ON δ-GENERALIZED LOCALLY CLOSED SETS AND δGLC-CONTINUOUS FUNCTIONS

δ-일반화된 국소 폐집합과 δGLC-연속함수에 대하여

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A Dissertation

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요 약

위상공간에서의 분리공리, 컴팩트성, 연결성 및 함수의 연속성은 일반위상수학 뿐만 아니라 수학의 여러분야에서의 기본적이고 중요한 주제들로서 많은 수학자들에 의해서 현재까지 연구되어져 오고 있다. 최근에는 폐집합보다 일반화된 집합을 찾고 그 집합들의 특성을 조사함으로서 폐집합의 성질, 나아가 위 주제들의 연구에 보다 활발한 발전이 이루어지고 있다. Kuratowski와 Sierpinski는 폐집합보다 약한 국소 폐집합을 소개하고 분리공리와 컴팩트성에서 폐집합의 결과를 개선하였다.

Ganster와 Reilly에 의해서 이들 집합이 보다 연구되어 졌으며, Balachandran, Nasef 및 Park 등에 의해서 이들 집합들의 일반화와 이들 집합을 이용한 다양한 연속함수에 대한 연구가 진행되었다.

본 논문에서는 Velicko가 소개한 δ-개집합과 δ-폐집합 그리고 Dontchev와 Ganster가 소개한 δg-폐집합과 δg-개집합등을 이용하여 Balachandran등이 소개한 glc-집합보다 강아며, Park이 소개한 $l\delta c$-집합보다 약한 δglc^*-집합과 δglc^{**}-집합을 소개하고 기본적인 성질을 조사하였다. 또한, 이 집합들을 이용하여 δGLC-연속함수, δGLC^*-연속함수 및 δGLC^{**}-연속함수 등을 소개하여 composition, restriction 및 combination과 같은 기본적인 성질을 밝히고, Balachandran 과 Park 등이 소개한 일반화된 여러 가지 LC-연속함수들과의 관계를 조사하였다.

1. INTRODUCTION

Continuity, compactness, connectedness and separation axioms on topological spaces, as important and basic subjects in studies of General Topology and several branches of mathematics, have been researched by many mathematicians. Recent progress in the study of characterizations and generalizations of these subjects has been done by means of several "generalized closed sets".

In 1968, Veličko [18] introduced δ-open sets, which are stronger than open sets, in order to investigate the characterization of H-closed spaces and showed that τ_δ(= the collection of all δ-open sets) is the topology in X such that $\tau_\delta \subset \tau$ and, in particular, τ_δ equal to the semi-regularization topology τ_s.

The approach of the concept of semi-regularization from a different perspective via the concept of generalized closedness was done by Dontchev and Ganster [5]. They consider sets whose closure in the semi-regularization are contained in every superset which is open in the original topology. They called these sets δ-generalized closed (for short, δg-closed) and studied their basic properties.

The notion of a locally closed set in a topological space was implicitly introduced by Kuratowski and Sierpienski [11]. According to Bourbaki [2] a subset of a topological space X is locally closed in X if it is the intersection of an open set and a closed set in X. Ganster and Reilly [9] continued the study of locally closed sets. Intensive research on the field of locally closed sets was done in the past ten years as theory developed by Balanchandran et al. [1], Ganster and Reilly [7], Ganster et al. [9], Nasef [13] and Park and Park [15].

The purpose of this paper is to introduce three new classes of sets called δglc-set, δglc^*-set and δglc^{**}-set, which contained in the class of glc-sets and contain the class of $l\delta c$-sets by using the notion of δ-open and δ-closed sets due to Veličko [18] or δg-open and δg-closed sets due to Dontchev and Ganster [5]. Also, we introduce some different classes of continuity and irresoluteness and study some of their properties.

2. PRELIMINARIES

All topological spaces considered in this paper lack any separation axioms unless explicitly stated. The topology of a space is denoted by τ and (X,τ) will be replaced by X if there is no chance for confusion. For a subset $A \subset X$, the closure and the interior of A in X are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. The δ-interior [18] of a subset A of X is the union of all regular open sets of X contained in A and is denoted by δ-$\mathrm{Int}(A)$. A subset A is called δ-open [18] if $A = \delta$-$\mathrm{Int}(A)$, i.e. a set is δ-open if it is the union of regular open sets. The complement of a δ-open set is called δ-closed. Alternatively, a set $A \subset (X,\tau)$ is called δ-closed [18] if $A = \delta$-$\mathrm{Cl}(A)$, where $\delta\text{-}\mathrm{Cl}(A) = \{x \in X : \mathrm{Int}(\mathrm{Cl}(U)) \cap A \neq \phi$, for every open set U containing $x\}$. The family of all δ-open sets form a topology on X and is denoted τ_δ.

Since the intersection of two regular open sets is regular open, the collection of all regular open sets forms a base for a coarser topology τ_s than the original one τ. The family τ_s is called semi-regularization of τ. A topological space (X,τ) is called semi-regular if $\tau = \tau_s$.

Proposition 2.1 [18]. *For subsets A and B of X, the following are valid:*

(a) *$Cl(A) \subset \delta$-$Cl(A)$.*

(b) *If $A \subset B$, then δ-$Cl(A) \subset \delta$-$Cl(B)$.*

(c) *A is δ-closed in X if and only if $A = \delta$-$Cl(A)$.*

(d) *δ-$Cl(A)$ are closed on the space X.*

(e) *If a set A is open in X, then δ-$Cl(A) = Cl(A)$ and $Cl(A)$ is δ-closed in X, i.e. δ-$Cl(Cl(A)) = Cl(A)$.*

(f) *$x \in \delta$-$Cl(A)$ if and only if $A \cap U \neq \phi$ for every δ-open set U containing x.*

Proposition 2.2 [18]. (a) *Any intersection of δ-closed sets is δ-closed.*

(b) *The union of a finite number of δ-closed sets is δ-closed.*

Proposition 2.3 [18]. *Let A be a subset of X. Then,*

(a) *For each δ-open G in X, δ-Cl$(A) \cap G \subset \delta$-Cl$(A \cap G)$.*

(b) *For each δ-closed F in X, δ-Int$(A \cup F) \subset \delta$-Int$(A) \cup F$.*

Definition 2.4. A subset A of a space X is called:

(a) *g-closed* [18] if $\mathrm{Cl}(A) \subset G$ whenever $A \subset G$ and G is open in X;

(b) *δg-closed* [1] if δ-$\mathrm{Cl}(A) \subset G$ whenever $A \subset G$ and G is open in X;

(c) *δg^*-closed* [4] if $\mathrm{Cl}(A) \subset G$ whenever $A \subset G$ and G is δ-open in X;

(d) *g-open* (resp. *δg-open, δg^*-open*) if the complement of A is g-closed (resp. δg-closed, δg^*-closed).

Proposition 2.5 [5]. *Let (X, τ) be a space. Then the following are true:*

(a) *Every δ-closed set is δg-closed.*

(b) *Every δg-closed set in (X, τ) is g-closed in (X, τ_s).*

Proposition 2.6 [5]. (a) *Finite union of δg-closed sets is always a δg-closed set.*

(b) *Countable union of δg-closed sets need not be a δg-closed set.*

(c) *Finite intersection of δg-closed sets may fail to be a δg-closed set.*

Proposition 2.7. *The intersection of a δg-closed set and a δ-closed set is always δg-closed.*

Proof. Let A be δg-closed and let F be δ-closed. If G is an open set with $A \cap F \subset G$, then $A \subset G \cup (X \setminus F)$ and so δ-$\mathrm{Cl}(A) \subset U \cup (X \setminus F)$. Now δ-$\mathrm{Cl}(A \cap F) \subset \delta$-$\mathrm{Cl}(A) \cap F \subset G$ and so $A \cap F$ is δg-closed. $\square$

Definition 2.8. A subset A of (X,τ) is called:

(a) locally closed set [9] (briefly, lc-set) if $A = G \cap F$ where $G \in \tau$ and F is closed in (X,τ).

(b) locally δ-closed set [14] (briefly, $l\delta c$-set) if $A = G \cap F$ where G is δ-open and F is δ-closed in (X,τ).

(c) generalized locally closed set [1] (briefly, glc-set) if $A = G \cap F$ where G is g-open in (X,τ) and F is g-closed in (X,τ).

(d) glc^*-set [9] if there exist a g-open set G and a closed set F of (X,τ) such that $A = G \cap F$.

(e) glc^{**}-set [9] if there exist an open set G and a g-closed set F of (X,τ) such that $A = G \cap F$.

From the results of Balanchandran et al. [1], Ganster and Reilly [9] and Park [14], we have the following the diagram for some kinds of locally closed sets:

$$\begin{array}{ccccccc} & & & & glc^*\text{-set} & & \\ & & & \nearrow & & \searrow & \\ l\delta c\text{-set} & \rightarrow & lc\text{-set} & & & & glc\text{-set} \\ & & & \searrow & & \nearrow & \\ & & & & glc^{**}\text{-set} & & \end{array}$$

Proposition 2.9. *Let A and B be subset of (X,τ).*

(a) *If $A \in GLC^*(X,\tau)$ and $B \in GLC^*(X,\tau)$, then $A \cap B \in GLC^*(X,\tau)$.*

(b) *If $A \in GLC^{**}(X,\tau)$ and B is closed or open, then $A \cap B \in GLC^{**}(X,\tau)$.*

(c) *If $A \in GLC(X,\tau)$ and B is g-open or closed,. then $A \cap B \in GLC(X,\tau)$.*

The collection of all locally closed sets (resp. glc^*-sets, glc^{**}-sets) of (X,τ) will be denoted by $LC(X,\tau)$ (resp. $GLC^*(X,\tau)$, $GLC^{**}(X,\tau)$) (cf. p.19 in Bourbaki [2]).

Definition 2.10. A function $f:(X,\tau)\rightarrow(Y,\sigma)$ is called:

(a) $L\delta C$-continuous if $f^{-1}(V)\in L\delta C(X,\tau)$ for each $V\in\sigma$.

(b) GLC^*-continuous if $f^{-1}(V)\in GLC^*(X,\tau)$ for each $V\in\sigma$.

(c) GLC^{**}-continuous if $f^{-1}(V)\in GLC^{**}(X,\tau)$ for each $V\in\sigma$.

(d) GLC-continuous if $f^{-1}(V)\in GLC(X,\tau)$ for each $V\in\sigma$.

Definition 2.11. A function $f:(X,\tau)\rightarrow(Y,\sigma)$ is called:

(a) sub-$L\delta C$-continuous [14] if there is a basis $\mathcal{B}$ for (Y,σ) such that $f^{-1}(U)\in L\delta C(X,\tau)$ for each $U\in\mathcal{B}$.

(b) sub-GLC^*-continuous [1] if there is a basis $\mathcal{B}$ for (Y,σ) such that $f^{-1}(U)\in GLC^*(X,\tau)$ for each $U\in\mathcal{B}$.

From the results of Balanchandran et al.[1] and Ganster and Reilly[8], we have the following the diagram for kinds of LC-continuous functions.

$$\begin{array}{ccccc}
\text{Continuity} & \rightarrow & LC\text{-continuity} & \leftarrow & L\delta C\text{-continuity} \\
\downarrow & & & & \downarrow \\
GLC^*\text{-continuity} & \rightarrow & GLC\text{-continuity} & \leftarrow & GLC^{**}\text{-continuity}
\end{array}$$

3. δ-GENERALIZED LOCALLY CLOSED SETS

Definition 3.1. A subset A of (X, τ) is called δ-generalized locally closed set (briefly, δglc-set) if $A = G \cap F$ where G is δg-open in (X, τ) and F is δg-closed in (X, τ).

Definition 3.2. Let A be a subset of (X, τ). Then, A is called:

(a) δglc^*-set if there exist a δg-open set G and a δ-closed set F of (X, τ) such that $A = G \cap F$.

(b) δglc^{**}-set if there exist a δ-open set G and a δg-closed set F of (X, τ) such that $A = G \cap F$.

Every δg-closed set (resp. δg-open set) is δglc-set. The collection of all δglc-sets (resp. δglc^*-sets, δglc^{**}-sets) of (X, τ) will be denoted by $\delta GLC(X, \tau)$ (resp. $\delta GLC^*(X, \tau)$, $\delta GLC^{**}(X, \tau)$).

Proposition 3.3. *Let A be a subset of (X, τ).*

(a) *If $A \in L\delta C(X, \tau)$, then $A \in \delta GLC^*(X, \tau)$ and $A \in \delta GLC^{**}(X, \tau)$.*

(b) *If $A \in \delta GLC^*(X, \tau)$ or $A \in \delta GLC^{**}(X, \tau)$, then $A \in \delta GLC(X, \tau)$.*

(c) *If $A \in \delta GLC^*(X, \tau)$, then $A \in GLC^*(X, \tau)$.*

(d) *If $A \in \delta GLC^{**}(X, \tau)$, then $A \in GLC^{**}(X, \tau)$.*

(e) *If $A \in \delta GLC(X, \tau)$, then $A \in GLC(X, \tau)$.*

Proof. It follows from definitions.

From Proposition 3.3, we have the following diagram:

$$\begin{array}{ccccccc}
 & & \delta glc^*\text{-set} & \rightarrow & glc^*\text{-set} & & \\
 & \nearrow & & \searrow & & \searrow & \\
l\delta c\text{-set} & & & & \delta glc\text{-set} & \rightarrow & glc\text{-set} \\
 & \searrow & & \nearrow & & \nearrow & \\
 & & \delta glc^{**}\text{-set} & \rightarrow & glc^{**}\text{-set} & &
\end{array}$$

The converses of Proposition 3.3 need not be true as from the following examples.

Example 3.4. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$. Then the collection of δ-open sets is $\{X, \phi, \{a\}, \{b, c\}\}$, the collection of δ-closed sets is $\{X, \phi, \{a\}, \{b, c\}\}$, the collection of g-open sets is $\{X, \phi, \{b, c\}, \{a, c\}, \{c\}, \{a\}\}$, the collection of g-closed sets is $\{X, \phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$, the collection of δg-open sets is $\{X, \phi, \{a\}, \{c\}, \{b, c\}\}$ and the collection of δg-closed sets is $\{X, \phi, \{a\}, \{b, c\}, \{a, b\}\}$.

(a) $\{c\} \in \delta GLC^*(X, \tau)$, but $\{c\} \notin L\delta C(X, \tau)$ and $\{a, b\} \in \delta GLC^{**}(X, \tau)$, but $\{a, b\} \notin L\delta C(X, \tau)$.

(b) $\{a, b\} \in \delta GLC(X, \tau)$, but $\{a, b\} \notin \delta GLC^*(X, \tau)$ and $\{c\} \in \delta GLC(X, \tau)$, but $\{c\} \notin \delta GLC^{**}(X, \tau)$.

(c) $\{a, c\} \in GLC^*(X, \tau)$, but $\{a, c\} \notin \delta GLC^*(X, \tau)$.

(d) $\{c\} \in GLC^{**}(X, \tau)$, but $\{c\} \notin \delta GLC^{**}(X, \tau)$.

Example 3.5. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Then $\{c\} \in GLC(X, \tau)$, but $\{c\} \notin \delta GLC(X, \tau)$.

Now, we obtain a characterization for δglc^*-sets as follows (cf. Proposition 2.8 in [1]):

Theorem 3.6. *For a subset A of (X, τ), the following are equivalent:*

(a) *$A \in \delta GLC^*(X, \tau)$.*

(b) *$A = G \cap \delta\text{-Cl}(A)$ for some δg-open set G.*

(c) *$\delta\text{-Cl}(A) \setminus A$ is δg-closed.*

(d) *$A \cup (X \setminus \delta\text{-Cl}(A))$ is δg-open.*

Proof. (a)$\Rightarrow$(b) There exist a δg-open set G and a δ-closed set F such that $A = G \cap F$. Since $A \subset G$ and $A \subset \delta\text{-Cl}(A)$, $A \subset G \cap \delta\text{-Cl}(A)$. Conversely, since δ-Cl$(A) \subset F$ we have $G \cap \delta$-Cl$(A) \subset G \cap F = A$. Therefore, we have $A = G \cap \delta$-Cl(A).

(b)$\Rightarrow$(a) Since G is δg-open and δ-Cl(A) is δ-closed, $G \cap \delta$-Cl$(A) \in \delta GLC^*(X, \tau)$ by Definition 3.2.

(b)$\Rightarrow$(c) It follows from assumption and the Theorem 3.12 in [5] that δ-Cl$(A) \setminus A = \delta$-Cl$(A) \cap (X \setminus G)$ is δg-closed.

(c)$\Rightarrow$(b) Let $U = X \setminus (\delta$-Cl$(A) \setminus A)$. By assumption, U is δg-open and $A = U \cap \delta$-Cl(A) holds.

(c)$\Rightarrow$(d) Let $F = \delta$-Cl$(A) \setminus A$. Then, $A \cup (X \setminus \delta$-Cl$(A))$ is δg-open, since $X \setminus F = A \cup (X \setminus \delta$-Cl$(A))$ holds and $X \setminus F$ is δg-open.

(d)$\Rightarrow$(c) Let $U = A \cup (X \setminus \delta$-Cl$(A))$. Then, $X \setminus U$ is δg-closed and $X \setminus U = \delta$-Cl$(A) \setminus A$ holds. It completes the proof.

However, it is not true that $A \in \delta GLC^*(X, \tau)$ if and only if δ-Int$(A \cup (X \setminus \delta Cl(A))) \supset A$. In fact, let $A = \{a, b\}$ be a subset of (X, τ) given in Example 3.5. Then δ-Int$(A \cup (X \setminus \delta$-Cl$(A))) = \delta$-Int$(\{a, b\}) = \{a\} \not\supset A$ and $A \in \delta GLC^*(X, \tau)$.

We need the following definition to get a corollary to this theorem.

Definition 3.7. A topological space (X, τ) is called δg-submaximal if every δ-dense subset is δg-open.

Every δg-submaximal is g-submaximal but the converse is not true as seen the following example.

Example 3.8. Let (X, τ) be the topological space given in Example 3.4. Then (X, τ) is g-submaximal but not δg-submaximal, since $\{a, c\}$ is δ-dense (even dense) and g-open but not δg-open.

Corollary 3.9. A topological space (X, τ) is δg-submaximal if and only if $P(X) = \delta GLC^*(X, \tau)$ holds.

Proof. (*Necessity*)- Let $A \in P(X)$ and let $U = A \cup (X \setminus \delta\text{-Cl}(A))$. Then, it is easily verified that $X = \delta\text{-Cl}(U)$, i.e. U is a δ-dense subset of (X, τ). By assumption, U is δg-open. Therefore, it follows from Theorem 3.6 that $U \in \delta GLC^*(X, \tau)$, and hence $P(X) = \delta GLC^*(X, \tau)$ holds.

(*Sufficiency*)- Let A be a δ-dense subset of (X, τ). Then, it follows from assumptions and Theorem 3.6 (d) that $A \cup (X \setminus \delta\text{-Cl}(A)) = A$ holds, $A \in \delta GLC^*(X, \tau)$ and A is δg-open. This implies (X, τ) is δg-submaximal.

Proposition 3.10. *For a subset A of (X, τ), if $A \in \delta GLC^{**}(X, \tau)$ then there exists a δ-open set P such that $A = P \cap \delta\text{-Cl}(A)$.*

Proof. There exist a δ-open set P and a δg-closed set F such that $A = P \cap F$. Since $A \subset P$ and $A \subset \delta\text{-Cl}(A)$ we have $A \subset P \cap \delta\text{-Cl}(A)$, and hence $A = P \cap \delta\text{-Cl}(A)$.

The following results are basic properties of "δ-generalized locally closed sets".

Proposition 3.11. *For subsets A and B of (X, τ) the following are true:*

(a) *If $A \in \delta GLC^*(X, \tau)$ and $B \in \delta GLC^*(X, \tau)$ then $A \cap B \in \delta GLC^*(X, \tau)$.*

(b) *If $A \in \delta GLC^{**}(X, \tau)$ and B is δ-closed (or δ-open), then $A \cap B \in \delta GLC^{**}(X, \tau)$.*

(c) *If $A \in \delta GLC(X, \tau)$ and B is δg-open (or δ-closed), then $A \cap B \in \delta GLC(X, \tau)$.*

Proof. (a) It follows from Theorem 3.6 (b) that there exist δg-open sets G and U such that $A = G \cap \delta\text{-Cl}(A)$ and $B = U \cap \delta\text{-Cl}(B)$. Then, $A \cap B \in \delta GLC^*(X, \tau)$ since $G \cap U$ is δg-open by Theorem 3.11 (i) in [5] and $\delta\text{-Cl}(A) \cap \delta\text{-Cl}(B)$ is δ-closed.

(b) It follows from Definition 3.2 that there exist a δ-open set G and a δg-closed set F such that $A \cap B = G \cap F \cap B$. First suppose that B is δ-open.

Then, it is shown that $A \cap B \in \delta GLC^{**}(X, \tau)$. Next suppose that B is δ-closed. By using Theorem 3.12 in [5] it is proved that $F \cup B$ is δg-closed and so $A \cap B \in \delta GLC^{**}(X, \tau)$.

(c) It follows from Definition 3.1 that there exist a δg-open set G and a δg-closed set F such that $A \cap B = G \cap F \cap B$. First suppose that B is δg-open. Then, by using Theorem 3.11 in [5], it is shown that $A \cap B \in \delta GLC(X, \tau)$. Next suppose that B is δ-closed. By using Theorem 3.12 in [5], it is proved that $F \cap B$ is δg-closed and so $A \cap B \in \delta GLC(X, \tau)$.

Proposition 3.12. *Let A and B be subsets of (X, τ), $B \subset A$ and A be δg-closed in X. If B is δg-closed in A then B is δg-closed in X.*

Proof. Let G be an open set in X such that $B \subset G$. Then $B \subset A \cap G$ since $A \cap G$ is an open set in A. By Assumption, $\delta\text{-Cl}(B) \subset A \cap G$. And then $A \cap \delta\text{-Cl}(B) \subset A \cap G$ and $A \subset G \cup (X \setminus \delta\text{-Cl}(B))$. Since A is δg-closed in X, $\delta\text{-Cl}(B) \subset \delta g\text{-Cl}(A) \subset G \cup (X \setminus \delta\text{-Cl}(B))$. Therefore B is δg-closed in X.

Theorem 3.13. *Let Z be open subset of (X, τ) and let $A \subset Z$.*

(a) *If Z is δg-open in (X, τ) and $A \in \delta GLC^*(Z, \tau|Z)$, then $A \in \delta GLC^*(X, \tau)$.*

(b) *If Z is δg-closed in (X, τ) and $A \in \delta GLC^{**}(Z, \tau|Z)$, then $A \in \delta GLC^{**}(X, \tau)$.*

(c) *If Z is δg-closed and δg-open in (X, τ) and $A \in \delta GLC(Z, \tau|Z)$, then $A \in \delta GLC(X, \tau)$.*

Proof. (a) It follows from Theorem 3.6 that there exists a δg-open set G of $(Z, \tau \mid Z)$ such that $A = G \cap \delta\text{-Cl}_Z(A)$, where $\delta\text{-Cl}_Z(A) = Z \cap \delta\text{-Cl}(A)$. By using Theorem 3.11 in [5] and Definition 3.2, it is proved that $A = (Z \cap G) \cap \delta\text{-Cl}(A) \in \delta GLC^*(X, \tau)$.

(b) There exist a δ-open set G of $(Z, \tau|Z)$ and a δg-closed set F of $(Z, \tau|Z)$ such that $A = G \cap F$. By Proposition 3.12 F is δg-closed in (X, τ). Since

$G = B \cap Z$ for some δ-open set B of (X, τ), $A = (Z \cap B) \cap F = F \cap B$. Thus have $A \in \delta GLC^{**}(X, \tau)$.

(c) There exist a δg-open set G of $(Z, \tau|Z)$ and a δg-closed set F of $(Z, \tau|Z)$ such that $A = G \cap F$. It is proved that $A \in \delta GLC(X, \tau)$.

Remark 3.14. The following example shows that one of the assumptions of Theorem 3.13 (a), i.e. Z is δg-open, cannot be removed.

Example 3.15. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{a, b\}\}$. Let $\mathcal{V}$ denote the collection of all δg-open sets of (X, τ). Then we have $\mathcal{V} = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Put $Z = A = \{a, c\}$. It is shown that Z is not δg-open and $A \in \delta GLC^*(Z, \tau|Z)$. However, $A \notin \delta GLC^*(X, \tau)$ since $\delta GLC^*(X, \tau) = P(X) \setminus \{a, c\}$.

Proposition 3.16. *Suppose that the collection of all δg-open sets of (X, τ) is closed under finite unions. Let $A \in \delta GLC^*(X, \tau)$ and $B \in \delta GLC^*(X, \tau)$. If A and B are δ-separated, i.e. $A \cap \delta\text{-}Cl(B) = \phi$ and $B \cap \delta\text{-}Cl(A) = \phi$, then $A \cup B \in \delta GLC^*(X, \tau)$.*

Proof. By using Theorem 3.6 there exist δg-open sets G and S of (X, τ) such that $A = G \cap \delta\text{-Cl}(A)$ and $B = S \cap \delta\text{-Cl}(B)$. Put $U = G \cap (X \setminus \delta\text{-Cl}(B))$ and $V = S \cap (X \setminus \delta\text{-Cl}(A))$. Then, $A = U \cap \delta\text{-Cl}(A)$, $B = V \cap \delta\text{-Cl}(B)$, $U \cap \delta\text{-Cl}(B) = \phi$, $V \cap \delta\text{-Cl}(A) = \phi$. It follows from Theorem 3.11 in [5] that U and V are δg-open sets of (X, τ). Therefore, since $A \cup B = (U \cup V) \cap (\delta\text{-Cl}(A \cup B))$ and $U \cup V$ is δg-open by assumption, we have $A \cup B \in \delta GLC^*(X, \tau)$.

The following example shows that one of assumption of Proposition 3.16, i.e. A and B are δ-separated, cannot be removed.

Example 3.17. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$. Then $\{a\} \in \delta GLC^*(X, \tau)$ and $\{d\} \in \delta GLC^*(X, \tau)$. However, $\{a\}$ and $\{d\}$ are not δ-separated and $\{a, d\} \notin \delta GLC^*(X, \tau)$.

Proposition 3.18. *Let* $\{Z_i : i \in \Gamma\}$ *be a finite* δg*-closed cover of* (X, τ), *i.e.* $X = \cup\{Z_i : i \in \Gamma\}$, *and let* A *be a subset of* (X, τ). *If* $A \cap Z_i \in \delta GLC^{**}(Z_i, \tau|Z_i)$ *for each* $i \in \Gamma$, *then* $A \in \delta GLC^{**}(X, \tau)$.

Proof. For each $i \in \Gamma$ there exist a δ-open set U_i of (X, τ) and a δg-closed set F_i of $(Z_i, \tau|Z_i)$ such that $A \cap Z_i = U_i \cap (Z_i \cap F_i)$. Then, $A = \cup\{A \cap Z_i : i \in \Gamma\} = [\cup\{U_i : i \in \Gamma\}] \cap [\cup\{Z_i \cap F_i : i \in \Gamma\}]$, and hence $A \in \delta GLC^{**}(X, \tau)$ by Theorem 3.11 in [5].

4. δGLC-FUNCTIONS AND SOME OF THEIR PROPERTIES

In [14], Park defined three distinct notions of $L\delta C$-continuity, i.e. $L\delta C$-irresoluteness, $L\delta C$-continuity and sub-$L\delta C$-continuity. In this section we use δglc-, δglc^*- and δglc^{**}-sets to generalize $L\delta C$-irresolute, $L\delta C$-continuous and sub-$L\delta C$-continuous functions and study properties of such functions.

Definition 4.1. A function $f : (X, \tau) \to (Y, \sigma)$ is called:

(a) δGLC-continuous if $f^{-1}(V) \in \delta GLC(X, \tau)$ for each $V \in \sigma$.

(b) δGLC^*-continuous if $f^{-1}(V) \in \delta GLC^*(X, \tau)$ for each $V \in \sigma$.

(c) δGLC^{**}-continuous if $f^{-1}(V) \in \delta GLC^{**}(X, \tau)$ for each $V \in \sigma$.

Definition 4.2. A function $f : (X, \tau) \to (Y, \sigma)$ is called:

(a) δGLC-irresolute if $f^{-1}(V) \in \delta GLC(X, \tau)$ for each $V \in \delta GLC(Y, \sigma)$.

(b) δGLC^*-irresolute if $f^{-1}(V) \in \delta GLC^*(X, \tau)$ for each $V \in \delta GLC^*(Y, \sigma)$.

(c) δGLC^{**}-irresolute if $f^{-1}(V) \in \delta GLC^{**}(X, \tau)$ for each $V \in \delta GLC^{**}(Y, \sigma)$.

Proposition 4.3. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function.*

(a) *If f is $L\delta C$-continuous, then it is δGLC^*-continuous and δGLC^{**}-continuous.*

(b) *If f is δGLC^*-continuous or δGLC^{**}-continuous, then it is δGLC-continuous.*

(c) *If f is δGLC^*-continuous, then it is GLC^*-continuous.*

(d) *If f is δGLC^{**}-continuous, then it is GLC^{**}-continuous.*

(e) *If f is δGLC^*-continuous, then it is δGLC-continuous.*

(f) *If f is δGLC^{**}-continuous, then it is δGLC-continuous.*

(g) *If f is δGLC-continuous, then it is GLC-continuous.*

Proof. We prove the only (a). Suppose that f is $L\delta C$-continuous. Let V be an open set of (Y, σ). Then $f^{-1}(V)$ is locally δ-closed in (X, τ) by definition. By

Proposition 3.3 f is δGLC^*-continuous and δGLC^{**}-continuous.

From Proposition 4.3, we have the following diagram, which is an enlargement of Balachandran et al. [2] and Park [14].

$$\begin{array}{ccccc} & & L\delta C\text{-continuity} & & \\ & \swarrow & & \searrow & \\ \delta GLC^*\text{-continuity} & \rightarrow & \delta GLC\text{-continuity} & \leftarrow & \delta GLC^{**}\text{-continuity} \\ \downarrow & & \downarrow & & \downarrow \\ GLC^*\text{-continuity} & \rightarrow & GLC\text{-continuity} & \leftarrow & GLC^{**}\text{-continuity} \end{array}$$

The converses of Proposition 4.3 need not be true as from he following examples.

Example 4.4. Let $X = Y = \{a, b, c\}$ and (X, τ) be the topological space given in Example 3.8. Let $\sigma = \{X, \phi, \{c\}\}$ be a topology on Y. If a function $f : (X, \tau) \to (Y, \sigma)$ is the identity, then f is δGLC^*-continuous but it is not $L\delta C$-continuous, since $f^{-1}(\{c\}) \in \delta GLC^*(X, \tau)$ but $f^{-1}(\{c\}) \notin L\delta C(X, \tau)$.

Example 4.5. Let $X = Y = \{a, b, c\}$ and (X, τ) be the topological space given in Example 3.8. Let $\sigma = \{X, \phi, \{a, b\}\}$ be a topology on Y. If a function $f : (X, \tau) \to (Y, \sigma)$ is the identity, then f is δGLC^{**}-continuous but it is not $L\delta C$-continuous, since $f^{-1}(\{a, b\}) \in \delta GLC^{**}(X, \tau)$ but $f^{-1}(\{a, b\}) \notin L\delta C(X, \tau)$.

Example 4.6. Let $X = Y = \{a, b, c\}$ and (X, τ) be the topological space given in Example 3.8. Let $\sigma = \{X, \phi, \{c\}\}$ be a topology on Y. If a function $f : (X, \tau) \to (Y, \sigma)$ is the identity, then f is GLC^{**}-continuous but it is not δGLC^{**}-continuous, since $f^{-1}(\{c\}) \in GLC^{**}(X, \tau)$ but $f^{-1}(\{c\}) \notin \delta GLC^{**}(X, \tau)$.

Example 4.7. Let $X = Y = \{a, b, c\}$ and (X, τ) be the topological space given in Example 3.8. Let $\sigma = \{X, \phi, \{c\}\}$ be a topology on of Y. If a function $f : (X, \tau) \to (Y, \sigma)$ is the identity, then f is δGLC-continuous but it is not δGLC^{**}-continuous, since $f^{-1}(\{c\}) \in \delta GLC(X, \tau)$ but $f^{-1}(\{c\}) \notin \delta GLC^{**}(X, \tau)$.

Example 4.8. Let $X = Y = \{a, b, c\}$ and (X, τ) be the topological space given in Example 3.8. Let $\sigma = \{X, \phi, \{a, b\}\}$ be a topology on Y. If a function $f : (X, \tau) \to (Y, \sigma)$ is the identity, then f is δGLC-continuous but it is not δGLC^{*}-continuous, since $f^{-1}(\{a, b\}) \in \delta GLC(X, \tau)$ but $f^{-1}(\{a, b\}) \notin \delta GLC^{*}(X, \tau)$.

The following result is an immediate consequence of Corollary 3.13 (cf. Ganster and Reilly, Proposition 6).

Proposition 4.9. *A topological space (X, τ) is δg-submaximal if and only if every function having (X, τ) as its domain is δGLC^{*}-continuous.*

Proof. (*Necessity*)- Suppose that $f : (X, \tau) \to (Y, \sigma)$ is a function. By Corollary 3.13 we have that $f^{-1}(V) \in \delta GLC^{*}(X, \tau) = P(X)$ for each open set V of (Y, σ). Therefore, f is δGLC^{*}-continuous.

(*Sufficiency*)- Let $Y = \{0, 1\}$ be the Sierpinski space with topology $\sigma = \{Y, \phi, \{0\}\}$. Let V be a subset of (X, τ) and $f : (X, \tau) \to (Y, \sigma)$ a function defined by $f(x) = 0$ for every $x \in V$ and $f(x) = 1$ for every $x \notin V$. It follows from assumption that f is δGLC^{*}-continuous and hence $f^{-1}(\{0\}) = V \in \delta GLC^{*}(X, \tau)$. Therefore we have $P(X) = \delta GLC^{*}(X, \tau)$ and so (X, τ) is δg-submaximal by Corollary 3.19.

Proposition 4.10. *If $f : (X, \tau) \to (Y, \sigma)$ is δGLC^{**}-continuous and a subset B is δ-closed in (X, τ), then the restriction of f to B, say $f|B : (B, \tau|B) \to (Y, \sigma)$ is δGLC^{**}-continuous.*

Proof. Let V be an open set of (Y, σ). Then, $f^{-1}(V) = G \cap F$ for some δ-open set G and δg-closed set F of (X, τ). By proposition 3.12, $(f|B)^{-1}(V) = (G \cap B) \cap (F \cap B) \in \delta GLC^{**}(B, \tau|B)$. This implies that $f|B$ is δGLC^{**}-continuous.

We recall the definition of the combination of two functions: let $X = A \cup B$ and $f : A \to Y$ and $h : B \to Y$ be two functions. We say that f and h are compatible if $f|A \cap B = h|A \cap B$. Then, we can define a function $f \nabla h : X \to Y$ as follows: $(f \nabla h)(x) = f(x)$ for every $x \in A$ and $(f \nabla h)(x) = h(x)$ for every $x \in B$. The function $f \nabla h : X \to Y$ is called the combination of f and h.

Theorem 4.11. *Let $X = A \cup B$, where A and B are δg-closed sets of (X, τ), and $f : (A, \tau|A) \to (Y, \sigma)$ and $h : (B, \tau|B) \to (Y, \sigma)$ be compatible functions.*

(a) *If f and h are δGLC^{**}-continuous, then $f \nabla h : (X, \tau) \to (Y, \sigma)$ is δGLC^{**}-continuous.*

(b) *If f and h are δGLC^{**}-irresolute, then $f \nabla h : (X, \tau) \to (Y, \sigma)$ is δGLC^{**}-irresolute.*

Proof. (a) Let $V \in \sigma$. Then, $(f \nabla h)^{-1}(V) \cap A = f^{-1}(V)$ and $(f \nabla h)^{-1}(V) \cap B = h^{-1}(V)$. By assumptions we have $(f \nabla h)^{-1}(V) \cap A \in \delta GLC^{**}(A, \tau|A)$ and $(f \nabla h)^{-1}(V) \cap B \in \delta GLC^{**}(B, \tau|B)$. Therefore, it follows from Proposition 3.18 that $(f \nabla h)^{-1}(V) \in \delta GLC^{**}(X, \tau)$ and hence $f \nabla h$ is δGLC^{**}-continuous.

(b) Let $V \in \delta GLC^{**}(Y, \sigma)$. Then we have $(f \nabla h)^{-1}(V) \cap A = f^{-1}(V)$ and $(f \nabla h)^{-1}(V) \cap B = h^{-1}(V)$. By assumptions we have $(f \nabla h)^{-1}(V) \cap A \in \delta GLC^{**}(A, \tau|A)$ and $(f \nabla h)^{-1}(V) \cap B \in \delta GLC^{**}(B, \tau|B)$. Therefore, it follows from Proposition 3.18 that $(f \nabla h)^{-1}(V) \in \delta GLC^{**}(X, \tau)$ and hence $f \nabla h$ is δGLC^{**}-irresolute.

Concerning compositions of functions we have the following:

Remark 4.12. Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be the mappings.

(a) The composition $f \circ g$ of two δGLC-irresolute (resp. δGLC^*-irresolute, δGLC^{**}-irresolute) functions are clearly δGLC-irresolute (resp. δGLC^*-irresolute, δGLC^{**}-irresolute).

(b) If f is δGLC-irresolute(resp. δGLC^*-irresolute, δGLC^{**}-irresolute) and g is δGLC-continuous(resp. δGLC^*-continuous, δGLC^{**}-continuous), then the composition $f \circ g$ is clearly δGLC-continuous (resp. δGLC^*-continuous, δGLC^{**}-continuous).

Definition 4.13. A function $f : (X, \tau) \to (Y, \sigma)$ is called sub-δGLC^*-continuous if there is a basis $\mathcal{B}$ for (Y, σ) such that $f^{-1}(U) \in \delta GLC^*(X, \tau)$ for each $U \in \mathcal{B}$.

Proposition 4.14. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function.*

(a) *f is sub-δGLC^*-continuous if and only if there is a subbasis $\mathcal{C}$ of (Y, σ) such that $f^{-1}(U) \in \delta GLC^*(X, \tau)$ for each $U \in \mathcal{C}$.*

(b) *If f is sub-$L\delta C$-continuous, then f is sub-δGLC^*-continuous.*

(c) *If f is sub-δGLC^*-continuous, then f is sub-GLC^*-continuous.*

Proof. (a) *(Necessity)-* It follows from assumption that there is a basis $\mathcal{B}$ for (Y, σ) such that $f^{-1}(U) \in \delta GLC^*(X, \tau)$ for each $U \in \mathcal{B}$. Since $\mathcal{B}$ is also a subbasis for (Y, σ), the proof is obvious.

(Sufficiency)- For a subbasis $\mathcal{C}$, let $\mathcal{C}_\delta = \{A \subset Y : A$ is an intersection of finitely many sets belonging to $\mathcal{C}\}$. Then, $\mathcal{C}_\delta$ is a basis for (Y, σ). For $U \in \mathcal{C}_\delta$, $U = \cap\{F_i : F_i \in \mathcal{C}, i \in \Lambda\}$ where Λ is a finite set, By using Proposition 3.11 and assumption we have $f^{-1}(U) = \cap\{f^{-1}(F_i) : i \in \Lambda\} \in \delta GLC^*(X, \tau)$.

(b) Let $\mathcal{B}$ be a base of (Y, σ) and $B \in \mathcal{B}$. Then, by assumption, $f^{-1}(B) \in L\delta C(X, \tau)$ and since every δ-open is δg-open, $f^{-1}(B) \in \delta GLC^*(X, \tau)$. Therefore, f is sub-δGLC^*-continuous.

(c) Similar to (b).

Example 4.15. Let $X = Y = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$ and σ is the topology induced by a base $\mathcal{B}$ of Y. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function.

(a) If $\mathcal{B} = \{Y, \{a\}, \{c\}\}$, then f is sub-δGLC^*-continuous, but it is not δGLC^*-continuous, since $f^{-1}(\{a, c\}) \notin \delta GLC^*(X, \tau)$ for $\{a, c\} \in \sigma$.

(b) If $\mathcal{B} = \{Y, \{c\}\}$, then f is sub-δGLC^*-continuous, but it is not sub-$L\delta C$-continuous, since $f^{-1}(\{c\}) = \{c\} \in \delta GLC^*(X\tau)$ but $f^{-1}(\{c\}) \notin L\delta C(X, \tau)$.

(c) If $\mathcal{B} = \{Y, \{a\}, \{c\}, \{a, c\}\}$, then f is sub-GLC^*-continuous, but it is not sub-δGLC^*-continuous, since $f^{-1}(\{a, c\}) \in GLC^*(X, \tau)$ but $f^{-1}(\{a, c\}) \notin \delta GLC^*\ (X, \tau)$.

From Proposition 4.3 (a), Examples 4.4 - 4.8 and Example 3.17 of Balachandran [1], we have the following diagram:

$$\begin{array}{ccccc}
L\delta C\text{-continuity} & \to & \delta GLC^*\text{-continuity} & \to & GLC^*\text{-continuity} \\
\downarrow & & \downarrow & & \downarrow \\
\text{sub-}L\delta C\text{-continuity} & \to & \text{sub-}\delta GLC^*\text{-continuity} & \to & \text{sub-}GLC^*\text{-continuity}
\end{array}$$

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