$

By separation theorem ([71], Theorem 11.7), we can prove

$$\sum_{i=1}^{p} \xi_i q_i \in int(0^+\triangle)^+ - (\sum_{i=1}^{p} \xi_i M_i)\triangle,$$

where $(0^+\triangle)^+$ is the positive polar cone of $0^+\triangle$.
Thus $\sum_{i=1}^{p} \xi_i q_i \in (0^+\triangle)^+ - (\sum_{i=1}^{p} \xi_i M_i)\triangle$ and hence there exists $x \in \triangle$ such that $\langle \sum_{i=1}^{p} \xi_i M_i x + \sum_{i=1}^{p} \xi_i q_i,\ v\rangle \geqq 0$ for all $v \in 0^+\triangle$. By Proposition 4.1.3, $sol(VI)_\xi \neq \emptyset$ for all $\xi \in \Lambda$. So, by Proposition 4.1.4, $sol(VVI) \neq \emptyset$ and $sol(VVI)^w \neq \emptyset$.

Now we will prove that $sol(VVI)^w$ is bounded. Assume to the contrary that $sol(VVI)^w$ is not bounded. Then there exists $x^k \in sol(VVI)^w$ such that $\| x^k \| \to +\infty$ as $k \to +\infty$. By Proposition 4.1.4, for each $k \in N$, there exists $\xi^k \in \Lambda$ such that

$$x^k \in sol(VI)_{\xi^k}. \tag{4.1.7}$$

We may assume that $\frac{x^k}{\|x^k\|} \to \bar{v}$, $\| \bar{v} \| = 1$, and $\xi^k \to \bar{\xi} \in \Lambda$. By (4.1.7), $\left\langle \sum_{i=1}^p \xi_i^k M_i x^k + \sum_{i=1}^p \xi_i^k q_i,\ x - x^k \right\rangle \geqq 0$ for all $x \in \triangle$. Hence we have

$$\left\langle \sum_{i=1}^p \xi_i^k M_i x^k + \sum_{i=1}^p \xi_i^k q_i,\ x \right\rangle \geqq \left\langle \sum_{i=1}^p \xi_i^k M_i x^k,\ x^k \right\rangle$$

$$+ \left\langle \sum_{i=1}^p \xi_i^k q_i,\ x^k \right\rangle \quad \text{for all } x \in \triangle. \tag{4.1.8}$$

Since $\sum_{i=1}^p \xi_i^k M_i$ is positive semidefinite, from (4.1.8),

$$\left\langle \sum_{i=1}^p \xi_i^k M_i x^k + \sum_{i=1}^p \xi_i^k q_i,\ x \right\rangle \geqq \left\langle \sum_{i=1}^p \xi_i^k q_i,\ x^k \right\rangle \quad \text{for all } x \in \triangle. \tag{4.1.9}$$

Dividing (4.1.9) by $\| x^k \|$ and letting $k \to \infty$, we have

$$\left\langle \sum_{i=1}^p \bar{\xi}_i M_i \bar{v},\ x \right\rangle \geqq \left\langle \sum_{i=1}^p \bar{\xi}_i q_i,\ \bar{v} \right\rangle \quad \text{for all } x \in \triangle. \tag{4.1.10}$$

Since $Ax^k \geqq b$, we have $A\bar{v} \geqq 0$, i.e., $\bar{v} \in 0^+\triangle$. Dividing (4.1.8) by $\| x^k \|^2$ and letting $k \to \infty$, we have

$$\left\langle \sum_{i=1}^p \bar{\xi}_i M_i \bar{v},\ \bar{v} \right\rangle \leqq 0.$$

Since $\sum_{i=1}^p \bar{\xi}_i M_i$ is positive semidefinite,

$$\left\langle \sum_{i=1}^p \bar{\xi}_i M_i \bar{v},\ \bar{v} \right\rangle = 0.$$

Thus, from Lemma 4.1.2, we have

$$(\sum_{i=1}^{p} \bar{\xi}_i M_i + \sum_{i=1}^{p} \bar{\xi}_i M_i^T)\bar{v} = 0. \tag{4.1.11}$$

From (4.1.9) and (4.1.10), we have

$$\left\langle \sum_{i=1}^{p} \bar{\xi}_i M_i x + \sum_{i=1}^{p} \bar{\xi}_i q_i,\ \bar{v} \right\rangle \leq 0 \quad \text{for all } x \in \triangle.$$

This contradicts (4.1.6). Hence $sol(VVI)^w$ is bounded. Since $sol(VVI)^w$ is closed, $sol(VVI)^w$ is compact. Since $sol(VVI) \subset sol(VVI)^w$, $sol(VVI)$ is bounded. By the similar argument with the proof of Theorem 4.1.1, we can prove that $sol(VVI)$ and $sol(VVI)^w$ are connected. □

Now we give examples clarifying and illustrating Theorems 4.1.1 and 4.1.2.

Example 4.1.1. Let $M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $q_1 = q_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $A = \begin{pmatrix} 1 & -1 \\ -\frac{1}{2} & 1 \end{pmatrix}$, $b = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, $p = n = 2$.

We consider (VVI) and $(VVI)^w$ for M_i, q_i, $(i = 1, 2)$, A and b. Since M_1 and M_2 are positive semidefinite and hence M_1 and M_2 are monotone on $\triangle$. For any $(v_1, v_2) \in 0^+\triangle \setminus \{(0,0)\}$, $\left\langle (v_1, v_2), M_i \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle = v_i^2 > 0$, $i = 1, 2$. Thus all the assumptions of Theorems 4.1.1 and 4.1.2 are satisfied. By Proposition 4.1.4, $sol(VVI) = \bigcup_{\xi \in \mathring{\Lambda}} sol(VI)_\xi$. Moreover $(\bar{x}_1, \bar{x}_2) \in sol_{\xi \in \mathring{\Lambda}}(VI)_\xi$ if and only if $(\bar{x}_1, \bar{x}_2)$ is an optimal solution of the following linear optimization problem.

$$(LP)_\xi \quad \begin{array}{ll} \text{Minimize} & \xi_1 x_1 \bar{x}_1 + \xi_2 x_2 \bar{x}_2 \\ \text{subject to} & x_1 - x_2 \geqq -1, -\frac{1}{2}x_1 + x_2 \geqq -1. \end{array}$$

$\Longleftrightarrow$ There exist $\mu_1 \geqq 0, \mu_2 \geqq 0$ such that

$$(1) \qquad \begin{cases} \begin{pmatrix} \xi_1 \bar{x}_1 \\ \xi_2 \bar{x}_2 \end{pmatrix} + \mu_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \mu_2 \begin{pmatrix} \frac{1}{2} \\ -1 \end{pmatrix} = 0 \\ \mu_1(-\bar{x}_1 + \bar{x}_2 - 1) = 0, \ \mu_2(\frac{1}{2}\bar{x}_1 - \bar{x}_2 - 1) = 0. \end{cases}$$

(i) in the case of $\mu_1 = 0$ and $\mu_2 = 0$:

$\xi_1 \bar{x}_1 = 0, \xi_2 \bar{x}_2 = 0$, where $(\bar{x}_1, \bar{x}_2) \in \Delta$. Then the solution of (1) is $\{(0, 0)\}$

(ii) in the case of $\mu_1 = 0$ and $\mu_2 > 0$:

$\xi_1 \bar{x}_1 + \frac{1}{2}\mu_2 = 0, \xi_2 \bar{x}_2 - \mu_2 = 0, \frac{1}{2}\bar{x}_1 - \bar{x}_2 - 1 = 0$, where $(\bar{x}_1, \bar{x}_2) \in \Delta$. $\left(\frac{2\xi_2}{4\xi_1+\xi_2}, \frac{-4\xi_1}{4\xi_1+\xi_2}\right)$ satisfies three equations of (1). But $\mu_2 = \xi_2 \bar{x}_2 = \frac{-4\xi_1\xi_2}{4\xi_1+\xi_2} < 0$, so there does not exist a solution of (1).

(iii) in the case of $\mu_1 > 0$ and $\mu_2 = 0$:

$\xi_1 \bar{x}_1 - \mu_1 = 0, \ \xi_2 \bar{x}_2 + \mu_1 = 0, \ -\bar{x}_1 + \bar{x}_2 - 1 = 0$, where $(\bar{x}_1, \bar{x}_2) \in \Delta$. Then $(-\xi_2, \xi_1)$ satisfies three equations of (1). But $\mu_1 = -\xi_2 \bar{x}_2 = -\xi_2 \xi_1 < 0$, so there does not exist a solution of (1).

(iv) in the case of $\mu_1 > 0$ and $\mu_2 > 0$:

$\xi_1 \bar{x}_2 - \mu_1 + \frac{1}{2}\mu_2 = 0, \ \xi_2 \bar{x}_2 + \mu_1 - \mu_2 = 0, \ -\bar{x}_1 + \bar{x}_2 - 1 = 0, \ \frac{1}{2}\bar{x}_1 - \bar{x}_2 - 1 = 0$, where $(\bar{x}_1, \bar{x}_2) \in \Delta$. Then $(-4, -3)$ satisfies the latter three parts. But in this case $\mu_1 = -8\xi_1 - 3\xi_2 < 0,\ \mu_2 = -8\xi_1 - 6\xi_2 < 0$. So there does not exist a solution of (1).

From (i) $\sim$ (iv), we conclude that the solution of (VVI) is $\{(0,0)\}$.

Next we consider $sol(VVI)^w$. By Proposition 4.1.4

$$sol(VVI)^w = \bigcup_{\xi \in \Lambda} sol(VI)_\xi.$$

So, it is sufficient to consider the cases of ($\xi_1 = 0$ and $\xi_2 = 1$) and ($\xi_1 = 1$ and $\xi_2 = 0$).

(v) in the case of $\xi_1 = 0$ and $\xi_2 = 1$:

$$(2) \qquad \begin{cases} -\mu_1 + \frac{1}{2}\mu_2 = 0,\ \bar{x}_2 + \mu_1 - \mu_2 = 0, \\ \mu_1(-\bar{x}_1 + \bar{x}_2 - 1) = 0,\ \mu_2(\frac{1}{2}\bar{x}_1 - \bar{x}_2 - 1) = 0. \end{cases}$$

• $\mu_1 = 0,\ \mu_2 = 0 :\ \bar{x}_2 = 0$, where $(\bar{x}_1, \bar{x}_2) \in \Delta$. Hence $-1 \leqq \bar{x}_1 \leqq 2$. So $sol(VVI)^w = \{(\bar{x}_1, 0) | -1 \leqq \bar{x}_1 \leqq 2\}$.

• ($\mu_1 = 0,\ \mu_2 > 0$) and ($\mu_1 > 0, \mu_2 = 0$): there does not exist a solution of (2).

• $\mu_1 > 0,\ \mu_2 > 0 :\ -\mu_1 + \frac{1}{2}\mu_2 = 0,\ \ \bar{x}_2 + \mu_2 - \mu_2 = 0,$

$-\bar{x}_1 + \bar{x}_2 - 1 = 0, \frac{1}{2}\bar{x}_1 - \bar{x}_2 - 1 = 0, (\bar{x}_1, \bar{x}_2) \in \Delta$. Then $(-4, -3)$ satisfies the latter three parts. But in this case $\mu_1 < 0, \mu_2 < 0$, there does not exist solution.

(vi) in the case of $\xi_1 = 1$ and $\xi_2 = 0$:

$$(3) \qquad \begin{cases} \bar{x}_1 - \mu_1 + \frac{1}{2}\mu_2 = 0,\ \mu_1 - \mu_2 = 0, \\ \mu_1(-\bar{x}_1 + \bar{x}_2 - 1) = 0,\ \mu_2(\frac{1}{2}\bar{x}_1 - \bar{x}_2 - 1) = 0 \end{cases}$$

From the second part of (3), we only consider $\mu_1 = \mu_2 = 0$ and $\mu_1 > 0$, $\mu_2 > 0$.

• $\mu_1 = 0,\ \mu_2 = 0$: $\bar{x}_1 = 0, (\bar{x}_1, \bar{x}_2) \in \Delta$. Hence $-1 \leqq \bar{x}_2 \leqq 1$ so $sol(VVI)^w = \{(0, \bar{x}_2) | -1 \leqq \bar{x}_2 \leqq 1\}$.

• $\mu_1 > 0,\ \mu_2 > 0$: Similarly to the case (v), we see that there does not exist solution of (3).

From (v),(vi), $sol(VVI)^w = \{(\bar{x}_1, 0) | -1 \leqq \bar{x}_1 \leqq 2\} \cup \{(0, \bar{x}_2) | -1 \leqq \bar{x}_2 \leqq 1\}$. Hence $sol(VVI)$ and $sol(VVI)^w$ are nonempty, bounded and connected.

Consequently, all the assumptions of Theorems 4.1.1 and 4.1.2 are satisfied. Actually, $sol(VVI) = \{(0,0)\}$, $sol(VVI)^w = \{(x_1, 0) \in \mathbb{R}^2 \mid -1 \leqq x_1 \leqq 2\} \cup \{(0, x_2) \in \mathbb{R}^2 \mid -1 \leqq x_2 \leqq 1\}$. □

Example 4.1.2. Let $M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $q_1 = q_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $p = n = 2$.

We consider (VVI) and $(VVI)^w$ for $M_i,\ q_i,\ (i = 1, 2),\ A$ and b. Clearly, M_i is monotone on $\Delta = \{(x_1, x_2) \in \mathbb{R}^2 \mid -x_1 \geqq 0,\ -x_2 \geqq 0\}$. But for all $v = (v_1, v_2) \in 0^+\Delta \setminus \{(0,0)\}$, $\left\langle \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, M_i \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle = v_i^2 \geqq 0,\ i = 1, 2.$

Since the assumptions of Theorem 4.1.1 do not hold, Theorem 4.1.1 cannot be applied to this example. But for any $v = (v_1, v_2) \in 0^+\Delta \setminus \{(0,0)\}$, there exists $x \in \Delta$ such that $\langle M_i x + q_i,\ v_i \rangle > 0$, i.e. all the assumptions of Theorem 4.1.2 are satisfied. By Proposition 4.1.4., $sol(VVI) = \bigcup_{\xi \in \mathring{\Lambda}} sol(VI)_\xi$.

Hence $(\bar{x}_1, \bar{x}_2) \in \bigcup_{\xi \in \overset{\circ}{\Lambda}} sol(VI)_\xi$ if and only if $(\bar{x}_1, \bar{x}_2)$ is an optimal solution of the following linear optimization problem.

$$
(LP)_\xi \qquad \begin{array}{ll} \text{Minimize} & \xi_1 x_1 \bar{x}_1 + \xi_2 x_2 \bar{x}_2 - x_1 - x_2 \\ \text{subject to} & -x_1 \geqq 0, -x_2 \geqq 0. \end{array}
$$

$\Longleftrightarrow$ there exists $\mu_1 \geqq 0,\ \mu_2 \geqq 0$ such that

$$
(4) \qquad \left\{ \begin{array}{l} \begin{pmatrix} \xi_1 \bar{x}_1 - 1 \\ \xi_2 \bar{x}_2 - 1 \end{pmatrix} + \mu_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0, \\ \mu_1 \bar{x}_1 = 0,\ \mu_2 \bar{x}_2 = 0. \end{array} \right.
$$

(i) in the case of $\mu_1 = 0$ and $\mu_2 = 0$:

$\xi_1 \bar{x}_1 - 1 = 0,\ \xi_2 \bar{x}_2 - 1 = 0, (\bar{x}_1, \bar{x}_2) \in \Delta$. In this case $(\bar{x}_1, \bar{x}_2) = (\frac{1}{\xi_1}, \frac{1}{\xi_2}) \notin \Delta$.

So there does not exist a solution of (4).

(ii) in the case of $\mu_1 = 0$ and $\mu_2 > 0$:

$\xi_1 \bar{x}_1 - 1 = 0, \xi_2 \bar{x}_2 - 1 + \mu_2 = 0, \bar{x}_2 = 0, \quad (\bar{x}_1, \bar{x}_2) \in \Delta,\ (\bar{x}_1, \bar{x}_2) = (\frac{1}{\xi_1}, 0) \notin \Delta$.

So there does not exist a solution of (4).

(iii) in the case of $\mu_1 > 0$ and $\mu_2 = 0$:

$\xi_1 \bar{x}_1 + \mu_1 = 0,\ \xi_2 \bar{x}_2 - 1 = 0, \bar{x}_1 = 0, \quad (\bar{x}_1, \bar{x}_2) \in \Delta,\ (\bar{x}_1, \bar{x}_2) = (0, \frac{1}{\xi_2}) \notin \Delta$.

So there does not exist a solution of (4).

(iv) in the case of $\mu_1 > 0$ and $\mu_2 > 0$:

$\xi_1 \bar{x}_1 - 1 + \mu_1 = 0, \xi_2 \bar{x}_2 - 1 + \mu_2 = 0, \bar{x}_1 = 0, \bar{x}_2 = 0,\ (\bar{x}_1, \bar{x}_2) \in \Delta$,

$(\bar{x}_1, \bar{x}_2) = (0, 0) \in \Delta$. Hence the solution of (4) is $\{(0, 0)\}$.

From (i)$\sim$(iv), $sol(VVI) = \{(0, 0)\}$.

Next we consider $sol(VVI)^w$. By Proposition 4.1.4

$$sol(VVI)^w = \bigcup_{\xi \in \Lambda} sol(VI)_\xi.$$

So, it is sufficient to consider the cases of ($\xi_1 = 0$ and $\xi_2 = 1$) and ($\xi_1 = 1$ and $\xi_2 = 0$).

(v) in the case of $\xi_1 = 0$ and $\xi_2 = 1$:

$$(5) \qquad \begin{cases} -1 + \mu_1 = 0, \ \bar{x}_2 - 1 + \mu_2 = 0, \\ \mu_1 \bar{x}_1 = 0, \ \mu_2 \bar{x}_2 = 0. \end{cases}$$

From the first part of (5), we only consider $\mu_1 > 0$,

• $\mu_1 > 0, \ \mu_2 = 0 : \ \bar{x}_1 = 0, \bar{x}_2 - 1 = 0, \ (\bar{x}_1, \bar{x}_2) \in \Delta, \ (0,1) \notin \Delta$. There does not exist a solution of (5).

• $\mu_1 > 0, \ \mu_2 > 0 : \ \mu_1 = 1, \ \bar{x}_2 - 1 + \mu_2 = 0, \bar{x}_1 = 0, \bar{x}_2 = 0, (\bar{x}_1, \bar{x}_2) \in \Delta$ $(0,0) \in \Delta$. Hence the solution of (5) is $\{(0,0)\}$.

(vi) in the case of $\xi_1 = 1$ and $\xi_2 = 0$:

$$(6) \qquad \begin{cases} \bar{x}_1 - 1 + \mu_1 = 0, \ -1 + \mu_2 = 0, \\ \mu_1 \bar{x}_1 = 0, \ \mu_2 \bar{x}_2 = 0. \end{cases}$$

From the second part of (6), we only consider $\mu_2 > 0$.

• $\mu_1 = 0, \ \mu_2 > 0 : \ \bar{x}_1 - 1 = 0, \mu_2 = 1, \bar{x}_2 = 0, (\bar{x}_1, \bar{x}_2) \in \Delta$. $(1,0) \notin \Delta$. There does not exist a solution of (6).

• $\mu_1 > 0, \ \mu_2 > 0$: $\bar{x}_1 - 1 + \mu_1 = 0, \mu_2 = 1, \bar{x}_1 = 0, \bar{x}_2 = 0, (\bar{x}_1, \bar{x}_2) \in \Delta$, $(0,0) \in \Delta$. Hence the solution of (6) is $\{(0,0)\}$.

From (v),(vi), $sol(VVI)^w = \{(0,0)\}$.

Consequently, Theorem 4.1.1 can not be applied to this example. But all the assumptions of Theorem 4.1.2 hold and hence Theorem 4.1.2 can be applied to this example. Actually, $sol(VVI) = sol(VVI)^w = \{(0,0)\}$. So, $sol(VVI)$ and $sol(VVI)^w$ are nonempty, bounded and connected. □

Example 4.1.3. Let $M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $q_1 = q_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $p = n = 2$. We consider (VVI) and $(VVI)^w$ for M_i, q_i, $(i = 1,2)$, A and b.

Clearly M_i is monotone on $\Delta = \{(x_1, x_2) | -x_1 \geqq 0, -x_2 \geqq 0\}$. But for all $v = (v_1, v_2) \in 0^+\Delta \setminus \{(0,0)\}$, $\left\langle \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, M_i \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle = v_i^2 \geqq 0$, $i = 1,2$.

Hence all the assumptions of Theorem 4.1.1 do not hold. So Theorem 4.1.1 cannot be applied to this example. And since $v = (0, v_2) \in 0^+\Delta \setminus \{(0,0)\}$ or $v = (v_1, 0) \in 0^+\Delta \setminus \{(0,0)\}$, there does not exist $x = (x_1, x_2)$ such that $\langle M_i x + q_i, v \rangle > 0$, that is, all the assumptions of Theorem 4.1.2 do not hold. Hence Theorem 4.1.2 cannot be applied to this example.

By Proposition 4.1.4, $sol(VVI) = \bigcup_{\xi \in \mathring{\Lambda}} sol(VI)_\xi$. Hence $(\bar{x}_1, \bar{x}_2) \in \bigcup_{\xi \in \mathring{\Lambda}} sol(VI)_\xi$.

$$\iff \quad (LP)_\xi \quad \text{Minimize} \quad \xi_1 x_1 \bar{x}_1 + \xi_2 x_2 \bar{x}_2 + x_1 + x_2$$
$$\text{subject to} \quad -x_1 \geqq 0, -x_2 \geqq 0.$$

$\iff$ There exists $\mu_1 \geqq 0, \mu_2 \geqq 0$ such that

$$\begin{pmatrix} \xi_1\bar{x}_1 + 1 \\ \xi_2\bar{x}_2 + 1 \end{pmatrix} + \mu_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0, \ \mu_1\bar{x}_1 = 0, \ \mu_2\bar{x}_2 = 0.$$

(i) in the case of $\mu_1 = 0$ and $\mu_2 = 0$:

$\xi_1\bar{x}_1 + 1 = 0, \ \xi_2\bar{x}_2 + 1 = 0, \ (\bar{x}_1, \bar{x}_2) \in \Delta$, then $(\bar{x}_1, \bar{x}_2) = (-\frac{1}{\xi_1}, -\frac{1}{\xi_2}) \in \Delta$,

$\xi_1 + \xi_2 = 1$. Hence $sol(x_1, x_2) \ni y = \frac{-x}{x+1}$.

(ii) in the case of $\mu_1 = 0$ and $\mu_2 > 0$:

$\xi_1\bar{x}_1 + 1 = 0, \ \xi_2\bar{x}_2 + 1 + \mu_2 = 0, \ \bar{x}_2 = 0, \ \ (\bar{x}_1, \bar{x}_2) \in \Delta$, which is impossible.

(iii) in the case of $\mu_1 > 0$ and $\mu_2 = 0$:

$\xi_1\bar{x}_1 + 1 + \mu_1 = 0, \ \xi_2\bar{x}_2 + 1 = 0, \ \bar{x}_1 = 0, \ \ (\bar{x}_1, \bar{x}_2) \in \Delta$, which is impossible.

(iv) in the case of $\mu_1 > 0$ and $\mu_2 > 0$:

$\xi_1\bar{x}_1 + 1 + \mu_1 = 0, \ \xi_2\bar{x}_2 + 1 + \mu_2 = 0, \ \bar{x}_1 = 0, \ \bar{x}_2 = 0, \ \ (\bar{x}_1, \bar{x}_2) \in \Delta$.

$(\bar{x}_1, \bar{x}_2) = (0, 0) \Rightarrow \mu_1 < 0, \ \mu_2 < 0$, which is impossible.

From (i) $\sim$ (iv), $sol(VVI) = \{(\bar{x}_1, \bar{x}_2) | \ \bar{x}_2 = \frac{-\bar{x}_1}{\bar{x}_1+1}\}$.

Next we consider $sol(VVI)^w$. By Proposition 4.1.4,

$$sol(VVI)^w = \bigcup_{\xi \in \Lambda} sol(VI)_\xi.$$

So, it is sufficient to consider the cases of ($\xi_1 = 0$ and $\xi_2 = 1$) and ($\xi_1 = 1$ and $\xi_2 = 0$).

(v) in the case of $\xi_1 = 0$ and $\xi_2 = 1$:

$1 + \mu_1 = 0, \ \bar{x}_2 + 1 + \mu_2 = 0, \ \mu_1\bar{x}_1 = 0, \ \mu_2\bar{x}_2 = 0$, which is impossible.

(vi) in the case of $\xi_1 = 1$ and $\xi_2 = 0$:

$\bar{x}_1+1+\mu_1 = 0, 1+\mu_2 = 0, \mu_1\bar{x}_1 = 0, \mu_2\bar{x}_2 = 0$. It is not occured since $\mu_2 > 0$.

Consequently, all the assumptions of Theorem 4.1.1 and 4.1.2 do not hold. Actually, $sol(VVI) = sol(VVI)^w = \{(\bar{x}_1, \bar{x}_2) |\ \bar{x}_2 = \frac{-\bar{x}_1}{\bar{x}_1+1}\}$. Thus $sol(VVI)$ and $sol(VVI)^w$ are not bounded and connected. □

Example 4.1.4. Let $M_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $q_1 = q_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

$$A = \begin{pmatrix} \frac{1}{2} & -1 \\ -\frac{1}{2} & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, p = n = 2.$$

We consider (VVI) and $(VVI)^w$ for M_i and q_i, $(i = 1, 2)$, A and b.

First we have $\Delta = \{(x_1, x_2) | \frac{1}{2}x_1 - x_2 \geqq 0, -\frac{1}{2}x_1 + x_2 \geqq 0, -x_1 \geqq 0, -x_2 \geqq 0\} = \{(x_1, x_2) | x_2 = \frac{1}{2}x_1, x_1 \leqq 0, x_2 \leqq 0\}$ for all $x = (x_1, \frac{1}{2}x_1), y = (y_1, \frac{1}{2}y_1)$ in Δ.

Since $(y-x)^T M_1 (y-x) = (y_1 - x_1, y_2 - x_2) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 - x_1 \\ y_2 - x_2 \end{pmatrix}$

$= (y_1 - x_1)^2 - (y_2 - x_2)^2 = (y_1 - x_1)^2 - \frac{1}{4}(y_1 - x_1)^2 \geqq 0$,

we have $(y-x)^T M_2 (y-x) = (y_2 - x_2)^2 \geqq 0$.

So M_i is monotone, $i = 1, 2$. For all $v = (v_1, v_2) \in 0^+\Delta \setminus \{(0,0)\}$,

$$\left\langle \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, M_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ -v_2 \end{pmatrix} \right\rangle = v_1^2 - v_2^2 = v_1^2 - \frac{1}{4}v_1^2 > 0,$$

$$\left\langle \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, M_2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \right\rangle = v_2^2 > 0.$$

Hence the assumptions of Theorem 4.1.1 are satisfied. Since M_1 is not positive semidefinite, Theorem 4.1.2 cannot be applied to this example. By Proposition 4.1.4,

$$(\bar{x}_1, \bar{x}_2) \in \bigcup_{\xi \in \mathring{\Lambda}} sol(VI)_\xi.$$

$\Longleftrightarrow \quad (LP)_\xi$

$$\begin{array}{ll} \text{Minimize} & \xi_1 x_1 \bar{x}_1 + (\xi_2 - \xi_1) x_2 \bar{x}_2 \\ \text{subject to} & \frac{1}{2}x_1 - x_2 \geqq 0, -\frac{1}{2}x_1 + x_2 \geqq 0, -x_1 \geqq 0, -x_2 \geqq 0. \end{array}$$

$\Longleftrightarrow$ there exists $\mu_1 \geqq 0, \mu_2 \geqq 0, \mu_3 \geqq 0, \mu_4 \geqq 0$ such that

$$\begin{pmatrix} \xi_1 \bar{x}_1 \\ (\xi_2 - \xi_1)\bar{x}_2 \end{pmatrix} + \mu_1 \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} + \mu_2 \begin{pmatrix} \frac{1}{2} \\ -1 \end{pmatrix} + \mu_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mu_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$\mu_1(-\frac{1}{2}\bar{x}_1 + \bar{x}_2) = 0, \;\; \mu_2(\frac{1}{2}\bar{x}_1 - \bar{x}_2) = 0, \;\; \mu_3 \bar{x}_1 = 0, \;\; \mu_4 \bar{x}_2 = 0.$$

(i) in the case of $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0$:

$\xi_1 \bar{x}_1 = 0, (\xi_2 - \xi_1)\bar{x}_2 = 0, (\bar{x}_1, \bar{x}_2) \in \Delta$, $(\bar{x}_1, \bar{x}_2) = (0, 0) \in \Delta$. Then the solution is $\{(0,\ 0)\}$

(ii) in the case of $\mu_1 = \mu_2 = \mu_3 = 0$ and $\mu_4 > 0$:

$\xi_1 \bar{x}_1 = 0, (\xi_2 - \xi_1)\bar{x}_2 + \mu_4 = 0,\ \mu_4 \bar{x}_2 = 0, (\bar{x}_1, \bar{x}_2) \in \Delta$, which is impossible.

(iii) in the case of $\mu_1 = \mu_2 = \mu_4 = 0$ and $\mu_3 > 0$. This is impossible

(iv) in the case of $\mu_1 = \mu_3 = \mu_4 = 0$ and $\mu_2 > 0$:

$\xi_1 \bar{x}_1 + \frac{1}{2}\mu_2 = 0, (\xi_2 - \xi_1)\bar{x}_2 - \mu_2 = 0,\ \mu_2(\frac{1}{2}\bar{x}_1 - \bar{x}_2) = 0,\ (\bar{x}_1, \bar{x}_2) \in \Delta, \bar{x}_2 = \frac{1}{2}\bar{x}_1, \bar{x}_1 = \frac{3\mu_2}{2\xi_1} > 0$. This is impossible.

(v) in the case of $\mu_1 > 0$ and $\mu_2 = \mu_3 = \mu_4 = 0$: $\xi_1\bar{x}_1 - \frac{1}{2}\mu_1 = 0, (\xi_2 - \xi_1)\bar{x}_2 + \mu_1 = 0,\ \mu_1(-\frac{1}{2}\bar{x}_1 + \bar{x}_2) = 0$. This is impossible.

(vi) in the case of $\mu_1 = \mu_2 = 0,\ \mu_3 > 0$ and $\mu_4 > 0$:

$\xi_1\bar{x}_1 + \mu_3 = 0, (\xi_2 - \xi_1)\bar{x}_2 + \mu_4 = 0,\ \mu_3\bar{x}_1 = 0, \mu_4\bar{x}_2 = 0, (\bar{x}_1, \bar{x}_2) \in \Delta$. This is impossible.

(vii) in the case of $\mu_1 = \mu_4 = 0,\ \mu_2 > 0$ and $\mu_3 > 0$:

$\xi_1\bar{x}_1 + \frac{1}{2}\mu_2 + \mu_3 = 0, (\xi_2 - \xi_1)\bar{x}_2 - \mu_2 = 0,\ \mu_2(\frac{1}{2}\bar{x}_1 - \bar{x}_2) = 0, \mu_3\bar{x}_1 = 0, (\bar{x}_1, \bar{x}_2) \in \Delta$. This is impossible.

(viii) in the case of $\mu_1 = \mu_3 = 0,\ \mu_2 > 0$ and $\mu_4 > 0$:

$\xi_1\bar{x}_1 + \frac{1}{2}\mu_2 = 0, (\xi_2 - \xi_1)\bar{x}_2 - \mu_2 + \mu_4 = 0,\ \mu_2(\frac{1}{2}\bar{x}_1 - \bar{x}_2) = 0, \mu_4\bar{x}_2 = 0, (\bar{x}_1, \bar{x}_2) \in \Delta$. This is impossible.

(ix) in the case of $\mu_2 = \mu_3 = 0,\ \mu_1 > 0$ and $\mu_4 > 0$:

$\xi_1\bar{x}_1 - \frac{1}{2}\mu_1 = 0, (\xi_2 - \xi_1)\bar{x}_2 + \mu_1 + \mu_4 = 0,\ \mu_1(-\frac{1}{2}\bar{x}_1 + \bar{x}_2) = 0, \mu_4\bar{x}_2 = 0, (\bar{x}_1, \bar{x}_2) \in \Delta$. This is impossible.

(x) in the case of $\mu_2 = \mu_4 = 0,\ \mu_1 > 0$ and $\mu_3 > 0$:

$\xi_1\bar{x}_1 - \frac{1}{2}\mu_1 + \mu_3 = 0, (\xi_2 - \xi_1)\bar{x}_2 + \mu_1 = 0,\ \mu_1(-\frac{1}{2}\bar{x}_1 + \bar{x}_2) = 0, \mu_3\bar{x}_1 = 0, (\bar{x}_1, \bar{x}_2) \in \Delta$. This is impossible.

(xi) in the case of $\mu_3 = \mu_4 = 0,\ \mu_1 > 0$ and $\mu_2 > 0$:

$\xi_1\bar{x}_1 - \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2 = 0, (\xi_2 - \xi_1)\bar{x}_2 + \mu_1 - \mu_2 = 0,\ \mu_1(-\frac{1}{2}\bar{x}_1 + \bar{x}_2) = 0,\ \mu_2(\frac{1}{2}\bar{x}_1 - \bar{x}_2) = 0,\ (\bar{x}_1, \bar{x}_2) \in \Delta,\ \bar{x}_1 = 0, \bar{x}_2 = 0,\ (0, 0) \in \Delta$. In this case, the solution is $\{(0, 0)\}$.

(xii) in the case of $\mu_1 = 0,\ \mu_2 > 0,\ \mu_3 > 0$ and $\mu_4 > 0$:

$\xi_1\bar{x}_1 + \frac{1}{2}\mu_2 + \mu_3 = 0, (\xi_2 - \xi_1)\bar{x}_2 - \mu_2 + \mu_4 = 0,\ \mu_2(\frac{1}{2}\bar{x}_1 - \bar{x}_2) = 0,\ \mu_3\bar{x}_1 = 0, \mu_4\bar{x}_2 = 0,\ (\bar{x}_1, \bar{x}_2) \in \Delta$. This is impossible.

(xiii) in the case of $\mu_2 = 0,\ \mu_1 > 0,\ \mu_3 > 0$ and $\mu_4 > 0$:

$\xi_1\bar{x}_1 - \frac{1}{2}\mu_1 + \mu_3 = 0, (\xi_2 - \xi_1)\bar{x}_2 + \mu_1 + \mu_4 = 0,\ \mu_1(-\frac{1}{2}\bar{x}_1 + \bar{x}_2) = 0,\ \mu_3\bar{x}_1 = 0, \mu_4\bar{x}_2 = 0,\ (\bar{x}_1, \bar{x}_2) \in \Delta$. This is impossible.

(xiv) in the case of $\mu_3 = 0,\ \mu_1 > 0,\ \mu_2 > 0$ and $\mu_4 > 0$:

$\xi_1\bar{x}_1 - \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2 = 0, (\xi_2 - \xi_1)\bar{x}_2 + \mu_1 - \mu_2 + \mu_4 = 0,\ \mu_1(-\frac{1}{2}\bar{x}_1 + \bar{x}_2) = 0,\ \mu_2(\frac{1}{2}\bar{x}_1 - \bar{x}_2) = 0, \mu_4\bar{x}_2 = 0,\ (\bar{x}_1, \bar{x}_2) \in \Delta$. In this case, the solution is $\{(0,0)\}$.

(xv) in the case of $\mu_4 = 0,\ \mu_1 > 0,\ \mu_2 > 0$ and $\mu_3 > 0$:

$\xi_1\bar{x}_1 - \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2 + \mu_3 = 0, (\xi_2 - \xi_1)\bar{x}_2 + \mu_1 - \mu_2 = 0,\ \mu_1(-\frac{1}{2}\bar{x}_1 + \bar{x}_2) = 0,\ \mu_2(\frac{1}{2}\bar{x}_1 - \bar{x}_2) = 0, \mu_3\bar{x}_1 = 0,\ (\bar{x}_1, \bar{x}_2) \in \Delta$. In this case, the solution is $\{(0,0)\}$.

(xvi) in the case of $\mu_1 > 0,\ \mu_2 > 0,\ \mu_3 > 0$ and $\mu_4 > 0$: In this case, the solution is $\{(0,0)\}$.

From (i) $\sim$ (xvi), $sol(VVI)$=$\{(0,0)\}$.

Next we consider $sol(VVI)^w$. By Proposition 4.1.4. $sol(VVI)^w = \bigcup_{\xi\in\Lambda} sol(VI)_\xi$.

Enough to solve the cases ($\xi_1 = 0$ and $\xi_2 = 1$) and ($\xi_1 = 1$ and $\xi_2 = 0$).

(xvii) in the case of $\xi_1 = 0$ and $\xi_2 = 1$:

$-\frac{1}{2}\mu_1 + \frac{1}{2}\mu_2 + \mu_3 = 0,\ \bar{x}_2 + \mu_1 - \mu_2 + \mu_4 = 0,\ \mu_1(-\frac{1}{2}\bar{x}_1 + \bar{x}_2) = 0,\ \mu_2(\frac{1}{2}\bar{x}_1 - \bar{x}_2) = 0, \mu_3\bar{x}_1 = 0,\ \mu_4\bar{x}_2 = 0,\ (\bar{x}_1, \bar{x}_2) \in \Delta$. In this case, the solution is $\{(0,0)\}$.

(xviii) in the case of $\xi_1 = 1$ and $\xi_2 = 0$:

$\bar{x}_1 - \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2 + \mu_3 = 0$, $-\bar{x}_2 + \mu_1 - \mu_2 + \mu_4 = 0$, $\mu_1(-\frac{1}{2}\bar{x}_1 + \bar{x}_2) = 0$, $\mu_2(\frac{1}{2}\bar{x}_1 - \bar{x}_2) = 0, \mu_3\bar{x}_1 = 0$, $\mu_4\bar{x}_2 = 0$, $(\bar{x}_1, \bar{x}_2) \in \Delta$. In this case, the solution is $\{(0,0)\}$. Hence $sol(VVI) = sol(VVI)^w = \{(0,0)\}$.

Consequently, all the assumptions of Theorem 4.1.1 hold, but, since M_1 is not positive semidefinite, Theorem 4.1.2 can not be applied to this example. Actually, $sol(VVI) = sol(VVI)^w = \{(0,0)\}$. So, $sol(VVI)$ and $sol(VVI)^w$ are nonempty, bounded and connected. □

4.2. Applications to Multiobjective Optimization Problems

Now we apply Theorem 4.1.2 to a multiobjective linear fractional optimization problem and a multiobjective convex linear-quadratic optimization problem. Let $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \cdots, p$ be linear fractional functions, that is, $f_i(x) = \frac{a_i^T x + \alpha_i}{b_i^T x + \beta_i}$ for some $a_i = (a_{i_1}, \cdots, a_{i_n})^T \in \mathbb{R}^n$, $b_i = (b_{i_1}, \cdots, b_{i_n})^T \in \mathbb{R}^n$, $\alpha_i \in \mathbb{R}$ and $\beta_i \in \mathbb{R}$. Let $\triangle = \{x \in \mathbb{R}^n \mid Ax \geq b\}$, where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Assume that, for every $i \in \{1, \cdots, m\}$ and every $x \in \triangle$, $b_i^T x + \beta_i > 0$. Define $f(x) = (f_1(x), \cdots, f_p(x))$.

Consider the following multiobjective linear fractional optimization problem:

$$\begin{aligned} (FP) \qquad & \text{Minimize} \qquad f(x) \\ & \text{subject to} \quad x \in \triangle. \end{aligned}$$

Denote by $E(FP)$ the set of all the efficient points (Pareto solutions) of (FP). By definition, $x \in E(FP)$ if and only if there does not exist $y \in \triangle$ satisfying $f(y) - f(x) \in -\mathbb{R}^p_+ \setminus \{0\}$. Denote by $E^w(FP)$ the set of all the weakly efficient points of (FP). By definition, $x \in E^w(FP)$ if and only if there does not exist $y \in \triangle$ satisfying $f(y) - f(x) \in -int\mathbb{R}^p_+$.

Let $M_i = \begin{pmatrix} a_{i_1} & a_{i_1} & \cdots & a_{i_1} \\ a_{i_2} & a_{i_2} & \cdots & a_{i_2} \\ \cdot & \cdot & \cdots & \cdot \\ a_{i_n} & a_{i_n} & \cdots & a_{i_n} \end{pmatrix} \cdot \begin{pmatrix} b_{i_1} & 0 & \cdots & 0 \\ 0 & b_{i_2} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & b_{i_n} \end{pmatrix}$

$$-\begin{pmatrix} b_{i_1} & b_{i_1} & \cdots & b_{i_1} \\ b_{i_2} & b_{i_2} & \cdots & b_{i_2} \\ \cdot & \cdot & \cdots & \cdot \\ b_{i_n} & b_{i_n} & \cdots & b_{i_n} \end{pmatrix} \cdot \begin{pmatrix} a_{i_1} & 0 & \cdots & 0 \\ 0 & a_{i_2} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & a_{i_n} \end{pmatrix} \text{ and}$$

$$q_i = \beta_i \begin{pmatrix} a_{i_1} \\ a_{i_2} \\ \vdots \\ a_{i_n} \end{pmatrix} - \alpha_i \begin{pmatrix} b_{i_1} \\ b_{i_2} \\ \vdots \\ b_{i_n} \end{pmatrix}, \quad i = 1, \cdots, p.$$

Then we can easily check that $M_i^T = -M_i$ and that $\langle x,\ M_i x\rangle = 0$ for any $x \in \mathbb{R}^n$. Thus for each $i \in \{1, \cdots, p\}$, M_i is positive semidefinite.

Consider problems $(VI)_\xi$, (VVI) and $(VVI)^w$ by using the above M_i's and q_i's. By the results in [64], $E(FP) = \bigcup_{\xi\in\overset{\circ}{\Lambda}} sol(VI)_\xi$ and $E^w(FP) = \bigcup_{\xi\in\Lambda} sol(VI)_\xi$. Thus we have, by Proposition 4.1.4, $E(FP) = sol(VVI)$ and $E^w(FP) = sol(VVI)^w$.

Hence we have the following theorem from Theorem 4.1.2:

Theorem 4.2.1. If for all $v \in 0^+\triangle \setminus \{0\}$, there exists $x \in \triangle$ such that $\langle M_i x + q_i,\ v\rangle > 0$ for all $i \in \{1, \cdots, p\}$, then $E(FP)$ and $E^w(FP)$ are nonempty, bounded and connected. In particular, $E^w(FP)$ is compact.

Let $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \cdots, p$ be linear-quadratic functions, that is, $f_i(x) = \frac{1}{2}x^T M_i x + q_i^T x + \alpha_i$ where $M_i \in \mathbb{R}^{n\times n}$, $q_i \in \mathbb{R}^n$, and $\alpha_i \in \mathbb{R}$ for $i = 1, \cdots, p$. If M_i is symmetric and positive semidefinite, then f_i is convex. Assume that M_i, $i = 1, \cdots, p$ are symmetric and positive semidefinite. Let $\triangle = \{x \in \mathbb{R}^n \mid Ax \geq b\}$, where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Define $f(x) = (f_1(x), \cdots, f_p(x))$.

Consider the following multiobjective convex linear-quadratic programming problem:

$$
\begin{array}{lll}
(QP) & \text{Minimize} & f(x) \\
 & \text{subject to} & x \in \triangle.
\end{array}
$$

By the same manners as in (FP), we define the efficient point of (QP) and the weakly efficient point of (QP). We denote by $E(QP)$ the set of all the efficient points of (QP) and by $E^w(QP)$ the set of all the weakly efficient points of (QP). Also, $x \in \triangle$ is called a properly efficient point of (QP) if $x \in E(QP)$ and there exists $M > 0$ such that for each $i \in \{1, \cdots, p\}$, we have

$$\frac{f_i(x) - f_i(y)}{f_j(y) - f_j(x)} \leq M$$

for some j such that $f_j(y) > f_j(x)$ whenever $y \in \triangle$ and $f_i(y) < f_i(x)$. We denote by $E^{Pr}(QP)$ the set of all the properly efficient points of (QP).

The quantity $\frac{f_i(x)-f_i(y)}{f_j(y)-f_j(x)}$ may be interpreted as the marginal trade-off for the objective functions f_i and f_j between x and y. Geoffrion ([32]) considered the concept of the proper efficiency to eliminate the unbounded trade-off between the objective functions of (QP).

Consider problems $(VI)_\xi$, (VVI) and $(VVI)^w$ by using the above M_i's and q_i's. Then by Theorem 2 in ([53]), $E^{Pr}(QP) = sol(VVI)$, and by Proposition 5 in ([35]), $E^w(QP) = sol(VVI)^w$. Hence we have the following theorem from Theorem 4.1.2:

Theorem 4.2.2 If for all $v \in 0^+\triangle \setminus \{0\}$, there exists $x \in \triangle$ such that

$$\langle M_i x + q_i,\ v\rangle > 0 \quad \text{for all} \quad i \in \{1, \cdots, p\},$$

then $E^{Pr}(QP)$ and $E^w(QP)$ are nonempty, bounded and connected. In particular, $E^w(QP)$ is compact.

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