Operations on Supra-Topological Spaces

초-위상공간상에서의 연산

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Operations on Supra-Topological Spaces

A Dissertation

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요약

S. Kasahara는 주어진 집합상의 위상 τ에서 그 집합의 멱집합으로의 연산 γ를 이용하여 컴팩트성을 일반화함으로서 다양한 컴팩트공간에서의 결과를 통합적으로 관찰하였고, H. Ogata와 T. Fukutaka 등은 연산 γ를 이용하여 위상공간상에서의 집합의 openness와 closedness, 함수의 연속성 및 분리공리 등과 같은 여러 가지 개념을 일반화하고 주어진 위상 τ와 이와 관련된 위상 τ_γ에서의 위상적 성질을 조사하였다.

또한, A.S. Masshour등은 주어진 집합에 대한 초-위상 τ^*을 정의하고 주어진 위상 τ와 관련된 초-위상공간의 분리공리와 함수의 연속성에 대한 결과들을 조사함으로서, 위상공간상에서의 개집합보다 약한 반-개집합, 전-개집합 및 반전-개집합 등과 관련된 결과에 대한 통합적인 접근방법을 제시하였다.

본 논문에서는 주어진 집합상의 초-위상에서 그 집합의 멱집합으로의 연산 γ^*를 정의하여 초-위상공간을 연구함으로서, 위상공간에서의 여러 가지 개념에 대한 S. Kasahara와 H. Ogata 등의 연산-접근방법과 A.S. Masshour 등의 초-위상을 이용한 통합적 접근방법보다 일반화된 γ^*-연산 접근방법을 소개하였다.

특히, 초-위상 τ^*와 이와 관련된 초-위상 $\tau^*_{\gamma^*}$에서의 초-위상적 성질을 밝히고, 주어진 위상 τ와 이와 관련된 위상 τ_γ과의 관계를 정립하였고, 연산 γ^*의 성질을 이용하여 sg-폐집합, gs-폐집합 등과 같은 일반화된 폐집합의 결과들을 통합적으로 다루었고, γ^*-sup T_i 공간 (i=0,1/2,1,2)들의 특성을 밝혔으며 연산 γ와 관련된 연산 γ^* (associated operation with γ)을 이용하여 네트의 γ^*-수렴성 및 이를 이용한 초-위상공간상의 함수의 약 (γ^*, β^*)-연속성의 특성을 조사하였고, 또한 위상공간에서의 함수의 폐-그래프 정리를 초-위상공간으로 일반화하였다.

Chapter 1

Introduction

Separation axioms, compactness, connectedness and continuity on topological spaces, as important and basic subjects in study fields of General Topology and several branches of Mathematics, have been researched many mathematicians. Recent tendency of research activities in General Topology are the study of ideal topological spaces which are topological spaces having the structure of ideals, the study of operation functions on topological spaces and the research of generalized closed sets in topological spaces.

In particular, Kasahaha([48]) in 1973 first treated an operation approach method on a topological spaces as defined by the concept of an operation γ(or operation α) that is a mapping from the topology τ of a given topological spaces (X, τ) into the power sets $P(X)$ of X such that $U \subset U^\gamma$ for each $U \in \tau$, where U^γ denotes the value of γ at U. The several topological results can be unified by the theory (i.e. Several known characterizations of compact spaces, nearly compact spaces and H-closed spaces can be unified under general treatment of the notion of compactness).

In 1983 Janković ([44]) continued the investigations of topological properties with help of a certain operation. In detail, he defined the concept of γ-closed (i.e. for $A \subset X$, $\mathrm{Cl}_\gamma(A) \subset A$) and further studied functions with γ-closed graphs.

Recently, by similar method which was utilized by Kasahara ([48]) and Janković ([44]), Ogata introduced the concept of operation-openness (i.e. γ-open sets) into a topological space and investigated the related topological properties of the associated operation-topology and the original topology in [84], [85] and [86]. Furthermore, in [85] he introduced operation-separation axioms (i.e. γ-T_i, $i = 0, 1/2, 1, 2$), which generalize one of separation axioms (i.e. T_i, $i = 0, 1/2, 1, 2$).

On the other hand, the notion of closed sets in General Topology is fundamental. In 1970, Levine ([53]) first introduced the concept of generalized closed sets in a topological space by comparing the closure of subset with its open supersets. Recall that a subset A of a topological space (X, τ) is a *generalized closed* (briefly, g-closed) set if $\mathrm{Cl}(A) \subset U$ whenever U is an open set containing A. Note that this definition uses both the "closure operator" and "openness" of the superset. By considering other generalized closure operators or classes of generalized open sets, various notions analogous to Levine's g-closed sets have been studied; see to [15] for more detail.

Thereafter, Mashhour et al. ([70]) in 1983, introduced the concept of supra open sets in supra-topological spaces, which was considered as a generalization of open sets, semiopen sets [52], preopen sets [67], α-open sets [76] and β-open [1] (or semipre open [3]) sets. Moreover, they showed that many results obtained in topological spaces can be considered as special cases of those found in supra-topological spaces and then, they defined the notion of supra-R_0 and supra-R_1

spaces in 1985 using the concept of supra open sets. It was shown that many results in the previous papers can be also reflected as special cases of their results.

In this thesis, using the operation approach methods on supra-topological spaces that is the similar method which is introduced by Kasahara ([48]) and Ogata ([84]), we shall unify the concepts of several supra open sets, supra mappings and supra separation axioms defined in the given supra-topological spaces or topological spaces (see, [1], [12], [19], [21], [47], [84] [52], [53], [56], [60], [67], [70], [73], [74] and [76]), and consider those as spacial cases of our results that the various properties have been investigated by many authors in the previous papers, and obtained by choosing some special mappings such as the identity mapping, the supra closure operator, or the supra interior-supra closure operator being stated in terms of a certain mapping of a supra topology τ^* into itself the power set of $\cup\tau^*$. We discuss some related notions and we further investigate operations approaches of closed graphs of mappings. It should be noticed that similar mappings were already studied in [48] and [84]

This thesis consists of five chapters. We briefly summarize the contents of each chapter.

In Chapter 2, we present some general facts that will be useful in this thesis. No originality is claimed for any of the results here. Some basic concepts and notations are also defined.

In Chapter 3, we introduce the concept of γ^*-supra open sets and the concept closure and interior of a set via the various γ^*-operation on supra-topological space (X, τ^*), and investigate their basic properties.

In Chapter 4, we first introduce the concepts of (γ^*, β^*)-supra continuous, weakly (γ^*, β^*)-supra continuous mappings and γ^*-supra convergences defined by various operations γ^* and β^* in a supra-topological space (X, τ^*) and a supra topological space (Y, σ^*), respectively. We also define the concepts of γ^*-supra closed, γ^*-supra open mappings and their weaker forms and study their basic properties and relationships.

Finally, Chapter 5 deals with the concepts of γ^*-supra T_i spaces, where $i = 0$, $1/2, 1$ or 2, defined by various operations γ^* on a supra space (X, τ^*) and their mutual relationships among γ^*-supra T_i $(i = 0, 1/2, 1, 2)$. We also introduce the concepts of generalized γ^*-supra closed sets weaker than γ^*-supra closed sets and study their basic properties and relationships. Moreover, we investigate the closed graphs properties of mappings and preserving the properties of γ^*-sup T_i spaces and generalized forms of γ^*-supra sets (namely, γ^*-supra closed sets or γ^*-supra open sets), where $i = 0, 1/2, 1$ or 2.

Chapter 2

Preliminaries

In this chapter we present some general facts that will be useful in this thesis. No originality is claimed for any of the results here. Some basic concepts and notation are also defined.

Throughout this thesis we consider spaces on which no separation axioms are assumed unless explicitly stated. The word a "supra space" will always mean a supra topological space. The topological spaces (resp. supra topological spaces) are denoted by (X, τ) (resp. (X, τ^*)) if there is no danger for confusion.

2.1 Background concerning Generalized Closed Sets

Recall next the following definition of weaker forms than open and closed sets which will be used often throughout the thesis. For a subset $A \subset X$, the closure and the interior of A in (X, τ) are denoted by $\mathrm{Cl}(A)$, and $\mathrm{Int}(A)$ respectively.

Definition 2.1.1. A subset A of (X, τ) is called :

(a) semiopen [52] if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$ and semiclosed if $\mathrm{Int}(\mathrm{Cl}(A)) \subset A$,

(b) preopen [67] if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$ and preclosed if $\mathrm{Cl}(\mathrm{Int}(A)) \subset A$,

(c) α-open [76] if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ and α-closed if $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))) \subset A$,

(d) semi-preopen [3] (= β-open [1]) if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$ and semi-preclosed (=β-closed) if $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A))) \subset A$.

Recall that the semi-interior $\mathrm{sInt}(A)$ (resp. pre-interior $\mathrm{pInt}(A)$) of a subset of $A \subset X$ is the union of all semiopen sets (resp. preopen sets) contained in A. The semi-closure $\mathrm{sCl}(A)$ (resp. pre-closure $\mathrm{pCl}(A)$) is the intersection of all semiclosed sets (resp. preclosed sets) that contain A.

Note that [46] $\mathrm{sCl}(A)=A \cup \mathrm{Int}(\mathrm{Cl}(A))$ and $\mathrm{sInt}(A)=A \cap \mathrm{Cl}(\mathrm{Int}(A))$. Observe also that [3] $\mathrm{pCl}(A)=A \cup \mathrm{Cl}(\mathrm{Int}(A))$ and $\mathrm{pInt}(A)=A \cap \mathrm{Int}(\mathrm{Cl}(A))$. The semi-preclosure sp-$\mathrm{Cl}(A)$(resp. semi-preinterior sp-$\mathrm{Int}(A)$) and the α-closure $\alpha\mathrm{Cl}(A)$ (resp. α-interior $\alpha\mathrm{Int}(A)$) of a subset $A \subset X$ are analogously defined and it is observed that sp-$\mathrm{Cl}(A)=A \cup \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ and $\alpha\mathrm{Cl}(A) = A \cup \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$.

Note here that the family τ_α of all α-open sets in (X, τ) forms always a topology on X finer than τ [76]. The family of all semiopen (resp. preopen, semi-preopen) subsets of (X, τ) will be as always denoted by $SO(X, \tau)$ (resp. $PO(X, \tau)$, $SPO(X, \tau)$) and a space (X, τ) is called *extremally disconnected* if the closure of every open set is open. In 1965 Njåstad [76] showed that $SO(X, \tau)$ is a topology on X if and only if (X, τ) is extremally disconnected. The reader can refer to [38] to see when $PO(X, \tau)$ is a topology on X. However $SPO(X, \tau)$

is a topology on X if and only if both $SO(X,\tau)$ and $SPO(X,\tau)$ are topologies on X [40].

We review some concepts of generalized forms of open and closed sets that will be used throughout the dissertation.

Definition 2.1.2. A subset A of a topological space (X,τ) is called:

(a) *a generalized closed set* (briefly, *g-closed*) [52]) if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is open,

(b) *a semi-generalized closed set* (briefly, *sg-closed*) [11] if $\mathrm{sCl}(A) \subset U$ whenever $A \subset U$ and U is semi-open,

(c) *a generalized α-closed set* (briefly, *$g\alpha$-closed*) [62] if $\alpha\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is α-open,

(d) *a generalized semi-closed set* (briefly, *gs-closed*) [5] if $\mathrm{sCl}(A) \subset U$ whenever $A \subset U$ and U is open,

(e) *a α-generalized closed set* (briefly, *αg-closed*) [63] if $\alpha\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is open,

(f) *a regular generalized closed set* (briefly, *rg-closed*) [87] if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is regular open.

(g) *g-open* (resp. *sg-open*, *$g\alpha$-open*, *gs-open*, *αg-open*, *rg-open*) if $X \setminus A$ is g-closed (resp. sg-closed, $g\alpha$-closed, gs-closed, αg-closed, rg-closed).

In 1980 Dunham and Levine ([34]) published the following theorem and corollary.

Theorem 2.1.3. [34, Theorem 2.2] *The following conditions are equivalent:*

(a) *A is g-closed,*

(b) *for each* $x \in \mathrm{Cl}(A)$, $\mathrm{Cl}(\{x\}) \cap A \neq \phi$,

(c) $\mathrm{Cl}(A) \setminus A$ *contains non-empty closed set.*

As an immediate consequence we have the familiar:

Theorem 2.1.4. [34, Corollary 2.3] *A set* A *is* g*-closed if and only if* $A = F \setminus N$, *where* F *is closed and* N *contains no non-empty closed subsets.*

The following are well-known properties of Levine [53].

Theorem 2.1.5. [53, Theorem 4.8] *If* $\mathrm{Cl}(A) \subset B \subset A$ *and if* A *is* g*-open, then* B *is* g*-open.*

Theorem 2.1.6. [53, Theorem 4.9] *A set* A *is* g*-open if and only if* $\mathrm{Cl}(A) \setminus A$ *is* g*-open.*

We recall some definitions of various kinds of continuous functions that will be useful this thesis.

Definition 2.1.7. A function $f : (X, \tau) \to (Y, \sigma)$ is called:

(a) *semi-continuous* [52] if $f^{-1}(U) \in SO(X, \tau)$ for every open set U of (Y, σ),

(b) *pre-continuous* [67] if $f^{-1}(U) \in PO(X, \tau)$ for every open set U of (Y, σ),

(c) α*-continuous* [69] if $f^{-1}(U) \in \alpha(X, \tau)$ for every open set U of (Y, σ),

(d) β*-continuous* [5] or *semi-precontinuous* [75] if $f^{-1}(U) \in SPO(X, \tau)$ for every open set U of (Y, σ).

Review that several notions of various kinds of irresolute functions that will be used in sequel.

Definition 2.1.8. A function $f : (X, \tau) \to (Y, \sigma)$ is called:

(a) *irresolute* [20] if $f^{-1}(U) \in SO(X, \tau)$ for every $U \in SO(Y, \sigma)$,

(b) *pre-irresolute* [93] if $f^{-1}(U) \in PO(X, \tau)$ for every $U \in PO(Y, \sigma)$,

(c) α*-irresolute* [58] if $f^{-1}(U) \in \alpha(X, \tau)$ for every $U \in \alpha(Y, \sigma)$,

(d) β*-irresolute* [59] if $f^{-1}(U) \in SPO(X, \tau)$ for every $U \in SPO(Y, \sigma)$.

Recall that some concepts of various kinds of open and closed functions that will be used throughout this thesis.

Definition 2.1.9. A function $f : (X, \tau) \to (Y, \sigma)$ is called:

(a) *semi-open* (resp. *semi-closed*) [12] if $f(U)$ is semiopen (resp. semiclosed) in (Y, σ) for every open (resp. closed) set U of (X, τ),

(b) *pre-open* [67] (resp. *pre-closed*) [35] if $f(U)$ is preopen (resp. preclosed) in (Y, σ) for every open (resp. closed) set U of (X, τ),

(c) α*-open* (resp. α*-closed*) [69] if $f(U)$ is α-open (resp. α-closed) in (Y, σ) for every open (resp. closed) set U of (X, τ),

(d) *semi-preopen* [14] or *pre-*β*-open* [59]) (resp. *semi-preclosed* [75] or *pre-*β*-open* [59])) if $f(U)$ is semi-preopen (resp. semi-preclosed) in (Y, σ) for every semiopen (resp. semiclosed) set U of (X, τ),

(e) *pre-semiopen* [21] (resp. *pre-semiclosed*) [96] if $f(U)$ is semiopen (resp. semiclosed) in (Y, σ) for every semiopen (resp. semiclosed) set U of (X, τ),

(f) *p-open* [45] (=*M-preopen* [68]) (resp. *p-closed* [45] (=*M-preclosed*)) [68] if $f(U)$ is preopen in (Y, σ) for every preopen set U of (X, τ),

(g) *pre-*β*-open* (resp. *pre-*β*-closed* [59]) if $f(U)$is semi-preopen in (Y, σ) for every semi-preopen set U of (X, τ).

The following are well-known definition of semi-separation axioms [56] in topological spaces.

Definition 2.1.10. A topological space (X,τ) is called:

(a) *semi-T_0* if for each pair of distinct points in x there is a semiopen set in (X,τ) containing one point but not the other,

(b) *semi-T_1* if for each pair of distinct points x, y in X there is a semiopen set containing x but not y,

(c) *semi-T_2* if for each pair of distinct points x, y in X there are disjoint semiopen sets U, V such that $x \in U$ and $y \in V$.

Recall that Kar and Bhattacharyya ([47]) (resp. Navalagi ([73], [74]), Maheshwari and Tapi ([57])) defined pre-T_i (resp. α-T_i, semipre-T_i, feebly T_i) spaces $(i = 0,1,2)$ were replacing the expression "semiopen" in the definition of semi-T_i space $(i = 0,1,2)$ with "preopen" (resp. "α-open", "semipreopen", "feebly open") sets and it was shown that these spaces are weaker forms than general forms (i.e. T_i spaces $(i = 0,1,2)$) and that for each separation axioms, T_i-type implies T_j-type if $i > j$ where $i, j = 0,1,2$.

Remark 2.1.11. Noiri [83, Proof of Corollary 4.7] has given an elegant proof of the fact that (X,τ) is T_2 if and only if (X,τ_α) is T_2. Thus the notion of feebly T_2 coincides with the usual Hausdorff (or T_2) property.

2.2 On γ-Operation Approaches

In this section, we will review that the concepts and properties of operations on a topological space and the operation-closed graphs of a mapping, which were introduced by Kasahara ([48]) in 1979 and recall the notion of γ-open sets [84] and the concepts of closure and interior operators via the various γ operations on a topological space (X, τ) and their basic properties. Furthermore, we will also review the concept of operation-separation axioms (i.e. γ-T_i [84], $i = 0, 1/2, 1, 2$) and some topological properties on them and show some properties.

We first state the definitions introduced by Kasahara ([48]) and Ogata ([84] - [86]).

Definition 2.2.1. Let (X, τ) be a topological space. An *operation* γ [48] on the topology τ is a mapping from τ into the power set $P(X)$ of X such that $V \subset V^\gamma$ for each $V \in \tau$, where V^γ denotes the value of γ at V. It is denoted by $\gamma : \tau \to P(X)$.

For example, the mapping γ defined by $\gamma(G) = G^\gamma = G$ for each $G \in \tau$ is an operation on τ, which will be called the *identity operator* on τ in what follows. The closure operator in a topological space (X, τ) defines of course an operation on the topology τ, and the composition Int $\circ$ Cl of the closure operator Cl with the interior operator Int is also an operation on τ; we shall call the former the *closure operation* on τ and the latter the *interior-closure operator* on a topology τ.

Definition 2.2.2. Let (X, τ) be a space. An operation γ is said to be

(a) *regular* [84] if for every open neighborhood U and V of each $x \in X$, there exists an open neighborhood W of x such that $W^\gamma \subset U^\gamma \cap V^\gamma$;

(b) *open* [84] if for every open neighborhood U of each $x \in X$, there exists a γ-open set V such that $x \in V$ and $V \subset U^\gamma$.

Proposition 2.2.3. *Let $\gamma : \tau \to P(X)$ be a regular operation on τ.*

(a) [84, Proposition 2.9 (a)]. *If A and B are γ-open, then $A \cap B$ is γ-open.*

(b) [84, Proposition 2.9 (b)]. *τ_γ is a topology on X such that $\tau_\gamma \subset \tau$.*

Definition 2.2.4. A point x of X is in the *γ-closure* [84] of $A \subset X$, denoted by $\mathrm{Cl}_\gamma(A)$, if $U^\gamma \cap A \neq \phi$ for any open neighborhood U of x. A subset A of X is said to be *γ-closed* (in the sense of Janković [44]) if $\mathrm{Cl}_\gamma(A) = A$. A point x of X is in the *γ-interior* [84] of A, denoted by $\mathrm{Int}_\gamma(A)$, if $U^\gamma \subset A$ for some open neighborhood U of x.

Proposition 2.2.5. *Let $\gamma : \tau \to P(X)$ be an operation on τ.*

(a) $A \subset \mathrm{Cl}(A) \subset \mathrm{Cl}_\gamma(A) \subset \tau_\gamma\text{-}\mathrm{Cl}(A)$, *where τ_γ-$\mathrm{Cl}(A)$ is γ-closure of A in the sense of Ogata* ([84]).

(b) *If $A \subset B$, then* $\mathrm{Cl}_\gamma(A) \subset \mathrm{Cl}_\gamma(B)$.

(c) $\mathrm{Cl}_\gamma(A \cup B) = \mathrm{Cl}_\gamma(A) \cup \mathrm{Cl}_\gamma(B)$.

(d) *If γ is an open operator,* $\mathrm{Cl}_\gamma(\mathrm{Cl}_\gamma(A)) = \mathrm{Cl}_\gamma(A)$.

(e) $X \setminus \mathrm{Int}_\gamma(A) = \mathrm{Cl}_\gamma(X \setminus A)$.

The following is a well-known definition of γ-regular spaces introduced by Kasahara ([48]).

Definition 2.2.6. A topological space (X, τ) is said to be *γ-regular*, where γ is an operation on τ, if for each $x \in X$ and for each open neighborhood U of x, there exists an open neighborhood V of x such that $V^\gamma \subset U$.

By using the concept of γ-regular spaces due to Kasahara ([48]) the following proposition is well-known by Ogata ([84]).

Proposition 2.2.7. *Let $\gamma : \tau \to P(X)$ be an operation on a topological space (X, τ). Then (X, τ) is a γ-regular space if and only if $\tau = \tau_\gamma$.*

Now we introduce the notion of $sg.\gamma$-closed sets and investigate the relation between $sg.\gamma$-closed sets and γ-g.closed sets due to Ogata ([84]).

Definition 2.2.8. A subset A of (X, τ) is said to be:

(a) *γ-g.closed* [84] if $\mathrm{Cl}_\gamma(A) \subset U$ whenever $A \subset U$ and U is γ-open in (X, τ);

(b) *$sg.\gamma$-closed* if $\mathrm{Cl}_\gamma(A) \subset U$ whenever $A \subset U$ and U is open in (X, τ).

Remark 2.2.9. From above definition and Definition 2.1 of Levine ([53]), we obtain the following diagram:

$$\begin{array}{ccccc} \gamma\text{-}closed & \to & sg.\gamma\text{-}closed & \to & \gamma\text{-}g.closed \\ \downarrow & & \downarrow & & \\ closed & \to & g\text{-}closed & & \end{array}$$

Example 2.2.10. (a) Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Let $\gamma : \tau \to P(X)$ be an operation defined by $\{a, b\}^\gamma = \{a, b\}$ and $A^\gamma = \mathrm{Cl}(A)$ if $A(\neq \{a, b\}) \in \tau$. Then $\{a\}$ is γ-g.closed in (X, τ) but not g-closed. Also $\{b, c\}$ is $sg.\gamma$-closed and closed in (X, τ) but not γ-closed.

(b) Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b, c\}\}$. Let $\gamma : \tau \to P(X)$ be an operation defined by $\{a\}^\gamma = \{a, c\}$ and $A^\gamma = \mathrm{Cl}(A)$ if $A(\neq \{a\}) \in \tau$. Then $\{b, c\}$ is closed in (X, τ) but not γ-g.closed. Also, $\{a, b\}$ is $sg.\gamma$-closed in (X, τ) but not closed.

In 1991 Ogata ([84, Proposition 4.6]) proved that if τ_γ-$\mathrm{Cl}(\{x\}) \cap A \neq \phi$ for every $x \in \mathrm{Cl}_\gamma(A)$, then A is γ-g.closed in (X, τ). We need the following lemma to improve this result.

Lemma 2.2.11. (a) *If A is any subset and B is a γ-open set in (X, τ) with $A \cap B = \phi$, then τ_γ-$\mathrm{Cl}(A) \cap B = \phi$.*

(b) *For any subset A of (X, τ), τ_γ-$\mathrm{Cl}(A)$ is γ-closed.*

Proof. (a): The proof is straightforward by using Proposition 3.3 of Ogata ([84]).

(b): Let $x \notin \tau_\gamma$-$\mathrm{Cl}(A)$. By Proposition 3.3 of Ogata ([84]) there exists a γ-open set U containing x such that $U \cap A = \phi$ and thus by (a), $U \cap \tau_\gamma$-$\mathrm{Cl}(A) = \phi$. This shows that τ_γ-$\mathrm{Cl}(A)$ is γ-closed. $\square$

Proposition 2.2.12. *τ_γ-$\mathrm{Cl}(\{x\}) \cap A \neq \phi$ for every $x \in \mathrm{Cl}_\gamma(A)$ if and only if A is γ-g.closed in (X, τ).*

Proof. By Lemma 2.2.11 (b) τ_γ-$\mathrm{Cl}(\{x\})$ is γ-closed, the proof is same manner as Proposition 4.6 of Ogata ([84]). $\square$

In the following theorem, we obtain the part (a) replacing an "regular operator" condition due to Ogata in [84, Remark 4.8] with a "open operator" condition.

Theorem 2.2.13. *Let A be a subset of (X, τ) and $\gamma : \tau \to P(X)$ be an operation.*

(a) *If A is γ-g.closed in (X, τ), then $\mathrm{Cl}_\gamma(A) \setminus A$ contains no nonempty γ-closed set. The converse is true if γ is an open operator.*

(b) *A subset A is $sg.\gamma$-closed if and only if $\mathrm{Cl}_\gamma(A) \setminus A$ contains no nonempty closed set.*

Proof. (a): Let F be a γ-closed subset of $\mathrm{Cl}_\gamma(A) \setminus A$. Now, $A \subset X \setminus F$ and since A is γ-g.closed, we have $\mathrm{Cl}_\gamma(A) \subset X \setminus F$ or $F \subset X \setminus \mathrm{Cl}_\gamma(A)$. Thus $F \subset \mathrm{Cl}_\gamma(A) \cap X \setminus \mathrm{Cl}_\gamma(A) = \phi$ and F is empty.

Suppose that $A \subset U$ and U is γ-open. If $\mathrm{Cl}_\gamma(A) \not\subset U$, then by Theorem 3.6 (iii) of Ogata ([84]) $\mathrm{Cl}_\gamma(A) \cap X \setminus U$ is a nonempty γ-closed subset of $\mathrm{Cl}_\gamma(A) \setminus A$. This is a contradiction.

(b): The proof is similar to (a) by using Theorem 3.6 (i) of Ogata ([84]). □

Recall that (X, τ) is called a γ-regular space [48] if for each $x \in X$ and every open neighborhood U of x there exists an open neighborhood V of x such that $V^\gamma \subset U$.

Our next results follow easily from Theorem 2.2.13 and Theorem 3.6 (ii) of Ogata ([84]).

Corollary 2.2.14. (a) *Let $\gamma : \tau \to P(X)$ be an open operator. Then γ-g.closed set A is γ-closed if and only if $\mathrm{Cl}_\gamma(A) \setminus A$ is γ-closed.*

(b) *Let (X, τ) be a γ-regular space. Then $sg.\gamma$-closed set A is closed if and only if $\mathrm{Cl}_\gamma(A) \setminus A$ is closed.*

The following are well-known definitions of γ-T_i spaces introduced by Ogata ([84]), where $i = 0, 1/2, 1$ or 2.

Definition 2.2.15. A space (X, τ) is called

(a) γ-T_0 if for each distinct points $x, y \in X$, there exists an open set U such that either $x \in U$ and $y \notin U^\gamma$, or $y \in U$ and $x \notin U^\gamma$;

(b) γ-$T_{1/2}$ if every γ-g.closed set of (X, τ) is γ-closed;

(c) γ-T_1 if for each distinct points $x, y \in X$, there exist open sets U and V containing x and y, respectively, such that $y \notin U^\gamma$ and $x \notin V^\gamma$;

(d) γ-T_2 if for each distinct points $x, y \in X$, there exist open sets U and V such that $x \in U$, $y \in V$ and $U^\gamma \cap V^\gamma = \phi$.

Remark 2.2.16. From the above definition and Theorem 1 and Remark 4 of Ogata ([85]), we obtain the following diagram:

$$\begin{array}{ccccccc} \gamma\text{-}T_2 & \rightarrow & \gamma\text{-}T_1 & \rightarrow & \gamma\text{-}T_{1/2} & \rightarrow & \gamma\text{-}T_0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ T_2 & \rightarrow & T_1 & \rightarrow & T_{1/2} & \rightarrow & T_0 \end{array}$$

Throughout the rest of this section, let (X, τ) and (Y, σ) be spaces, and let $\gamma : \tau \rightarrow P(X)$ and $\beta : \sigma \rightarrow P(Y)$ be operations on τ and σ, respectively. Let id be the identity operator.

The following are well-known definitions of the (γ, β)-operator functions introduced by Ogata ([84]).

Definition 2.2.17. A mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ said to be

(a) (γ, β)-continuous if for each $x \in X$ and each open set V containing $f(x)$ there exists an open set U such that $x \in U$ and $f(U^\gamma) \subset V^\beta$;

(b) (γ, β)-closed if for any γ-closed F of (X, τ), $f(F)$ is β-closed in (Y, σ).

Proposition 2.2.18. *Suppose that $f : (X, \tau) \rightarrow (Y, \sigma)$ is an (id, β)-closed mapping.*

(a) *If A is sg.γ-closed in (X, τ) and if f is continuous, then $f(A)$ is sg.β-closed in (Y, σ).*

(b) *If A is γ-g.closed in (X,τ) and if f is (γ, id)-continuous, then $f(A)$ is sg.β-closed in (Y,σ).*

(c) *If A is sg.γ-closed in (X,τ) and if f is (id,β)-continuous, then $f(A)$ is β-g.closed in (Y,σ).*

Proof. (a): Let V be any open set of Y such $f(A) \subset V$. Since f is continuous and A is $sg.\gamma$-closed, $\mathrm{Cl}_\gamma(A) \subset f^{-1}(V)$ and hence $f(\mathrm{Cl}_\gamma(A)) \subset V$. By proposition 3.6 (i) of Ogata ([84]) and assumption, $f(\mathrm{Cl}_\gamma(A))$ is β-closed in (Y,σ). Thus we have $\mathrm{Cl}_\beta(f(A)) \subset \mathrm{Cl}_\beta(f(\mathrm{Cl}_\gamma(A))) = f(\mathrm{Cl}_\gamma(A)) \subset V$. This implies $f(A)$ is $sg.\beta$-closed in (Y,σ).

The proofs of (b) and (c) are similar to (a). □

Proposition 2.2.19. *Suppose that $f : (X,\tau) \to (Y,\sigma)$ is a (γ,β)-continuous mapping.*

(a) *If B is sg.β-closed in (Y,σ) and if f is closed, then $f^{-1}(B)$ is sg.γ-closed in (X,τ).*

(b) *If B is β-g.closed in (Y,σ) and if f is (id,β)-closed, then $f^{-1}(B)$ is sg.γ-closed in (X,τ).*

Proof. (a): Let U be any open subset of $(X,\tau$ such that $f^{-1}(B) \subset U$. Put $F = \mathrm{Cl}_\gamma(f^{-1}(B)) \cap (X \setminus U)$. Then by Theorem 3.6 (i) of Ogata ([84]), F is closed in (X,τ). Since f is closed, $f(F)$ is closed in (Y,σ) and $f(F) \subset \mathrm{Cl}_\beta(B) \setminus B$ by using Proposition 4.13 of Ogata ([84]). By Theorem 2.2.13, $f(F) = \phi$ and so $F = \phi$, i.e. $\mathrm{Cl}_\gamma(f^{-1}(B)) \subset U$. Hence $f^{-1}(B)$ is $sg.\gamma$-closed in (X,τ).

(b): The proof is similar to (a) by using Theorem 2.2.13. □

2.3 Supra-Topological Spaces

In this section we will review some concepts and results obtained in supra topological spaces that will be used throughout the thesis.

We first state the following definitions.

Definition 2.3.1. [70, Definition 1.1] A subclass $\tau^* \subset P(X)$ is called a supra topology on X if $X \in \tau^*$ and τ^* is closed under arbitrary union. (X, τ^*) is called a supra-topological space (briefly, supra space). The members of τ^* are called supra open sets.

Definition 2.3.2. [70, Definition 1.2] Let (X, τ) be a topological space and let τ^* be a supra topology on X. A supra topology τ^* is associated with τ if $\tau \subset \tau^*$.

Note that various notions like the interior and closure operators and derived set as well as set properties like the open, closed, dense and compact etc. can be defined in a supra-topological space in analogy with topological spaces. Many results of topological spaces remain valid in supra topological spaces, whereas some become false. The supra derived set (resp. supra closure, supra interior, supra boundary) of a subset A of a space (X, τ) will be denoted by sup-Der(A) (resp. sup-Cl(A), sup-Int(A), sup-Bd(A)).

The following is the main properties of such operations which give the derivations between these operations and that in topological spaces.

Theorem 2.3.3. [70, Theorem 1.1] *Let (X, τ^*) be a supra space and let A and B be a subset of (X, τ^*), then*

(a) sup-Der(A) $\cup$ sup-Der(B) $\subset$ sup-Der($A \cup B$);

(b) sup-Cl(A) $\cup$ sup-Cl(B) $\subset$ sup-Cl($A \cup B$);

(c) sup-Int($A \cap B$) $\subset$ sup-Int(A) $\cap$ sup-Int(B).

Remark 2.3.4. It can be easily shown, by example [70, Example in §1], that the inequalities in (a), (b) and (c) cannot be replaced, in general, by equalities as in the topological spaces.

Definition 2.3.5. [70, Definition 2.1 and 3.1] Let (X, τ) and (Y, σ) be topological spaces and let τ^* and σ^* be associated supra topologies with τ and σ respectively.

(a) A function $f : (X, \tau^*) \to (Y, \sigma)$ is supra continuous (briefly, S-continuous) if $f^{-1}(U)$ is supra open in (X, τ^*) for every open set U in (Y, σ).

(b) A function $f : (X, \tau^*) \to (Y, \sigma^*)$ is supra* continuous (briefly, S^*-continuous) if $f^{-1}(U)$ is supra open in (X, τ^*) for every supra open set U in (Y, σ^*).

From the above definitions, the following is a well-known diagram of Mashhour et al. ([70]).

$$\text{Continuity} \longrightarrow \text{Supra Continuity} \longleftarrow \text{Supra}^* \text{ Continuity}$$

The following is a well-known Theorem of Mashhour et al. ([70]).

Theorem 2.3.6. [70, see Remark(iv)] *Let (X, τ) and (Y, σ) be topological spaces, and let τ^* and σ^* be associated supra topologies with τ^* and σ^*, respectively. If one of the following holds:*

(a) $f^{-1}(\text{sup-Int}(A)) \subset \text{Int}(f^{-1}(A))$ *for each* $A \subset Y$,

(b) $\text{Cl}(f^{-1}(A)) \subset f^{-1}(\text{sup-Cl}(A))$ *for each* $A \subset Y$,

(c) $f(\text{Cl}(A)) \subset \text{sup-Cl}(f(A))$ *for each* $A \subset X$

then $f : (X, \tau^*) \to (Y, \sigma^*)$ is a continuous function.

Remark 2.3.7. (a) Many properties of semi-continuous [52] (resp. pre-continuous [67], α-continuous [69]) and β-continuous [5](or semi-precontinuous [75]) functions can be easily deduced from the previous results of supra continuity by setting $\tau^* = SO(X,\tau)$, (resp. $\tau^* = PO(X,\tau)$, $\tau^* = \alpha(X,\tau)$ and $\tau^* = \beta(X,\tau)$).

(b) Many properties of irresolute [20] (resp. M-precontinuous [67], resp. M-α-continuous [69]) functions are special cases of results [70, see Theorem 3.2] of supra* continuous functions by setting $\tau^* = SO(X,\tau)$ and $\sigma^* = SO(Y,\sigma)$ (resp. $\tau^* = PO(X,\tau)$ and $\sigma^* = PO(Y,\sigma)$, resp. $\tau^* = \alpha(X,\tau)$ and $\sigma^* = \alpha(Y,\sigma)$).

Next we recall that some concepts of separation axioms in supra-topological spaces that will be used throughout the dissertation.

Definition 2.3.8. [70, Definition 4.1] Let (X,τ) be a topological space and let τ^* be a supra topology associated with τ. Then the space (X,τ) is called:

(a) supra-T_0 (briefly, S-T_0) if every two distinct points of X, there exists a supra open neighborhood of one of them to which the other does not belong.

(b) supra-T_1 (briefly, S-T_1) if for every two distinct points x and y in X, there exist two supra open sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

(c) supra-T_2 (briefly, S-T_2) if for every two distinct points x and y in X, there exist two disjoint supra open sets U and V such that $x \in U$ and $y \in V$.

From the above definitions, one can draw the following diagram:

$$\begin{array}{ccccc} T_2 & \longrightarrow & T_1 & \longrightarrow & T_0 \\ \downarrow & & \downarrow & & \downarrow \\ S\text{-}T_2 & \longrightarrow & S\text{-}T_1 & \longrightarrow & S\text{-}T_0 \end{array}$$

The following are well-known theorems of Mashhour et al. ([70]).

Theorem 2.3.9. [70, Theorem 4.3] *The property of being a supra-T_i space is supra-topological property, $i = 0, 1$ or 2.*

Theorem 2.3.10. [70, Theorem 4.4] *The property of being supra-T_2 is preserved under the supra*-open bijection.*

Remark 2.3.11. Various properties of semi-T_i [56] (resp. pre-T_i [47], resp. α-T_i [73] (or feebly-T_i [57]), resp. semipre-T_i [74]) spaces can be easily obtained from the properties of supra-T_i spaces by setting $\tau^* = SO(X, \tau)$ (resp. $\tau^* = PO(X, \tau)$, resp. $\tau^* = \alpha(X, \tau)$, resp. $\tau^* = SPO(X, \tau)$), where $i = 0, 1$ or 2.

Chapter 3

Operations and New Open Sets

In this chapter, we first introduce the concept of operators γ^* and as using this concept, we also introduce the concepts of γ^*-supra open sets, the concept of closure operators and the concept of interior operators on the supra space (X, τ^*), and investigate their basic properties.

3.1 γ^*-Supra Open Sets

First of all we define the concept of γ^*-operations in this section and investigate the difference and relationship between the operation introduced by Kasahara [48] and a new operation which will be defined in this section.

Definition 3.1.1. Let (X, τ^*) be a supra space. An *operation* γ^* on the supra topology τ^* is a mapping from τ^* into the power set $P(X)$ of X such that $U \subset U^{\gamma^*}$ for each $U \in \tau^*$, where U^{γ^*} denotes the value of γ^* at U. It is denoted by $\gamma^* : \tau^* \to P(X)$.

For example, the mapping γ^* defined by $\gamma^*(G)(= G^{\gamma^*}) = G$ for each $G \in \tau^*$

is an operation on τ^* which will be called the *identity operator* on τ^* in what follows. The supra closure operator in a supra space defines of course an operation on the supra topology, and the composition sup-Int $\circ$ sup-Cl of the supra closure operator sup-Cl with the supra interior operator sup-Int is also an operator on τ^*; we shall call the former the *supra closure operator* on τ^* and the latter the *supra interior-supra closure operator* on τ^*.

Definition 3.1.2. A subset A of supra space (X, τ^*) is said to be γ^**-supra open* in (X, τ^*) if, for each $x \in A$, there exists a supra open set U such that $x \in U$ and $U^{\gamma^*} \subset A$. $\tau^*_{\gamma^*}$ will denote the set of all γ^*-supra open sets. A subset B of (X, τ^*) is said to be γ^*-supra closed in (X, τ^*) if $X \setminus B$ is γ^*-supra open.

From the definitions above, we obtain the following basic results:

Proposition 3.1.3. *Let τ^* be a supra topology on a set X; let $\gamma^* : \tau^* \to P(X)$ be an operation ; and let Λ be an index set. If $A_i \in \tau^*_{\gamma^*}$* (i.e., A_i is a γ^*-supra open set) *for each $i \in \Lambda$, then $\cup\{A_i | i \in \Lambda\} \in \tau^*_{\gamma^*}$.*

Proof. Let $x \in \cup_{i\in\Lambda} A_i$. Then $x \in A_j$ for some $j \in \Lambda$. Since A_j is γ^*-supra open, there exists a supra open set U such that $x \in U^{\gamma^*} \subset A_j \subset \cup_{i\in\Lambda} A_i$. This implies that $\cup\{A_i | i \in \Lambda\}$ is a γ^*-supra open set. □

The following proposition shows that τ^* is finer than $\tau^*_{\gamma^*}$.

Proposition 3.1.4. *Let (X, τ^*) be a supra space. If a subset A of (X, τ^*) is γ^*-supra open, then it is supra open.*

Proof. Let $x \in A$. Then there exists a supra open set U such that $x \in U$ and $U^{\gamma^*} \subset A$. Now,

$$A \subset \cup_{x\in A} U \subset \cup_{x\in A} U^{\gamma^*} \subset A$$

and hence A is the supra open set since $\cup_{x\in A}U$ is the supra open set for each $x \in A$. □

The reverse in the proposition above is not always true. Next we give an example of a supra open set which is not γ^*-supra open.

Example 3.1.5. Let $X = \{a, b, c\}$ and $\tau^* = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ be a supra topology on X. Define an operation $\gamma^* : \tau^* \to P(X)$ by $\gamma^*(A)$ =sup-Cl(A) for each A of (X, τ^*). It is easy to check that every non-empty proper supra open set in (X, τ^*) is not γ^*-supra open.

Based on the preceeding discussion (i.e. Proposition 3.1.3 and 3.1.4 and Example 3.1.5), we have the following theorem.

Theorem 3.1.6. *Let (X, τ^*) be a supra space. Then the class $\tau^*_{\gamma^*}$ of all γ^*-supra open sets is supra topology such that $\tau^*_{\gamma^*} \subset \tau^*$.*

Definition 3.1.7. A supra space (X, τ^*) is said to be γ^*-*regular*, where γ^* is an operation on τ^*, if for each $x \in X$ and for each supra open neighborhood U of x, there exists a supra open neighborhood V of x such that V^{γ^*} is contained in U.

Using the notion of the γ^*-regularity introduced in the definition above, it turns out that the supra topology τ^* on a set coincides with $\tau^*_{\gamma^*}$. We next prove this claim:

Proposition 3.1.8. *A supra space (X, τ^*) is γ^*-regular if and only if $\tau^* = \tau^*_{\gamma^*}$.*

Proof. For each $x \in X$ and for each supra open neighborhood U of x, since $U \in \tau^* = \tau^*_{\gamma^*}$, there exists a supra open neighborhood V of x such that $V^{\gamma^*} \subset U$. This implies that (X, τ^*) is γ^*-regular.

To see the converse, it is sufficient to discuss that $\tau^* \subset \tau^*_{\gamma^*}$. Let $A \in \tau^*$ and $x \in A$. Then there is a supra open neighborhood U of x such that $U \subset A$. By virtue of the γ^*-regularity, we can find a supra open neighborhood W of x such that $W^{\gamma^*} \subset U$ and so $W^{\gamma^*} \subset A$. Hence A is γ^*-supra open and thus we have $\tau^* \subset \tau^*_{\gamma^*}$, as desired. □

Next we introduce the concepts of regular operators and open operators on a supra topology τ^*.

Definition 3.1.9. Let (X, τ^*) be a supra space. An operation γ^* on τ^* is said to be *regular* if, for every supra open neighborhood U and V of each $x \in X$, there exists a supra open neighborhood W of x such that $W^{\gamma^*} \subset U^{\gamma^*} \cap V^{\gamma^*}$.

Definition 3.1.10. An operation γ^* on τ^* is said to be *open* if, for every supra open neighborhood U of each $x \in X$, there exists a γ^*-supra open set V such that $x \in V$ and $V \subset U^{\gamma^*}$.

Note that the notions of the regular operator and the open operator on τ^* are independent from each other as the following examples.

Example 3.1.11. Consider again the supra space (X, τ^*) and the operation γ^* defined in Example 3.1.5. Then clearly, $\tau^*_{\gamma^*} = \{\phi, X\}$. It is easy to see that γ^* is a regular operator but not an open operator on τ^*.

Example 3.1.12. Let $X = \{a, b, c\}$ and let $\tau^* = \{\phi, \{b\}, \{a, b\}, \{a, c\}, X\}$ be a supra topology on X. Define an operation $\gamma^* : \tau^* \to P(X)$ by

$$\gamma^*(A) = A^{\gamma^*} = \begin{cases} A & \text{if } b \in A \\ \text{sup-Cl}(A) & \text{if } b \notin A. \end{cases}$$

Then the operation γ^* is not regular on τ^*. In fact, let $U = \{a, b\}$ and $V = \{a, c\}$ be supra open neighborhoods of $\{a\}$ then $U^{\gamma^*} \cap V^{\gamma^*} = \{a\}$ and $W^{\gamma^*} \not\subset U^{\gamma^*} \cap V^{\gamma^*}$ for any supra open neighborhood of $\{a\}$. Moreover, we can show that γ^* is an open operator on τ^*.

The following proposition provides a useful criterion for proving that $\tau^*_{\gamma^*}$ is a topology on a set.

Proposition 3.1.13. *Let $\gamma^* : \tau^* \to P(X)$ be a regular operator on τ^*. If A and B are γ^*-supra open, then $A \cap B$ is γ^*-supra open as well.*

Proof. Let $x \in A \cap B$. Then there exist supra open neighborhoods U and V of x such that $U^{\gamma^*} \subset A$ and $V^{\gamma^*} \subset B$, so that $U^{\gamma^*} \cap V^{\gamma^*} \subset A \cap B$. Since γ^* is regular, $W^{\gamma^*} \subset U^{\gamma^*} \cap V^{\gamma^*}$ for some supra open neighborhood W of x, and so $W^{\gamma^*} \subset A \cap B$. This shows that $A \cap B$ is γ^*-supra open. □

In the proposition above, if it is removing the condition that γ^* is a regular operator, then the finite intersection of γ^*-supra open sets may fail to be γ^*-supra open as follows:

Example 3.1.14. By the supra space (X, τ^*) and the operation γ^* of Example 3.1.12, it may be observed that the set $\{a\}$ of the intersection $\{a, b\}$ and $\{a, c\}$ is not γ^*-supra open, where $\{a, b\}$ and $\{a, c\}$ are γ^*-supra open, respectively.

In general, the collection $\tau^*_{\gamma^*}$ of all γ^*-supra open sets is not a topology on X. Consider again the operation γ^* on τ^* with a regular operator. Then we have the following result.

Theorem 3.1.15. *If $\gamma^* : \tau^* \to P(X)$ is a regular operator on τ^*, then $\tau^*_{\gamma^*}$ is a topology on X.*

Proof. By Definition 3.1.2 and Proposition 3.1.13, the proof is trivial. □

Remark 3.1.16. The concept of the operation γ introduced by Kasahara ([48]) is defined on a topology τ and the class τ_γ of all γ-open sets need not be a topology but the implication $\tau_\gamma \subset \tau$ is well known. However, the notion of γ^* operations is defined on a supra topology τ^* and the class τ_{γ^*} of all γ^*-open sets is a supra topology but not a topology.

We conclude this section with a brief discussion of the operation on an associated supra topology with a topology. This discussion provides the additional relation between the family of all γ-open sets and that of all γ^*-supra open sets.

Now, we shall present the concept of operations γ^* on a supra topology τ^* associated with operations γ on a topology τ at hand as follows:

Definition 3.1.17. Let $\gamma : \tau \to P(X)$ be any operation and let τ^* be a supra topology on X. An operation γ^* on τ^* is called *associated with* γ having the following properties:

(a) τ^* is associated with τ (i.e. $\tau \subset \tau^*$).

(b) $\gamma^*|_\tau = \gamma$ (i.e. γ is the restriction of γ^*).

From the definition above and Proposition 3.1.4, we have the following diagram:

$$\begin{array}{ccc} \gamma\text{-open sets} & \rightarrow & \gamma^*\text{-supra open sets} \\ \downarrow & & \downarrow \\ \text{open sets} & \rightarrow & \text{supraopen sets} \end{array}$$

In the diagram above, none of them of course is reversible as the following remark and examples show:

Remark 3.1.18. Let γ^* be an associated operation with γ. Then Example 3.1.19 provides an example of γ^*-supra open sets that is not γ-open, and Example 3.1.20 does one of supra open sets that is not γ^*-supra open. The rest examples are showed by Kasahara ([48]) (i.e. every open set is not always γ-open) and Mashhour et al. ([70]) (i.e. every supra open set is not always open).

Example 3.1.19. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, \{d\}, \{a, b\}, \{c, d\}, \{a, b, d\}\}$,

$\tau^* = \{\phi, \{d\}, \{a, b\}, \{c, d\}, \{a, b, d\}, \{a, c\}, \{b, c, d\}, \{a, c, d\}, \{a, b, c\}\}$.

Define an operation $\gamma^* : \tau^* \rightarrow P(X)$ by the identity (briefly, id) and define $\gamma : \tau \rightarrow P(X)$ by $\gamma(A) = \text{Int}(A)$ for $A \in \tau$. We then see that not every γ^*-supra open set is γ-open.

Example 3.1.20. Consider again (X, τ^*) is supra space and γ^* is operation in Example 3.1.5. If $\tau = \{\phi, \{a\}, X\}$, then the operation γ^* is associated with γ. It is easy to see that the sets $\{a\}$ and $\{a, b\}$ are supra open but not γ^*-supra open on (X, τ^*).

3.2 γ^*-Supra Closed Sets and Closure Operators

Recall that a γ^*-supra open set in a supra space (X, τ^*) is a subset of X that is a member of $\tau^*_{\gamma^*}$. We have introduced a supra space in terms of the properties that the γ^*-supra open sets must satisfy. In this section we study γ^*-supra closed sets and the γ^*-supra closure and $\tau^*_{\gamma^*}$-supra closure operators using various kinds of γ^*-operations.

We begin with the following two types definition of closure operators via various kinds of γ^*-operations.

Definition 3.2.1. A point $x \in X$ is in the γ^**-supra closure* of a subset A of the supra space (X, τ^*) if for each supra open neighborhood U of x, $U^{\gamma^*} \cap A \neq \phi$. The γ^*-supra closure of the set A is denoted by $\mathrm{Cl}_{\gamma^*}(A)$.

Notation. The point described above is called a γ^**-supra closure point* of A.

Definition 3.2.2. For the family $\tau^*_{\gamma^*}$, the $\tau^*_{\gamma^*}$*-supra closure* of a subset A of a supra space (X, τ^*) is defined as the intersection of all the γ^*-supra closed sets containing A. The $\tau^*_{\gamma^*}$-supra closure of the set A is denoted by $\tau^*_{\gamma^*}$-$\mathrm{Cl}(A)$.

Remark 3.2.3. $\tau^*_{\gamma^*}$-$\mathrm{Cl}(A)$ is the "smallest" γ^*-supra closed set that contains A; that is, if B is any γ^*-supra closed set such that $A \subset B$, then $\tau^*_{\gamma^*}$-$\mathrm{Cl}(A) \subset B$. By the compliment relation of Proposition 3.1.3, the intersection of any number of γ^*-supra closed sets is γ^*-supra closed. Therefore the $\tau^*_{\gamma^*}$-supra closure of a subset of a supra space is a γ^*-supra closed set.

The following theorem provides a useful method of determining whether a point belongs to the $\tau^*_{\gamma^*}$-supra closure of a set.

Theorem 3.2.4. *Let A be a subset of the supra space (X, τ^*) and let $x \in X$. Then $x \in \tau^*_{\gamma^*}$-Cl(A) if and only if every γ^*-supra open set U containing x has a nonempty intersection with A.*

Proof. Our theorem is easy to prove. Suppose that $x \notin \tau^*_{\gamma^*}$-Cl$(A)$ for each $x \in X$. Then $x \in X \setminus \tau^*_{\gamma^*}$-Cl$(A)$. If we set $U = X \setminus \tau^*_{\gamma^*}$-Cl$(A)$, it is a γ^*-supra open set containing x and $U \cap A = \phi$ as desired.

Conversely, suppose that there exists a γ^*-supra open set U containing x such that $U \cap A = \phi$. Then $X \setminus U$ is a γ^*-supra closed set containing A. By definition of $\tau^*_{\gamma^*}$-supra closure $\tau^*_{\gamma^*}$-Cl(A), the set $X \setminus U$ must contain $\tau^*_{\gamma^*}$-Cl(A). Hence x cannot be in $\tau^*_{\gamma^*}$-Cl(A). The proof is therefore completed. □

Notions. The point described above is called a *$\tau^*_{\gamma^*}$-supra closure point* of A.

We shall represent the statement "U is a γ^*-supra open set containing x" to the phrase "U is a *γ^*-supra open neighborhood* of x" in the sequel.

From the above definitions and theorem we have:

Proposition 3.2.5. *Let A be any subset of (X, τ^*) and $x \in X$.*

(a) *If x is in* $\mathrm{Cl}_{\gamma^*}(A)$, *then it belongs to* $\tau^*_{\gamma^*}$-Cl(A).

(b) *If x is in* sup-Cl(A), *then it belongs to* $\mathrm{Cl}_{\gamma^*}(A)$.

Proof. (a): Suppose that $x \notin \tau^*_{\gamma^*}$-Cl$(A)$. Then there exists γ^*-supra open V with $x \in V$ such that $V \cap A = \phi$. Since $V \in \tau^*_{\gamma^*}$ and $x \in V$, for each supra neighborhood U of x, $U^{\gamma^*} \subset V$ and hence $U^{\gamma^*} \cap A = \phi$. Therefore $x \notin \mathrm{Cl}_{\gamma^*}(A)$.

(b): The proof is similar to that of (a). □

However the reverse in the proposition above is not always true. Next we give examples of the concept of the γ^*-supra closure (resp. supra closure) of a set which is properly contained in the concept of the $\tau^*_{\gamma^*}$-supra closure (resp. γ^*-supra closure).

Example 3.2.6. Let $X = \{a, b, c, d\}$ and $\tau^* = \{\phi, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Define an operation $\gamma^* : \tau^* \to P(X)$ by $\gamma^*(A) = \text{sup-Int}(A)$ for a subset A of (X, τ^*). If $A = \{a, d\}$ then $\tau^*_{\gamma^*}\text{-Cl}(A) = X$ and $\text{Cl}_{\gamma^*}(A) = \{a, d\}$.

Clearly a point $b \in \tau^*_{\gamma^*}\text{-Cl}(A)$ but $b \notin \text{Cl}_{\gamma^*}(A)$. Note that the concept of the γ^*-supra closure of a set which is properly contained in the concept of the $\tau^*_{\gamma^*}$-supra closure.

Example 3.2.7. Consider again (X, τ^*) defined in Example 3.1.19. Define an operation $\gamma^* : \tau^* \to P(X)$ by $\gamma^*(A) = \text{sup-Cl}(A)$ for a subset A of (X, τ^*). If $A = \{b, d\}$ then $\text{Cl}_{\gamma^*}(A) = X$ and $\text{sup-Cl}(A) = \{b, d\}$. It is obvious that a point $a \notin \text{Cl}_{\gamma^*}(A)$ but $a \notin \text{sup-Cl}(A)$.

In the Proposition 3.2.5 and Example 3.2.6 and 3.2.7 we have the following diagram with implications of which none is reversible:

$$A \subset \text{sup-Cl}(A) \subset \text{Cl}_{\gamma^*}(A) \subset \tau^*_{\gamma^*}\text{-Cl}(A).$$

There is yet another way of describing the closure of a set via γ^*-operations. Such a way is that involves the important concept of the limit point via γ^*-operations, which we consider now.

Definition 3.2.8. Let A be a subset of the supra space (X, τ^*) and let x be a point in X. We say that:

(a) x is a γ^*-*limit point* of A if for each supra open neighborhood U of x, $U^{\gamma^*} \cap A \setminus \{x\} \neq \phi$;

(b) x is a $\tau^*_{\gamma^*}$-*limit point* of A if for each γ^*-supra open neighborhood U of x, $U \cap A \setminus \{x\} \neq \phi$.

Differently put for the definitions above, x is a γ^*-limit point (resp. $\tau^*_{\gamma^*}$-limit point) of A if it belongs to the γ^*-closure (resp. $\tau^*_{\gamma^*}$-closure) of $A \setminus \{x\}$.

We let $\mathrm{Der}_{\gamma^*}(A)$ (resp. $\tau^*_{\gamma^*}$-$\mathrm{Der}(A)$) denote the set of all γ^*-limit points (resp. $\tau^*_{\gamma^*}$-limit points) of A.

Straight from the definition above, we see that the following results.

Proposition 3.2.9. *Let x be any point of X and A be a subset of (X, τ^*). If x is in $\mathrm{Der}_{\gamma^*}(A)$, then x belongs to $\tau^*_{\gamma^*}$-$\mathrm{Der}(A)$.*

Proof. We obtain that every γ^*-supra open set U contains the V^{γ^*} for each $x \in U$, where $V \in \tau^*_{\gamma^*}$ and thus the proof is obvious. □

Proposition 3.2.10. *Let A and B be a subset of (X, τ^*).*

(a) *If A is a subset of B, then $\mathrm{Der}_{\gamma^*}(A) \subset \mathrm{Der}_{\gamma^*}(B)$.*

(b) *If A is a subset of B, then $\tau^*_{\gamma^*}$-$\mathrm{Der}(A) \subset \tau^*_{\gamma^*}$-$\mathrm{Der}(B)$.*

Proof. (a): Let $x \in \mathrm{Der}_{\gamma^*}(A)$. Then for every supra open neighborhood U of x, $U^{\gamma^*} \cap A \setminus \{x\} \neq \phi$. But since $A \subset B$, $U^{\gamma^*} \cap B \setminus \{x\} \neq \phi$ and hence $x \in \mathrm{Der}_{\gamma^*}(B)$. This completes the proof.

(b): This follows similarly to (a). □

A relationship between the γ^*-closure (resp. $\tau^*_{\gamma^*}$-closure) operators and the γ^*-limit points (resp. $\tau^*_{\gamma^*}$-limit points) of a set is given in the following theorem:

Theorem 3.2.11. *Let A be a subset of the supra space (X, τ^*).*

(a) $\mathrm{Cl}_{\gamma^*}(A) = A \cup \mathrm{Der}_{\gamma^*}(A)$.

(b) $\tau^*_{\gamma^*}\text{-}\mathrm{Cl}(A) = A \cup \tau^*_{\gamma^*}\text{-}\mathrm{Der}(A)$.

Proof. (a): Suppose that $x \in A \cup \mathrm{Der}_{\gamma^*}(A)$. If $x \in \mathrm{Der}_{\gamma^*}(A)$, then for every supra open neighborhood U of x, the set U^{γ^*} contains a point of A that is different from x. Therefore, by Definition 3.2.1, $x \in \mathrm{Cl}_{\gamma^*}(A)$. Hence $\mathrm{Der}_{\gamma^*}(A) \subset \mathrm{Cl}_{\gamma^*}(A)$. Since by Proposition 3.2.5(b), $A \subset \mathrm{Cl}_{\gamma^*}(A)$, it follows that $A \cup \mathrm{Der}_{\gamma^*}(A) \subset \mathrm{Cl}_{\gamma^*}(A)$.

To demonstrate the reverse, we let $x \in \mathrm{Cl}_{\gamma^*}(A)$ and we wish to show that $x \in A \cup \mathrm{Der}_{\gamma^*}(A)$. If x happens to lie in A, it is trivial that $x \in A \cup \mathrm{Der}_{\gamma^*}(A)$. Suppose that x does not lie in A. Since $x \in \mathrm{Cl}_{\gamma^*}(A)$, we know that for every supra open neighborhood U of x, $U^{\gamma^*} \cap A \neq \phi$. Because $x \notin A$, $U^{\gamma^*} \cap A \setminus \{x\} \neq \phi$. Then $x \in \mathrm{Der}_{\gamma^*}(A)$, so that $x \in A \cup \mathrm{Der}_{\gamma^*}(A)$, as desired.

(b): This proof is similar to the part (a). □

The following corollary slightly improves Proposition 3.2.10.

Corollary 3.2.12. *Let A and B be a subset of (X, τ^*). Then the following statements are true.*

(a) *If A is a subset of B, then $\mathrm{Cl}_{\gamma^*}(A) \subset \mathrm{Cl}_{\gamma^*}(B)$.*

(b) *If A is a subset of B, then $\tau^*_{\gamma^*}\text{-}\mathrm{Cl}(A) \subset \tau^*_{\gamma^*}\text{-}\mathrm{Cl}(B)$.*

Proof. This is immediate in view of the Theorem 3.2.11. □

Next we have a characterization of γ^*-supra closed sets on a supra space (X, τ^*).

Theorem 3.2.13. *A subset of a supra space is γ^*-supra closed if and only if it contains all its $\tau^*_{\gamma^*}$-limit points.*

Proof. We first show: if A is a γ^*-supra closed subset, then it must contain all its $\tau^*_{\gamma^*}$-limit points. We argue by contradiction. Suppose that A is a γ^*-supra closed subset and there is a $\tau^*_{\gamma^*}$-limit point x of A with $x \in X \setminus A$. But $X \setminus A$ is a γ^*-supra open subset containing x,which does not contain points of A, contradicting the assumption that x is a $\tau^*_{\gamma^*}$-limit point of A.

Secondly, suppose that every $\tau^*_{\gamma^*}$-limit point of A is contained in A. Let $x \in X \setminus A$. Since x is not a $\tau^*_{\gamma^*}$-limit point of A, there must be a γ^*-supra open subset of (X, τ^*), call it U_x, such that $U_x \cap A \setminus \{x\} = \phi$. Then $U_x \cap A = \phi$. But consider $U = \cup\{U_x | x \in X \setminus A\}$. Since U is the union of γ^*-supra open sets, it is γ^*-supra open. Therefore $A = X \setminus U$ is γ^*-supra closed. □

Another way of the describing γ^*-supra closed sets is given in the following theorem.

Theorem 3.2.14. *For a subset A of a supra topological space (X, τ^*), the following statements are equivalent:*

(a) *A is γ^*-supra closed in (X, τ^*).*

(b) $\mathrm{Cl}_{\gamma^*}(A) = A$.

(c) $\tau^*_{\gamma^*}\text{-Cl}(A) = A$.

Proof. (a)⇒(b): To see this, it is sufficient to show that $\mathrm{Cl}_{\gamma^*}(A) \subset A$. Indeed, suppose that $x \notin A$. Then, since $x \in X \setminus A$ and $X \setminus A$ is γ^*-supra open, there exists a supra open set U containing x such that $U^{\gamma^*} \subset X \setminus A$ and hence $U^{\gamma^*} \cap A = \phi$. Therefore $x \notin \mathrm{Cl}_{\gamma^*}(A)$, as desired.

(b)$\Rightarrow$(c): We wish only to show that $\tau^*_{\gamma^*}$-Cl$(A) \subset A$. Suppose that $x \notin A$. By (b), since $x \notin \mathrm{Cl}_{\gamma^*}(A)$, there exists a supra open set U containing x such that $U^* \cap A = \phi$ and hence $X \setminus A$ is γ^*-supra open. Then, we have $A \cap (X \setminus A) = \phi$. This implies $x \notin \tau^*_{\gamma^*}$-Cl$(A)$.

(c)$\Rightarrow$(a): By Definition 3.2.2 and Remark 3.2.3, the proof is obvious. □

For later use, we state the following lemma:

Lemma 3.2.15. *For any supra open subset U and any subset A of (X, τ^*), if $U^{\gamma^*} \cap A = \phi$ then $U \cap \mathrm{Cl}_{\gamma^*}(A) = \phi$.*

Proof. Suppose that $U \cap \mathrm{Cl}_{\gamma^*}(A) \neq \phi$. Then there exists a point $x \in X$ such that $x \in U \cap \mathrm{Cl}_{\gamma^*}(A)$ and thus $x \in U$ and $x \in \mathrm{Cl}_{\gamma^*}(A)$. But since $x \in \mathrm{Cl}_{\gamma^*}(A)$, a subset U^{γ^*} containing x has a nonempty intersection with A, contradicting the hypothesis that $U^{\gamma^*} \cap A = \phi$. □

We proceed to establish several fundamental results related to the concept of the γ^*-closure and the $\tau^*_{\gamma^*}$-closure of a set. Some additional results will also be obtained in the following section.

Theorem 3.2.16. *Let $\gamma^* : \tau^* \to P(X)$ be an associated operation with γ, where $\gamma : \tau \to P(X)$ and let A be a subset of (X, τ^*).*

(a) *The subset A is γ^*-supra closed if and only if $\tau^*_{\gamma^*}$-Cl$(A) = A$ holds.*

(b) *The subset $\mathrm{Cl}_{\gamma^*}(A)$ is supra closed in (X, τ^*).*

(c) *If (X, τ^*) is a γ^*-regular space, then $\mathrm{Cl}_{\gamma^*}(A) = \text{sup-Cl}(A)$ holds.*

(d) *If γ^* is open, then $\mathrm{Cl}_{\gamma^*}(A) = \tau^*_{\gamma^*}$-Cl$(A)$ and $\mathrm{Cl}_{\gamma^*}(\mathrm{Cl}_{\gamma^*}(A)) = \mathrm{Cl}_{\gamma^*}(A)$ hold, and furthermore $\mathrm{Cl}_{\gamma^*}(A)$ is also γ^*-supra closed in (X, τ^*).*

Proof. (a): By Remark 3.2.3 and Theorem 3.2.14 , the proof is obvious.

(b): To prove this statement, we wish to show that sup-Cl($\mathrm{Cl}_{\gamma^*}(A)$) $\subset \mathrm{Cl}_{\gamma^*}(A)$. Indeed, suppose that $x \notin \mathrm{Cl}_{\gamma^*}(A)$. Then there exists a supra open set U such that $x \in U$ and $U^{\gamma^*} \cap A = \phi$. By Lemma 3.2.15, $U \cap \mathrm{Cl}_{\gamma^*}(A) = \phi$ and hence $U \cap A = \phi$. Therefore $x \notin$ sup-Cl(A), as desired.

(c): By Proposition 3.2.5(b), we assert the following fact that the containment relation sup-Cl(A) $\subset \mathrm{Cl}_{\gamma^*}(A)$. To show this, it is sufficient to discuss $\mathrm{Cl}_{\gamma^*}(A) \subset$ sup-Cl(A). Let $x \in \mathrm{Cl}_{\gamma^*}(A)$ and let U be any supra open neighborhood of x. By the γ^*-regularity, there exists a supra open set V such that $x \in V$ and $V^{\gamma^*} \subset U$. Since $x \in \mathrm{Cl}_{\gamma^*}(A)$, $V^{\gamma^*} \cap A \neq \phi$ and thus $U \cap A \neq \phi$. Therefore $x \in$ sup-Cl(A).

(d): Suppose that $x \notin \mathrm{Cl}_{\gamma^*}(A)$. Then there exists a supra open neighborhood of x such that $U^{\gamma^*} \cap A = \phi$. Since γ^* is open, for U and $x (\in U)$, there exists a γ^*-supra open set V such that $x \in V \subset U^{\gamma^*}$. Then, we have $V \cap A = \phi$. This shows that $x \notin \tau^*_{\gamma^*}$-Cl($A$), and hence $\tau^*_{\gamma^*}$-Cl(A) $\subset \mathrm{Cl}_{\gamma^*}(A)$. By Proposition 3.2.5(a), we have the equality $\mathrm{Cl}_{\gamma^*}(A) = \tau^*_{\gamma^*}$-Cl($A$). Then we obtain $\mathrm{Cl}_{\gamma^*}(\mathrm{Cl}_{\gamma^*}(A)) = \tau^*_{\gamma^*}\text{-Cl}(\tau^*_{\gamma^*}\text{-Cl}(A)) = \tau^*_{\gamma^*}\text{-Cl}(A) = \mathrm{Cl}_{\gamma^*}(A)$. □

The following examples show that the assumption in (c) and (d) of Theorem 3.2.16 cannot be removed.

Example 3.2.17. With respect to the supra space (X, τ^*) and the operation γ^* defined in Example 3.2.6, it is seen that (X, τ^*) is not γ^*-regular. Note that $\mathrm{Cl}_{\gamma^*}(A) \neq$ sup-Cl(A) for the set A given in Example 3.2.6.

Example 3.2.18. Let (X, τ^*) be the supra space and γ^* be the operation in

Example 3.1.5. Then it is shown that the operation γ^* is not open in Example 3.1.11. If $A = \{a\}$, then it is observed that $\mathrm{Cl}_{\gamma^*}(A) \neq \tau^*_{\gamma^*}\text{-Cl}(A)$.

The next examples, respectively, show that $\mathrm{Cl}_{\gamma^*}(A \cup B) \neq \mathrm{Cl}_{\gamma^*}(A) \cup \mathrm{Cl}_{\gamma^*}(B)$ and $\tau^*_{\gamma^*}\text{-Cl}(A \cup B) \neq \tau^*_{\gamma^*}\text{-Cl}(A) \cup \tau^*_{\gamma^*}\text{-Cl}(B)$ for any subset A and B of (X, τ^*).

Example 3.2.19. Let (X, τ^*) be the supra space and let γ^* be the operation from Example 3.2.6. If $A = \{b\}$ and $B = \{c\}$, then $\tau^*_{\gamma^*}\text{-Cl}(A) = \{b\}$ and $\tau^*_{\gamma^*}\text{-Cl}(B) = \{c\}$. It is observed that $X = \tau^*_{\gamma^*}\text{-Cl}(A \cup B) \neq \tau^*_{\gamma^*}\text{-Cl}(A) \cup \tau^*_{\gamma^*}\text{-Cl}(B)$.

Example 3.2.20. Consider again the supra space (X, τ^*) and the operation γ^* in Example 3.1.19 and Example 3.2.7. If $A = \{b\}$ and $B = \{d\}$, it is easily seen that $\mathrm{Cl}_{\gamma^*}(A) = \{a, b\}$ and $\mathrm{Cl}_{\gamma^*}(B) = \{d\}$. Note that $X = \mathrm{Cl}_{\gamma^*}(A \cup B) \neq \mathrm{Cl}_{\gamma^*}(A) \cup \mathrm{Cl}_{\gamma^*}(B)$.

However, the following lemma hold:

Lemma 3.2.21. *Let (X, τ^*) be a supra space. If an operation $\gamma^* : \tau^* \to P(X)$ is regular, then the following statements are true.*

(a) $\mathrm{Cl}_{\gamma^*}(A \cup B) = \mathrm{Cl}_{\gamma^*}(A) \cup \mathrm{Cl}_{\gamma^*}(B)$.

(b) $\tau^*_{\gamma^*}\text{-Cl}(A \cup B) = \tau^*_{\gamma^*}\text{-Cl}(A) \cup \tau^*_{\gamma^*}\text{-Cl}(B)$.

Proof. It is straightforward. □

Corollary 3.2.22. *Let (X, τ^*) be a supra space and $\gamma^* : \tau^* \to P(X)$.*

(a) *If an operation γ^* is regular and open on (X, τ^*), then the γ^*-closure operation Cl_{γ^*} satisfies the Kuratowski closure axiom.*

(b) *If an operation γ^* is regular on (X, τ^*), then the $\tau^*_{\gamma^*}$-closure operation $\tau^*_{\gamma^*}$-Cl also satisfies the Kuratowski closure axiom.*

Proof. The statements (a) and (b) are proved by using Theorem 3.2.16, Definition 3.2.1 and 3.2.2 and Lemma 3.2.21. □

We conclude this section by establishing further results with diagram and then giving an illustrative example.

From Definition 3.2.1 and 3.2.2, and the property (3.4) in Ogata [84], we have the following diagram:

$$\begin{array}{ccccc} \text{sup-Cl}(A) & \subset & \text{Cl}_{\gamma^*}(A) & \subset & \tau^*_{\gamma^*}\text{-Cl}(A) \\ \cap & & \cap & & \cap \\ \text{Cl}(A) & \subset & \text{Cl}_{\gamma}(A) & \subset & \tau_{\gamma}\text{-Cl}(A) \end{array}$$

, where γ^* is an operation on a supra topology τ^* associated with a γ operation on a topology τ.

In the diagram above, none of them of course is reversible as the following example shows:

Example 3.2.23. Consider again (X, τ^*) is the supra space and γ^* is the operation in Example 3.1.19. Define an operation γ also is the identity mapping.

(a) If $A = \{b\}$ then $\text{Cl}_{\gamma}(A) = \{a, b\}$ and $\text{Cl}_{\gamma^*}(A) = \{b\}$. Hence $a \in \text{Cl}_{\gamma}(A)$ but $a \notin \text{Cl}_{\gamma^*}(A)$.

(b) Since $\tau = \tau_{\gamma}$ and $\tau^* = \tau^*_{\gamma^*}$, this is the same fashion of the (a) above.

3.3 Interiors Operators and their Relationships

We now change our point of view as following two types definition:

Definition 3.3.1. A point $x \in X$ is in the γ^**-supra interior* of a subset A of the supra space (X, τ^*) if there exits a supra open neighborhood U of x, $U^{\gamma^*} \subset A$. The γ^*-supra interior of the set A is denoted by $\text{Int}_{\gamma^*}(A)$.

Notation. The point described above is called a γ^**-supra interior point* of A.

Definition 3.3.2. A $\tau^*_{\gamma^*}$*-supra interior* of a subset A in a supra space (X, τ^*) is defined as the union of all the γ^*-supra open sets contained in A. The $\tau^*_{\gamma^*}$-supra interior of the set A is denoted by $\tau^*_{\gamma^*}\text{-Int}(A)$.

Remark 3.3.3. (a) $\tau^*_{\gamma^*}\text{-Int}(A)$ is the "largest" γ^*-supra open set that contains A; that is, if B is any γ^*-supra open set such that $B \subset A$, then $B \subset \tau^*_{\gamma^*}\text{-Int}(A)$. By Proposition 3.1.3, the arbitrary union of any γ^*-supra open sets is γ^*-supra open. Therefore the $\tau^*_{\gamma^*}$-supra interior of a subset of a supra space is a γ^*-supra open set.

(b) $\text{Int}_{\gamma^*}(A)$ is a supra open set, for if $x \in \text{Int}_{\gamma^*}(A)$, then there is a supra open neighborhood U of x with $U^{\gamma^*} \subset A$. By the Definition of an operation γ^*, $x \in U \subset U^{\gamma^*}$ and thus $x \in \text{sup-Int}(\text{Int}_{\gamma^*}(A))$. This shows that $\text{Int}_{\gamma^*}(A)$ is supra open, as desired.

The following theorem provides a useful method of determining whether a point belongs to the $\tau^*_{\gamma^*}$-supra interior of a set.

Theorem 3.3.4. *Let A be a subset of the supra space (X, τ^*) and let $x \in X$. Then $x \in \tau^*_{\gamma^*}\text{-Int}(A)$ if and only if there exists a γ^*-supra open set U such that $x \in U$ and $U \subset A$.*

Proof. Let $x \in \tau^*_{\gamma^*}$-Int(A) for each $x \in X$. Then, by Remark 3.3.3(a), $\tau^*_{\gamma^*}$-Int(A) is γ^*-supra open itself. If we set $U = \tau^*_{\gamma^*}$-Int(A), then it is a γ^*-supra open set containing x and by Definition 3.3.4, $\tau^*_{\gamma^*}$-Int$(A) = U \subset A$, as we wished to show.

The converse follows easily from the in view of Remark 3.3.3(a) and hence this is trivial. □

Notation. The point described above is called a $\tau^*_{\gamma^*}$*-supra interior point* of A.

We next establish three lemmas, which will be useful in the section that follows.

Lemma 3.3.5. *Let A be a subset of a supra space (X, τ^*). Then we have:*

(a) $\mathrm{Int}_{\gamma^*}(A) = A \setminus \mathrm{Der}_{\gamma^*}(X \setminus A)$.

(b) $\tau^*_{\gamma^*}$-Int$(A) = A \setminus \tau^*_{\gamma^*}$-Der$(X \setminus A)$

Proof. (a): Let $x \in A \setminus \mathrm{Der}_{\gamma^*}(X \setminus A)$. Then $x \in A$ and $x \notin \mathrm{Der}_{\gamma^*}(X \setminus A)$. Since $x \notin \mathrm{Der}_{\gamma^*}(X \setminus A)$, there exists a supra open set U such that $x \in U$ and $U^{\gamma^*} \cap (X \setminus A) = \phi$. Then we have $U^{\gamma^*} \subset A$. This show that $x \in \mathrm{Int}_{\gamma^*}(A)$ by Definition 3.3.1. Therefore $A \setminus \mathrm{Der}_{\gamma^*}(X \setminus A) \subset \mathrm{Int}_{\gamma^*}(A)$.

On the another hand, let $x \in \mathrm{Int}_{\gamma^*}(A)$. Then, by Remark 3.3.3(b), $\mathrm{Int}_{\gamma^*}(A)$ is a supra open set itself containing x and since $\mathrm{Int}_{\gamma^*}(A) \cap (X \setminus A) = \phi$ and $\mathrm{Int}_{\gamma^*}(A) \subset A$ by Definition 3.3.1, $x \notin \mathrm{Int}_{\gamma^*}(A)$. This shows that $x \in A$. Therefore $\mathrm{Int}_{\gamma^*}(A) \subset A \setminus \mathrm{Der}_{\gamma^*}(X \setminus A)$.

(b): This proof is similar to that of (a) □

The γ^*-supra interior (resp. $\tau^*_{\gamma^*}$-supra interior) operator is closely connected with the γ^*-supra closure (resp. $\tau^*_{\gamma^*}$-supra closure) operator, as are shown by the following results.

Lemma 3.3.6. *Let A be a subset of a supra space (X, τ^*). Then we have:*

(a) $X \setminus \mathrm{Int}_{\gamma^*}(A) = \mathrm{Cl}_{\gamma^*}(X \setminus A)$.

(b) $X \setminus \tau^*_{\gamma^*}\text{-Int}(A) = \tau^*_{\gamma^*}\text{-Cl}(X \setminus A)$.

Proof. (a):
$$\begin{aligned} X \setminus \mathrm{Int}_{\gamma^*}(A) &= X \setminus (A \setminus \mathrm{Der}_{\gamma^*}(X \setminus A)) \text{ (by Lemma 3.3.5)} \\ &= (X \setminus A) \cup \mathrm{Der}_{\gamma^*}(X \setminus A) \\ &= \mathrm{Cl}_{\gamma^*}(X \setminus A) \text{ (by Theorem 3.2.11(a))}. \end{aligned}$$

(b): This follows similarly to (a). □

Lemma 3.3.7. *Let A be a subset of a supra space (X, τ^*). Then we have:*

(a) $\mathrm{Int}_{\gamma^*}(A) = X \setminus \mathrm{Cl}_{\gamma^*}(X \setminus A)$.

(b) $\tau^*_{\gamma^*}\text{-Int}(A) = X \setminus \tau^*_{\gamma^*}\text{-Cl}(X \setminus A)$.

Proof. (a):
$$\begin{aligned} X \setminus \mathrm{Cl}_{\gamma^*}(X \setminus A) &= X \setminus (X \setminus \mathrm{Int}_{\gamma^*}(A)) \text{ (by Lemma 3.3.6)} \\ &= \mathrm{Int}_{\gamma^*}(A). \end{aligned}$$

(b): The proof is similar to the fashion of (a). □

For future reference, let us state the following result which is an immediate consequence of Lemma 3.3.7.

Proposition 3.3.8. *Let A be a subset of a supra space (X, τ^*). Then we have:*

(a) $X \setminus \mathrm{Cl}_{\gamma^*}(A) = \mathrm{Int}_{\gamma^*}(X \setminus A)$.

(b) $X \setminus \tau^*_{\gamma^*}\text{-Cl}(A) = \tau^*_{\gamma^*}\text{-Int}(X \setminus A)$.

Proof. (a): Putting A for $X \setminus A$ in the Lemma 3.3.7, this shows that the proof is trivial.

(b): The proof is the same fashion as (a). □

We now give a characterization of a γ^*-supra open set in terms of the γ^*-interior operators and the $\tau^*_{\gamma^*}$-interior operators as following theorem.

Theorem 3.3.9. *For a subset A of a supra space (X, τ^*), the following conditions are equivalent:*

(a) *A is γ^*-supra open in (X, τ^*).*

(b) $\mathrm{Int}_{\gamma^*}(A) = A$.

(c) $\tau^*_{\gamma^*}\text{-}\mathrm{Int}(A) = A$.

Proof. Since $X \setminus A$ is γ^*-supra closed, by the virtue of Theorem 3.2.14 and Lemma 3.3.7, the proof is obvious. □

Let us use these results just above established to prove the following proposition.

Proposition 3.3.10. *Let A be any subset of (X, τ^*) and $x \in X$.*

(a) *If x is in $\tau^*_{\gamma^*}\text{-}\mathrm{Int}(A)$, then it belongs to $\mathrm{Int}_{\gamma^*}(A)$.*

(b) *If x is in $\mathrm{Int}_{\gamma^*}(A)$, then it belongs to $\text{sup-}\mathrm{Int}(A)$.*

Proof. The proofs of (a) and (b) are the immediate consequence of Definition 3.3.1 and 3.3.4, Lemma 3.3.6 and 3.3.7 and Proposition 3.3.8. □

In the notion of Example 3.2.6 and 3.2.7 and Proposition 3.3.10, we have the following diagram with implications of which none is reversible:

$$\tau^*_{\gamma^*}\text{-}\mathrm{Int}(A) \subset \mathrm{Int}_{\gamma^*}(A) \subset \text{sup-}\mathrm{Int}(A) \subset A.$$

We have now established the most fundamental properties of γ^*-interior and $\tau^*_{\gamma^*}$-interior operators in the supra space (X, τ^*) associated with topological space (X, τ).

Theorem 3.3.11. *Let $\gamma^* : \tau^* \to P(X)$ be an associated operation with γ, where $\gamma : \tau \to P(X)$ and let A be a subset of (X, τ^*).*

(a) *The subset A is γ^*-supra open if and only if $\tau^*_{\gamma^*}$-$\mathrm{Int}(A) = A$.*

(b) *If a supra space (X, τ^*) is γ^*-regular, then $\mathrm{Int}_{\gamma^*}(A) = \text{sup-Int}(A)$.*

(c) *If γ^* is open, then $\mathrm{Int}_{\gamma^*}(A) = \tau^*_{\gamma^*}$-$\mathrm{Int}(A)$ and $\mathrm{Int}_{\gamma^*}(\mathrm{Int}_{\gamma^*}(A)) = \mathrm{Int}_{\gamma^*}(A)$ hold, and furthermore $\mathrm{Int}_{\gamma^*}(A)$ is also γ^*-supra open in (X, τ^*).*

Proof. (a): By Remark 3.3.3, the proof is obvious.

(b): By Proposition 3.3.10(b), we assure the fact that $\mathrm{Int}_{\gamma^*}(A) \subset \text{sup-Int}(A)$. There remains only to prove that $\text{sup-Int}(A) \subset \mathrm{Int}_{\gamma^*}(A)$. Let $x \in \text{sup-Int}(A)$. Then there exists a supra open neighborhood U of x such that $U \subset A$. By the γ^*-regularity, there exists a supra open set V such that $x \in V$ and $V^{\gamma^*} \subset U$. This implies that $x \in \mathrm{Int}_{\gamma^*}(A)$.

(c): Let $x \in \mathrm{Int}_{\gamma^*}(A)$, as desired. Then there exists a supra open neighborhood of x such that $U^{\gamma^*} \subset A$. Since γ^* is open, for U and $x (\in U)$, there exists a γ^*-supra open set V such that $x \in V \subset U^{\gamma^*}$. Then, we have $V \subset A$. This shows that $x \in \tau^*_{\gamma^*}$-$\mathrm{Int}(A)$, and hence $\tau^*_{\gamma^*}$-$\mathrm{Int}(A) \subset \mathrm{Int}_{\gamma^*}(A)$. By Proposition 3.3.10(a), we have the equality $\mathrm{Int}_{\gamma^*}(A) = \tau^*_{\gamma^*}$-$\mathrm{Int}(A)$. Then we obtain $\mathrm{Int}_{\gamma^*}(\mathrm{Int}_{\gamma^*}(A)) = \tau^*_{\gamma^*}$-$\mathrm{Int}(\tau^*_{\gamma^*}$-$\mathrm{Int}(A)) = \tau^*_{\gamma^*}$-$\mathrm{Int}(A) = \mathrm{Int}_{\gamma^*}(A)$. □

Remark 3.3.12. Consider again the space (X, τ^*) and the operation γ^* in Example 3.2.17 and 3.2.18. By Lemma 3.3.6 and 3.3.7 and Proposition 3.3.8, it is easily observed that the assumption in (c) and (d) of Theorem 3.3.11 cannot be removed.

We conclude this section by establishing further results with diagram and then giving an illustrative example.

From Definition 3.3.1 and 3.3.4, and the property (3.4) in Ogata [84], we have the following diagram:

$$\begin{array}{ccccc} \tau^*_{\gamma^*}\text{-Int}(A) & \subset & \text{Int}_{\gamma^*}(A) & \subset & \text{sup-Int}(A) \\ \cup & & \cup & & \cup \\ \tau_\gamma\text{-Int}(A) & \subset & \text{Int}_\gamma(A) & \subset & \text{Int}(A) \end{array}$$

, where γ^* is an operation on a supra topology τ^* associated with a γ operation on a topology τ.

In the diagram above, none of them of course is reversible as the following example shows:

Example 3.3.13. Consider again (X, τ^*) is the supra space and γ^* and γ are the operations in Example 3.2.23. It is easy to give examples similarly to the conditions (a) and (b) of Example 3.2.23 to show that the mutual inverse implications don't hold.

Chapter 4

Mappings via Operations and γ^*-Supra Convergences

In this chapter, we introduce the concepts of (γ^*, β^*)-supra continuous and weakly (γ^*, β^*)-supra continuous mappings defined by various operations γ^* and β^* in a supra space (X, τ^*) and a supra space (Y, σ^*), respectively. We also define the concepts of γ^*-supra closed, γ^*-supra open mappings and their weaker forms and study their basic properties and relationships.

4.1 (γ^*, β^*)-Supra Continuous Mappings

The present section is devoted to the description of the mappings on supra spaces. These functions need from simpler spaces to more complicated ones. Throughout the rest of this section, let (X, τ^*) and (Y, σ^*) be supra spaces, and let $\gamma^* : \tau^* \to P(X)$ and $\beta^* : \sigma^* \to P(Y)$ be operations on topologies τ^* and σ^*, respectively.

We begin with the following definition.

Definition 4.1.1. A mapping $f : (X, \tau^*) \to (Y, \sigma^*)$ is said to be (γ^*, β^*)-*supra continuous* if for each point $x \in X$ and each supra open set V containing $f(x)$, there exists a supra open set U such that $x \in U$ and $f(U^{\gamma^*}) \subset V^{\beta^*}$.

From the above definition, we obtain some of representation methods of various kinds of continuous mappings as following remark.

Remark 4.1.2. Let $f : (X, \tau^*) \to (Y, \sigma^*)$ be an (id, id)-supra continuous mapping and let $\gamma^* : \tau^* \to P(X)$ and $\beta^* : \sigma^* \to P(Y)$. Then we have:

(a) If τ^* and σ^* are the class of semiopen [52] (resp. preopen [67], α-open [76], semi-preopen [3] (or β-open [1])) sets, then f is the irresolute [20] (resp. preirresolute [93], α-irresolute [58], β-irresolute [59]) mapping.

(b) If τ^* is the class of semiopen (resp. preopen, α-open, semi-preopen (or β-open)) sets and $\sigma^* = \sigma$, then f is the semi-continuous [52] (resp. precontinuous [67], α-continuous [69], semi-precontinuous [75] (or β-continuous [5])) mapping.

(c) If a supra topology τ^* on X and $\sigma^* = \sigma$, then f is supra continuous [70] (briefly S-continuous).

(d) If a supra topology τ^* and σ^* on X and Y respectively, then f is supra* continuous [70] (briefly S^*-continuous).

The following results generalize the concept of the (γ, β)-continuous mappings of Ogata [84] to arbitrary supra spaces associated with topological spaces via the operations γ^* and β^* associated with γ and β operations, respectively.

Proposition 4.1.3. *Let $f : (X, \tau^*) \to (Y, \sigma^*)$ be a (γ^*, β^*)-supra continuous mapping. Then we have:*

(a) *For every subset A of (X, τ^*), $f(\mathrm{Cl}_{\gamma^*}(A)) \subset \mathrm{Cl}_{\beta^*}(f(A))$ holds.*

(b) *For every β^*-supra closed subset F of (Y, σ^*), $f^{-1}(F)$ is γ^*-supra closed in (X, τ^*), i.e. for any $U \in \sigma^*_{\beta^*}$, $f^{-1}(U) \in \tau^*_{\gamma^*}$ holds.*

Proof. (a) Let $y \in f(\mathrm{Cl}_{\gamma^*}(A))$ and let V be any supra open neighborhood of y. Then there exists a point $x \in X$ and a supra open neighborhood U of x such that $f(x) = y$ and $f(U^{\gamma^*}) \subset V^{\beta^*}$. Since $x \in \mathrm{Cl}_{\gamma^*}(A)$, we have $U^{\gamma^*} \cap A \neq \phi$, and hence

$$\phi \neq f(U^{\gamma^*} \cap A) \subset f(U^{\gamma^*}) \cap f(A) \subset V^{\gamma^*} \cap f(A).$$

This implies $y \in \mathrm{Cl}_{\beta^*}(f(A))$. Therefore we obtain (a).

(b) It is sufficient to prove that (a) implies (b). Let F be a β^*-supra closed set of (Y, σ^*), i.e. $\mathrm{Cl}_{\beta^*}(F) = F$. Since $f^{-1}(F)$ is any subset of (X, τ^*), it follows from (a) that the below inclusion relations hold, that is,

$$f(\mathrm{Cl}_{\gamma^*}(f^{-1}(F))) \subset \mathrm{Cl}_{\beta^*}(f(f^{-1}(F))) \subset \mathrm{Cl}_{\beta^*}(F) = F.$$

Then we also obtain the following the relations:

$$\mathrm{Cl}_{\gamma^*}(f^{-1}(F)) \subset f^{-1}(f(\mathrm{Cl}_{\beta^*}(f^{-1}(F)))) \subset f^{-1}(F).$$

Therefore we have $\mathrm{Cl}_{\gamma^*}(f^{-1}(F)) = f^{-1}(F)$ and hence $f^{-1}(F)$ is γ^*-supra closed in (X, τ^*). The half part of (b) is obtained by Definition 3.1.2 and Theorem 3.3.9. □

In the notion of Proposition 4.1.3, the converse claim is not true in general as it can be seen from the following example.

Example 4.1.4. Let $Y = \{a, b, c\}$ and let $\sigma^* = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, Y\}$ be a supra topology on Y. For $b \in Y$, we define an operation $\beta^* : \sigma^* \to P(Y)$ by

$$\beta^*(B) = B^{\beta^*} = \begin{cases} B & \text{if } b \in B \\ \text{sup-Cl}(B) & \text{if } b \notin B. \end{cases}$$

Consider again the supra space (X, τ^*) defined in Example 3.1.19 and an operation $\gamma^* : \tau^* \to P(X)$ defined by $A^{\gamma^*} = \text{sup-Cl}(A)$ if A is a subset of (X, τ^*). Define a mapping $f : (X, \tau^*) \to (Y, \sigma^*)$ by $f(a) = f(b) = b, f(c) = a$ and $f(d) = c$. Then the mapping f satisfies the condition (b) of Proposition 4.1.3 but does not (γ^*, β^*)-supra continuous.

However the following proposition hold:

Proposition 4.1.5. In Proposition 4.1.3, *suppose that* (Y, σ^*) *is a* β^**-regular space. Then the* (γ^*, β^*)*-supra continuity of* f, (a) *and* (b) *are equivalent to each other.*

Proof. In proving the discuss, it is sufficient to show that (b) implies a (γ^*, β^*)-supra continuity of f. Indeed, let V be a supra open set containing $f(x)$ for each $x \in X$. Since (Y, σ^*) is a β^*-regular space, $\sigma^* = \sigma^*_{\beta^*}$ and hence the set V is a β^*-supra open set with $f(x) \in V$. By (b), $f^{-1}(V)$ is a γ^*-supra open set containing $x \in X$ and so there exists a supra open set U such that $x \in U$ and $U^{\gamma^*} \subset f^{-1}(V)$. Therefore $f(U^{\gamma^*}) \subset f(f^{-1}(V)) \subset V \subset V^{\beta^*}$. This completes the proof. □

In the proposition above, if it is removing the assumption that (Y, σ^*) is a β^*-regular space, then the conditions (a) and (b) of Proposition 4.1.3 may fail to be equivalent to each other.

Example 4.1.6. Let $X = Y = \{a, b, c\}$ and let $\tau^* = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$ and $\sigma^* = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ be a supra topology on X and Y, respectively. For $b \in X$, we define an operation $\gamma^* : \tau^* \to P(X)$ by

$$A^{\gamma^*} = \begin{cases} A & \text{if } b \in A \\ \text{sup-Cl}(A) & \text{if } b \notin A. \end{cases}$$

and for a subset B of (Y, σ^*), $\beta^* : \sigma^* \to P(Y)$ by $\beta^*(B) = \text{sup-Cl(B)}$. Define a mapping $f : (X, \tau^*) \to (Y, \sigma^*)$ by $f(a) = b, f(b) = c$ and $f(c) = a$. Then the mapping f satisfies the condition (b) of Proposition 4.1.3 but not (a).

However the following statement is true:

Proposition 4.1.7. In Proposition 4.1.3, *suppose that* $\beta^* : \sigma^* \to P(Y)$ *is an open operator. Then,* (b) *implies* (a) *and hence* (a) *and* (b) *are equivalent to each other.*

Proof. To see this, we wish only to show that (b) implies (a). Let A be any subset of (X, τ^*). Then, since an operation β^* is open, $\text{Cl}_{\beta^*}(f(A))$ is β^*-supra closed and hence by (b), $f^{-1}(\text{Cl}_{\beta^*}(f(A)))$ is γ^*-supra closed. Now we have

$$f^{-1}(\text{Cl}_{\beta^*}(f(A))) = \text{Cl}_{\gamma^*}(f^{-1}(\text{Cl}_{\beta^*}(f(A)))) \supset \text{Cl}_{\gamma^*}(f^{-1}(f(A))) \supset \text{Cl}_{\gamma^*}(A).$$

Therefore $f(\text{Cl}_{\gamma^*}(A)) \subset \text{Cl}_{\beta^*}(f(A))$. □

The following definitions are the concepts of closed and open mappings via the operations γ^* and β^*.

Definition 4.1.8. A mapping $f : (X, \tau^*) \to (Y, \sigma^*)$ is said to be:

(a) (γ^*, β^*)-supra closed if for any γ^*-supra closed set A of (X, τ^*), $f(A)$ is β^*-supra closed of (Y, σ^*);

(b) (γ^*, β^*)-supra open if for any γ^*-supra open set A of (X, τ^*), $f(A)$ is β^*-supra open of (Y, σ^*).

The following example shows that the concepts of a (γ^*, β^*)-supra closed mapping and a (γ^*, β^*)-supra open mapping are independent from each other.

Example 4.1.9. Consider again the supra spaces (X, τ^*) and (Y, σ^*) and operation $\gamma^* : \tau^* \to P(X)$ defined in Example 4.1.6. Let $\beta^* : \sigma^* \to P(Y)$ be an operation defined by $A^{\beta^*} = \text{sup-Int}(\text{sup-Cl}(A))$ for a subset A of (Y, σ^*).

(a) If $f : (X, \tau^*) \to (Y, \sigma^*)$ defined by $f(a) = b$ and $f(b) = f(c) = a$, then f is (γ^*, β^*)-supra open but not (γ^*, β^*)-supra closed.

(b) If $f : (X, \tau^*) \to (Y, \sigma^*)$ defined by $f(a) = b$ and $f(b) = f(c) = c$, then f is (γ^*, β^*)-supra closed but not (γ^*, β^*)-supra open.

The next examples show that the concepts of a (γ^*, β^*)-supra open mapping and a (γ^*, β^*)-supra closed mapping are independent of the concept of a (γ^*, β^*)-supra continuous mapping.

Example 4.1.10. Let $X = Y = \{a, b, c\}$ and let $\tau^* = \{\phi, \{a\}, \{a, b\}, X\}$ and $\sigma^* = \{\phi, \{a\}, \{a, b\}, Y\}$. For a subset A of (X, τ^*) and a subset B of (Y, σ^*), we define an operation $\gamma^* : \tau^* \to P(X)$ by $A^{\gamma^*} = A^{id} = A$ and an operation $\beta^* : \sigma^* \to P(Y)$ by $B^{\beta^*} = \text{sup-Cl}(B)$. Define a mapping $f : (X, \tau^*) \to (Y, \sigma^*)$ by $f(a) = b$, $f(b) = c$ and $f(c) = a$. Then f is (γ^*, β^*)-supra continuous neither (γ^*, β^*)-supra open nor (γ^*, β^*)-supra closed.

Example 4.1.11. In the Example 4.1.9 above, the mapping f defined in (a) is (γ^*, β^*)-supra open but not (γ^*, β^*)-supra continuous and the mapping f defined in (b) is (γ^*, β^*)-supra closed but not (γ^*, β^*)-supra continuous.

Next we consider the case when an operation $\gamma^* : \tau^* \to P(X)$ is the identity operator.

Proposition 4.1.12. *For a supra space (X, τ^*) and a supra space (Y, σ^*), the following are true:*

(a) *If f is (id, β^*)-supra closed, then $f(F)$ is a β^*-supra closed set for any supra closed set F of (X, τ^*).*

(b) *If f is bijective and $f^{-1} : (Y, \sigma^*) \to (X, \tau^*)$ is (β^*, id)-supra continuous, then f is (id, β^*)-supra closed.*

Proof. The proof of this proposition are straightforward, so we omit it. □

For the (γ^*, β^*)-supra closed mappings and the (γ^*, β^*)-supra open mappings, the following two theorems hold.

Theorem 4.1.13. *Let (X, τ^*) and (Y, σ^*) be supra spaces and let $f : (X, \tau^*) \to (Y, \sigma^*)$ be a bijection. A mapping f is (γ^*, β^*)-supra open if and only if f is a (γ^*, β^*)-supra closed mapping.*

Proof. Suppose that f is a (γ^*, β^*)-supra open mapping. Let A be a γ^*-supra closed subset of (X, τ^*). Then $X \setminus A$ is γ^*-supra open. Since f is (γ^*, β^*)-supra open and bijective, $f(X \setminus A)$ is β^*-supra open and also $f(X \setminus A) = Y \setminus f(A)$ and hence $f(A)$ is β^*-supra closed. Therefore f is (γ^*, β^*)-supra closed.

The proof of converse is obvious as replacing "A is γ^*-supra closed" in the previous first proof with "A is γ^*-supra open". □

We give a characterization with respect to the (γ^*, β^*)-supra closed mappings and the (γ^*, β^*)-supra open mappings, respectively.

Theorem 4.1.14. *Let (X, τ^*) and (Y, σ^*) be supra spaces and let $f : (X, \tau^*) \to (Y, \sigma^*)$ be a mapping. The following statements hold:*

(a) *f is a (γ^*, β^*)-supra open mapping if and only if for every subset A of (X, τ^*), $f(\tau^*_{\gamma^*}\text{-Int}(A)) \subset \sigma^*_{\beta^*}\text{-Int}(f(A))$.*

(b) *f is a (γ^*, β^*)-supra closed mapping if and only if for every subset A of (X, τ^*), $\sigma^*_{\beta^*}\text{-Cl}(f(A)) \subset f(\tau^*_{\gamma^*}\text{-Cl}(A))$.*

Proof. (a): Suppose that f is a (γ^*, β^*)-supra open mapping. Let A be a subset of (X, τ^*). Then $\tau^*_{\gamma^*}\text{-Int}(A) \subset A$ and thus $f(\tau^*_{\gamma^*}\text{-Int}(A)) \subset f(A)$. Since $f(\tau^*_{\gamma^*}\text{-Int}(A))$ is β^*-supra open by the hypothesis, $f(\tau^*_{\gamma^*}\text{-Int}(A)) \subset \sigma^*_{\beta^*}\text{-Int}(f(A))$ as desired.

Conversely, suppose that $f(\tau^*_{\gamma^*}\text{-Int}(A)) \subset \sigma^*_{\beta^*}\text{-Int}(f(A))$ for every subset A of (X, τ^*). Let U be a γ^*-supra open subset of (X, τ^*). Then $U = \tau^*_{\gamma^*}\text{-Int}(U)$ and so $f(U) = f(\tau^*_{\gamma^*}\text{-Int}(U)) \subset \sigma^*_{\beta^*}\text{-Int}(f(U))$. Since $\sigma^*_{\beta^*}\text{-Int}(f(U)) \subset f(U)$, $f(U) = \sigma^*_{\beta^*}\text{-Int}(f(U))$ and hence $f(U)$ is β^*-supra open, which shows that f is (γ^*, β^*)-supra open.

(b): This proof is similar to that of (a) and is omitted, which is dual to the proof of the statement (a). □

The following many properties and concepts of various kinds of open and closed mappings are special cases of Definition 4.1.8 and previous results of (γ^*, β^*)-supra open mappings and (γ^*, β^*)-supra closed mappings.

Remark 4.1.15. (a) Let $f : (X, \tau^*) \to (Y, \sigma^*)$ be an (id, id)-supra open mapping and let $\gamma^* : \tau^* \to P(X)$ and $\beta^* : \sigma^* \to P(Y)$. Then we have:

(i) If $\tau^* = \tau$ and σ^* is the class of semiopen [52] (resp. preopen [67], α-open [76], semi-preopen [3](or β-open [1])) sets, then f is a semiopen [12](resp. preopen [67], α-open [69], semi-preopen [14] (or β-open [1])) mapping.

(ii) If τ^* and σ^* are the class of semiopen (resp. preopen, α-open, semi-preopen(or β-open)) sets, then f is a pre-semiopen [21] (resp. p-open [45] (or M-preopen [68]), strongly α-open [98], pre β-open [59]) mapping.

(b) Let $f : (X, \tau^*) \to (Y, \sigma^*)$ be an (id, id)-supra closed mapping and let $\gamma^* : \tau^* \to P(X)$ and $\beta^* : \sigma^* \to P(Y)$. Then many properties and concepts of various kinds of closed mappings (e.g. semiclosed [12], preclosed [35], α-closed [69], semi-preclosed [14] (or β-open [1]), presemiclosed [96], p-closed [45] (or M-preclosed [68]), strongly α-closed [98] and pre β-closed [59]) are similarly to the special cases of (i).

4.2 Weakly (γ^*, β^*)-Supra Continuous Mappings

In this section, we introduce the concept of weaker forms than (γ^*, σ^*)-supra continuous, (γ^*, σ^*)-supra closed and (γ^*, σ^*)-supra open mappings and investigate their basic properties and mutual relationships.

Definition 4.2.1. A mapping $f : (X, \tau^*) \to (Y, \sigma^*)$ is said to be *weakly* (γ^*, σ^*)-*supra continuous* if for each point $x \in X$ and each β^*-supra open set V containing $f(x)$, there exists a γ^*-supra open set U such that $x \in U$ and $f(U) \subset V$.

The next remark gives mutual relationships of weakly (γ^*, β^*)-supra continuous mappings, (γ^*, β^*)-supra open mappings and (γ^*, β^*)-supra closed mappings.

Remark 4.2.2. In the notion of Example 4.1.9 and 4.1.10, the concepts of (γ^*, β^*)-supra open mappings and (γ^*, β^*)-supra closed mappings are independent of the concepts of weakly (γ^*, β^*)-supra continuous mappings.

The following results generalize the concept of (γ^*, β^*)-supra continuous mappings in the previous section and the notion of (γ, β)-continuous mappings in the sense of Ogata ([84]) to arbitrary supra spaces associated with topological specs via the operations γ^* and β^* associated with the operations γ and β, respectively.

Theorem 4.2.3. *Let $f : (X, \tau^*) \to (Y, \sigma^*)$ be a mapping. The following are equivalent:*

(a) *f is a weakly (γ^*, β^*)-supra continuous mapping.*

(b) *For every β^*-supra open subset U of (Y, σ^*), $f^{-1}(U)$ is γ^*-supra open on (X, τ^*).*

(c) *For every subset* F *of* (Y, σ^*) *is a* β^**-supra closed set,* $f^{-1}(F)$ *is* γ^**-supra closed on* (X, τ^*).

Proof. (a)$\Rightarrow$(b): Let $U \in \sigma^*_{\beta^*}$ and let $x \in f^{-1}(U)$. Then $f(x) \in U$. Since f is weakly (γ^*, β^*)-supra continuous, there exists a γ^*-supra open set V_x containing x such that $f(V_x) \subset U$ and so $V_x \subset f^{-1}(U)$. Let $V = \cup\{V_x : x \in f^{-1}(U)\}$. Then V is γ^*-supra open and $V = f^{-1}(U)$ and hence $f^{-1}(U) \in \tau^*_{\gamma^*}$.

(b)$\Rightarrow$(c): Let F be a β^*-supra closed subset of (Y, σ^*). Then $Y \setminus F \in \sigma^*_{\beta^*}$. By (b), $f^{-1}(Y \setminus F) \in \tau^*_{\gamma^*}$. Since $f^{-1}(F) = X \setminus f^{-1}(Y \setminus F)$, $f^{-1}(F)$ is a γ^*-supra closed subset on (X, τ^*).

(c)$\Rightarrow$(a): Let $x \in X$ and let V be a β^*-supra open neighborhood of $f(x)$. Then $Y \setminus V$ is a β^*-supra closed subset of (Y, σ^*) and hence $f^{-1}(Y \setminus V)$ is γ^*-supra closed on (X, τ^*). Let $U = X \setminus f^{-1}(Y \setminus V)$. Then U is a γ^*-supra open neighborhood of x and $f(U) \subset V$. □

From the notions of Theorem 4.2.3, Definition 4.2.1 and Definition 4.1.4, we see that every (γ^*, β^*)-supra continuous mapping is weakly (γ^*, β^*)-supra continuous, but the converse is not true as seen the following example.

Example 4.2.4. Consider again the supra spaces (X, τ^*) defined in Example 3.1.19 and the supra space (Y, σ^*) and the operation $\beta^* : \sigma^* \to P(Y)$ defined in Example 4.1.4. For a subset A of (X, τ^*), we define an operation $\gamma^* : \tau^* \to P(X)$ by $\gamma^*(A) = A^{\gamma^*} = \text{sup-Int}(A)$. Define a mapping $f : (X, \tau^*) \to (Y, \sigma^*)$ by $f(a) = (b) = b$, $f(c) = a$ and $f(d) = c$. Then f is weakly (γ^*, β^*)-supra continuous but not (γ^*, β^*)-supra continuous.

The following theorem is other equivalent formulation of the weakly (γ^*, β^*)-supra continuous mappings.

Theorem 4.2.5. *Let $f : (X, \tau^*) \to (Y, \sigma^*)$ be a mapping. The following are equivalent:*

(a) *f is a weakly (γ^*, β^*)-supra continuous mapping.*

(b) *$f(\tau^*_{\gamma^*}\text{-Cl}(A)) \subset \sigma^*_{\beta^*}\text{-Cl}(f(A))$ for each subset A of (X, τ^*).*

(c) *If x is a $\tau^*_{\gamma^*}$-limit point of A, $f(x) \in \sigma^*_{\beta^*}\text{-Cl}(f(A))$ for each subset A of (X, τ^*).*

(d) *$\tau^*_{\gamma^*}\text{-Cl}(f^{-1}(B)) \subset f^{-1}(\sigma^*_{\beta^*}\text{-Cl}(B))$ for every subset B of (Y, σ^*).*

Proof. (a)$\Rightarrow$(b): Let $y \in f(\tau^*_{\gamma^*}\text{-Cl}(A))$. Then there exits a point $x \in \tau^*_{\gamma^*}\text{-Cl}(A)$ such that $f(x) = y$. Let V be a β^*-supra open neighborhood of y. Since f is weakly (γ^*, β^*)-supra continuous, there is a γ^*-supra open neighborhood U of x such that $f(U) \subset V$. Since $x \in \tau^*_{\gamma^*}\text{-Cl}(A)$, there is a point $p \in U \cap A$ and hence $f(p) \in f(U \cap A) \subset f(U) \cap f(A)$ and so $V \cap f(A) \neq \phi$ because $f(U) \subset V$. Therefore $y \in \sigma^*_{\beta^*}\text{-Cl}(f(A))$.

(b)$\Rightarrow$(c): Let $A \subset X$ and let x be a $\tau^*_{\gamma^*}$-limit point of A. Then $x \in \tau^*_{\gamma^*}\text{-Cl}(A)$ and hence by (b), $f(x) \in \sigma^*_{\beta^*}\text{-Cl}(f(A))$.

(c)$\Rightarrow$(d): Let $x \in \tau^*_{\gamma^*}\text{-Cl}(f^{-1}(B))$. Then by Theorem 3.2.11 (b), $x \in f^{-1}(B)$ or an x is a $\tau^*_{\gamma^*}$-limit point of $f^{-1}(B)$. If $x \in f^{-1}(B)$, then $x \in f^{-1}(\sigma^*_{\beta^*}\text{-Cl}(B))$. If an x is a $\tau^*_{\gamma^*}$-limit point of $f^{-1}(B)$, then $f(x) \in \sigma^*_{\beta^*}\text{-Cl}(f(f^{-1}(B)))$ and so $f(x) \in \sigma^*_{\beta^*}\text{-Cl}(B)$. Therefore $x \in f^{-1}(\sigma^*_{\beta^*}\text{-Cl}(B))$.

(d)$\Rightarrow$(a): We assume (d) and prove (a) by showing that condition (c) of Theorem 4.2.3 is satisfied. Let F be a β^*-supra closed subset of (Y, σ^*). Then by (d), $\tau^*_{\gamma^*}\text{-Cl}(f^{-1}(F)) \subset f^{-1}(\sigma^*_{\beta^*}\text{-Cl}(F))$. Since F is β^*-supra closed, $f^{-1}(\sigma^*_{\beta^*}\text{-Cl}(F)) = f^{-1}(F)$. Since $\tau^*_{\gamma^*}\text{-Cl}(f^{-1}(F)) \subset f^{-1}(F)$, $f^{-1}(F)$ is a γ^*-supra closed subset of (X, τ^*). □

We now define a (γ^*, β^*)-supra homeomorphism via operations $\gamma^* : \tau^* \to P(X)$

and $\beta^* : \sigma^* \to P(Y)$.

Definition 4.2.6. Let (X, τ^*) and (Y, σ^*) be supra spaces. A bijective mapping $f : (X, \tau^*) \to (Y, \sigma^*)$ is said to be a (γ^*, β^*)-*supra homeomorphism* if f is weakly (γ^*, β^*)-supra continuous and f^{-1} is weakly (β^*, γ^*)-supra continuous. Supra spaces (X, τ^*) and (Y, σ^*) are said to be (γ^*, β^*)-*supra homeomorphic* if there is a (γ^*, β^*)-supra homeomorphism $f : (X, \tau^*) \to (Y, \sigma^*)$. A property of supra spaces preserved by the (γ^*, β^*)-supra homeomorphisms is called a supra topological property.

We now state three results that involve characterization of a (γ^*, β^*)-supra homeomorphism.

Theorem 4.2.7. *Let (X, τ^*) and (Y, σ^*) be supra spaces. For a mapping $f : (X, \tau^*) \to (Y, \sigma^*)$, the following statements are equivalent:*

(a) *f is a (γ^*, β^*)-supra homeomorphism.*

(b) *f is (γ^*, β^*)-supra open and f^{-1} is (β^*, γ^*)-supra open.*

(c) *f is (γ^*, β^*)-supra closed and f^{-1} is (β^*, γ^*)-supra closed .*

Proof. The proof follows the facts that f is weakly (γ^*, β^*)-supra continuous, f^{-1} is weakly (β^*, γ^*)-supra continuous and $(f^{-1})^{-1}(A) = f(A)$ for every subset A of (X, τ^*). □

The following two corollaries result immediately from the Theorem 4.2.7.

Corollary 4.2.8. *Let (X, τ^*) and (Y, σ^*) be supra spaces. For a mapping $f : (X, \tau^*) \to (Y, \sigma^*)$, the following statements are equivalent:*

(a) *f^{-1} is weakly (β^*, γ^*)-supra continuous.*

(b) f *is* (γ^*, β^*)*-supra open.*

(c) f *is* (γ^*, β^*)*-supra closed.*

Corollary 4.2.9. *Let* (X, τ^*) *and* (Y, σ^*) *be supra spaces. For a mapping* $f : (X, \tau^*) \to (Y, \sigma^*)$, *the following statements are equivalent:*

(a) f *is a* (γ^*, β^*)*-supra homeomorphism.*

(b) f *is* (γ^*, β^*)*-supra open and weakly* (γ^*, β^*)*-supra continuous.*

(c) f *is* (γ^*, β^*)*-supra closed and weakly* (γ^*, β^*)*-supra continuous.*

Next we consider the case when a operator $\gamma^* : \tau^* \to P(X)$ and an operation $\beta^* : \sigma^* \to P(Y)$ are the identity.

Remark 4.2.10. Let (X, τ^*) and (Y, σ^*) be supra spaces and let $f : (X, \tau^*) \to (Y, \sigma^*)$ be a (γ^*, β^*)-supra homeomorphism.

(a) If supra spaces (X, τ^*) and (Y, σ^*) are topological spaces (X, τ) and (Y, σ) respectively, then f is a (γ, β)-homeomorphism (in the sense of Ogata ([84])).

(b) If supra topologies τ^* and σ^* are the class of semiopen sets respectively, then f is a semihomeomorphism (in the sense of Crossely and Hildebrand ([21])).

4.3 γ^*-Supra Convergences

In this section, we introduce the concept of γ^*-supra convergences and discuss certain related subsets. Using γ^*-supra convergences, we may present in a unified way the various concepts of limit that are used in the published papers by many authors. Let D be a directed set (with the order relation denoted $\geq$), in symbols $(D, \geq)$ or D.

Recall that if $(x_\delta)_{\delta \in D}$ (briefly, (x_δ)) is a net in a set X and A is a subset of (X, τ^*), we say that $x_\delta \in A$ *eventually* if there exists $\delta_0 \in D$ such that $x_\delta \in A$ for all $\delta \geq \delta_0$.

Definition 4.3.1. Let $(x_\delta)_{\delta \in D}$ be a net in a supra space (X, τ^*). The net $(x_\delta)_{\delta \in D}$ *γ^*-supra converges to a point* $x(\in X)$, in symbol $x_\delta \xrightarrow{\gamma^*} x$, if for every γ^*-supra open set U containing x, $x_\sigma \in U$ eventually. When this happens we also say that x is γ^*-supra limit of the net $(x_\delta)_{\delta \in D} = (x_\delta)$ in (X, τ^*).

Lemma 4.3.2. *Let (X, τ^*) be a supra space and γ^* be a regular operator. If $\mathcal{N}_{\gamma^*}(x) = \{U | U$ is a γ^* -supra open neighborhood of $x(\in X)\}$ is a subset of (X, τ^*), then $(\mathcal{N}_{\gamma^*}(x), \subset)$ is a directed set, where $\subset$ is the order relation by inclusion.*

Proof. For all the subsets U of $\mathcal{N}_{\gamma^*}(x)$, the inclusion relation $\subset$ is reflexive and transitive, and since γ^* is a regular operator, for given any subsets $U \subset \mathcal{N}_{\gamma^*}(x)$ and $V \subset \mathcal{N}_{\gamma^*}(x)$, $U \cap V$ is a γ^*-supra open set containing x and then $U \cap V \subset U$ and $U \cap V \subset V$. Therefore $\mathcal{N}_{\gamma^*}(x)$ is a directed set. $\square$

Theorem 4.3.3. *Let (X, τ^*) be a supra space and γ^* be a regular operator. Let $(x_\delta)_{\delta \in \mathcal{N}_{\gamma^*}(x)}$ be a net in a set X and let $U_1, U_2, \cdots, U_n$ be (finitely many) subsets* *of* (X,τ^*). *If* $x_\delta \in U_k$ *eventually for each* $k = 1,2,\cdots,n$, *then* $x_\delta \in \cap U_k$ *eventually.*

Proof. For each k, choose $\delta_k \in \mathcal{N}_{\gamma^*}(x)$ such that $\delta \geq \delta_k$ implies $x_\delta \in U_k$. Since $\mathcal{N}_{\gamma^*}(x)$ is a directed set, there exists δ_0 satisfying $\delta_0 \geq \delta_k$ for each k. Then $\delta \geq \delta_0$ implies $\delta \geq \delta_k$ for each k and hence $x_\delta \in U_k$ for each k. □

Proposition 4.3.4. *Let* (X,τ^*) *be a supra space and* $x \in X$ *and let* γ^* *be a regular operator. For each* γ^**-supra open set* U *containing* x *choose some* $x_U \in U$. *Then* $(x_U)_{U\in\mathcal{N}_{\gamma^*}(x)}$ *is a net which* γ^**-converges to* x.

Proof. Let U be any γ^*-supra open neighborhood of x. Then U includes some member U_0 of $\mathcal{N}_{\gamma^*}(x)$. Let $\delta_0 = U_0$. Then $\delta \geq \delta_0$ implies $x_\delta \in U$ (i.e. $\delta \geq \delta_0$ mean $\delta \subset A$, thus $x_\delta \in \delta \subset A \subset U$). Therefore $x_\delta \xrightarrow{\gamma^*} x$. □

Theorem 4.3.5. *Let* (X,τ^*) *be a supra space,* $x \in X$ *and* A *be a subset of* (X,τ^*). *Let* γ^* *be a regular operator. Then* $x \in \tau^*_{\gamma^*}$*-*$\mathrm{Cl}(A)$ *if and only if there exists a net* (x_δ) *in* A *such that* $x_\delta \xrightarrow{\gamma^*} x$.

Proof. Suppose that there exists a net (x_δ) in A such that $x_\delta \xrightarrow{\gamma^*} x$. Let U be a γ^*-supra open neighborhood of x. Then there exists $\delta_0 \in D$ such that $\delta \geq \delta_0$ implies $x_\delta \in U$ and hence $x_\delta \in U \cap A \neq \phi$. Therefore $x \in \tau^*_{\gamma^*}$-$\mathrm{Cl}(A)$.

Conversely, suppose that $x \in \tau^*_{\gamma^*}$-$\mathrm{Cl}(A)$. Then, by Lemma 4.3.2, there exists a directed set $(\mathcal{N}_{\gamma^*}(x),\subset)$. Let $U \in \mathcal{N}_{\gamma^*}(x)$. Then U is γ^*-supra open neighborhood of x and $U \cap A \neq \phi$. Choosing $x_U \in U \cap A$. Then, by Proposition 4.3.4, $(x_U)_{U\in\mathcal{N}_{\gamma^*}(x)}$ is a net in A which converges to x. □

Theorem 4.3.6. *Let* (X,τ^*) *and* (Y,δ^*) *be a supra space and let* γ^* *and* β^* *be regular operators. A mapping* $f:(X,\tau^*)\to(Y,\sigma^*)$ *is weakly* (γ^*,β^*) *continuous at a point* $x \in X$ *if and only if* $x_\delta \xrightarrow{\gamma^*} x$ *implies* $f(x_\delta) \xrightarrow{\beta^*} f(x)$.

Proof. Suppose that f is weakly (γ^*, β^*) continuous at a point $x \in X$ and $x_\delta \xrightarrow{\gamma^*} x$. Let U be a β^*-supra open neighborhood of $f(x)$. Then $f^{-1}(U)$ is a γ^*-supra open neighborhood of x. Since $x_\delta \xrightarrow{\gamma^*} x$, there exists $\delta_0 \in D$ such that $\delta \geq \delta_0$ implies $x_\delta \in f^{-1}(U)$ and hence $f(x_\delta) \in U$. Therefore $f(x_\delta) \xrightarrow{\beta^*} f(x)$.

Conversely, Suppose that f is not weakly (γ^*, β^*)-supra continuous at a point $x \in X$. Let U be a γ^*-supra open neighborhood of x. Then there exists a β^*-supra open neighborhood of $f(x)$ such that $U \not\subset f^{-1}(V)$. For $U \in \mathcal{N}_{\gamma^*}(x)$, choosing x_U such that $x_U \in U \setminus f^{-1}(V)$ then (x_U) is a net and by Proposition 4.3.4, $x_U \xrightarrow{\gamma^*} x$. But, since $f(x_U) \in f(U) \setminus V$ and $f(x_U) \notin V$, $f(x_\delta) \overset{\beta^*}{\not\to} f(x)$. □

The following properties and concepts of some convergences are special cases of Definition 4.3.1 and previous results of γ^*-supra convergences.

Remark 4.3.7. Let (X, τ^*) be a supra space, (X, τ) be a topological space and $x \in X$. Let γ^* be operations on a supra topology τ^* associated with operations γ on a topology τ of a set X.

(i) If $\tau^* = \tau$, γ^* is the identity operator, then

(a) a net $(\mathcal{N}_{\gamma^*}(x), \subset)$ of (X, τ^*) is a net $(\mathcal{N}_x, \subset)$ of (X, τ), where $\mathcal{N}_x = \{U \mid U$ is an open neighborhood of $x\}$ is a neighborhood system.

(b) The γ^*-supra convergences on (X, τ^*) are the convergences on (X, τ).

(ii) If $\tau^* = \tau$, then the γ^*-convergences on (X, τ^*) are the γ-convergences on (X, τ) [99].

Chapter 5

γ^*-Supra Separation Axioms

In this chapter, we introduce the concepts of γ^*-supra T_i spaces, where $i = 0$, $1/2, 1$ or 2, defined by various operations γ^* on a supra space (X, τ^*) and investigate their mutual relationships among γ^*-supra T_i $(i = 0, 1/2, 1, 2)$.

We also define the concept of generalized γ^*-supra closed sets (resp. generalized γ^*-supra open sets) which is weaker than that of γ^*-supra closed sets (resp. γ^*-supra open sets), and study their basic properties and mutual relationships. Moreover, we investigate the closed graphs properties via the operations γ^* and β^* on supra spaces (X, τ^*) and (Y, σ^*), respectively and preserving properties of γ^*-sup T_i spaces, where $i = 0, 1/2, 1$ or 2 and generalized γ^*-supra closed sets.

5.1 γ^*-Supra T_i Spaces $(i = 0,\ 1/2,\ 1,\ 2)$

In this section, we investigate general operator approaches of γ^*-supra T_i spaces, where $i = 0, 1/2, 1$ or 2 and their relationships. Let (X, τ^*) be supra spaces, and let $\gamma^* : \tau^* \to P(X)$ be operations on a supra topology τ^*.

We begin with the following definitions:

Definition 5.1.1. A supra space (X, τ^*) is called a γ^**-supra* T_0 if for each distinct points x and y of X, there exists a supra open set U such that either $x \in U$ and $y \notin U^{\gamma^*}$, or $y \in U$ and $x \notin U^{\gamma^*}$.

Definition 5.1.2. A supra space (X, τ^*) is called a γ^**-supra* T_1 if for each distinct points $x, y \in X$, there exist supra open sets U and V containing x and y, respectively such that $y \notin U^{\gamma^*}$ and $x \notin V^{\gamma^*}$.

Definition 5.1.3. A supra space (X, τ^*) is called a γ^**-supra* T_2 if for each distinct points $x, y \in X$, there exist supra open sets U and V such that $x \in U$, $y \in V$ and $U^{\gamma^*} \cap V^{\gamma^*} = \phi$.

To define a γ^*-supra $T_{1/2}$ spaces we introduce the concept of generalized γ^*-supra closed sets (c.f. [53]).

Definition 5.1.4. A subset A of (X, τ^*) is said to be *generalized* γ^* *-supra closed* (briefly, $g.\gamma^*$*-supra closed*) if $\mathrm{Cl}_{\gamma^*}(A) \subset U$ whenever $A \subset U$ and U is γ^*-supra open in (X, τ^*).

Definition 5.1.5. (c.f. [53, Defintion 5.11]). A space (X, τ^*) is said to be γ^*-*supra* $T_{1/2}$ if every $g.\gamma^*$-supra closed set of (X, τ^*) is γ^*-supra closed.

From the above the definitions, we examine the following the diagram:

Remark 5.1.6. For a supra space (X, τ^*), we obtain the below implications

$$\gamma^*\text{-supra } T_2 \to \gamma^*\text{-supra } T_1 \to \gamma^*\text{-supra } T_{1/2} \to \gamma^*\text{-supra } T_0$$

, where $A \to B$ represents that A implies B.

However the converse of the above diagram is not true as following examples.

Example 5.1.7. Let X be a set and $\tau^* = \{U | X \setminus U \text{ is finite } \} \cup \{\phi\}$. Let $\gamma^* : \tau^* \to P(X)$ be an operation on τ^* defined by the supra interior operator. Then the supra space (X, τ^*) is not γ^*-supra T_2 but γ^*-supra T_1.

Example 5.1.8. Let $X = \{a, b, c, d\}$ and $\tau^* = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}$. For $a \in X$, we define an operation $\gamma^* : \tau^* \to P(X)$ by

$$A^{\gamma^*} = \begin{cases} A & \text{if } a \in A (\in \tau^*) \\ \text{sup-Cl}(A) & \text{if } a \notin A (\in \tau^*). \end{cases}$$

Then it is obtained that $\tau^*_{\gamma^*} = \tau^*$ (i.e. every supra open set is γ^*-supra open). Therefore (X, τ^*) is not γ^*-supra T_1 but γ^*-supra $T_{1/2}$.

Example 5.1.9. Let $X = \{a, b, c\}$ and $\tau^* = \{\phi, \{a\}, \{a, b\}, X\}$. Let $\gamma^* : \tau^* \to P(X)$ be an operation defined by the identity operator. Then, since $\{a, c\}$ is $g.\gamma^*$-supra closed but not γ^*-supra closed, this supra space (X, τ^*) is not γ^*-supra $T_{1/2}$ but γ^*-supra T_0.

The following some results are the standard properties of γ^*-supra T_i spaces where $i = 0, 1/2,\ 1$ or 2.

Theorem 5.1.10. *If a supra space (X, τ^*) is γ^*-supra T_0, then for each pair of distinct points x, y of X, $y \notin \mathrm{Cl}_{\gamma^*}(\{x\})$ or $x \notin \mathrm{Cl}_{\gamma^*}(\{y\})$.*

Proof. Let x and y be any two distinct points of X. Then there exists a supra open set U and V containing x and y, respectively such that $y \notin U^{\gamma^*}$ and $x \notin V^{\gamma^*}$. If there exists a supra open neighborhood U of x such that

$y \notin U^{\gamma^*}$, then since $U^{\gamma^*} \cap \{y\} = \phi$, $x \notin \mathrm{Cl}_{\gamma^*}(\{y\})$ and there exists a supra open neighborhood V of y such that $x \notin V^{\gamma^*}$, then since $V^{\gamma^*} \cap \{x\} = \phi$, $y \notin \mathrm{Cl}_{\gamma^*}(\{x\})$. □

Theorem 5.1.11. (a) *If a supra space (X, τ^*) is γ^*-supra $T_{1/2}$, then for each $x \in X$, one point set $\{x\}$ is γ^*-supra closed or γ^*-supra open in (X, τ^*).*

(b) *If $\gamma^* : \tau^* \to P(X)$ is a regular operator, then converse of* (i) *is true.*

Proof. (a): Let (X, τ^*) be a γ^*-supra $T_{1/2}$ space. Supposed that $\{x\}$ is not γ^*-supra closed. Then its compliment $\{x\}^c (= X \setminus \{x\})$ is not γ^*-supra open. Since the supra space (X, τ^*) is the only γ^*-supra open set containing $X \setminus \{x\}$, $X \setminus \{x\}$ is $g.\gamma^*$-supra closed and thus closed. Hence $\{x\}$ is γ^*-supra open.

(b): Let $A \subset X$ be $g.\gamma^*$-supra closed with $x \in \mathrm{Cl}_{\gamma^*}(A)$. If $\{x\}$ is γ^*-supra open, $\{x\} \cap A \neq \phi$. Otherwise $\{x\}$ is γ^*-supra closed and $\phi \neq \mathrm{Cl}_{\gamma^*}\{x\} \cap A = \{x\} \cap A$ by Theorem 3.2.14. In either case $x \in A$ and so A is γ^*-supra closed since γ^* is a regular operator. □

Corollary 5.1.12. (a) *If a supra space (X, τ^*) is γ^*-supra $T_{1/2}$ then every subset of X is the intersection of all γ^*-supra closed sets and all γ^*-supra open sets containing it.*

(b) *If $\gamma^* : \tau^* \to P(X)$ is a regular operator, then converse of* (a) *is true.*

Proof. (a): Let (X, τ^*) be a γ^*-supra $T_{1/2}$ space with $B \subset X$ arbitrary. Then $B = \cap\{X \setminus \{x\} | x \notin B\}$, the intersection of γ^*-supra closed and γ^*-supra open sets by Theorem 5.1.11 (a). The result follows.

(b): For each $x \in X$, $X \setminus \{x\}$ is the intersection of all γ^*-supra closed sets and all γ^*-supra open sets containing it. Thus $X \setminus \{x\}$ is either γ^*-supra closed or γ^*-supra open and so (X, τ^*) is a γ^*-supra $T_{1/2}$ space by Theorem 5.1.11 (b). □

Theorem 5.1.13. *Let (X, τ^*) be a finite supra space, and let $\gamma^* : \tau^* \to P(X)$ be the supra closure operation. A supra space (X, τ^*) is γ^*-supra $T_{1/2}$ if and only if the supra topology τ^* is discrete.*

Proof. Let (X, τ^*) be a γ^*-supra $T_{1/2}$ space. By Theorem 5.1.11, for each $x \in X$, $\{x\}$ is γ^*-supra closed or γ^*-supra open.

Case I. Suppose that $\{x\}$ is γ^*-supra open. Then $\{x\}$ is supra open by Proposition 3.1.5.

Case II. Suppose that $\{x\}$ is γ^*-supra closed. Since $X \setminus \{x\}$ is γ^*-supra open and X is finite, we show that $X \setminus \{x\} = \cup_{i=1}^{n} U_i^{\gamma^*}$ for some supra open set U_i $(1 \leq i \leq n)$. Since γ^* is the supra closure operator, $\{x\}$ is supra open. Therefore in both cases, every singleton in X is supra open and hence τ^* is discrete.

Conversely, the proof follows from Theorem 5.1.11 and assumptions. □

Theorem 5.1.14. *A supra space (X, τ^*) is γ^*-supra T_1 if and only if any singleton $\{x\}$ of X is γ^*-supra closed.*

Proof. Suppose that (X, τ^*) be a γ^*-supra T_1 space. For each $x \in X$, we claim that $X \setminus \{x\}$ is γ^*-supra open. Indeed, let $y \in X \setminus \{x\}$. Then $y \neq x$ and so there exists a supra open set U such that $y \in U$ but $x \notin U^{\gamma^*}$. Consequently $y \in U^{\gamma^*} \subset X \setminus \{x\}$. Hence $X \setminus \{x\}$ is γ^*-supra open, as we wished to show.

To prove the converse, suppose that each singleton in X is a γ^*-supra closed set. Let $x \neq y (\in X)$. Then $X \setminus \{x\}$ and $X \setminus \{y\}$ are γ^*-supra open and so there exist supra open sets U and V containing x and y, respectively such that

$U^{\gamma^*} \subset X \setminus \{y\}$ and $V^{\gamma^*} \subset X \setminus \{x\}$. Consequently $y \notin U^{\gamma^*}$ and $x \notin V^{\gamma^*}$. Hence (X, τ^*) is a γ^*-supra T_1 space. □

Theorem 5.1.15. *If a supra space (X, τ^*) is γ^*-supra T_2, then the intersection of all the supra closed neighborhoods of each point of the supra space is reduced to that point.*

Proof. Let (X, τ^*) be supra T_2 and let $x \in X$. Then for each $y \in X$, distinct from x, there exist supra open sets U and V such that $x \in U$, $y \in V$ and $U^{\gamma^*} \cap V^{\gamma^*} = \phi$. Since $U \cap V = \phi$, $x \in U \subset X \setminus V$ and thus $X \setminus V$ is a supra closed neighborhood of x to which y does not belong. Therefore the intersection of all the supra closed neighborhoods of (X, τ^*) is reduced to $\{x\}$. □

Remark 5.1.16. By using Definition 4.1 (i)-(iii) and Theorem 4.1 in [70] and Definition 5.1.1- 5.1.3 and the basic properties of γ^*-supra T_i spaces, it is shown that if (X, τ^*) is γ^*-supra T_i then S-T_i $(i = 0, 1, 2)$, where an operation $\gamma^* : \tau^* \to P(X)$ is associated with $\gamma : \tau \to P(X)$.

The following examples show that above implications are not reversible.

Example 5.1.17. Let $X = \{a, b\}$ and $\tau^* = \{\phi, \{a\}, X\}$. Let $\gamma^* : \tau^* \to P(X)$ be the supra closure operator. Then the supra space (X, τ^*) is not γ^*-supra T_0 but S-T_0.

Example 5.1.18. Let X be the set of all real numbers. The τ^* is a supra topology of Example 5.1.7. Let $\gamma^* : \tau^* \to P(X)$ be an operation on τ^* defined by the following conditions: for a particular point $p \in X$, and every supra open set U of (X, τ^*), $U^{\gamma^*} = U$ if $p \in U$ and $U^{\gamma^*} =$ sup-Cl(U) if $p \notin U$. Then (X, τ^*) is not γ^*-supra T_1 but S-T_1.

Example 5.1.19. Let $X = \{a, b, c\}$ and $\tau^* = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. For $a \in X$, we define an operation $\gamma^* : \tau^* \to P(X)$ by

$$A^{\gamma^*} = \begin{cases} X & \text{if } a \in A(\in \tau^*) \\ \text{sup-Cl}(A) & \text{if } a \notin A(\in \tau^*). \end{cases}$$

Then (X, τ^*) is not γ^*-supra T_2 but S-T_2.

From Remark 5.1.16, [70, Definition 4.1(i)-(iii) and Theorem 4.1], Example 5.1.7- 5.1.9 and Example 5.1.17- 5.1.19 we have the following diagram:

$$\begin{array}{ccccc} \gamma^*\text{-supra } T_2 & \to & \gamma^*\text{-supra } T_1 & \to & \gamma^*\text{-supra } T_0 \\ \downarrow & & \downarrow & & \downarrow \\ S\text{-}T_2 & \to & S\text{-}T_1 & \to & S\text{-}T_0 \end{array}$$

The following many the notions of various kinds of separation axioms are special cases of Definition 5.1.1- 5.1.3. Now, we consider the case when an operation $\gamma^* : \tau^* \to P(X)$ is the identity, where γ^* is associated with the operation $\gamma : \tau \to P(X)$.

Remark 5.1.20. Let (X, τ^*) be a supra space; and let (X, τ) be a topological space; and let γ^* be the identity operator associated with γ, where $i = 0, 1$ or 2. Then the following cases are true.

(a) If $\tau^* = SO(X, \tau)$, then the supra space (X, τ^*) is semi-T_i [56].

(b) If $\tau^* = PO(X, \tau)$, then the supra space (X, τ^*) is pre-T_i [47].

(c) If $\tau^* = SPO(X, \tau)$, then the supra space (X, τ^*) is semipre-T_i [74].

(d) If $\tau^* = \alpha(X, \tau)$, then the supra space (X, τ^*) is α-T_i [73] (or feebly T_i [57]).

5.2 Generalized γ^*-Supra Closed Sets

In this section, we study the basic properties of generalized γ^*-supra closed sets introduced in the previous section and their applications. Moreover we introduce the concept that is strictly between the γ^*-supra closed sets and the $g.\gamma^*$-supra closed sets and inquire their properties. Also, we further investigate supra operator approaches of closed graphs of mappings.

Recall that a subset A of (X, τ^*) is said to be generalized γ^*-supra closed (briefly, $g.\gamma^*$-supra closed). if $\mathrm{Cl}_{\gamma^*}(A) \subset U$ whenever $A \subset U$ and U is γ^*-supra open in (X, τ^*).

Definition 5.2.1. A subset of A is said to be *strongly generalized* γ^* *-supra closed* (briefly, *sg.*γ^* *-supra closed*) if $\mathrm{Cl}_{\gamma^*}(A) \subset U$ whenever $A \subset U$ and U is supra open in (X, τ^*).

From the above definitions, Definition 3.1.2 and Definition 5.1.4, we obtain the following diagram:

γ^*-supra closed sets $\rightarrow$ $sg.\gamma^*$-supra closed sets $\rightarrow$ $g.\gamma^*$-supra closed sets

The following examples show that above implications are not reversible.

Example 5.2.2. Consider again (X, τ^*) defined in Example 3.2.6. Let $\gamma^* : \tau^* \rightarrow P(X)$ be a supra closure operator. Then $\tau^*_{\gamma^*} = \{\phi, \{b\}, \{c\}, \{b, c\}, X\}$. If $A = \{a, c\}$ is supra open itself , then $\mathrm{Cl}_{\gamma^*}(A) = \{a, c, d\} \not\subset A$ and hence A is not $sg.\gamma^*$-supra closed but $g.\gamma^*$-supra closed.

Example 5.2.3. Let $X = \{a, b, c\}$ and $\tau^* = \{\phi, \{a\}, \{b, c\}, X\}$. Let $\gamma^* : \tau^* \rightarrow P(X)$ be an operation defined by $\{a\}^{\gamma^*} = \{a, c\}$ and $A^{\gamma^*} = \mathrm{Cl}(A)$ if $A(\neq \{a\}) \in \tau^*$. Then $\{a, b\}$ is $sg.\gamma^*$-supra closed in (X, τ^*) but not γ^*-supra closed.

The following two theorems are the characterization of $g.\gamma^*$-supra closed and $sg.\gamma^*$-supra closed sets.

Theorem 5.2.4. *Let $\gamma^* : \tau^* \to P(X)$ be an operation and A be a subset of (X, τ^*). A set A is $g.\gamma^*$-supra closed in (X, τ^*) if and only if for each $x \in \mathrm{Cl}_{\gamma^*}(A)$, $\tau^*_{\gamma^*}$-$\mathrm{Cl}(\{x\}) \cap A \neq \phi$.*

Proof. Suppose that $x \in \mathrm{Cl}_{\gamma^*}(A)$ but $\tau^*_{\gamma^*}$-$\mathrm{Cl}(\{x\}) \cap A = \phi$. Then $X \setminus \tau^*_{\gamma^*}$-$\mathrm{Cl}(\{x\})$ is γ^*-supra open and $A \subset X \setminus \tau^*_{\gamma^*}$-$\mathrm{Cl}(\{x\})$. Since A is $g.\gamma^*$-supra closed in (X, τ^*), $\mathrm{Cl}_{\gamma^*}(A) \subset X \setminus \tau^*_{\gamma^*}$-$\mathrm{Cl}(\{x\})$ and hence $x \notin \mathrm{Cl}_{\gamma^*}(A)$. This is a contradiction.

Conversely, let U be any γ^*-supra open set such that $A \subset U$ and let $x \in \mathrm{Cl}_{\gamma^*}(A)$. Since $\tau^*_{\gamma^*}$-$\mathrm{Cl}(\{x\}) \cap A \neq \phi$, there exists a point z such that $z \in \tau^*_{\gamma^*}$-$\mathrm{Cl}(\{x\})$ and $z \in A \subset U$. It is follows from Proposition 3.2.4 that $U \cap \{x\} \neq \phi$ and hence $x \in U$. Therefore A is $g.\gamma^*$-supra closed in (X, τ^*). □

Corollary 5.2.5. *Let $\gamma^* : \tau^* \to P(X)$ be an open operator and A be a subset of (X, τ^*). A set A is $g.\gamma^*$-supra closed in (X, τ^*) if and only if for each $x \in \mathrm{Cl}_{\gamma^*}(A)$, $\mathrm{Cl}_{\gamma^*}(\{x\}) \cap A \neq \phi$.*

Proof. The proof is the same fashion as (a) by using the open operator. □

Theorem 5.2.6. *Let $\gamma^* : \tau^* \to P(X)$ be an operation and A be a subset of (X, τ^*). A set A is $sg.\gamma^*$-supra closed in (X, τ^*) if and only if $\mathrm{Cl}_{\gamma^*}(A) \setminus A$ contains no nonempty supra closed set.*

Proof. Let F be a supra closed subset of $\mathrm{Cl}_{\gamma^*}(A) \setminus A$. Now $A \subset X \setminus F$ and since A is $sg.\gamma^*$-supra closed, we have $\mathrm{Cl}_{\gamma^*}(A) \subset X \setminus F$ or $F \subset X \setminus \mathrm{Cl}_{\gamma^*}(A)$. Thus $F \subset \mathrm{Cl}_{\gamma^*}(A) \cap X \setminus \mathrm{Cl}_{\gamma^*}(A) = \phi$ and F is empty.

Conversely, suppose that $A \subset U$ and U is supra open. If $\mathrm{Cl}_{\gamma^*}(A) \not\subset U$, then $\mathrm{Cl}_{\gamma^*}(A) \cap X \setminus U$ is a nonempty supra closed subset of $\mathrm{Cl}_{\gamma^*}(A) \setminus A$. This is a contradiction. □

Theorem 5.2.7. *Let $\gamma^* : \tau^* \to P(X)$ be an operation and A be a subset of (X, τ^*).*

(a) *If s subset A is $sg.\gamma^*$-supra closed in (X, τ^*), then $Cl_{\gamma^*}(\{x\}) \cap A \neq \phi$ for each $x \in Cl_{\gamma^*}(A)$.*

(b) *The converse is true if γ^* is an identity operator.*

Proof. (a): Suppose that $x \in \mathrm{Cl}_{\gamma^*}(A)$ but $Cl_{\gamma^*}(\{x\}) \cap A = \phi$. Then $X \setminus \mathrm{Cl}_{\gamma^*}(\{x\})$ is supra open and $A \subset X \setminus \mathrm{Cl}_{\gamma^*}(\{x\})$. Since A is $sg.\gamma^*$-supra closed in (X, τ^*), $\mathrm{Cl}_{\gamma^*}(A) \subset X \setminus \mathrm{Cl}_{\gamma^*}(\{x\})$ and hence $x \notin \mathrm{Cl}_{\gamma^*}(A)$ contradicting to $x \in \mathrm{Cl}_{\gamma^*}(A)$.

(b): Let U be any supra open set such that $A \subset U$ and let $x \in \mathrm{Cl}_{\gamma^*}(A)$. Since $\mathrm{Cl}_{\gamma^*}(\{x\}) \cap A \neq \phi$, there exists a point z such that $z \in \mathrm{Cl}_{\gamma^*}(\{x\})$ and $z \in A \subset U$. $U \cap \{x\} \neq \phi$ and hence $x \in U$. Therefore A is $sg.\gamma^*$-supra closed in (X, τ^*). □

Theorem 5.2.8. *Let $\gamma^* : \tau^* \to P(X)$ be an operation and A be a subset of (X, τ^*). The following are true.*

(a) *If A is $g.\gamma^*$-supra closed in (X, τ^*), then $\mathrm{Cl}_{\gamma^*}(A) \setminus A$ contains no non-empty γ^*-supra closed subsets.*

(b) *The converse is true if γ^* is an open operator.*

Proof. (a): Let F be a γ^*-supra closed subset of $\mathrm{Cl}_{\gamma^*}(A) \setminus A$. Now $A \subset X \setminus F$ and since A is $g.\gamma^*$-supra closed, we have $\mathrm{Cl}_{\gamma^*}(A) \subset X \setminus F$ or $F \subset X \setminus \mathrm{Cl}_{\gamma^*}(A)$. Thus $F \subset \mathrm{Cl}_{\gamma^*}(A) \cap X \setminus \mathrm{Cl}_{\gamma^*}(A) = \phi$ and F is empty.

(b): Suppose that $A \subset U$ and U is γ^*-supra open. If $\text{Cl}_{\gamma^*}(A) \not\subset U$, then by Theorem 3.2.16 (d) $\text{Cl}_{\gamma^*}(A) \cap X \setminus U$ is a nonempty γ^*-supra closed subset of $\text{Cl}_{\gamma^*}(A) \setminus A$. This is a contradiction. □

Corollary 5.2.9. *Let $\gamma^* : \tau^* \to P(X)$ be an open operator. Then a $g.\gamma^*$-supra closed set A is γ^*-supra closed if and only if $\text{Cl}_{\gamma^*}(A) \setminus A$ is γ^*-supra closed.*

Proof. If A is γ^*-supra closed, then $\text{Cl}_{\gamma^*}(A) = \tau^*_\gamma\text{-Cl}(A) \setminus A = \phi$ by the open operator γ^*.

Conversely, suppose that $\text{Cl}_{\gamma^*}(A) \setminus A$ is γ^*-supra closed. Let A be a $g.\gamma^*$-supra closed set. Since $\text{Cl}_{\gamma^*}(A) \setminus A$ is γ^*-supra closed subset of itself, by Theorem 5.2.8 (a), $\text{Cl}_{\gamma^*}(A) \setminus A = \phi$ and hence $\text{Cl}_{\gamma^*}(A) = \tau^*_\gamma\text{-Cl}(A) = A$ since γ^* is an open operator. □

Corollary 5.2.10. *Let (X, τ^*) be a γ^*-regular space. Then a $sg.\gamma^*$-supra closed set A is supra closed if and only if $\text{Cl}_{\gamma^*}(A) \setminus A$ is supra closed.*

Proof. By using Theorem 3.2.16 (c) and Theorem 5.2.6, the proof is similar to (a) . □

Proposition 5.2.11. *For each $x \in X$, $\{x\}$ is γ^*-supra closed or $X \setminus \{x\}$ is $g.\gamma^*$-supra closed in (X, τ^*)*

Proof. Suppose that $\{x\}$ is not γ^*-supra closed. Then the complement $X \setminus \{x\}$ is not γ^*-supra open by Theorem 3.2.14. Let U be any γ^*-supra open set such that $X \setminus \{x\} \subset U$. Then since $U = X$, $\text{Cl}_{\gamma^*}(X \setminus \{x\}) \subset U$. Therefore $X \setminus \{x\}$ is $g.\gamma^*$-supra closed.

Proposition 5.2.12. *Let $\gamma^* : \tau^* \to P(X)$ be an operation and let A and B be the subset of (X, τ^*).*

(a) *If γ^* is open and if A is $g.\gamma^*$-supra closed and $A \subset B \subset \mathrm{Cl}_{\gamma^*}(A)$, then B is $g.\gamma^*$-supra closed.*

(b) *If A is $sg.\gamma^*$-supra closed and $A \subset B \subset \mathrm{Cl}_{\gamma^*}(A)$, then B is $sg.\gamma^*$-supra closed.*

Proof. (a): Since $\mathrm{Cl}_{\gamma^*}(B) \setminus B \subset \mathrm{Cl}_{\gamma^*}(A) \setminus A$ and $\mathrm{Cl}_{\gamma^*}(A) \setminus A$ contains no nonempty γ^*-supra closed sets, $\mathrm{Cl}_{\gamma^*}(B) \setminus B$ does not, either. By Theorem 5.2.8, B is $g.\gamma^*$-supra closed.

(b): The proof is similar to that of (a) by using Theorem 5.2.6. □

The following forms are the generalization of γ^*-supra open sets.

Definition 5.2.13. A subset A of (X, τ^*) is said to be *$g.\gamma^*$-supra open* (resp. *$sg.\gamma^*$-supra open*) in (X, τ^*) if the complement $X \setminus A$ is g.γ^*-supra open (resp. sg.γ^*-supra open).

From the above definition, we find the following results.

Theorem 5.2.14. *Let (X, τ^*) be a supra space. Then we have:*

(a) *A subset A is $g.\gamma^*$-supra open in (X, τ^*) if and only if $F \subset \mathrm{Int}_{\gamma^*}(A)$ whenever F is γ^*-supra closed and $F \subset A$.*

(b) *A subset A is $sg.\gamma^*$-supra open in (X, τ^*) if and only if $F \subset \mathrm{Int}_{\gamma^*}(A)$ whenever F is supra closed and $F \subset A$.*

Proof. Straightforward. □

Theorem 5.2.15. *Let* (X, τ^*) *be a supra space. The following are true.*

(a) *If a subset* A *is* $g.\gamma^*$*-supra open in* (X, τ^*)*, then* $U = X$ *whenever* U *is* γ^**-supra open and* $\mathrm{Int}_{\gamma^*}(A) \cup (X \setminus A) \subset U$. *The converse is true if* γ^* *is an open operator.*

(b) *A subset* A *is* $sg.\gamma^*$*-supra open if and only if* $U = X$ *whenever* U *is supra open and* $\mathrm{Int}_{\gamma^*}(A) \cup (X \setminus A) \subset U$.

Proof. (a): Let U be γ^*-supra open in (X, τ^*) and $\mathrm{Int}_{\gamma^*}(A) \cup (X \setminus A) \subset U$. Then $X \setminus U \subset \mathrm{Cl}_{\gamma^*}(X \setminus A) \cap A = \mathrm{Cl}_{\gamma^*}(X \setminus A) \setminus (X \setminus A)$. Since $X \setminus A$ is $g.\gamma^*$-supra closed and $X \setminus U$ is γ^*-supra closed, by Theorem 5.2.8, $X \setminus U = \phi$ and hence $X = U$.

Conversely, let F be γ^*-supra closed in (X, τ^*) and $F \subset A$. By Theorem 5.2.14 (a), it is sufficient to show that $F \subset \mathrm{Int}_{\gamma^*}(A)$. Indeed, since $\mathrm{Int}_{\gamma^*}(A) \cup X \setminus F$ is γ^*-supra open and $\mathrm{Int}_{\gamma^*}(A) \cup X \setminus A \subset \mathrm{Int}_{\gamma^*}(A) \cup X \setminus F$, we have $\mathrm{Int}_{\gamma^*}(A) \cup X \setminus F = X$.

(b): Using Theorem 5.2.6 and 5.2.14 (b), the proof is similar to that of (a). □

Proposition 5.2.16. *Let* $\gamma^* : \tau^* \to P(X)$ *be an operation and let* A *and* B *be the subset of* (X, τ^*).

(a) *If* $\gamma^* : \tau^* \to P(X)$ *is an open operator and if* $\mathrm{Int}_{\gamma^*}(A) \subset B \subset A$ *and* A *is* $g.\gamma^*$*-supra open, then* B *is* $g.\gamma^*$*-supra open.*

(b) *If* A *is* $sg.\gamma^*$*-supra open and if* $\mathrm{Int}_{\gamma^*}(A) \subset B \subset A$*, then* B *is* $sg.\gamma^*$*-supra open.*

Proof. The proof is similar to that of Proposition 5.2.12 (a) and (b) respectively by Lemma 3.3.6 (a). □

Theorem 5.2.17. *Let (X, τ^*) be a supra space. The following are hold.*

(a) *If a subset A is $g.\gamma^*$-supra closed in (X, τ^*), then $\mathrm{Cl}_{\gamma^*}(A) \setminus A$ is $g.\gamma^*$-supra open. The converse is true if γ^* is an open operator.*

(b) *A subset A is $sg.\gamma^*$-supra closed if and only if $\mathrm{Cl}_{\gamma^*} \setminus A$ is $sg.\gamma^*$-supra open.*

Proof. (a): Let A be $g.\gamma^*$-supra closed and $F \subset \mathrm{Cl}_{\gamma^*} \setminus A$, where F is γ^*-supra closed. By Theorem 5.2.8, $F = \phi$ and hence $F \subset \mathrm{Int}_{\gamma^*}(\mathrm{Cl}_{\gamma^*} \setminus A)$. By Theorem 5.2.14 (a), $\mathrm{Cl}_{\gamma^*}(A) \setminus A$ is $g.\gamma^*$-supra open.

Conversely, let U be γ^*-supra open and $A \subset U$. Now $\mathrm{Cl}_{\gamma^*}(A) \cap (X \setminus U) \subset \mathrm{Cl}_{\gamma^*}(A) \setminus A$ and since $\mathrm{Cl}_{\gamma^*}(A) \cap (X \setminus U)$ is γ^*-supra closed and $\mathrm{Cl}_{\gamma^*}(A) \setminus A$ is $g.\gamma^*$-supra open, it follows that $\mathrm{Cl}_{\gamma^*}(A) \cap (X \setminus U) \subset \mathrm{Int}_{\gamma^*}(\mathrm{Cl}_{\gamma^*}(A) \setminus A) = \phi$. Then $\mathrm{Cl}_{\gamma^*}(A) \subset U$ and hence A is $g.\gamma^*$-supra closed.

(b): Using Theorem 5.2.6, the proof is similar to (a). □

The following generalized properties of weaker forms (i.e. semiclosed, pre-closed, semipre-closed and α-closed sets) arising naturally in topology are special cases of Definition 5.1.4, and 5.2.1.

Remark 5.2.18. Let $\gamma^* : \tau^* \to P(X)$ is the identity; and let γ^* is associated with the operation $\gamma : \tau \to P(X)$; and let A be a $g.\gamma^*$-supra closed (or $sg.\gamma^*$-supra closed) subset of (X, τ^*).

(a) If $\tau^* = \tau$, then A is g-closed in (X, τ) [53].

(b) If $\tau^* = SO(X, \tau)$, then A is sg-closed in (X, τ) [11].

(c) If $\tau^* = PO(X, \tau)$, then A is pg-closed in (X, τ) [9].

(d) If $\tau^* = SPO(X, \tau)$, then A is $spspg$-closed in (X, τ) [87].

(e) If $\tau^* = \alpha(X, \tau)$, then A is $g\alpha$-closed in (X, τ^*) [64].

5.3 Preservation Theorems and Operation-Closed Graphs

In this section we obtain the preservation theorems of γ^*-supra T_i spaces ($i = 0,\ 1/2,\ 1,\ 2$) and investigate the supra operator (i.e. supra interior operatior or supra closure operatior) approaches of closed garphs of mappings.

Throughout the rest of this section, let (X, τ^*) and (Y, σ^*) be supra spaces and let $\gamma^* : \tau^* \to P(X)$ and $\beta^* : \sigma^* \to P(Y)$ be operations and let $f : (X, \tau^*) \to (Y, \sigma^*)$ be a mapping.

We begin with standard properties of preserving the inverse images of β^*-supra T_i spaces, where $i = 0, 1/2, 1$ or 2.

Theorem 5.3.1. *Let $f : (X, \tau^*) \to (Y, \sigma^*)$ be an injective (γ^*, β^*)-supra continuous mapping. If (Y, σ^*) is β^*-supra T_i, where $i = 0, 1,$ or 2, then (X, τ^*) is γ^*-supra T_i, respectively.*

Proof. Case I. Suppose that $i = 0$ (namely, (Y, σ^*) is β^*-supra T_0).

Let $x_1 \neq x_2 \in X$. Then, since f is injective, $f(x_1) = y_1$ and $f(x_2) = y_2$ for some distinct points $y_1, y_2 \in Y$. But since (Y, σ^*) is β^*-supra T_0, there exist supra open sets in (Y, σ^*), V_1 and V_2 such that either $f(x_1) \in V_1$ and $f(x_2) \notin V_1^{\beta^*}$ or $f(x_2) \in V_2$ and $f(x_1) \notin V_2^{\beta^*}$. Furthermore, since f is (γ^*, β^*)-supra continuous, for V_1, say $y_1 \in V_1$ but $y_2 \notin V_1^{\beta^*}$, there exist supra open sets U_1 and U_2 in (X, τ^*) such that $x_1 \in U_1$ and $f(U_1^{\gamma^*}) \subset V_1^{\beta^*}$ and then $f(x_2) = y_2 \notin f(U_1^{\gamma^*})$ since (Y, σ^*) is β^*-supra T_0. Therefore (X, τ^*) is γ^*-supra T_0.

Case II. Suppose that $i = 1$ (namely, (Y, σ^*) is β^*-supra T_1).

Let $x \in X$. We claim that $\{x\}$ is a γ^*-supra closed set. Indeed, since f is injective, there exists $y \in Y$ such that $f(x) = y$. But since (Y, σ^*) is β^*-supra T_1, $\{y\}$ is β^*-supra closed. Moreover, $f^{-1}(\{y\})$ is also γ^*-supra closed since f is (γ^*, β^*)-supra continuous. Therefore $\{x\}$ is γ^*-supra closed.

Case III. Suppose that $i = 2$ (namely, (Y, σ^*) is β^*-supra T_2).

Let $x_1 \neq x_2 \in X$. Then since f is injective, $f(x_1) \neq f(x_2)$. But since (Y, σ^*) is β^*-supra T_2, there exist supra open sets V_1 and V_2 such that $f(x_1) \in V_1$, $f(x_2) \in V_2^{\beta^*}$ and $V_1^{\beta^*} \cap V_2^{\beta^*} = \phi$. Furthermore, since f is (γ^*, β^*)-supra continuous, for V_1 and V_2, there exist supra open sets U_1 and U_2 such that $x_1 \in U_1$, $x_2 \in U_2$, $f(U_1^{\gamma^*}) \subset V_1^{\beta^*}$ and $f(U_2^{\gamma^*}) \subset V_2^{\beta^*}$ and then $U_1^{\gamma^*} \subset f^{-1}(V_1^{\beta^*})$ and $U_2^{\gamma^*} \subset f^{-1}(V_2^{\beta^*})$. Now we have that

$$U_1^{\gamma^*} \cap U_2^{\gamma^*} \subset f^{-1}(V_1^{\beta^*}) \cap f^{-1}(V_2^{\beta^*}) = f^{-1}(V_1^{\beta^*} \cap V_2^{\beta^*}) = \phi.$$

Therefore (X, τ^*) is γ^*-supra T_2. □

To see the property of preserving the images γ^*-supra $T_{1/2}$ spaces and inverse images of β^*-supra $T_{1/2}$ spaces, we prepare a proposition.

Proposition 5.3.2. *Let f be (γ^*, β^*)-supra continuous and (id, β^*)-supra closed.*

(a) *For every $g.\gamma^*$-supra closed set A of (X, τ^*), the image $f(A)$ is β^*-supra closed.*

(b) *If $\beta^* : \sigma^* \to P(Y)$ is a regular operator, then for every $g.\beta^*$-supra closed set B of (Y, σ^*), the inverse image set $f^{-1}(B)$ is $g.\gamma^*$-supra closed.*

Proof. (a): Let V be any β^*-supra open set of (Y, σ^*) with $f(A) \subset V$. By using Proposition 4.1.3 (b), $f^{-1}(V)$ is γ^*-supra open. Since A is $g.\gamma^*$-supra closed and $A \subset f^{-1}(V)$, we have $\mathrm{Cl}_{\gamma^*}(A) \subset f^{-1}(V)$, and hence $f(\mathrm{Cl}_{\gamma^*}(A)) \subset V$. It follows from Theorem 3.2.16 (b) and assumption that $f(\mathrm{Cl}_{\gamma^*}(A))$ is β^*-supra closed. Therefore we have $\text{Cl}_{\beta^*}(f(A)) \subset \text{Cl}_{\beta^*}(f(\text{Cl}_{\gamma^*}\ (A))) = f(\text{Cl}_{\gamma^*}(A)) \subset V$. This implies $f(A)$ is $g.\beta^*$-supra closed.

(b): Let U be a γ^*-supra open set of (X, τ^*) with $f^{-1}(B) \subset U$. Put $F = \text{Cl}_{\gamma^*}(f^{-1}(B)) \cap X \setminus U$. It follows from Theorem 3.2.16 (b) and Definition 3.1.2 that F is supra closed in (X, τ^*). Since f is (id, β^*)-supra closed, $f(F)$ is β^*-supra closed in (Y, σ^*). By using Theorem 5.2.8 and the following inclusion $f(F) \subset \text{Cl}_{\beta^*}(B) \setminus B$, it is obtained that $f(F) = \phi$ and hence $F = \phi$.Therefore $\text{Cl}_{\gamma^*}(f^{-1}(B)) \subset U$, as desired. □

Remark 5.3.3. In view of the above Proposition 5.3.2, we have the following results.

(a) If γ^* and β^* are regular identity operations, then theorems due to Levine ([53, Theore 6.1and 6.3]) are obtained.

(b) If γ^* and β^* are identity operatiors and if $\tau^* = SO(X, \tau)$ and $\sigma^* = SO(Y, \sigma)$, then theorems due to Sundaram et al. ([97, Theorem 3.7]) are obtained.

Theorem 5.3.4. *Let $f : (X, \tau^*) \to (Y, \sigma^*)$ be a (γ^*, β^*)-supra continuous injection and (id, β^*)-supra closed mapping. If (Y, σ^*) is β^*-supra $T_{1/2}$, then (X, τ^*) is γ^*-supra $T_{1/2}$.*

Proof. Let A be $g.\gamma^*$-supra closed in (X, τ^*). We wish only to show that A is γ^*-supra closed. Indeed, by Proposition 5.3.2 (a) and assumption, it is obtained that $f(A)$ is $g.\beta^*$-supra closed and hence β^*-supra closed. Since f is a (γ^*, β^*)-supra continuous injection, $f^{-1}(f(A)) = A$ is γ^*-supra closed. Therefore, the inverse image (X, τ^*) of f is γ^*-supra $T_{1/2}$. □

Now we have properties of preserving the images of γ^*-supra T_1 and γ^*-supra T_2 spaces in the following.

Theorem 5.3.5. *Let $f : (X, \tau^*) \to (Y, \sigma^*)$ be a bijective (γ^*, β^*)-supra open mapping and let $\gamma^* : \tau^* \to P(X)$ be an open operator. If (X, τ^*) is γ^*-supra T_1 (resp. γ^*-supra T_2), then (Y, σ^*) is β^*-supra T_1 (resp. β^*-supra T_2).*

Proof. Suppose that (X, γ^*) is γ^*-supra T_2. Let $y_1 \neq y_2 \in Y$. Then since f is injective and surjective, $f^{-1}(y_1) = x_1$ and $f^{-1}(y_2) = x_2$ for some distinct points x_1 and x_2 in X. Since (X, γ^*) is γ^*-supra T_2, there exist supra open sets U_1 and U_2 such that $x_1 \in U_1$, $x_2 \in U_2$ and $U_1^{\gamma^*} \cap U_2^{\gamma^*} = \phi$. Furthermore, since γ^* is open, there exist γ^*-supra open sets G_1 and G_2 such that $x_1 \in G_1$, $x_2 \in G_2$, $G_1^{\gamma^*} \subset U_1$ and $G_2^{\gamma^*} \subset U_2$ and then since f is (γ^*, β^*)-supra open, $f(G_1)$ and $f(G_2)$ is β^*-supra open and $f(G_1) \cap f(G_2) = f(G_1 \cap G_2) = \phi$. This imply that there exist supra open sets W_1 and W_2 in (Y, σ^*) such that $y_1 \in W_1$, $y_2 \in W_2$ and $W_1^{\beta^*} \subset f(G_1)$ and $W_2^{\beta^*} \subset f(G_2)$. Therefore $W_1^{\beta^*} \cap W_2^{\beta^*} = \phi$ and hence (Y, σ^*) is β^*-supra T_2. The proof of the case of γ^*-supra T_1 is proved similarly. □

Theorem 5.3.6. *Let $f : (X, \tau^*) \to (Y, \sigma^*)$ be a (γ^*, β^*)-supra continuous surjection and (id, β^*)-supra closed mapping. Suppose that $\beta^* : \sigma^* \to P(Y)$ be a regular operator. If (X, τ^*) is γ^*-supra $T_{1/2}$, then (Y, σ^*) is β^*-supra $T_{1/2}$.*

Proof. By using Proposition 5.3.2 (ii), this proof is similar to that of Theorem 5.3.4. □

As an immediate consequence of Theorem 5.3.6, we obtain the follow corollary.

Corollary 5.3.7. *Let $f : (X, \tau^*) \to (Y, \sigma^*)$ be (γ^*, β^*)-homeomorpic and let $\beta^* : \sigma^* \to P(Y)$ be a regular operator. If (X, τ^*) is γ^*-supra $T_{1/2}$, (Y, σ^*) is β^*-supra $T_{1/2}$.*

Remark 5.3.8. In Theorem 5.3.6 and Corollary 5.3.7, let γ^* and β^* be the identy regular operators. Then, we have Theorem 3.3 and Corollary 3.5 due to Dunham ([32]).

By Theorem 3.1.15, $\tau^*_{\gamma^*}$ is a topology of X if $\gamma^* : \tau^* \to P(X)$ is a regular operator. Then, we have the following:

Theorem 5.3.9. *Let $\gamma^* : \tau^* \to P(X)$ is a regular operator. A supra space (X, τ^*) is γ^*-supra $T_{1/2}$ if and only if $(X, \tau^*_{\gamma^*})$ is $T_{1/2}$.*

Proof. It is proved by Theorem 5.1.11 and 3.1.15 and Theorem 2.5 due to Dunham [32]. □

Theorem 5.3.10. *Let $\gamma^* : \tau^* \to P(X)$ is a regular and open operator. A supra space (X, τ^*) is γ^*-supra T_2 if and only if $(X, \tau^*_{\gamma^*})$ is T_2.*

Proof. Suppose that $(X, \tau^*_{\gamma^*})$ is T_2. Let $x \neq y$ be points of X. Then there exist open sets U and V in $(X, \tau^*_{\gamma^*})$ such that $x \in U$, $y \in V$ and $U \cap V = \phi$. Since U and V is γ^*-supra open in (X, τ^*), there exist supra open sets U_0 and V_0 in (X, τ^*) such that $x \in U_0$, $y \in V_0$, $U_0^{\gamma^*} \subset U$ and $V_0^{\gamma^*} \subset V$ and hence $U_0^{\gamma^*} \cap V_0^{\gamma^*} = \phi$. Therefore (X, τ^*) is γ^*-supra T_2.

Conversely, Suppose that (X, τ^*) is γ^*-supra T_2. Let $x \neq y$ be points of X. Then there exist supra open sets U and V in (X, τ^*) such that $x \in U$, $y \in V$ and $U^{\gamma^*} \cap V^{\gamma^*} = \phi$. Since γ^* is open, there exist γ^*-supra open sets U_0 and V_0 in (X, τ^*) such that $x \in U_0$, $y \in V_0$ and $U_0 \subset U^{\gamma^*}$ and $V_0 \subset V^{\gamma^*}$ and hence $U_0 \cap V_0 = \phi$. Therefore $(X, \tau^*_{\gamma^*})$ is T_2 since U_0 and V_0 are open in $(X, \tau^*_{\gamma^*})$. □

Theorem 5.3.11. *Let γ^* and β^* be regular operators. If f, $g : (X, \tau^*) \to (Y, \sigma^*)$ are (γ^*, β^*)-supra continuous and if (Y, σ^*)-supra T_2, then the sets $A = \{x \in X | f(x) = g(x)\}$ is γ^*-supra closed in (X, τ^*).*

Proof. Let $x \in X \setminus A$. Then $f(x) \neq g(x)$ and there exist supra open sets V_1 and V_2 such that $f(x) \in V_1$, $g(x) \in V_2$ and $V_1^{\beta^*} \cap V_2^{\beta^*}$, and since f and g are (γ^*, β^*)-supra continuous, there are supra open sets U_1 and U_2 such that $x \in U_1$, $x \in U_2$ and $U_1^{\gamma^*} \subset f^{-1}(V_1^{\beta^*})$ and $U_2^{\gamma^*} \subset g^{-1}(V_2^{\beta^*})$. But since γ^* is regular, there exists supra open set $x \in G$ such that $G^{\gamma^*} \subset U_1^{\gamma^*} \cap U_2^{\gamma^*} \subset f^{-1}(V_1^{\beta^*}) \cap g^{-1}(V_2^{\beta^*})$ and hence $G^{\gamma^*} \subset X \setminus A$. Therefore A is γ^*-supra closed. □

Now we investigate properties of the images of $g.\gamma^*$-supra closed and $sg.\gamma^*$-supra closed sets in (X, τ^*) and the inverse image of $g.\beta^*$-supra closed and $sg.\beta^*$-supra closed sets in (Y, σ^*) in the following.

Theorem 5.3.12. *Let f be an (id, β^*)-supra closed mapping.*

(a) *If A is $sg.\gamma^*$-supra closed in (X, τ^*) and if f is S^*-continuous [70], then $f(A)$ is $sg.\beta^*$-supra closed in (Y, σ^*), where τ^* and σ^* are associated with the topologies τ and σ, respectively.*

(b) *If A is $g.\gamma^*$-supra closed in (X, τ^*) and if f is (γ^*, id)-supra continuous, then $f(A)$ is $sg.\beta^*$-supra closed in (Y, σ^*).*

(c) *If A is $sg.\gamma^*$-supra closed in (X, τ^*) and if f is (id, β^*)-supra continuous, then $f(A)$ is $g.\beta^*$-supra closed in (Y, σ^*).*

Proof. (a): Let V be any supra open of (Y, σ^*) such that $f(A) \subset V$. Since f is S^*-continuous and A is $sg.\gamma^*$-supra closed, $\mathrm{Cl}_{\gamma^*}(A) \subset f^{-1}(V)$ and hence $f(\mathrm{Cl}_{\gamma^*}(A)) \subset V$. By Theorem 3.2.16 (b) and assumption, $f(\mathrm{Cl}_{\gamma^*}(A))$ is β^*-supra closed in (Y, σ^*). Thus we have $\mathrm{Cl}_{\beta^*}(f(A)) \subset \mathrm{Cl}_{\beta^*}(f(\mathrm{Cl}_{\gamma^*}(A))) = f(\mathrm{Cl}_{\gamma^*}(A)) \subset V$. This implies $f(A)$ is $sg.\beta^*$-supra closed in (Y, σ^*).

The proofs of (b) and (c) are similar to that of (a). □

Theorem 5.3.13. *Let f be a (γ^*, β^*)-supra continuous mapping.*

(a) *If B is $sg.\beta^*$-supra closed in (Y, σ^*) and if f is S^*-closed [70], then $f^{-1}(B)$ is $sg.\gamma^*$-supra closed in (X, τ^*), where τ^* and σ^* are associated with the topologies τ and σ, respectively.*

(b) *If B is $g.\beta^*$-supra closed in (Y, σ^*) and if f is (id, β^*)-supra closed, then $f^{-1}(B)$ is $sg.\gamma^*$-supra closed in (X, τ^*).*

Proof. (a): Let U be any supra open of (X, τ^*) with $f^{-1}(B) \subset U$. Put $F = \mathrm{Cl}_{\gamma^*}(f^{-1}(B)) \cap X \setminus U$. Then by Theorem 3.2.16 (b), F is supra closed in (X, τ^*). Since f is S^*-closed, $f(F)$ is supra closed in (Y, σ^*) and by Proposition 4.1.3, $f(F) \subset \mathrm{Cl}_{\beta^*}(B) \setminus B$. By Theorem 5.2.6, $f(F) = \phi$ and so $F = \phi$, that is, $\mathrm{Cl}_{\gamma^*}(f^{-1}(B)) \subset U$. Therefore $f^{-1}(B)$ is $sg.\gamma^*$-supra closed in X, τ^*.

(b): The proof is similar to (a) by using Theorem 5.2.8. □

Now, we further investigate general supra operators, namely supra interior and supra closure operatiors, approaches of closed garphs of mappings.

Throughout this section, let $(X \times Y, \tau^* \times \sigma^*)$ be the product supra space and let $\rho^* : \tau^* \times \sigma^* \to P(X \times Y)$ be an operation on $\tau^* \times \sigma^*$.

We define the new class of operations ρ^* associated with γ^* and β^* in the following:

Definition 5.3.14. An operation $\rho : \tau^* \times \sigma^* \to P(X \times Y)$ is said to be *associated with* γ^* and β^*, if $(U \times V)^{\rho^*} = U^{\gamma^*} \times V^{\beta^*}$ holds for each $(\phi \neq) U \in \tau^*$ and $(\phi \neq) V \in \sigma^*$. $\rho : \tau^* \times \sigma^* \to P(X \times Y)$ is said to be *regular with repect to* γ^* and β^*, if for each point $(x, y) \in X \times Y$ and each supra open neighborhood W of (x, y), there exist supra open sets $U \in \tau^*$ and $V \in \sigma^*$ such that $x \in U$, $y \in V$ and $U^{\gamma^*} \times V^{\beta^*} \subset W^{\rho^*}$.

Theorem 5.3.15. *Let $\rho^* : \tau^* \times \tau^* \to P(X \times X)$ be an associated operation with γ^* and γ^*. If $f : (X, \tau^*) \to (Y, \sigma^*)$ is (γ^*, β^*)-supra continuous and (Y, σ^*) is a β^*-supra T_2 space, then the set $A = \{(x, y) \in X \times X : f(x) = f(y)\}$ is ρ^*-supra closed set of $(X \times X, \tau^* \times \tau^*)$.*

Proof. We show that $\mathrm{Cl}_{\rho^*}(A) \subset A$. Let $(x, y) \in X \times X \setminus A$. Then, there exist two supra open sets V_1 and V_2 in (Y, σ^*) such that $f(x) \in V_1$, $f(y) \in V_2$ and $V_1^{\beta^*} \cap V_2^{\beta^*} = \phi$. Moreover, for V_1 and V_2, there exist supra open sets U_1 and U_2 in (X, τ^*) such that $x \in U_1$, $y \in U_2$ and $f(U_1^{\gamma^*}) \subset V_1^{\beta^*}$ and $f(U_2^{\gamma^*}) \subset V_2^{\beta^*}$. Therefore we have $(U_1 \times U_2)^{\rho^*} \cap A = \phi$. This shows that $(x, y) \notin \mathrm{Cl}_{\rho^*}(A)$. □

In view of the above Definition 5.3.14 and Theorem 5.3.15, we have the following result holds:

Corollary 5.3.16. *Let $\rho^* : \tau^* \times \tau^* \to P(X \times X)$ be an associated operation with γ^* and γ^* and let ρ^* be regular with repect to γ^* and γ^*. A space (X, τ^*) is a γ^*-supra T_2 if and only if the diagonal set $\triangle = \{(x, x) | x \in X\}$ is ρ^*-supra closed in $(X \times X, \tau^* \times \tau^*)$.*

Proposition 5.3.17. *Let $\rho^* : \tau^* \times \sigma^* \to P(X \times Y)$ be an associated operation with γ^* and β^*. If $f : (X, \tau^*) \to (Y, \sigma^*)$ is (γ^*, β^*)-supra continuous and (Y, σ^*) is β^*-supra T_2, then the graph of f, $G(f) = \{(x, f(x)) \in X \times Y\}$ is a ρ^*-supra closed set of $(X \times Y, \tau^* \times \sigma^*)$.*

Proof. The proof is similar to that of Proposition 5.3.15. □

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