Optimality and Duality for Multiobjective Programming Problems with Various Generalized Convexity

다양한 일반화된 볼록성을 가지는 다목적 계획 문제들의 최적성과 쌍대성

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Optimality and Duality for Multiobjective Programming Problems with Various Generalized Convexity

A dissertation

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CONTENTS

다양한 일반화된 볼록성을 가지는 다목적 계획 문제들의 최적성과 쌍대성

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요 약

다양한 일반화된 볼록 함수 조건하에서 약 효율해(weakly efficient solution)에 대한 다목적 계획 문제에서의 필요 충분 최적 정리와 쌍대 정리를 정립하였다.

지지함수(support function)항을 포함하고 있는 미분 불가능한 목적함수를 가지는 다목적 계획 문제에서 Fritz John형 및 Kuhn-Tucker형 필요 충분 최적 정리를 얻고, 약 효율해에 대한 일반화된 볼록 함수 조건 아래에서 쌍대 정리를 얻었다. 또한 일반화된 인벡스티 조건에서 지지함수를 포함하는 미분 불가능한 분수 다목적 계획 문제의 필요 충분 최적 정리와 쌍대 정리들을 얻었다. 그리고 추 제약식을 가지는 Mond-Weir형 및 Wolfe형 다목적 대칭 쌍대 문제를 만들고 각각 Pseudo-invex 함수조건과 K-Preinvex 함수조건 아래에서 두 문제 사이의 대칭 쌍대 정리가 성립함을 보였다. 마지막으로 두 점이 일반화된 호로 연결되는 연결함수(arcwise connected function)를 정의하고, 일반화된 연결함수를 가지는 다목적 계획 문제와 다목적 분수 계획 문제를 정형화하고 약 효율해에 대한 필요 충분 최적 정리와 이에 대한 쌍대 정리를 얻었다.

Chapter 1

Introduction and Preliminaries

Multiobjective programming problems consist of conflicting objective functions and constraint sets, and are intended to optimize the objective functions over the constraint sets under some concepts of solution. Their optimums form the solution concepts that appear to be the natural extension of the optimum of a single objective to one of multiple objectives. In economic analysis([7]), game ([15]) and system science, such optimums are effective for treating such a multiplicity of values.

Optimality and duality are very important topics in investigating optimization problems. There are a large number of papers discussing optimality and duality for optimization problems ([19], [35], [40], [45], [44], [46], [50], [66]).

In 1948, John ([31]) gave a necessary optimality theorem for an optimization problem with inequality constraints and without any constraint qualification, which is now called the Fritz John necessary optimality theorem. In 1961, Kuhn and Tucker ([41]) proved another necessary optimality theorem for an optimization problem with inequality constraints under a constraint qualification, which is now called the Kuhn-Tucker necessary optimality theorem. Of course, such optimality theorems are closely related. Fritz John and Kuhn Tucker necessary optimality theorems have been extended to nondifferentiable optimization problems ([49, 81]) and multiobjective optimization problems. Conditions in two optimality theorems are sufficient ones for feasible points to be optimal under generalized convexity or generalized invexity assumptions on functions. Kuhn-Tucker sufficient optimality theorems have been extended to several kinds of optimization problems under generalized convexity and generalized invexity. On the other hand, many authors have studied Fritz John type sufficient optimality theorems for single objective(i.e, scalar) optimization problems under generalized convexity.

In many aspects of mathematical programming problems including sufficient optimality conditions, duality theorems and alternative theorems, convexity plays a vital role ([1], [12], [24], [25], [27], [34], [50], [52]). Hanson and Mond ([26]) introduced type-1 and type-2 invexities which have been further generalized by many researchers and applied to nonlinear programming problems in different settings. To relax convexity assumptions imposed on the functions in theorems on sufficient optimality conditions and duality, various generalized convexity notions have been proposed. Vial ([71]), Preda ([63]), Ben-Isral and Mond ([11]), Hanson and Mond ([26]), Jeyakumar and Mond ([33]), Ye ([77]), Antezak ([2]) and many others have studied some properties, applications and further generalizations of these functions. Recently, Liang *et al.* ([46]) introduced (F, α, ρ, d)-convexity which is one of such generalizations of invex functions.

Under certain convexity assumptions and suitable constraint qualifications, the primal and dual problems have equal optimal objective values and hence it is possible to solve the primal problem indirectly by solving the dual problem. In 1961, Wolfe ([74]) formulated a dual problem for a single objective optimization problem on the basis of the Kuhn-Tucker necessary optimality conditions, which is now called the Wolfe dual problems, and proved weak and strong duality theorems. In 1981, Mond and Weir ([59])

gave another type of dual problem for a single objective optimization problem on the basis of the Kuhn-Tucker necessary optimality condition, which is now called the Mond-Weir dual problem. They also proved weak, strong and converse duality theorems. Until now many authors have formulated Wolfe type dual problems and Mond-Weir type dual problems for several kinds of optimization problems and have studied these problems for duality theorems. Duality theorems for fractional scalar(single-objective) minimization problems have been of much interest in the past ([4], [13], [28], [67], [68]).

Recently there has been of growing intrest in studying duality for fractional multiobjective minimization problems ([6], [14], [21], [32] [42], [45], [44], [48], [54], [58], [72], [73]). Also several authors have been interested in optimality conditions and duality theorms for nondifferentiable mulitobjective programming problems ([49], [53], [56], [57], [78], [81]). Bhatia and Jain ([9]) considered a nonlinear nondifferentiable multiobjective fractional programming problem in which numerator of each component of the objective function contains a term involving square root of a certain positive semi-definite quadratic form. We shall introduce a class of nondifferentiable mulitobjective programming problems in this paper.

Symmetric duality is one of the major branches approaching to multiobjective optimization and it plays a useful role in the theory and computational algorithm of multiobjective optimization. Symmetric duality in nonlinear programming in which the dual of the dual is the primal was first introduced by Dorn ([17]) by defining a symmetric dual program for quadratic programs. Dantzig, Eisenberg and Cottle ([18]) first formulated a pair of symmetric dual nonlinear programs involving a scalar function. Recently Suneja *et al.* ([69]) formulated multiobjective symmetric dual programming problems over arbitary cones and Yang *et al.*([79], [80]) formulated symmetric duality for a class of nonlinear, nondifferentiable multiobjective fractional programming programs under generalized invexity. We consider symmetric dualities for nonlinear multiobjective programming problems with cone constraints which are applications of multiobjective optimization.

Now we discuss solution concepts for multiobjective optimization problems and introduce some fundamental properties of solutions. Optimal solutions to multiobjective optimization are not trivial and in itself debatable. It is closely related to the preference attitudes of the decision makers ([70]).

We formulate the standard form of the multiobjective optimization problems.

$$\textbf{(VP)} \qquad \text{Minimize} \quad f(x) := (f_1(x), \cdots, f_p(x))$$
$$\text{subject to} \quad x \in S,$$

where $f_i : X \to \mathbb{R},\ i = 1, \cdots, p$, are functions and S is a subset of X.

This problem is also called a vector optimization. For multiobjective optimization problems, there are three kinds of solution. We call them properly efficient, efficient and weakly efficient solution. The most fundamental solution concept is that of efficient solutions (also called parato optimal solutions or noninferior solutions) with respect to the domination structure of the decision maker.

Optimization of **(VP)** is finding (properly, weakly) efficient solutions defined as follows:

Definition 1.1. A point $\bar{x} \in S$ is said to be an efficient solution of **(VP)** if for any $x \in S$,

$$(f_1(x) - f_1(\bar{x}), \cdots, f_p(x) - f_p(\bar{x})) \notin -\mathbb{R}^p_+ \backslash \{0\},$$

where $\mathbb{R}^p_+ = \{y \in \mathbb{R}^p \ : \ y \geqq 0\}$ is the nonnegative orthant of $\mathbb{R}^p$.

Definition 1.2. A point $\bar{x} \in S$ is said to be a properly efficient solution of **(VP)** if $\bar{x} \in S$ is an efficient solution of **(VP)** and there exists a constant $M > 0$ such that for each $i = 1, \cdots, p$, we have

$$\frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} \leqq M$$

for some j such that $f_j(x) > f_j(\bar{x})$ whenever $x \in S$ and $f_i(x) < f_i(\bar{x})$.

The quantity $\frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})}$ may be interpreted as the marginal trade-off for objective functions f_i and f_j between x and $\bar{x}$. Geoffrion ([23]) considered the concept of the proper efficiency to eliminate unbounded trade-off between objective functions of **(VP)**.

Definition 1.3. A point $\bar{x} \in S$ is said to be a weakly efficient solution of **(VP)** if for any $x \in S$,

$$(f_1(x) - f_1(\bar{x}), \cdots, f_p(x) - f_p(\bar{x})) \notin -int\mathbb{R}^p_+,$$

where $int\mathbb{R}^p_+$ is the interior of $\mathbb{R}^p_+$.

In this paper we shall use the concept of weakly efficient solution.

Now we introduce notations and theorem that will be used later.
Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space. We use the following notations for vectors x, y in $\mathbb{R}^n$:

$$x < y \iff x_i < y_i, \quad i = 1, 2, \cdots, n;$$

$$x \not< y \quad \text{is the negation of} \quad x < y;$$

$$x \leqq y \iff x_i \leqq y_i, \quad i = 1, 2, \cdots, n;$$

$$x \leq y \iff x_i \leqq y_i, \quad i = 1, 2, \cdots, n \quad \text{but} \quad x \neq y;$$

$$x \not\leq y \quad \text{is the negation of} \quad x \leq y.$$

Next is a generalization of Gordan's theorem ([51]) to a convex function over an arbitary convex set in $\mathbb{R}^n$.

Theorem 1.1 (Gordan Theorem). Let f be an n-dimensional convex vector function on the convex set $S \subset \mathbb{R}^n$. Then

I. $f(x) < 0$ has a solution $x \in S$ or

II. $pf(x) \geqq 0$ for all $x \in S$ for some $p \geq 0$, $p \in \mathbb{R}^n$.

but never both.

The purpose of this dissertation is to establish necessary and sufficient optimality conditions and duality theorems for multiobjective programming problems under various generalized convexity conditions involving differentiable or nondifferentiable functions. In particular, we prove necessary optimaltity conditions, sufficient optimality conditions and duality theorems for the weakly efficient solution. We show that the weak and strong duality hold between primal problems and dual problems.

This dissertation is organized as follows :

In Chapter 2, we introduce the concepts of (F, α, ρ, d)-convexity and generalized (F, α, ρ, d)-convexity of differentiable function ϕ for sublinear functional F. We formulate necessary and sufficient optimality conditions for weakly efficient solutions of nondifferentiable multiobjective programming problem **(NVP)**, in which each component of the objective function contains a term involving the support function $s(x|C)$ of a compact convex set C of $\mathbb{R}^n$. And we formulate the generalized dual Programming problem **(NVD)** for weakly efficient solution under generalized (F, α, ρ, d)-convexity assumptions. As special cases of our duality results, we give Mond-Weir type and Wolfe type duality theorems.

In Chapter 3, we introduce the concept of (V, ρ)-ratio invexity and we present a nondifferentiable multiobjective fractional problem **(NFP)**, in which each component of the objective function contains a term involving the support function of a compact convex set. We obtain the the Fritz John and Kuhn-Tucker necessary and sufficient optimality conditions for weakly efficient solutions. We formulate Mond-Weir type dual problem and Wolfe type dual problem of problem **(NFP)** under (V, ρ)-ratio invexity assumptions. Also we introduce weak and strong duality theorems for each problems.

In Chapter 4, we formulate Mond-Weir type symmetric dual problems and Wolfe type symmetric dual problems with cone constraints. We obtain duality results under weakly efficient solution involving pseudo-invex functions for Mond-Weir type symmetric duality. Also we obtain duality results involving K-preinvex functions for Wolfe type symmetric duality. And we establish the weak, strong, converse and self duality for each symmetric dual problems.

In Chapter 5, we introduce arcwise connected functions defined on arcwise connected sets by replacing a line segment joining two points by a continuous arc, and we present some basic properties of arcwise connected sets and functions. We introduce multiobjective programming problem **(MOP)** and its geneal dual **(MOD)** involving generalized arcwise connected functions. We obtain the necessary and sufficient optimality conditions for **(MOP)** and prove the weak, strong duality theorems for **(MOD)**, based on the weakly efficiency. Also we obtain the multiobjective fractional programming problem **(MFP)** and its general dual **(MFD)**. We introduce parametric multiobjective optimization problem $(\mathbf{MFP})_\lambda$ to obtain the optimality conditions and duality theorems by establishing equivalent relationship between **(MFP)** and $(\mathbf{MFP})_\lambda$.

Chapter 2

Optimality and Duality for Nondifferentiable Programming Problems with $(\mathbf{F}, \alpha, \rho, \mathbf{d})$-Convexity

2.1. Introduction

Mond and Schechter ([56]) were first to introduce nondifferentiable symmetric duality, in which the objective function contains a support function. Lal *et al.* ([49]) obtained duality theorems for nondifferentiable static programming problem with the square root term. Jeyakumar ([29]) defined ρ-invexity for nonsmooth optimization problems, and Kuk *et al.* ([39]) defined the concept of V-ρ-invexity for vector valued functions, which is a generalization of the V-invex function ([33, 55]). Recently, Yang *et al.* ([78]) studied a class of nondifferentiable multiobjective programming problems. They replaced the objective function by the support funtion of a compact convex set. And they have constructed a more general dual model for a class of nondifferentiable multiobjective programs and established only weak duality theorems for efficient solutions under suitable weak convexity conditions. Very recently, Liang *et al.* ([46]) introduced the concept of (F, α, ρ, d)-convexity and presented optimality and duality results for a class of nonlinear fractional programming problems. In this chapter, we introduce (F, α, ρ, d)-convexity and generalized (F, α, ρ, d)-convexity of differentiable function ϕ for sublinear functional F. We formulate necessary and sufficient optimality conditions for weakly efficient solutions of nondifferentiable multiobjective programming problem **(NVP)**, in which each component of the objective function contains a term involving the support function $s(x|C)$ of a compact convex set C of $\mathbb{R}^n$. We shall formulate the generalized dual programming problem for weakly efficient solution under generalized (F, α, ρ, d)-convexity assumptions. As special cases of our duality results, we give Mond-Weir type and Wolfe type duality theorems.

Next we recall the following definitions of the generalized (F, ρ)-convexity defined by Preda ([63]).

Definition 2.1.1. A functional $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is sublinear if for any $x, u \in \mathbb{R}^n$,

$$F(x, u; a_1 + a_2) \leqq F(x, u; a_1) + F(x, u; a_2) \quad \text{for all } a_1, a_2 \in \mathbb{R}^n$$

and

$$F(x, u; \alpha a) = \alpha F(x, u; a) \quad \text{for all } \alpha \in \mathbb{R},\ \alpha \geqq 0, \quad \text{and} \quad a \in \mathbb{R}^n.$$

Let $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be a sublinear functional, the function $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be differentiable at $u \in \mathbb{R}^n$, $\alpha(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}_+ \backslash \{0\}$, $\rho \in \mathbb{R}$ and $d(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ ([46]).

Definition 2.1.2. The function ϕ is said to be (F, α, ρ, d)-convex at u if

$$\phi(x) - \phi(u) \geqq F(x, u; \alpha(x, u)\nabla\phi(u)) + \rho d^2(x, u) \quad \text{for all } x \in \mathbb{R}^n.$$

Definition 2.1.3. The function ϕ is (F, α, ρ, d)-quasiconvex at u if

$$\phi(x) \leqq \phi(u) \Rightarrow F(x, u; \alpha(x, u)\nabla\phi(u)) \leqq -\rho d^2(x, u) \quad \text{for all } x \in \mathbb{R}^n.$$

Definition 2.1.4. The function ϕ is (F, α, ρ, d)-pseudoconvex at u if

$$F(x, u; \alpha(x, u)\nabla\phi(u)) \geqq -\rho d^2(x, u) \Rightarrow \phi(x) \geqq \phi(u) \quad \text{for all } x \in \mathbb{R}^n.$$

Remark 2.1.1. (i) When $\alpha(x, u) = 1$, the concept of (F, α, ρ, d)-convexity is the same as that of (F, ρ)-convexity in [63].

(ii) When $F(x, u; \alpha(x, u)\nabla\phi(u)) = \alpha(x, u)\nabla\phi(u)\eta(x, u)$ for a certain function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $\rho = 0$, the same concept appeared in the definition of V-invex functon in [33].

We consider the following multiobjective programming problem,

$$\textbf{(NVP)} \quad \text{Minimize} \quad (f_1(x) + s(x|C_1), \cdots, f_p(x) + s(x|C_p))$$
$$\text{subject to} \quad g(x) \geqq 0,$$

where f and g are differentiable functions $\mathbb{R}^n \to \mathbb{R}^p$ and $\mathbb{R}^n \to \mathbb{R}^m$, respectively; C_i, for each $i \in P = \{1, 2, \cdots, p\}$, is a compact convex set of $\mathbb{R}^n$ and $s(x|C_i) = \max\{\langle x, y\rangle \mid y \in C_i\}$. Further let $S := \{x \in \mathbb{R}^n \mid g_i(x) \geqq 0, \ i = 1, \cdots, m\}$ and $I(x) := \{i \mid g_i(x) = 0\}$ for any $x \in \mathbb{R}^n$.

Let $h_i(x) = s(x|C_i)$, $i = 1, \cdots, p$. Then h_i is a convex function and $\partial h_i(x) = \{w \in C_i | \langle w, x\rangle = s(x|C_i)\}$, where ∂h_i is the subdifferential of h_i ([56]).

2.2. Optimality Conditions

In this section, we establish Fritz John and Kuhn-Tucker necessary and sufficient conditions for weakly efficient solutions of **(NVP)**.

Theorem 2.2.1 (Fritz John Necessary Optimality Condition). Suppose that $f_i, g_j : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \cdots, p$, $j = 1, \cdots, m$, are differentiable. If $\bar{x} \in S$ is a weakly efficient solution of **(NVP)**, then there exist $\lambda_i, i = 1, \cdots, p$, $\mu_j, j = 1, \cdots, m$ and $w_i \in C_i$, $i = 1, \cdots, p$, such that

$$\sum_{i=1}^{p} \lambda_i \nabla f_i(\bar{x}) + \sum_{i=1}^{p} \lambda_i w_i - \sum_{j=1}^{m} \mu_j \nabla g_j(\bar{x}) = 0,$$

$$\langle w_i, \bar{x} \rangle = s(\bar{x}|C_i), \ i = 1, \cdots, p,$$

$$\sum_{j=1}^{m} \mu_j g_j(\bar{x}) = 0,$$

$$(\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m) \geqq 0, \ (\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m) \neq 0.$$

Proof. Let $h_i(x) = s(x|C_i)$, $i = 1, \cdots, p$. Since C_i is convex and compact, $h_i : \mathbb{R}^n \to \mathbb{R}$ is a convex function and hence for all $d \in \mathbb{R}^n$,

$$h_i'(\bar{x}; d) = \lim_{\lambda \to 0+} \frac{h_i(\bar{x} + \lambda d) - h_i(\bar{x})}{\lambda}$$

is finite. Also, for all $d \in \mathbb{R}^n$,

$$\begin{aligned}
(f_i + h_i)'(\bar{x}; d) &= \lim_{\lambda \to 0+} \frac{f_i(\bar{x} + \lambda d) + h_i(\bar{x} + \lambda d) - f_i(\bar{x}) - h_i(\bar{x})}{\lambda} \\
&= \lim_{\lambda \to 0+} \frac{f_i(\bar{x} + \lambda d) - f_i(\bar{x})}{\lambda} + \lim_{\lambda \to 0+} \frac{h_i(\bar{x} + \lambda d) - h_i(\bar{x})}{\lambda} \\
&= f_i'(\bar{x}; d) + h_i'(\bar{x}; d) \\
&= \langle \nabla f_i(\bar{x}), d \rangle + h_i'(\bar{x}; d).
\end{aligned}$$

Since $\bar{x}$ is a weakly efficient solution of **(NVP)**,

$$\left(\begin{array}{c} \langle \nabla f_i(\bar{x}), d \rangle + h_i'(\bar{x}; d) < 0, \ i = 1, \cdots, p \\ -\langle \nabla g_j(\bar{x}), d \rangle < 0, \ j \in I(\bar{x}) \end{array} \right)$$

has no solution $d \in \mathbb{R}^n$. By Gordan Theorem for convex functions, there exist $\lambda_i \geqq 0, \ i = 1, \cdots, p$ and $\mu_j \geqq 0, \ j \in I(\bar{x})$ which are not all zero such that for any $d \in \mathbb{R}^n$,

$$\sum_{i=1}^{p} \lambda_i \ \langle \nabla f_i(\bar{x}), d \rangle + \sum_{i=1}^{p} \lambda_i h_i'(\bar{x}; d) - \sum_{j \in I(\bar{x})} \mu_j \ \langle \nabla g_j(\bar{x}), d \rangle \geqq 0. \qquad (2.1)$$

Let $A = \{\sum_{i=1}^{p} \lambda_i [\nabla f_i(\bar{x}) + \xi_i] - \sum_{j \in I(\bar{x})} \mu_j \nabla g_j(\bar{x}) \mid \xi_i \in \partial h_i(\bar{x}), \ i = 1, \cdots, p\}$. Then $0 \in A$. Ab absurdo, suppose that $0 \notin A$. By separation theorem ([51]), there exists $d^* \in \mathbb{R}^n$, $d^* \neq (0, \cdots, 0)$, such that for all $a \in A$, $\langle a, d^* \rangle < 0$, that is, $\sum_{i=1}^{p} \lambda_i \langle \nabla f_i(\bar{x}), d^* \rangle + \sum_{i=1}^{p} \lambda_i \langle \xi_i, d^* \rangle - \sum_{j \in I(\bar{x})} \mu_j \langle \nabla g_j(\bar{x}), d^* \rangle < 0$, for all $\xi_i \in \partial h_i(\bar{x})$. Hence

$$\sum_{i=1}^{p} \lambda_i \langle \nabla f_i(\bar{x}), d^* \rangle + \sum_{i=1}^{p} \lambda_i h_i'(\bar{x}; d) - \sum_{j \in I(\bar{x})} \mu_j \langle \nabla g_j(\bar{x}), d^* \rangle < 0,$$

which contradicts (2.1). Letting $\mu_j = 0$, for all $j \notin I(\bar{x})$, we have

$$0 \in \sum_{i=1}^{p} \lambda_i \nabla f_i(\bar{x}) + \sum_{i=1}^{p} \lambda_i \partial h_i(\bar{x}) - \sum_{j=1}^{m} \mu_j \nabla g_j(\bar{x})$$

and $\sum_{j=1}^{m} \mu_j g_j(\bar{x}) = 0, \ (\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m) \neq 0$.

Since $\partial h_i(\bar{x}) = \{w_i \mid \langle w_i, \bar{x} \rangle = s(\bar{x}|C_i)\}$, we obtain the desired result. □

Theorem 2.2.2 (Kuhn-Tucker Necessary Optimality Condition). Suppose that $f_i, g_j : \mathbb{R}^n \to \mathbb{R},\ i = 1, \cdots, p, j = 1, \cdots, m$ are differentiable and assume that there exists $z^* \in \mathbb{R}^n$ such that $\langle \nabla g_j(\bar{x}), z^* \rangle > 0,\ j \in I(\bar{x})$. If $\bar{x} \in S$ is a weakly efficient solution of **(NVP)**, then there exist $\lambda_i \geqq 0,\ i = 1, \cdots, p,\ \mu_j \geqq 0, j = 1, \cdots, m$ and $w_i \in C_i, i = 1, \cdots, p$ such that

$$\sum_{i=1}^{p} \lambda_i \nabla f_i(\bar{x}) + \sum_{i=1}^{p} \lambda_i w_i - \sum_{j=1}^{m} \mu_j \nabla g_j(\bar{x}) = 0,$$

$$\langle w_i, \bar{x} \rangle = s(\bar{x}|C_i),\ i = 1, \cdots, p,$$

$$\sum_{j=1}^{m} \mu_j g_j(\bar{x}) = 0,$$

$$(\lambda_1, \cdots, \lambda_p) \neq (0, \cdots, 0).$$

Proof. Since $\bar{x}$ is a weakly efficient solution of **(NVP)**, by Theorem 2.2.1, there exists $(\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m) \neq (0, \cdots, 0)$. Since $\mu_k g_k(\bar{x}) = 0$ and $\mu_k \geqq 0$, for all $k \in I(\bar{x}), \mu_k = 0$, for all $k \notin I(\bar{x})$. Hence

$$\sum_{i=1}^{p} \lambda_i \nabla f_i(\bar{x}) + \sum_{i=1}^{p} \lambda_i w_i - \sum_{j=1}^{m} \mu_j \nabla g_j(\bar{x}) = 0 \tag{2.2}$$

and $\langle w_i, \bar{x} \rangle = s(\bar{x}|C_i),\ i = 1, \cdots, p$. Assume that there exists $z^* \in \mathbb{R}^n$ such that $\langle \nabla g_j(\bar{x}), z^* \rangle > 0$, for all $j \in I(\bar{x})$. Then $(\lambda_1, \cdots, \lambda_p) \neq (0, \cdots, 0)$.

Ab absurdo, suppose that $(\lambda_1, \cdots, \lambda_p) = (0, \cdots, 0)$. Since $(\mu_1, \cdots, \mu_m) \neq (0, \cdots, 0)$, i.e., $\mu_j \geqq 0$, for all $j \in \{1, \cdots, m\}$ and hence $\mu_i > 0$ for some $i \in \{1, \cdots, m\}$. From (2.2), $0 = \sum_{j=1}^{m} \mu_j \nabla g_j(\bar{x})$.

However,

$$\begin{aligned} 0 &= < \sum_{j \in I(\bar{x})} \mu_j \nabla g_j(\bar{x}), z^* > \\ &= \sum_{j \in I(\bar{x})} \mu_j \langle \nabla g_j(\bar{x}), z^* \rangle \\ &> 0. \end{aligned}$$

This is a contradiction. Hence $(\lambda_1, \cdots, \lambda_p) \neq (0, \cdots, 0)$. □

Theorem 2.2.3 (Fritz John Sufficient Optimality Condition). Let $(\bar{x}, \lambda, w, \mu)$ satisfy the Fritz John conditions as follows:

$$\sum_{i=1}^{p} \lambda_i \nabla f_i(\bar{x}) + \sum_{i=1}^{p} \lambda_i w_i - \sum_{j=1}^{m} \mu_j \nabla g_j(\bar{x}) = 0,$$

$$\langle w_i, \bar{x} \rangle = s(\bar{x}|C_i), \; i = 1, \cdots, p, \quad w_i \in C_i,$$

$$\sum_{j=1}^{m} \mu_j g_j(\bar{x}) = 0,$$

$$(\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m) \geqq 0, \quad (\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m) \neq 0.$$

If one of the following conditions holds :

(a) $f_i(\cdot) + (\cdot)^T w_i$ is (F, α, ρ_i, d)-pseudoconvex at $\bar{x}$, and $-\sum_{j=1}^{m} \mu_j g_j(\cdot)$ is strictly (F, α, β, d)-pseudoconvex at $\bar{x}$, with $\beta + \sum_{i=1}^{p} \lambda_i \rho_i \geqq 0$;

(b) $\sum_{i=1}^{p} \lambda_i (f_i(\cdot) + (\cdot)^T w_i)$ is (F, α, ρ, d)-quasiconvex at $\bar{x}$, and $-\sum_{j=1}^{m} \mu_j g_j(\cdot)$ is strictly (F, α, β, d)-pseudoconvex at $\bar{x}$, with $\beta + \rho \geqq 0$, then $\bar{x}$ is a weakly efficient solution of **(NVP)**.

Proof. (a) Suppose that $\bar{x}$ is not a weakly efficient solution of **(NVP)**. Then there exists an $x^* \in S$ such that $f_i(x^*) + s(x^*|C_i) < f_i(\bar{x}) + s(\bar{x}|C_i)$. Since $\langle w_i, \bar{x}\rangle = s(\bar{x}|C_i),\ i = 1, \cdots, p$,

$$\begin{aligned} f_i(x^*) + x^{*T}w_i &\leqq f_i(x^*) + s(x^*|C_i) \\ &< f_i(\bar{x}) + s(\bar{x}|C_i) \\ &= f_i(\bar{x}) + \bar{x}^T w_i. \end{aligned}$$

By the (F, α, ρ_i, d)-pseudoconvexity of $f_i(\bar{x}) + \bar{x}^T w_i$,

$$F(x^*, \bar{x}; \alpha(x^*, \bar{x})(\nabla f_i(\bar{x}) + w_i)) < -\rho_i d^2(x^*, \bar{x}).$$

By sublinearity, there exists $\lambda_i \geqq 0$,

$$F(x^*, \bar{x}; \alpha(x^*, \bar{x}) \sum_{i=1}^{p} \lambda_i(\nabla f_i(\bar{x}) + w_i)) \leqq -\sum_{i=1}^{p} \lambda_i \rho_i d^2(x^*, \bar{x}).$$

Since $\beta + \sum_{i=1}^{p} \lambda_i \rho_i \geqq 0$,

$$F(x^*, \bar{x}; -\alpha(x^*, \bar{x}) \sum_{j=1}^{m} \mu_i \nabla g_j(\bar{x})) \geqq -\beta d^2(x^*, \bar{x}).$$

Since $-\sum_{j=1}^{m} \mu_j g_j(\bar{x})$ is strictly (F, α, β, d)-pseudoconvex,

$$-\sum_{j=1}^{m} \mu_j g_j(x^*) > -\sum_{j=1}^{m} \mu_j g_j(\bar{x}).$$

Since $\mu_j g_j(\bar{x}) = 0,\ j = 1, \cdots, m$, we have

$$\sum_{j=1}^{m} \mu_j g_j(x^*) < 0,$$

which contradicts the condition $\mu_j \geqq 0$ and $g_j(x^*) \geqq 0$. □

(b) Suppose that $\bar{x}$ is not a weakly efficient solution of **(NVP)**. Then there exists an $x^* \in S$ such that $f_i(x^*) + s(x^*|C_i) < f_i(\bar{x}) + s(\bar{x}|C_i)$. Since $\langle w_i, \bar{x}\rangle = s(\bar{x}|C_i),\ i = 1, \cdots, p,$

$$f_i(x^*) + x^{*T} w_i < f_i(\bar{x}) + \bar{x}^T w_i.$$

Since $\lambda_i \geqq 0$, we have

$$\sum_{i=1}^{p} \lambda_i (f_i(x^*) + x^{*T} w_i) \leqq \sum_{i=1}^{p} \lambda_i (f_i(\bar{x}) + \bar{x}^T w_i).$$

By the (F, α, ρ, d)-quasiconvexity of $\sum_{i=1}^{p} \lambda_i (f_i(\bar{x}) + \bar{x}^T w_i)$,

$$F(x^*, \bar{x}; \alpha(x^*, \bar{x}) \sum_{i=1}^{p} \lambda_i (\nabla f_i(\bar{x}) + w_i)) \leqq -\rho d^2(x^*, \bar{x}).$$

Since $\beta + \rho \geqq 0$,

$$F(x^*, \bar{x}; -\alpha(x^*, \bar{x}) \sum_{j=1}^{m} \mu_i \nabla g_j(\bar{x})) \geqq -\beta d^2(x^*, \bar{x}).$$

Since $-\sum_{j=1}^{m} \mu_j g_j(\bar{x})$ is strictly (F, α, β, d)-pseudoconvex,

$$-\sum_{j=1}^{m} \mu_j g_j(x^*) > -\sum_{j=1}^{m} \mu_j g_j(\bar{x}).$$

Since $\mu_j g_j(\bar{x}) = 0,\ j = 1, \cdots, m$, we have

$$\sum_{j=1}^{m} \mu_j g_j(x^*) < 0,$$

which contradicts the condition $\mu_j \geqq 0$ and $g_j(x^*) \geqq 0$. □

Theorem 2.2.4 (Kuhn-Tucker Sufficient Optimality Condition). Let $(\bar{x}, \lambda, w, \mu)$ satisfy the Kuhn-Tucker conditions as follows:

$$\sum_{i=1}^{p} \lambda_i \nabla f_i(\bar{x}) + \sum_{i=1}^{p} \lambda_i w_i - \sum_{j=1}^{m} \mu_j \nabla g_j(\bar{x}) = 0,$$

$$\langle w_i, \bar{x} \rangle = s(\bar{x}|C_i), \ i = 1, \cdots, p, \quad w_i \in C_i,$$

$$\sum_{j=1}^{m} \mu_j g_j(\bar{x}) = 0,$$

$$(\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m) \geqq (0, \cdots, 0), \quad (\lambda_1, \cdots, \lambda_p) \neq (0, \cdots, 0).$$

If one of the following conditions holds :

(a) $f_i(\cdot) + (\cdot)^T w_i$ is (F, α, ρ_i, d)-pseudoconvex at $\bar{x}$, and $-\sum_{j=1}^{m} \mu_j g_j(\cdot)$ is (F, α, β, d)-quasiconvex at $\bar{x}$, with $\beta + \sum_{i=1}^{p} \lambda_i \rho_i \geqq 0$;

(b) $\sum_{i=1}^{p} \lambda_i (f_i(\cdot) + (\cdot)^T w_i)$ is (F, α, ρ, d)-pseudoconvex at $\bar{x}$, and $-\sum_{j=1}^{m} \mu_j g_j(\cdot)$ is (F, α, β, d)-quasiconvex at $\bar{x}$, with $\beta + \rho \geqq 0$, then $\bar{x}$ is a weakly efficient solution of **(NVP)**.

Proof. (a) Suppose that $\bar{x}$ is not a weakly efficient solution of **(NVP)**. Then there exists an $x^* \in S$ such that $f_i(x^*) + s(x^*|C_i) < f_i(\bar{x}) + s(\bar{x}|C_i)$. Since $\langle w_i, \bar{x} \rangle = s(\bar{x}|C_i), \ i = 1, \cdots, p$,

$$\begin{aligned} f_i(x^*) + x^{*T} w_i &\leqq f_i(x^*) + s(x^*|C_i) \\ &< f_i(\bar{x}) + s(\bar{x}|C_i) \\ &= f_i(\bar{x}) + \bar{x}^T w_i. \end{aligned}$$

By the (F, α, ρ_i, d)-pseudoconvexity of $f_i(\bar{x}) + \bar{x}^T w_i$,

$$F(x^*, \bar{x}; \alpha(x^*, \bar{x})(\nabla f_i(\bar{x}) + w_i)) < -\rho_i d^2(x^*, \bar{x}).$$

By sublinearity, there exists $\lambda_i \geq 0$,

$$F(x^*, \bar{x}; \alpha(x^*, \bar{x}) \sum_{i=1}^{p} \lambda_i(\nabla f_i(\bar{x}) + w_i)) < -\sum_{i=1}^{p} \lambda_i \rho_i d^2(x^*, \bar{x}).$$

Since $\beta + \sum_{i=1}^{p} \lambda_i \rho_i \geqq 0$,

$$F(x^*, \bar{x}; -\alpha(x^*, \bar{x}) \sum_{j=1}^{m} \mu_i \nabla g_j(\bar{x})) > -\beta d^2(x^*, \bar{x}).$$

Since $-\sum_{j=1}^{m} \mu_j g_j(\bar{x})$ is (F, α, β, d)-quasiconvex,

$$-\sum_{j=1}^{m} \mu_j g_j(x^*) > -\sum_{j=1}^{m} \mu_j g_j(\bar{x}).$$

Since $\mu_j g_j(\bar{x}) = 0, \ j = 1, \cdots, m$, we have

$$\sum_{j=1}^{m} \mu_j g_j(x^*) < 0,$$

which contradicts the condition $\mu_j \geqq 0$ and $g_j(x^*) \geqq 0$. □

(b) Suppose that $\bar{x}$ is not a weakly efficient solution of **(NVP)**. Then there exists $x^* \in S$ such that $f_i(x^*) + s(x^*|C_i) < f_i(\bar{x}) + s(\bar{x}|C_i)$. Since $\langle w_i, \bar{x} \rangle = s(\bar{x}|C_i), \ i = 1, \cdots, p$,

$$f_i(x^*) + x^{*T} w_i < f_i(\bar{x}) + \bar{x}^T w_i.$$

Since $\lambda_i \geq 0$, we have

$$\sum_{i=1}^{p} \lambda_i(f_i(x^*) + x^{*T} w_i) < \sum_{i=1}^{p} \lambda_i(f_i(\bar{x}) + \bar{x}^T w_i).$$

By the (F, α, ρ, d)-pseudoconvexity of $\sum_{i=1}^{p} \lambda_i(f_i(\bar{x}) + \bar{x}^T w_i)$,

$$F(x^*, \bar{x}; \alpha(x^*, \bar{x}) \sum_{i=1}^{p} \lambda_i(\nabla f_i(\bar{x}) + w_i)) < -\rho d^2(x^*, \bar{x}).$$

Since $\beta + \rho \geqq 0$,

$$F(x^*, \bar{x}; -\alpha(x^*, \bar{x}) \sum_{j=1}^{m} \mu_i \nabla g_j(\bar{x})) > -\beta d^2(x^*, \bar{x}).$$

Since $-\sum_{j=1}^{m} \mu_j g_j(\bar{x})$ is (F, α, β, d)-quasiconvex,

$$-\sum_{j=1}^{m} \mu_j g_j(x^*) > -\sum_{j=1}^{m} \mu_j g_j(\bar{x}).$$

Since $\mu_j g_j(\bar{x}) = 0, \ j = 1, \cdots, m$, we have

$$\sum_{j=1}^{m} \mu_j g_j(x^*) < 0,$$

which contradicts the condition $\mu_j \geqq 0$ and $g_j(x^*) \geqq 0$. □

2.3. Duality Theorems

In this section, we introduce the generalized dual programming problem for a weakly efficient solution under generalized (F, α, ρ, d)-convexity assumptions. Now we propose the following general dual **(NVD)** to **(NVP)**.

(NVD) Maximize

$$(f_1(u) + u^T w_1 - \sum_{i \in I_0} y_i g_i(u), \cdots, f_p(u) + u^T w_p - \sum_{i \in I_0} y_i g_i(u))$$

subject to
$$\sum_{i=1}^{p} \lambda_i (\nabla f_i(u) + w_i) - y^T \nabla g(u) = 0, \tag{2.3}$$

$$\sum_{i \in I_\alpha} y_i g_i(u) \leqq 0, \ \alpha = 1, \cdots, r, \tag{2.4}$$

$$y \geq 0,$$

$$w_i \in C_i, \ i = 1, \cdots, p, \ \lambda = (\lambda_1, \cdots, \lambda_p) \in \Lambda^+,$$

where $I_\alpha \subset M = \{1, \cdots, m\}$, $\alpha = 0, 1, \cdots, r$ with $\cup_{\alpha=0}^{r} I_\alpha = M$ and $I_\alpha \cap I_\beta = \emptyset$ if $\alpha \neq \beta$. Let $\Lambda^+ = \{\lambda \in \mathbb{R}^p : \ \lambda \geqq 0, \ \lambda^T e = 1, \ e = (1, \cdots, 1) \in \mathbb{R}^p\}$.

Theorem 2.3.1 (Weak Duality). Assume that for all feasible x of **(NVP)** and all feasible (u, λ, w, y) of **(NVD)**, $-\sum_{i \in I_\alpha} y_i g_i(\cdot)$ $(\alpha = 1, \cdots, r)$ is $(F, \alpha, \beta_\alpha, \rho)$-quasiconvex at u and assume that one of the following conditions holds:

(a) $f_i(\cdot) + (\cdot)^T w_i - \sum_{i \in I_0} y_i g_i(\cdot)$ is (F, α, ρ_i, d)-pseudoconvex at u, with $\sum_{\alpha=1}^{r} \beta_\alpha + \sum_{i=1}^{p} \lambda_i \rho_i \geqq 0$;

Now suppose, contrary to the result, that (2.5) holds. Since $x^T w_i \leq s(x|C_i)$, we have for all $i \in \{1, \cdots, p\}$

$$\begin{aligned} f_i(x) + x^T w_i - \sum_{i \in I_0} y_i g_i(x) &\leq f_i(x) + x^T w_i \\ &\leq f_i(x) + s(x|C_i) \\ &< f_i(u) + u^T w_i - \sum_{i \in I_0} y_i g_i(u). \end{aligned} \tag{2.9}$$

By (a), we get

$$F(x, u; \alpha(x, u)(\nabla f_i(u) + w_i - \sum_{i \in I_0} y_i \nabla g_i(u))) < -\rho_i d^2(x, u), \text{for all}\, i \in \{1, \cdots p\}. \tag{2.10}$$

From $\lambda \in \Lambda^+$, (2.10) and the sublinearity of F, we have

$$F(x, u; \alpha(x, u)(\sum_{i=1}^{p} \lambda_i (\nabla f_i(u) + w_i) - \sum_{i \in I_0} y_i \nabla g_i(u))) < (-\sum_{i=1}^{p} \lambda_i \rho_i) d^2(x, u). \tag{2.11}$$

Since $\sum_{\alpha=1}^{r} \beta_\alpha + \sum_{i=1}^{p} \lambda_i \rho_i \geq 0$, it follows from (2.11) that

$$F(x, u; \alpha(x, u)(\sum_{i=1}^{p} \lambda_i (\nabla f_i(u) + w_i) - \sum_{i \in I_0} y_i \nabla g_i(u))) < (\sum_{\alpha=1}^{r} \beta_\alpha) d^2(x, u),$$

which contradics (2.8). Hence (2.5) cannot hold.

Suppose now that (b) is satisfied. From $\lambda \in \Lambda^+$ and (2.9), it follows that

$$\sum_{i=1}^{p} \lambda_i (f_i(x) + x^T w_i) - \sum_{i \in I_0} y_i g_i(x) < \sum_{i=1}^{p} \lambda_i (f_i(u) + u^T w_i) - \sum_{i \in I_0} y_i g_i(u).$$

(b) $\sum_{i=1}^{p} \lambda_i(f_i(\cdot) + (\cdot)^T w_i) - \sum_{i \in I_0} y_i g_i(\cdot)$ is (F, α, ρ, d)-pseudoconvex at u, with $\sum_{\alpha=1}^{r} \beta_\alpha + \rho \geqq 0$, then the following holds:

$$f_i(x) + s(x|C_i) \not< f_i(u) + u^T w_i - \sum_{j \in I_0} y_j g_j(u), \text{ for all } i \in \{1, \cdots, p\}. \quad (2.5)$$

Proof. As x is feasible for **(NVP)** and (u, λ, w, y) is feasible for **(NVD)**, we have

$$\sum_{i \in I_\alpha} y_i g_i(x) \geqq 0 \geqq \sum_{i \in I_\alpha} y_i g_i(u), \ \alpha = 1, \cdots, r.$$

By the $(F, \alpha, \beta_\alpha, d)$-quasiconvexity of $-\sum_{i \in I_\alpha} y_i g_i(u), \ \alpha = 1, \cdots, r$, it follows that

$$F(x, u; -\alpha(x, u) \sum_{i \in I_\alpha} y_i \nabla g_i(u)) \leqq -\beta_\alpha d^2(x, u), \ \alpha = 1, \cdots, r. \quad (2.6)$$

On the other hand, by (2.3) and the sublinearity of F, we have

$$F(x, u; \alpha(x, u)(\sum_{i=1}^{p} \lambda_i(\nabla f_i(u) + w_i) - \sum_{i \in I_0} y_i \nabla g_i(u)))$$

$$+ \sum_{\alpha=1}^{r} F(x, u; -\alpha(x, u) \sum_{i \in I_\alpha} y_i \nabla g_i(u))$$

$$\geqq F(x, u; \alpha(x, u)(\sum_{i=1}^{p} \lambda_i(\nabla f_i(u) + w_i) - y^T \nabla g(u))) = 0. \quad (2.7)$$

From (2.6) and (2.7)

$$F(x, u; \alpha(x, u)(\sum_{i=1}^{p} \lambda_i(\nabla f_i(u) + w_i) - \sum_{i \in I_0} y_i \nabla g_i(u))) \geqq (\sum_{\alpha=1}^{r} \beta_\alpha) d^2(x, u). \quad (2.8)$$

Then, by the (F, α, ρ, d)-pseudoconvexity of $\sum_{i=1}^{p} \lambda_i(f_i(\cdot) + (\cdot)^T w_i) - \sum_{i \in I_0} y_j g_j(\cdot)$ at u,

$$F(x, u; \alpha(x, u)(\sum_{i=1}^{p} \lambda_i(\nabla f_i(u) + w_i) - \sum_{i \in I_0} y_i \nabla g_i(u))) < -\rho d^2(x, u). \quad (2.12)$$

Since $\sum_{\alpha=1}^{r} \beta_\alpha + \rho \geq 0$, it follows from (2.12) that

$$F(x, u; \alpha(x, u)(\sum_{i=1}^{p} \lambda_i(\nabla f_i(u) + w_i) - \sum_{i \in I_0} y_i \nabla g_i(u))) < (\sum_{\alpha=1}^{r} \beta_\alpha) d^2(x, u),$$

which contradicts (2.8). Hence (2.5) cannot hold. □

Remark 2.3.1. Theorem 2.3.1 is an extension of Theorem 2.1 in [63] for weakly efficient solutions under generalized (F, α, ρ, d)-convexity.

Theorem 2.3.2 (Strong Duality). If $\bar{x} \in S$ is a weakly efficient solution of **(NVP)**, and assume that there exists $z^* \in \mathbb{R}^n$ such that $\langle \nabla g_j(\bar{x}), z^* \rangle > 0$, for all $j \in I(\bar{x})$. Then there exists $\bar{\lambda} \in \mathbb{R}^p$, $\bar{w}_i \in C_i, i = 1, \cdots, p$, $\bar{y} \in \mathbb{R}^m$ such that $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{y})$ is feasible for **(NVD)** and $\bar{x}^T \bar{w}_i = s(\bar{x}|C_i)$, $i = 1, \cdots, p$. Moreover, if the weak duality holds, then $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{y})$ is a weakly efficient solution of **(NVD)**.

Proof. By Theorem 2.2.2, there exists $\bar{\lambda} \in \mathbb{R}^p$, $\bar{y} \in \mathbb{R}^m$ and $\bar{w}_i \in C_i, i = 1, \cdots, p$, such that $\sum_{i=1}^{p} \bar{\lambda}_i(\nabla f_i(\bar{x}) + \bar{w}_i) - \sum_{j=1}^{m} \bar{y}_j \nabla g_j(\bar{x}) = 0$, $\bar{y}_j g_j(\bar{x}) = 0$, $j = 1, \cdots, m$, and $\bar{w}_i \in C_i$, $i = 1, \cdots, p$. Thus $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{y})$ is a feasible for **(NVD)** and $\bar{x}^T \bar{w}_i = s(\bar{x}|C_i)$, $i = 1, \cdots, p$. Notice that $f_i(\bar{x}) + s(\bar{x}|C_i) = f_i(\bar{x}) + \bar{x}^T \bar{w}_i = f_i(\bar{x}) + \bar{x}^T \bar{w}_i - \sum_{i \in I_0} \bar{y}_i g_i(\bar{x})$. By weak duality, $(f_1(\bar{x}) +$

$s(\bar{x}|C_1), \cdots, f_p(\bar{x}) + s(\bar{x}|C_p)) \not< (f_1(u) + u^T w_1 - \sum_{i \in I_0} y_i g_i(u), \cdots, f_p(u) + u^T w_p - \sum_{i \in I_0} y_i g_i(u))$ where (u, λ, w, y) is any feasible solution of **(NVD)**. Since $\bar{x}^T \bar{w}_i = s(\bar{x}|C_i)$, we have

$$(f_1(\bar{x}) + \bar{x}^T \bar{w}_1 - \sum_{i \in I_0} \bar{y}_i g_i(\bar{x}), \cdots, f_p(\bar{x}) + \bar{x}^T \bar{w}_p - \sum_{i \in I_0} \bar{y}_i g_i(\bar{x}))$$

$$\not< (f_1(u) + u^T w_1 - \sum_{i \in I_0} y_i g_i(u), \cdots, f_p(u) + u^T w_p - \sum_{i \in I_0} y_i g_i(u)).$$

Since $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{y})$ is a **(NVD)** feasible solution, $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{y})$ is a weakly efficient solution of **(NVD)**. Hence the result holds. □

2.4. Special Cases

As special cases of our duality results between **(NVP)** and **(NVD)**, we give Mond-Weir type and Wolfe type duality theorems.

If $I_0 = \emptyset$, $I_\alpha = M$, then **(NVD)** reduced to the Mond-Weir type dual $\mathbf{(NVD)}_M$.

$\mathbf{(NVD)}_M$ Maximize $(f_1(u) + u^T w_1, \cdots, f_p(u) + u^T w_p)$

subject to

$$\sum_{i=1}^{p} \lambda_i (\nabla f_i(u) + w_i) - \sum_{j=1}^{m} y_j \nabla g_j(u) = 0, \quad (2.13)$$

$$\sum_{j=1}^{m} y_j g_j(u) \leqq 0, \quad (2.14)$$

$$y_j \geqq 0, \ j = 1, \cdots, m,$$

$$w_i \in C_i, \ i = 1, \cdots, p, \ \lambda = (\lambda_1, \cdots, \lambda_p) \in \Lambda^+,$$

where $\Lambda^+ = \{\lambda \in \mathbb{R}^p : \ \lambda \geqq 0, \ \lambda^T e = 1, \ e = (1, \cdots, 1) \in \mathbb{R}^p\}$.

Theorem 2.4.1 (Weak Duality). Assume that, for all feasible x of **(NVP)** and for all feasible (u, λ, w, y) of $\mathbf{(NVD)}_M$, $-\sum_{j=1}^m y_j g_j(\cdot)$ is (F, α, β, d)-quasiconvex at u and assume that the following conditions hold:

(a) $f_i(\cdot)+(\cdot)^T w_i$ is (F, α, ρ_i, d)-pseudoconvex at u, with $\beta+\sum_{i=1}^p \lambda_i \rho_i \geqq 0$;

(b) $\sum_{i=1}^p \lambda_i(f_i(\cdot)+(\cdot)^T w_i)$ is (F, α, ρ, d)-pseudoconvex at u, with $\beta+\rho \geqq 0$,

then the following holds:

$$f_i(x) + s(x|C_i) \not\leq f_i(u) + u^T w_i, \text{ for all } i \in \{1, \cdots, p\}. \tag{2.15}$$

Proof. As x is a feasible for **(NVP)** and (u, λ, w, y) is feasible for $\mathbf{(NVD)}_M$, we have

$$\sum_{j=1}^m \mu_j g_j(x) \geqq 0 \geqq \sum_{j=1}^m \mu_j g_j(u).$$

By the (F, α, β, d)-quasiconvexity of $-\sum_{j=1}^m y_j g_j(u)$, it follows

$$F(x, u; -\alpha(x, u) \sum_{j=1}^m y_j \nabla g_j(u)) \leqq -\beta d^2(x, u). \tag{2.16}$$

On the other hand, by (2.13) and the sublinearity of F, we have

$$F(x, u; \alpha(x, u) \sum_{i=1}^p \lambda_i(\nabla f_i(u) + w_i)) + F(x, u; -\alpha(x, u) \sum_{j=1}^m y_j \nabla g_j(u))$$

$$\geqq F(x, u; \alpha(x, u)(\sum_{i=1}^p \lambda_i(\nabla f_i(u) + w_i) - \sum_{j=1}^m y_j \nabla g_j(u))) \tag{2.17}$$

$$= 0.$$

Combination (2.16) and (2.17) gives

$$
\begin{aligned}
F(x,u;\alpha(x,u)\sum_{i=1}^{p}\lambda_i(\nabla f_i(u)+w_i)) &\geqq -F(x,u;-\alpha(x,u)\sum_{j=1}^{m}y_j\nabla g_j(u)) \\
&\geqq \beta d^2(x,u). \qquad (2.18)
\end{aligned}
$$

Now, suppose that contrary to the result, (2.15) holds. From (2.15) and $x^T w_i \leqq s(x|C_i)$, we have

$$
\begin{aligned}
f_i(x)+x^T w_i &\leqq f_i(x)+s(x|C_i) \\
&< f_i(u)+u^T w_i. \qquad (2.19)
\end{aligned}
$$

By (a), we get

$$F(x,u;\alpha(x,u)(\nabla f_i(u)+w_i)) < -\rho_i d^2(x,u). \qquad (2.20)$$

From, $\lambda \in \Lambda^+$, (2.20) and the sublinearity of F, we have

$$
\begin{aligned}
F(x,u;\alpha(x,u)\sum_{i=1}^{p}\lambda_i(\nabla f_i(u)+w_i)) &= \sum_{i=1}^{p}\lambda_i F(x,u;\alpha(x,u)(\nabla f_i(u)+w_i)) \\
&< (-\sum_{i=1}^{p}\lambda_i\rho_i)d^2(x,u). \qquad (2.21)
\end{aligned}
$$

Since $\beta+\sum_{i=1}^{p}\lambda_i\rho_i \geqq 0$, it follows from (2.21) that

$$F(x,u;\alpha(x,u)\sum_{i=1}^{p}\lambda_i(\nabla f_i(u)+w_i)) < \beta d^2(x,u),$$

which contradicts (2.18). Hence (2.15) cannot hold.

Suppose now that (b) is satisfied. From, $\lambda \in \Lambda^+$ and (2.19), it follows that

$$\sum_{i=1}^{p} \lambda_i(f_i(x) + x^T w_i) < \sum_{i=1}^{p} \lambda_i(f_i(u) + u^T w_i).$$

Then, by the (F, α, ρ, d)-pseudoconvexity of $\sum_{i=1}^{p} \lambda_i(f_i(\cdot) + (\cdot)^T w_i)$ at u,

$$F(x, u; \alpha(x, u) \sum_{i=1}^{p} \lambda_i(\nabla f_i(u) + w_i)) < -\rho d^2(x, u). \tag{2.22}$$

Since $\beta + \rho \geqq 0$, it follows from (2.22) that

$$F(x, u; \alpha(x, u) \sum_{i=1}^{p} \lambda_i(\nabla f_i(u) + w_i)) < \beta d^2(x, u),$$

which contradicts (2.18). Hence (2.15) cannot hold. □

Theorem 2.4.2 (Strong Duality). If $\bar{x} \in S$ is a weakly efficient solution of **(NVP)**, and assume that there exists $z^* \in \mathbb{R}^n$ such that $\langle \nabla g_j(\bar{x}), z^* \rangle > 0$, for all $j \in I(\bar{x})$, then there exists $\bar{\lambda} \in \mathbb{R}^p$, $\bar{w}_i \in C_i, i = 1, \cdots, p$, $\bar{y} \in \mathbb{R}^m$ such that $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{y})$ is feasible for $\mathbf{(NVD)}_M$ and $\bar{x}^T \bar{w}_i = s(\bar{x}|C_i)$, $i = 1, \cdots, p$. Moreover, if the weak duality holds, then $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{y})$ is a weakly efficient solution of $\mathbf{(NVD)}_M$.

Proof. By Theorem 2.2.2, there exists $\bar{\lambda} \in \mathbb{R}^p$, $\bar{y} \in \mathbb{R}^m$ and $\bar{w}_i \in C_i, i = 1, \cdots, p$, such that $\sum_{i=1}^{p} \bar{\lambda}_i(\nabla f_i(\bar{x}) + \bar{w}_i) - \sum_{j=1}^{m} \bar{y}_j \nabla g_j(\bar{x}) = 0$, $\sum_{i=1}^{m} \bar{y}_j g_j(\bar{x}) = 0$, $\bar{y}_j \geqq 0$, $j = 1, \cdots, m$ and $\bar{w}_i \in C_i$, $i = 1, \cdots, p$. Thus $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{y})$ is a feasible for $\mathbf{(NVD)}_M$ and $\bar{x}^T \bar{w}_i = s(\bar{x}|C_i)$, $i = 1, \cdots, p$. By weak duality,

$(f_1(\bar{x})+s(\bar{x}|C_1),\cdots,f_p(\bar{x})+s(\bar{x}|C_p)) \not< (f_1(u)+u^Tw_1,\cdots,f_p(u)+u^Tw_p)$ for any $\mathbf{(NVD)}_M$ feasible solution (u,λ,w,μ). Since $\bar{x}^T\bar{w}_i = s(\bar{x}|C_i)$, we have

$$(f_1(\bar{x})+\bar{x}^T\bar{w}_1,\cdots,f_p(\bar{x})+\bar{x}^T\bar{w}_p) \not< (f_1(u)+u^Tw_1,\cdots,f_p(u)+u^Tw_p).$$

Since $(\bar{x},\bar{\lambda},\bar{w},\bar{y})$ is a $\mathbf{(NVD)}_M$ feasible solution, and $(\bar{x},\bar{\lambda},\bar{w},\bar{y})$ is a weakly efficient solution of $\mathbf{(NVD)}_M$. Hence the result holds. □

If $I_0 = M$, $I_\alpha = \emptyset$, then **(NVD)** is reduced to the Wolfe type dual $\mathbf{(NVD)}_W$.

$\mathbf{(NVD)}_W$ Maximize

$$(f_1(u)+u^Tw_1-\sum_{j=1}^m y_jg_j(u),\cdots,f_p(u)+u^Tw_p-\sum_{j=1}^m y_jg_j(u))$$

$$\text{subject to } \sum_{i=1}^p \lambda_i(\nabla f_i(u)+w_i)-\sum_{j=1}^m y_j\nabla g_j(u)=0, \quad (2.23)$$

$$y_j \geqq 0,\ j=1,\cdots,m,$$

$$w_i\in C_i,\ i=1,\cdots,p,\ \lambda=(\lambda_1,\cdots,\lambda_p)\in\Lambda^+,$$

where $\Lambda^+ = \{\lambda\in\mathbb{R}^p:\ \lambda\geqq 0,\ \lambda^Te=1,\ e=(1,\cdots,1)\in\mathbb{R}^p\}$.

Theorem 2.4.3 (Weak Duality). Let x be a feasible for **(NVP)** and (u,λ,w,y) a feasible for $\mathbf{(NVD)}_W$. Assume that

(a) $f_i(\cdot)+(\cdot)^Tw_i-\sum_{j=1}^m y_jg_j(\cdot)$ is (F,α,ρ_i,d)-pseudoconvex at u, with $\sum_{i=1}^p \lambda_i\rho_i \geqq 0$; or

(b) $\sum_{i=1}^p \lambda_i(f_i(\cdot)+(\cdot)^Tw_i)-\sum_{j=1}^m y_jg_j(\cdot)$ is (F,α,ρ,d)-pseudoconvex at u, with $\rho\geqq 0$.

Then the following holds:

$$f_i(x) + s(x|C_i) \not< f_i(u) + u^T w_i - \sum_{j=1}^{m} y_j g_j(u), \text{ for all } i \in \{1, \cdots, p\}.$$

Proof. Suppose contrary to the result, $f_i(x) + s(x|C_i) < f_i(u) + u^T w_i - \sum_{j=1}^{m} y_j g_j(u)$ holds. Then

$$\begin{aligned} f_i(x) + x^T w_i &\leqq f_i(x) + s(x|C_i) \\ &< f_i(u) + u^T w_i - \sum_{j=1}^{m} y_j g_j(u). \end{aligned} \tag{2.24}$$

Since $f_i(x) + x^T w_i - \sum_{j=1}^{m} y_j g_j(x) < f_i(u) + u^T w_i - \sum_{j=1}^{m} y_j g_j(u)$. By (a), we get

$$F(x, u; \alpha(x, u)(\nabla f_i(u) + w_i - \sum_{j=1}^{m} y_j \nabla g_j(u))) < -\rho_i d^2(x, u).$$

Sublinearity of F,

$$F(x, u; \alpha(x, u)(\sum_{i=1}^{p} \lambda_i(\nabla f_i(u) + w_i) - \sum_{j=1}^{m} y_j \nabla g_j(u))) < -(\sum_{i=1}^{p} \lambda_i \rho_i) d^2(x, u).$$

Since $0 > -(\sum_{i=1}^{p} \lambda_i \rho_i) d^2(x, u)$, which contradicts (2.23). Hence the result holds.

Suppose that (b) is satisfied. From, $\lambda \in \Lambda^+$ and (2.24), it follows that

$$\sum_{i=1}^{p} \lambda_i(f_i(x) + x^T w_i) - \sum_{j=1}^{m} y_j g_j(x) < \sum_{i=1}^{p} \lambda_i(f_i(u) + u^T w_i) - \sum_{j=1}^{m} y_j g_j(u).$$

Then, by the (F, α, ρ, d)-pseudoconvexity of $\sum_{i=1}^{p} \lambda_i(f_i(\cdot)+(\cdot)^T w_i)-\sum_{j=1}^{m} y_j g_j(\cdot)$ at u,

$$F(x, u; \alpha(x, u)(\sum_{i=1}^{p} \lambda_i(\nabla f_i(u) + w_i) - \sum_{j=1}^{m} y_j \nabla g_j(u))) < -\rho d^2(x, u).$$

Since $0 > -\rho d^2(x, u)$, which contradicts (2.23). Hence the result holds. □

Theorem 2.4.4 (Strong Duality). If $\bar{x} \in S$ is a weakly efficient solution of **(NVP)**, and assume that there exists $z^* \in \mathbb{R}^n$ such that $\langle \nabla g_j(\bar{x}), z^* \rangle > 0$, for all $j \in I(\bar{x})$, then there exists $\bar{\lambda} \in \mathbb{R}^p$, $\bar{w}_i \in C_i, i = 1, \cdots, p$, $\bar{y} \in \mathbb{R}^m$ such that $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{y})$ is feasible for **(NVD)**$_W$ and $\bar{x}^T \bar{w}_i = s(\bar{x}|C_i)$, $i = 1, \cdots, p$. Moreover, if the weak duality holds, then $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{y})$ is a weakly efficient solution of **(NVD)**$_W$.

Proof. By Theorem 2.2.2, there exists $\bar{\lambda} \in \mathbb{R}^p$, $\bar{y} \in \mathbb{R}^m$ and $\bar{w}_i \in C_i, i = 1, \cdots, p$, such that $\sum_{i=1}^{p} \bar{\lambda}_i(\nabla f_i(\bar{x})+\bar{w}_i)-\sum_{j=1}^{m} \bar{y}_j \nabla g_j(\bar{x}) = 0$, $\sum_{j=1}^{p} \bar{y}_j g_j(\bar{x}) = 0$, $\bar{y}_j \geqq 0$, $j = 1, \cdots, m$, and $\bar{w}_i \in C_i$, $i = 1, \cdots, p$. Thus $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{y})$ is a feasible for **(NVD)**$_W$ and $\bar{x}^T \bar{w}_i = s(\bar{x}|C_i)$, $i = 1, \cdots, p$. Notice that $f_i(\bar{x}) + s(\bar{x}|C_i) = f_i(\bar{x}) + \bar{x}^T \bar{w}_i = f_i(\bar{x}) + \bar{x}^T \bar{w}_i - \sum_{j=1}^{m} \bar{y}_j g_j(\bar{x})$. By weak duality,

$$(f_1(\bar{x}) + s(\bar{x}|C_1), \cdots, f_p(\bar{x}) + s(\bar{x}|C_p))$$

$$\not< (f_1(u) + u^T w_1 - \sum_{j=1}^{m} y_j g_j(u), \cdots, f_p(u) + u^T w_p - \sum_{j=1}^{m} y_j g_j(u))$$

where (u, λ, w, y) is any feasible solution of **(NVD)**$_W$. Since $\bar{x}^T \bar{w}_i = s(\bar{x}|C_i)$, we have

$$(f_1(\bar{x}) + \bar{x}^T\bar{w}_1 - \sum_{j=1}^{m} \bar{y}_j g_j(\bar{x}), \cdots, f_p(\bar{x}) + \bar{x}^T\bar{w}_p - \sum_{j=1}^{m} \bar{y}_j g_j(\bar{x}))$$

$$\not< (f_1(u) + u^T w_1 - \sum_{j=1}^{m} y_j g_j(u), \cdots, f_p(u) + u^T w_p - \sum_{j=1}^{m} y_j g_j(u)).$$

Since $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{y})$ is a feasible solution of $\mathbf{(NVD)}_W$, $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{y})$ is a weakly efficient solution of $\mathbf{(NVD)}_W$. Hence the result holds. □

Chapter 3

Optimality and Duality for Nondifferentiable Fractional Programming Problems with (V, ρ)-Ratio Invexity

3.1. Introduction

Duality and optimality for nondifferentiable multiobjective programming problems, in which the objective function contains a support function was studied by Mond and Schechter ([56]). Based on this results, Yang *et al.* ([78]) studied Wolf type and Mond-Weir type dual problems for a class of nondifferentiable multiobjective programs. Bector *et al.* ([5]) derived Fritz John and Karush-Kuhn-Tucker necessary and sufficient optimality conditions for a class of nondifferentiable convex multiobjective fractional programming problems and they established some duality theorems. Following the approaches of Bector *et al.* ([5]), Liu ([45, 44]) obtained necessary and sufficient conditions and derived duality theorems for a class of nonsmooth multiobjective fractional programming problems involving either pseudoinvex functions or (F, ρ)-convex functions. Jeyakumar ([29]) defined ρ-invexity for nonsmooth optimization problems, and Kuk *et al.* ([39]) defined the concept of (V, ρ)-invexity for vector valued functions, which is a generalization of the V-invex function ([33, 55]).

On the other hand Reddy and Mukherjee ([64]) applied a generalized ratio invexity concept for single objective fractional programming problems and

Kim *et al.* ([38]) obtain the necessary and sufficient optimality theorems and generalized duality theorems for weakly efficient solutions under generalized (F, α, ρ, d)-convexity assumptions. Very recently Liang et al. ([47]) establish sufficient conditions and duality theorems for multiobjective fractional programming problems, which is for an efficient solution under (F, α, ρ, d)-convexity assumptions. Kim and Kim ([36]) present nonsmooth fractional programming under suitable ρ-invexity assumptions. Also Kim *et al.* ([37]) present multiobjective fractional programming with generalized invexity.

In this chapter, we introduce a nondifferentiable multiobjective fractional programming problem **(NFP)** with (V, ρ)-ratio invexity. We formulate the concept of (V, ρ)-ratio invexity and establish Fritz John and Kuhn-Tucker necessary and sufficient optimality conditions for weakly efficient solutions of this problem, in which each component of the objective function contains a term involving the support function of a compact convex set. Also we establish Mond-Weir type dual problem $\textbf{(NFD)}_M$ and Wolfe type dual problem $\textbf{(NFD)}_W$ to the primal problem **(NFP)** and prove the weak and strong duality theorems.

Now we consider the following multiobjective fractional programming problem,

$$\textbf{(NFP)} \quad \text{Minimize} \quad \left(\frac{f_1(x) + s(x|C_1)}{g_1(x)}, \cdots, \frac{f_p(x) + s(x|C_p)}{g_p(x)} \right)$$

$$\text{subject to} \quad h(x) \leqq 0, \quad x \in X_0,$$

where X_0 is an open set of $\mathbb{R}^n$, $f := (f_1, \cdots, f_p) : X_0 \to \mathbb{R}^p$, $g := (g_1, \cdots, g_p) : X_0 \to \mathbb{R}^p$, and $h := (h_1, \cdots, h_m) : X_0 \to \mathbb{R}^m$ are continuously differentiable over X_0; for each $i \in P = \{1, 2, \cdots, p\}$, C_i is a compact convex set of $\mathbb{R}^n$ and $s(x|C_i) = \max\{\langle x, y\rangle \mid y \in C_i\}$. Further let, $S = \{x \in X_0 \ : \ h(x) \leqq 0\}$ be the set of all feasible solutions and $I(x) := \{i \ : \ h_i(x) = 0\}$ for any $x \in X_0$. Let $k_i(x) = s(x|C_i)$, $i = 1, \cdots, p$. Then k_i is a convex function and $\partial k_i(x) = \{w \in C_i | \langle w, x\rangle = s(x|C_i)\}$ ([56]), where ∂k_i is the subdifferential of k_i. We assume that $f(x) \geqq 0$ for all $x \in X_0$ and $g(x) > 0$ for all $x \in X_0$ whenever g is not linear.

We introduce the following definition due to Kuk *et al.* ([39]).

Definition 3.1.1. A vector function $f : X_0 \to \mathbb{R}^p$ is said to be (V, ρ)-invex at $u \in X_0$ with respect to functions η and $\theta : X_0 \times X_0 \to \mathbb{R}^n$ if there exists $\alpha_i : X_0 \times X_0 \to \mathbb{R}_+ \backslash \{0\}$ and $\rho_i \in \mathbb{R}$, $i = 1, \cdots p$ such that for any $x \in X_0$, and for $i = 1, 2, \cdots, p$,

$$\alpha_i(x, u)\Big[f_i(x) - f_i(u)\Big] \geqq \nabla f_i(u)\eta_i(x, u) + \rho_i \|\theta_i(x, u)\|^2.$$

The function f is (V, ρ)-invex on X_0 if it is (V, ρ)-invex at every point in X_0.

Theorem 3.1.1. Assume that f and g are vector-valued differentiable functions defined on X_0 and $f(x) + \langle w, x\rangle \geqq 0$, $g(x) > 0$ for all $x \in X_0$. If $f(\cdot) + \langle w, \cdot\rangle$ and $-g(\cdot)$ are (V, ρ)-invex at $x_0 \in X_0$, then $\frac{f(\cdot)+\langle w, \cdot\rangle}{g(\cdot)}$ is (V, ρ)-invex at x_0, where

$$\bar{\alpha}_i(x, x_0) = \frac{g_i(x)}{g_i(x_0)}\alpha_i(x, x_0), \quad \bar{\theta}_i(x, x_0) = \Big(\frac{1}{g_i(x_0)}\Big)^{1/2}\theta_i(x, x_0).$$

Proof. Let $k_i(x) = s(x|C_i)$, $i = 1, \cdots, p$. Choose $w_i \in \partial k_i(x_0)$. Let $x, x_0 \in X_0$. By the (V, ρ)-invexity of $f(\cdot) + \langle w, \cdot\rangle$ and $-g(\cdot)$,

$$\alpha_i(x,x_0)\Big[\frac{f_i(x)+\langle w_i,x\rangle}{g_i(x)}-\frac{f_i(x_0)+\langle w_i,x_0\rangle}{g_i(x_0)}\Big]$$

$$=\alpha_i(x,x_0)\Big[\frac{f_i(x)+\langle w_i,x\rangle-f_i(x_0)-\langle w_i,x_0\rangle}{g_i(x)}$$

$$-(f_i(x_0)+\langle w_i,x_0\rangle)\frac{g_i(x)-g_i(x_0)}{g_i(x)g_i(x_0)}\Big]$$

$$\geq\frac{1}{g_i(x)}\Big[(\nabla f_i(x_0)+w_i)\eta_i(x,x_0)+\rho_i\|\theta_i(x,x_0)\|^2\Big]$$

$$+\frac{f_i(x_0)+\langle w_i,x_0\rangle}{g_i(x)g_i(x_0)}\Big[(-\nabla g_i(x_0))\eta_i(x,x_0)+\rho_i\|\theta_i(x,x_0)\|^2\Big].$$

Since $g(x)>0$ for all $x\in X_0$, we see that

$$\alpha_i(x,x_0)\Big[\frac{f_i(x)+\langle w_i,x\rangle}{g_i(x)}-\frac{f_i(x_0)+\langle w_i,x_0\rangle}{g_i(x_0)}\Big]$$

$$\geq\frac{g_i(x_0)}{g_i(x)}\Big[\frac{\nabla f_i(x_0)+w_i}{g_i(x_0)}\eta_i(x,x_0)+\rho_i\|\Big(\frac{1}{g_i(x_0)}\Big)^{1/2}\theta_i(x,x_0)\|^2$$

$$-\frac{f_i(x_0)+\langle w_i,x_0\rangle}{(g_i(x_0))^2}\nabla g_i(x_0)\eta_i(x,x_0)+\rho_i\|\Big(\frac{f_i(x_0)+\langle w_i,x_0\rangle}{(g_i(x_0))^2}\Big)^{1/2}\theta_i(x,x_0)\|^2\Big].$$

Thus, we have

$$\alpha_i(x,x_0)\Big[\frac{f_i(x)+\langle w_i,x\rangle}{g_i(x)}-\frac{f_i(x_0)+\langle w_i,x_0\rangle}{g_i(x_0)}\Big]\geq$$

$$\frac{g_i(x_0)}{g_i(x)}\Big[\frac{\nabla f_i(x_0)g_i(x_0)+w_ig_i(x_0)-f_i(x_0)\nabla g_i(x_0)-\langle w_i,x_0\rangle\nabla g_i(\bar{x})}{(g_i(x_0))^2}\eta_i(x,x_0)$$

$$+\rho_i\|\Big(\frac{1}{g_i(x_0)}\Big)^{1/2}\theta_i(x,x_0)\|^2+\rho_i\|\Big(\frac{f_i(x_0)+\langle w_i,x_0\rangle}{(g_i(x_0))^2}\Big)^{1/2}\theta_i(x,x_0)\|^2\Big]$$

$$= \frac{g_i(x_0)}{g_i(x)}\Big[\nabla\Big(\frac{f_i(x_0)+\langle w_i, x_0\rangle}{g_i(x_0)}\Big)\eta_i(x, x_0)$$

$$+\rho_i\|\Big(\Big(\frac{1}{g_i(x_0)}\Big)\Big(1+\frac{f_i(x_0)+\langle w_i, x_0\rangle}{g_i(x_0)}\Big)\Big)^{1/2}\theta_i(x, x_0)\|^2\Big].$$

Considering that

$$1+\frac{f_i(x_0)+\langle w_i, x_0\rangle}{g_i(x_0)} \geqq 1, \; i=1,2,\cdots,p,$$

we have for all i,

$$\alpha_i(x, x_0)\Big[\frac{f_i(x)+\langle w_i, x\rangle}{g_i(x)} - \frac{f_i(x_0)+\langle w_i, x_0\rangle}{g_i(x_0)}\Big]$$

$$\geq \frac{g_i(x_0)}{g_i(x)}\Big[\nabla\Big(\frac{f_i(x_0)+\langle w_i, x_0\rangle}{g_i(x_0)}\Big)\eta_i(x, x_0) + \rho_i\|\Big(\frac{1}{g_i(x_0)}\Big)^{1/2}\theta_i(x, x_0)\|^2\Big].$$

Therefore, the function $\frac{f(x)+\langle w_i, x\rangle}{g(x)}$ is (V, ρ)-invex, where

$$\bar{\alpha}_i(x, x_0) = \frac{g_i(x)}{g_i(x_0)}\alpha_i(x, x_0),$$

$$\bar{\theta}_i(x, x_0) = \Big(\frac{1}{g_i(x_0)}\Big)^{1/2}\theta_i(x, x_0).$$

□

3.2. Optimality Conditions

In this section, we establish Fritz John and Kuhn-Tucker necessary and sufficient conditions for weakly efficient solutions of **(NFP)**.

Theorem 3.2.1 (Fritz John Necessary Optimality Condition). If $x_0 \in S$ is a weakly efficient solution of **(NFP)**, then there exists $\lambda_i, i = 1, \cdots, p,\ \mu_j,\ j = 1, \cdots, m$ such that

$$\sum_{i=1}^{p} \lambda_i \nabla\Big(\frac{f_i(x_0) + \langle w_i, x_0\rangle}{g_i(x_0)}\Big) + \sum_{j=1}^{m} \mu_j \nabla h_j(x_0) = 0,$$

$$\langle w_i, x_0\rangle = s(x_0|C_i),\ w_i \in C_i,\ i = 1, \cdots, p,$$

$$\sum_{j=1}^{m} \mu_j h_j(x_0) = 0,$$

$$(\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m) \geqq 0,\ (\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m) \neq 0.$$

Proof. Let $k_i(x) = s(x|C_i),\ i = 1, \cdots, p$. Since C_i is convex and compact, $k_i : \mathbb{R}^n \to \mathbb{R}$ is a convex function and hence for all $d \in \mathbb{R}^n$,

$$k_i'(x_0; d) = \lim_{\lambda \to 0+} \frac{k_i(x_0 + \lambda d) - k_i(x_0)}{\lambda}$$

is finite. Also, for all $d \in \mathbb{R}^n$,

$$\begin{aligned}\Big(\frac{f_i + k_i}{g_i}\Big)'(x_0; d) &= \frac{1}{(g_i(x_0))^2}\Big[g_i(x_0)\langle \nabla f_i(x_0), d\rangle + g_i(x_0)k_i'(x_0; d) \\ &\quad - f_i(x_0)\langle \nabla g_i(x_0), d\rangle - k_i(x_0)\langle \nabla g_i(x_0), d\rangle\Big].\end{aligned}$$

Since x_0 is a weakly efficient solution of **(NFP)**,

$$\left(\begin{array}{c} \frac{1}{(g_i(x_0))^2}\Big[g_i(x_0)\langle \nabla f_i(x_0), d\rangle + g_i(x_0)k_i'(x_0; d) - f_i(x_0)\langle \nabla g_i(x_0), d\rangle \\ - k_i(x_0)\langle \nabla g_i(x_0), d\rangle\Big] < 0,\ i = 1, \cdots, p \\ \\ \langle \nabla h_j(x_0), d\rangle < 0,\ j \in I(x_0) \end{array}\right)$$

has no solution $d \in \mathbb{R}^n$. By Gordan theorem for convex functions, there exists $\lambda_i \geqq 0,\ i = 1, \cdots, p$ and $\mu_j \geqq 0,\ j \in I(x_0)$ are not all zero such that for any $d \in \mathbb{R}^n$,

$$\sum_{i=1}^{p} \frac{\lambda_i}{(g_i(x_0))^2} \Big[g_i(x_0) \langle \nabla f_i(x_0), d \rangle + g_i(x_0) k_i'(x_0; d) - f_i(x_0) \langle \nabla g_i(x_0), d \rangle$$

$$- k_i(x_0) \langle \nabla g_i(x_0), d \rangle \Big] + \sum_{j \in I(x_0)} \mu_j \ \langle \nabla h_j(x_0), d \rangle \geqq 0. \qquad (3.1)$$

Let $A = \{ \sum_{i=1}^{p} \frac{\lambda_i}{(g_i(x_0))^2} \Big[g_i(x_0)(\nabla f_i(x_0) + w_i) - (f_i(x_0) + k_i(x_0)) \nabla g_i(x_0) \Big] + \sum_{j \in I(x_0)} \mu_j \nabla h_j(x_0) \mid w_i \in \partial k_i(x_0),\ i = 1, \cdots, p \}$. Then A is a nonempty closed convex set and $0 \in A$. Suppose to the contrary that $0 \notin A$. By separation theorem, there exists $d^* \in \mathbb{R}^n, d^* \neq (0, \cdots, 0)$, such that for all $a \in A$, $\langle a, d^* \rangle < 0$, that is,

$\sum_{i=1}^{p} \frac{\lambda_i}{(g_i(x_0))^2} \langle g_i(x_0)(\nabla f_i(x_0) + w_i) - (f_i(x_0) + k_i(x_0)) \nabla g_i(x_0), d^* \rangle$
$+ \sum_{j \in I(x_0)} \mu_j \langle \nabla h_j(x_0), d^* \rangle < 0$, for all $w_i \in \partial k_i(x_0)$. Hence
$\sum_{i=1}^{p} \frac{\lambda_i}{(g_i(x_0))^2} \Big[g_i(x_0) \langle \nabla f_i(x_0), d^* \rangle + g_i(x_0) k_i'(x_0; d^*) - f_i(x_0) \langle \nabla g_i(x_0), d^* \rangle$
$- k_i(x_0) \langle \nabla g_i(x_0), d^* \rangle \Big] + \sum_{j \in I(x_0)} \mu_j \langle \nabla h_j(x_0), d^* \rangle < 0$, which contradicts (3.1).
Since $0 \in A$, there exists $w_i \in \partial k_i(x_0),\ i = 1, \cdots, p$, such that

$$0 = \sum_{i=1}^{p} \frac{\lambda_i}{(g_i(x_0))^2} \Big[g_i(x_0)(\nabla f_i(x_0) + w_i) - (f_i(x_0) + \langle w_i, x_0 \rangle) \nabla g_i(x_0) \Big]$$

$$+ \sum_{j \in I(x_0)} \mu_j \nabla h_j(x_0).$$

Letting $\mu_j = 0,$ for all $j \notin I(x_0)$, we have

$$0 = \sum_{i=1}^{p} \lambda_i \nabla \Big(\frac{f_i(x_0) + \langle w_i, x_0 \rangle}{g_i(x_0)} \Big) + \sum_{j=1}^{m} \mu_j \nabla h_j(x_0)$$

and $\sum_{j=1}^{m} \mu_j h_j(x_0) = 0,\ (\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m) \neq 0$. Since $\partial k_i(x_0) = \{w_i \in C_i \mid \langle w_i, x_0 \rangle = s(x_0|C_i)\}$, we obtain the desired result. □

Theorem 3.2.2 (Kuhn-Tucker Necessary Optimality Condition). If $x_0 \in S$ is a weakly efficient solution of **(NFP)**, and assume that there exists $z^* \in \mathbb{R}^n$ such that $\langle \nabla h_j(x_0), z^* \rangle > 0,\ j \in I(x_0)$. Then there exist $\lambda_i \geqq 0,\ i = 1, \cdots, p,\ \mu_j \geqq 0, j = 1, \cdots, m$ and $w_i \in C_i, i = 1, \cdots, p$ such that

$$\sum_{i=1}^{p} \lambda_i \nabla \Big(\frac{f_i(x_0) + \langle w_i, x_0 \rangle}{g_i(x_0)} \Big) + \sum_{j=1}^{m} \mu_j \nabla h_j(x_0) = 0,$$

$$\langle w_i, x_0 \rangle = s(x_0|C_i),\ w_i \in C_i,\ i = 1, \cdots, p,$$

$$\sum_{j=1}^{m} \mu_j h_j(x_0) = 0,$$

$$(\lambda_1, \cdots, \lambda_p) \neq (0, \cdots, 0).$$

Proof. Since x_0 is a weakly efficient solution of **(NFP)**, by Theorem 3.2.1, there exists $(\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m) \neq (0, \cdots, 0)$. Since $\mu_k h_k(x_0) = 0$ and $\mu_k \geqq 0,$ for all $k \in I(x_0), \mu_k = 0,$ for all $k \notin I(x_0)$. Hence

$$\sum_{i=1}^{p} \lambda_i \nabla \Big(\frac{f_i(x_0) + \langle w_i, x_o \rangle}{g_i(x_0)} \Big) + \sum_{j=1}^{m} \mu_j \nabla h_j(x_0) = 0. \tag{3.2}$$

and $\langle w_i, x_0\rangle = s(x_0|C_i),\ i = 1, \cdots, p$. Assume that there exists $z^* \in \mathbb{R}^n$ such that $\langle \nabla h_j(x_0), z^*\rangle > 0$, for all $j \in I(x_0)$. Then $(\lambda_1, \cdots, \lambda_p) \neq (0, \cdots, 0)$. Ab absurdo, suppose that $(\lambda_1, \cdots, \lambda_p) = (0, \cdots, 0)$. Since $(\mu_1, \cdots, \mu_m) \neq (0, \cdots, 0)$, i.e., $\mu_j \geqq 0$, for all $j \in \{1, \cdots, m\}$ and hence $\mu_i > 0$ for some $i \in \{1, \cdots, m\}$. From (3.2), $0 = \sum_{j=1}^m \mu_j \nabla h_j(x_0)$. However,

$$
\begin{aligned}
0 &= < \sum_{j\in I(x_0)} \mu_j \nabla h_j(x_0), z^* > \\
&= \sum_{j\in I(x_0)} \mu_j \langle \nabla h_j(x_0), z^*\rangle \\
&> 0.
\end{aligned}
$$

This is a contradiction. Hence $(\lambda_1, \cdots, \lambda_p) \neq (0, \cdots, 0)$. □

In the following, we present some sufficient conditions for **(NFP)** under appropriate (V, ρ)-invexity assumptions.

Theorem 3.2.3 (Kuhn-Tucker Sufficient Optimality Condition). Let x_0 be a feasible solution of **(NFP)**. Suppose that there exists $\lambda = (\lambda_1, \cdots, \lambda_p) \in \mathbb{R}^p_+$, $\lambda > 0$, $\sum_{i=1}^p \lambda_i = 1$ and $\mu = (\mu_1, \cdots, \mu_m) \in \mathbb{R}^m_+$ such that

$$\sum_{i=1}^{p} \lambda_i \nabla\Big(\frac{f_i(x_0) + \langle w_i, x_0\rangle}{g_i(x_0)}\Big) + \sum_{j=1}^{m} \mu_j \nabla h_j(x_0) = 0, \tag{3.3}$$

$$\langle w_i, x_0\rangle = s(x_0|C_i),\ w_i \in C_i,\ i = 1, \cdots, p,$$

$$\sum_{j=1}^{m} \mu_j h_j(x_0) = 0.$$

If $f(\cdot) + \langle w, \cdot \rangle$ and $-g(\cdot)$ are (V, ρ)-invex at x_0 and h is (V, σ)-invex at x_0 with respect to the same η and $\sum_{i=1}^{p} \lambda_i \rho_i \geqq 0$ and $\sum_{j=1}^{m} \sigma_j \geqq 0$, where $\bar{\alpha}_i(x, x_0) = (\frac{g_i(x)}{g_i(x_0)})\alpha_i(x, x_0)$, $\bar{\theta}_i(x, x_0) = (\frac{1}{g_i(x_0)})^{1/2}\theta_i(x, x_0)$.
Then x_0 is a weakly efficient solution of **(NFP)**.

Proof. Suppose that x_0 is not a weakly efficient solution of **(NFP)**. Then there exists $x \in S$ such that $\frac{f_i(x)+s(x|C_i)}{g_i(x)} < \frac{f_i(x_0)+s(x_0|C_i)}{g_i(x_0)}$. Since $\langle w_i, x_0 \rangle = s(x_0|C_i)$, $i = 1, \cdots, p$,

$$\begin{aligned} \frac{f_i(x) + \langle w_i, x \rangle}{g_i(x)} &\leqq \frac{f_i(x) + s(x|C_i)}{g_i(x)} \\ &< \frac{f_i(x_0) + s(x_0|C_i)}{g_i(x_0)} \\ &= \frac{f_i(x_0) + \langle w_i, x_0 \rangle}{g_i(x_0)}. \end{aligned}$$

Since $\bar{\alpha}_i(x, x_0) > 0$, we have

$$\bar{\alpha}_i(x, x_0)\Big(\frac{f_i(x) + \langle w_i, x \rangle}{g_i(x)}\Big) < \bar{\alpha}_i(x, x_0)\Big(\frac{f_i(x_0) + \langle w_i, x_0 \rangle}{g_i(x_0)}\Big), \quad i = 1, \cdots, p.$$

By the (V, ρ)-invexity of $f(\cdot) + \langle w, \cdot \rangle$ and $-g$ at x_0, and Theorem 3.1.1, we have

$$\nabla\Big(\frac{f_i(x_0) + \langle w_i, x_0 \rangle}{g_i(x_0)}\Big)\eta(x, x_0) + \rho_i\|\bar{\theta}_i(x, x_0)\|^2 < 0.$$

Hence we have

$$\sum_{i=1}^{p} \lambda_i \nabla\Big(\frac{f_i(x_0) + \langle w_i, x_0 \rangle}{g_i(x_0)}\Big)\eta(x, x_0) + \sum_{i=1}^{p} \lambda_i \rho_i \|\bar{\theta}_i(x, x_0)\|^2 < 0.$$

Since $\sum_{i=1}^{p} \lambda_i \rho_i \|\bar{\theta}_i(x, x_0)\|^2 + \sum_{j=1}^{m} \sigma_j \|\bar{\theta}_j(x, x_0)\|^2 \geqq 0$, it follows from (3.3) that

$$\sum_{j=1}^{m} \mu_j \nabla h_j(x_0) \eta(x, x_0) + \sum_{j=1}^{m} \sigma_j \|\bar{\theta}_j(x, x_0)\|^2 > 0.$$

Then by the (V, σ)-invexity of h, we have

$$\sum_{j=1}^{m} \beta_j(x, x_0) \Big[\mu_j h_j(x) - \mu_j h_j(x_0)\Big] > 0.$$

Since $\mu_j h_j(x_0) = 0,\ j = 1, \cdots, m$, we have $\sum_{j=1}^{m} \beta_j(x, x_0) \mu_j h_j(x) > 0$, which contradicts the conditions $\beta_j(x, x_0) > 0,\ \mu_j \geqq 0$ and $h_j(x) \leqq 0$. Thus x_0 is a weakly efficient solution of **(NFP)**. □

3.3. Duality Theorems

In this section, we introduce dual problems for a weakly efficient solutions based on (V, ρ)-invexity assumptions. We propose the following Mond-Weir type dual problem to the primal problem **(NFP)**:

$(\mathbf{NFD})_M$ Maximize $\quad \Big(\dfrac{f_1(u) + \langle w_1, u\rangle}{g_1(u)}, \cdots, \dfrac{f_p(u) + \langle w_p, u\rangle}{g_p(u)}\Big)$

subject to

$$\sum_{i=1}^{p} \lambda_i \nabla \Big(\frac{f_i(u) + \langle w_i, u\rangle}{g_i(u)}\Big) + \sum_{j=1}^{m} \mu_j \nabla h_j(u) = 0, \quad (3.4)$$

$$\sum_{j=1}^{m} \mu_j h_j(u) \geqq 0,$$

$$w_i \in C_i,\ i = 1, \cdots, p,$$

$$(\mu_1, \cdots, \mu_m) \geqq 0,\ \lambda = (\lambda_1, \cdots, \lambda_p) \in \Lambda^+,$$

where $\Lambda^+ = \{\lambda \in \mathbb{R}^p : \lambda \geqq 0, \lambda^T e = 1, e = (1, \cdots, 1) \in \mathbb{R}^p\}$.

Theorem 3.3.1 (Weak Duality). Let $x \in S$ be a feasible for **(NFP)** and (u, λ, w, μ) be a feasible for $\mathbf{(NFD)}_M$. Assume that the functions $f(\cdot) + \langle w, \cdot \rangle, -g(\cdot)$ are (V, ρ)-invex functions over S, and if h is (V, σ)-invex at u with respect to the same η with $\sum_{i=1}^{p} \lambda_i \rho_i \geqq 0$ and $\sum_{j=1}^{m} \sigma_j \geqq 0$.
Then the following cannot hold,

$$\frac{f(x) + s(x|C)}{g(x)} < \frac{f(u) + \langle w, u \rangle}{g(u)}.$$

Proof. As x is feasible for **(NFP)** and (u, λ, w, μ) is feasible for $\mathbf{(NFD)}_M$, we have $\sum_{j=1}^{m} \mu_j h_j(x) \leqq 0 \leqq \sum_{j=1}^{m} \mu_j h_j(u)$. Since $\beta_j(x, u) > 0$, we have $\sum_{j=1}^{m} \beta_j(x, u)\mu_j h_j(x) \leqq \sum_{j=1}^{m} \beta_j(x, u)\mu_j h_j(u)$. By the (V, σ)-invexity of $h_j(u)$, $j = 1, \cdots, m$,

$$\sum_{j=1}^{m} \mu_j \nabla h_j(u)\eta(x, u) + \sum_{j=1}^{m} \mu_j \sigma_j \|\theta_j(x, u)\|^2 \leqq 0.$$

Using (3.4), we obtain

$$\sum_{i=1}^{p} \lambda_i \nabla\Big(\frac{f_i(u) + \langle w_i, u \rangle}{g_i(u)}\Big) \geqq \sum_{j=1}^{m} \mu_j \sigma_j \|\theta_j(x, u)\|^2. \tag{3.5}$$

Now suppose, contrary to the result, since $\langle w_i, x \rangle \leqq s(x|C_i)$, we have for all $i \in \{1, \cdots, p\}$

$$\begin{aligned} \frac{f_i(x) + \langle w_i, x \rangle}{g_i(x)} &\leqq \frac{f_i(x) + s(x|C_i)}{g_i(x)} \\ &< \frac{f_i(u) + \langle w_i, u \rangle}{g_i(u)}. \end{aligned}$$

By Theorem 3.1.1,

$$\bar{\alpha}_i(x,u)\Big[\frac{f_i(x)+\langle w_i,x\rangle}{g_i(x)}-\frac{f_i(u)+\langle w_i,u\rangle}{g_i(u)}\Big]$$

$$\geqq \nabla\Big(\frac{f_i(u)+\langle w_i,u\rangle}{g_i(u)}\Big)\eta(x,u)+\rho_i\|\bar{\theta}_i(x,u)\|^2.$$

By using $\lambda\in\Lambda^+$, we have

$$\sum_{i=1}^{p}\lambda_i\nabla\Big(\frac{f_i(u)+\langle w_i,u\rangle}{g_i(u)}\Big)\eta(x,u)<(-\sum_{i=1}^{p}\lambda_i\rho_i)\|\bar{\theta}_i(x,u)\|^2. \tag{3.6}$$

Since $\sum_{i=1}^{p}\lambda_i\rho_i\|\bar{\theta}_i(x,u)\|^2+\sum_{j=1}^{m}\mu_j\sigma_j\|\theta_j(x,u)\|^2\geqq 0$, it follows from (3.6) that

$$\sum_{i=1}^{p}\lambda_i\nabla\Big(\frac{f_i(u)+\langle w_i,u\rangle}{g_i(u)}\Big)\eta(x,u)<\sum_{j=1}^{m}\mu_j\sigma_j\|\theta_j(x,u)\|^2,$$

which contradicts (3.5). □

Theorem 3.3.2 (Strong Duality). If $\bar{x}$ is a weakly efficient solution of **(NFP)**, and assume that there exists $z^*\in\mathbb{R}^n$ such that $\langle\nabla h_j(\bar{x}),z^*\rangle>0,\ j\in I(\bar{x})$, then there exists $\bar{\lambda}\in\mathbb{R}^p,\ \bar{\mu}\in\mathbb{R}^m$ and $\bar{w}\in C$ such that $(\bar{x},\bar{\lambda},\bar{w},\bar{\mu})$ is feasible for **(NFD)**$_M$ and $\langle\bar{w},\bar{x}\rangle=s(\bar{x}|C)$. Moreover, if the weak duality holds, then $(\bar{x},\bar{\lambda},\bar{w},\bar{\mu})$ is a weakly efficient solution of **(NFD)**$_M$.

Proof. By Theorem 3.2.2, there exists $\bar{\lambda}\in\mathbb{R}^p,\ \bar{\mu}\in\mathbb{R}^m$ and $w_i\in C_i,\ i=1,\cdots,p$ such that $\sum_{i=1}^{p}\bar{\lambda}_i\nabla\Big(\frac{f_i(\bar{x})+\langle w_i,\bar{x}\rangle}{g_i(\bar{x})}\Big)+\sum_{j=1}^{m}\bar{\mu}_j\nabla h_j(\bar{x})=0,\ \ \langle\bar{w}_i,\bar{x}\rangle=s(\bar{x}|C_i),\ \bar{w}_i\in C_i,\ i=1,\cdots,p$ and $\sum_{j=1}^{m}\bar{\mu}_jh_j(\bar{x})=0$. Thus $(\bar{x},\bar{\lambda},\bar{w},\bar{\mu})$ is a feasible for **(NFD)**$_M$, $\langle\bar{w}_i,\bar{x}\rangle=s(\bar{x}|C_i),\ i=1,\cdots,p$. By weak duality,

$\frac{f(\bar{x})+s(\bar{x}|C)}{g(\bar{x})} \not< \frac{f(u)+\langle w,u\rangle}{g(u)}$ for any $(\mathbf{NFD})_M$ feasible solution (u, λ, w, μ). Since $\langle \bar{w}, \bar{x}\rangle = s(\bar{x}|C)$, we have $\frac{f(\bar{x})+\langle \bar{w},\bar{x}\rangle}{g(\bar{x})} \not< \frac{f(u)+\langle w,u\rangle}{g(u)}$. Since $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{\mu})$ is a feasible solution of $(\mathbf{NFD})_M$ and $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{\mu})$ is a weakly efficient solution of $(\mathbf{NFD})_M$. Hence the result holds. □

Now we propose the following Wolfe type dual problem to the primal problem **(NFP)**:

$$(\mathbf{NFD})_W \quad \text{Maximize} \quad \left(\frac{f_1(u)+\langle w_1,u\rangle}{g_1(u)} + \sum_{j=1}^{m} \mu_j h_j(u), \cdots, \frac{f_p(u)+\langle w_p,u\rangle}{g_p(u)} + \sum_{j=1}^{m} \mu_j h_j(u)\right)$$

$$\text{subject to} \quad \sum_{i=1}^{p} \lambda_i \nabla\Big(\frac{f_i(u)+\langle w_i,u\rangle}{g_i(u)}\Big) + \sum_{j=1}^{m} \mu_j \nabla h_j(u) = 0, \quad (3.7)$$

$$w_i \in C_i \; i = 1, \cdots, p,$$

$$(\mu_1, \cdots, \mu_m) \geqq 0, \quad \lambda = (\lambda_1, \cdots, \lambda_p) \in \Lambda^+,$$

where $\Lambda^+ = \{\lambda \in \mathbb{R}^p : \lambda \geqq 0, \lambda^T e = 1, e = (1, \cdots, 1) \in \mathbb{R}^p\}$.

Theorem 3.3.4 (Weak Duality). Let $x \in S$ be a feasible for **(NFP)** and (u, λ, w, μ) be a feasible for $(\mathbf{NFD})_W$. Assume that the functions $f(\cdot) + \langle w, \cdot\rangle$, $-g(\cdot)$ and $h(\cdot)$ are (V, ρ)-invex functions over S with respect to the $\sum_{i=1}^{p} \lambda_i \rho_i \geqq 0$.

Then the following cannot hold,

$$\frac{f(x)+s(x|C)}{g(x)} < \frac{f(u)+\langle w,u\rangle}{g(u)} + \sum_{j=1}^{m}\mu_j h_j(u)e.$$

Proof. As x is feasible for **(NFP)** and (u,λ,w,μ) is feasible for $\mathbf{(NFD)}_W$. Now suppose, contrary to the result, since $\langle w_i,x\rangle \leqq s(x|C_i)$, we have for all $i\in\{1,\cdots,p\}$

$$\begin{aligned}\frac{f_i(x)+\langle w_i,x\rangle}{g_i(x)} &\leqq \frac{f_i(x)+s(x|C_i)}{g_i(x)}\\ &< \frac{f_i(u)+\langle w_i,u\rangle}{g_i(u)} + \sum_{j=1}^{m}\mu_j h_j(u).\end{aligned}$$

Since $\sum_{j=1}^{m}\mu_j h_j(x)\leqq 0$, and for $i=1,\cdots,p$.

$$\frac{f_i(x)+\langle w_i,x\rangle}{g_i(x)} + \sum_{j=1}^{m}\mu_j h_j(x) < \frac{f_i(u)+\langle w_i,u\rangle}{g_i(u)} + \sum_{j=1}^{m}\mu_j h_j(u).$$

Since h is (V,ρ)-invex at u, it follows from Theorem 3.1.1 that

$$\bar{\alpha}_i(x,u)\Big[\frac{f_i(x)+\langle w_i,x\rangle}{g_i(x)} + \sum_{j=1}^{m}\mu_j h_j(u) - \frac{f_i(u)+\langle w_i,u\rangle}{g_i(u)} - \sum_{j=1}^{m}\mu_j h_j(u)\Big]$$

$$\geqq \Big[\nabla\Big(\frac{f_i(u)+\langle w_i,u\rangle}{g_i(u)}\Big) + \sum_{j=1}^{m}\mu_j\nabla h_j(u)\Big]\eta(x,u) + \rho_i\|\bar{\theta}_i(x,u)\|^2.$$

By using $\lambda\in\Lambda^+$, we have

$$\Big[\sum_{i=1}^{p}\lambda_i\nabla\Big(\frac{f_i(u)\,\langle w_i,u\rangle}{g_i(u)}\Big) + \sum_{j=1}^{m}\mu_j\nabla h_j(u)\Big]\eta(x,u) < (-\sum_{i=1}^{p}\lambda_i\rho_i)\|\bar{\theta}_i(x,u)\|^2. \tag{3.8}$$

Since $\sum_{i=1}^{p} \lambda_i \rho_i \|\bar{\theta}_i(x,u)\|^2 \geqq 0$, it follows from (3.8) that

$$\Big[\sum_{i=1}^{p} \lambda_i \nabla\Big(\frac{f_i(u) + \langle w_i, u\rangle}{g_i(u)}\Big) + \sum_{j=1}^{m} \mu_j \nabla h_j(u)\Big]\eta(x,u) < 0,$$

which contradicts (3.7). □

Theorem 3.3.5 (Strong Duality). If $\bar{x}$ is a weakly efficient solution of **(NFP)**, and assume that there exists $z^* \in \mathbb{R}^n$ such that $\langle \nabla h_j(\bar{x}), z^*\rangle > 0,\ j \in I(\bar{x})$, then there exists $\bar{\lambda} \in \mathbb{R}^p$, $\bar{\mu} \in \mathbb{R}^m$ and $\bar{w} \in C$ such that $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{\mu})$ is feasible for $\mathbf{(NFD)}_W$ and $\langle \bar{w}, \bar{x}\rangle = s(\bar{x}|C)$. Moreover, if the weak duality holds, then $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{\mu})$ is a weakly efficient solution of $\mathbf{(NFD)}_W$.

Proof. By Theorem 3.2.2, there exists $\bar{\lambda} \in \mathbb{R}^p$, $\bar{\mu} \in \mathbb{R}^m$ and $w_i \in C_i,\ i = 1, \cdots, p$ such that $\sum_{i=1}^{p} \bar{\lambda}_i \nabla\Big(\frac{f_i(\bar{x}) + \langle w_i, \bar{x}\rangle}{g_i(\bar{x})}\Big) + \sum_{j=1}^{m} \bar{\mu}_j \nabla h_j(\bar{x}) = 0$, $\langle \bar{w}_i, \bar{x}\rangle = s(\bar{x}|C_i)$, $\bar{w}_i \in C_i$, $i = 1, \cdots, p$ and $\sum_{j=1}^{m} \bar{\mu}_j h_j(\bar{x}) = 0$. Thus $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{\mu})$ is a feasible for $\mathbf{(NFD)}_W$, $\langle \bar{w}_i, \bar{x}\rangle = s(\bar{x}|C_i)$, $i = 1, \cdots, p$. By weak duality, $\frac{f(\bar{x}) + s(\bar{x}|C)}{g(\bar{x})} \not< \frac{f(u) + \langle w, u\rangle}{g(u)} + \sum_{j=1}^{m} \mu_j h_j(u) e$ for any $\mathbf{(NFD)}_W$ feasible solution (u, λ, w, μ). Notice that $\frac{f(\bar{x}) + s(\bar{x}|C)}{g(\bar{x})} = \frac{f(\bar{x}) + \langle \bar{w}, \bar{x}\rangle}{g(\bar{x})} = \frac{f(\bar{x}) + \langle \bar{w}, \bar{x}\rangle}{g(\bar{x})} + \sum_{j=1}^{m} \bar{\mu}_j h_j(\bar{x}) e$. Since $\frac{f(\bar{x}) + \langle \bar{w}, \bar{x}\rangle}{g(\bar{x})} + \sum_{j=1}^{m} \bar{\mu}_j h_j(\bar{x}) e \not< \frac{f(u) + \langle w, u\rangle}{g(u)} + \sum_{j=1}^{m} \mu_j h_j(u) e$. Since $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{\mu})$ is a feasible solution of $\mathbf{(NFD)}_W$ and $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{\mu})$ is a weakly efficient solution of $\mathbf{(NFD)}_W$. Hence the result holds. □

Chapter 4

Symmetric Duality for Multiobjective Programming Problems with Cone Constraints

4.1. Introduction

A nonlinear programming problem and its dual are said to be symmetric if the dual of the dual is the original problem. Symmetric duality in nonlinear programming in which the dual of the dual is the primal was first introduced by Dorn ([17]). Danzig, Eisenberg and Cottle ([18]) first formulated a pair of symmetric dual nonlinear programs involving a scalar function $f(x, y), x \in \mathbb{R}^n, y \in \mathbb{R}^m$ that is required to be convex in x for fixed y and concave in y for fixed x. Mond and Weir ([59]) have given a different pair of symmetric dual nonlinear programs requiring $f(x, y)$ to be pseudoconvex in x for fixed y and pseudoconcave in y for fixed x. Weir and Mond ([76]) discuss symmetric duality in multiple objective programming. The duals given there are reduced to those known for scalar valued symmetric programming and also some more recent results in multiobjective duality. The results were based on the concept of proper efficiency. Mond and Weir ([60]) presented two pairs of symmetric dual multiobjective programming problems for efficient solutions and obtained symmetric duality resutls concerning pseudoconvex/ pseudoconcave functions. Nanda and Das ([61]) also studied the symmetric dual fractional programming problem for arbitrary cones assuming the functions to be pseudo-invex. The symmetric duality result was generalized by Bazaraa and Goode ([8]) to arbitary cones. Kim *et al.* ([43]) studied a pair of mulitobjective symmetric dual programs for pseudo-invex functions and arbitary cones. Devi ([16]) formulated a pair of second-order symmetric dual programs and obtained duality results involving η-bonvex functions. Recently, Suneja *et al.* ([69]) formulated a pair of symmetric dual programs over arbitary cones. In this chapter, we formulate Mond-Weir type and Wolfe type multiobjective symmetric dual problems with cone constraints. We obtained duality results under weakly efficient solutions involving pseduo-invex functions for Mond-Weir model and K-preinvex functions for Wolfe model. We establish the weak, strong, converse and self duality theorems for these problems.

4.2. Notations and Preliminaries

We consider the following mulitobjective programming problem.

$$\textbf{(CP)} \quad \text{Minimize} \quad f(x)$$

$$\text{subject to} \quad -g(x) \in Q, \quad x \in C,$$

where $f : \mathbb{R}^n \to \mathbb{R}^p$, $g : \mathbb{R}^n \to \mathbb{R}^m$ and $C \subset \mathbb{R}^n$, Q is closed convex cone with nonempty interior in $\mathbb{R}^m$. Let $X^0 = \{x \in C | -g(x) \in Q\}$ be the set of all feasible solutions of **(CP)**.

Now we define generalized weakly efficient solution with respect to a closed convex cone K with nonempty interior in $\mathbb{R}^n$.

Definition 4.2.1. A point $\bar{x} \in X^0$ is a weakly efficient soluition of **(CP)** if there exist no other $x \in X^0$ such that $f(\bar{x}) - f(x) \in intK$.

Definition 4.2.2 ([11]). A differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is pseudo-invex with respect to the $\eta : C \times C \to \mathbb{R}^n$ if for all $(x, u) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$\eta(x,u)^T \nabla f(u) \geqq 0 \quad \Rightarrow \quad f(x) \geqq f(u).$$

Definition 4.2.3. The positive polar cone K^* of K is defined by

$$K^* = \{z \in \mathbb{R}^p \mid x^T z \geqq 0 \ \text{ for all } \ x \in K\}.$$

Definition 4.2.4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. Then f is K-preinvex with respect to the η if there exists a function $\eta : C \times C \to \mathbb{R}^n$ such that for any $x, y \in \mathbb{R}^n$ and $\alpha \in [0, 1]$,

$$\alpha f(x) + (1-\alpha) f(y) - f(y + \alpha \eta(x, y)) \in K.$$

When $K = \mathbb{R}_+$, the above definition reduced to one of the scalar preinvexity ([75]).

Remark 4.2.1. If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable and K-preinvex with respect to η then $f(x) - f(y) - \nabla f(y)^T \eta(x,y) \in K$. Moreover, for all $\lambda \in K^*$, $(\lambda^T f)(x) - (\lambda^T f)(y) - \nabla(\lambda^T f)(y)^T \eta(x,y) \geqq 0$.

4.3. Mond-Weir Type Duality Theorems

Now we formulate the following Mond-Weir type multiobjective symmetric dual problems:

$$
\begin{array}{llll}
(\mathbf{MSP})_M & K-\text{minimize} & f(x,y) & \\
& \text{subject to} & x \in C_1 & \\
& & -\nabla_y(\lambda^T f)(x,y) \in {C_2}^*, & (4.1)\\
& & y^T\nabla_y(\lambda^T f)(x,y) \geqq 0, & (4.2)\\
& & \lambda \in K^*,\ e \in intK,\ \lambda^T e = 1, & (4.3)
\end{array}
$$

and

$$
\begin{array}{llll}
(\mathbf{MSD})_M & K-\text{maximize} & f(u,v) & \\
& \text{subject to} & v \in C_2 & \\
& & \nabla_x(\lambda^T f)(u,v) \in {C_1}^*, & (4.4)\\
& & u^T\nabla_x(\lambda^T f)(u,v) \leqq 0, & (4.5)\\
& & \lambda \in K^*,\ e \in intK,\ \lambda^T e = 1, &
\end{array}
$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is a twice differentiable function, C_1 and C_2 are closed convex cones in $\mathbb{R}^n$, $\mathbb{R}^m$ with nonempty interiors, respectively. C_1^* and C_2^* are positive polar cones of C_1 and C_2, respectively, K is a closed convex cone in $\mathbb{R}^p$ such that $intK \neq \emptyset$.

Let $\nabla_x(\lambda^T f)(x,y)$ and $\nabla_y(\lambda^T f)(x,y)$ are gradients of $(\lambda^T f)(x,y)$ with respect to x and y. Similarly, $\nabla_{xx}(\lambda^T f)(x,y)$ and $\nabla_{yy}(\lambda^T f)(x,y)$ are the Hessian matrices of $(\lambda^T f)(x,y)$ with respect to x and y respectively.

We establish the symmetric duality theorems for $(\mathbf{MSP})_M$ and $(\mathbf{MSD})_M$.

Theorem 4.3.1 (Weak Duality). Let (x, y) be feasible for $\mathbf{(MSP)}_M$ and let (u, v) be feasible to $\mathbf{(MSD)}_M$. Let $(\lambda^T f)(\cdot, y)$ be pseudo-invex with respect to η_1 at u, and $-(\lambda^T f)(x, \cdot)$ be pseudo-invex with respect to η_2 at y. If $\eta_1(x, u) + u \in C_1$ and $\eta_2(v, y) + y \in C_2$, then

$$f(x, y) - f(u, v) \notin -intK.$$

Proof. From (4.4) and $\eta_1(x, u) + u \in C_1$,

$$(\eta_1(x, u) + u)^T \nabla_x (\lambda^T f)(u, v) \geqq 0.$$

From (4.5), $\eta_1(x, u) \nabla_x (\lambda^T f)(u, v) \geqq 0$. Since $(\lambda^T f)(\cdot, y)$ is pseudo-invex with respect to η_1 at u,

$$(\lambda^T f)(x, v) \geqq (\lambda^T f)(u, v). \tag{4.6}$$

From (4.1) and $\eta_2(v, y) + y \in C_2$,

$$(\eta_2(v, y) + y)^T \nabla_y (\lambda^T f)(x, y) \leqq 0.$$

From (4.2), $\eta_2(v, y)^T \nabla_y (\lambda^T f)(x, y) \leqq 0$. Since $-(\lambda^T f)(x, \cdot)$ is pseudo-invex with respect to η_2 at y,

$$(\lambda^T f)(x, y) \geqq (\lambda^T f)(x, v). \tag{4.7}$$

From (4.6) and (4.7),

$$(\lambda^T f)(x, y) \geqq (\lambda^T f)(u, v). \tag{4.8}$$

Suppose to the contrary that

$$f(x, y) - f(u, v) \in -intK.$$

Since $\lambda \in K^*$ and $\lambda \neq 0$, $(\lambda^T f)(x,y) < (\lambda^T f)(u,v)$, which contradicts (4.8).
□

In order to prove the strong duality theorem, we now obtain necessary optimality conditions for a point to be a weak minimum of **(CP)**.

Lemma 4.3.1([69]). If x^* is a weakly efficient solution of **(CP)**, then there exist $\alpha \in K^*$ and $\beta \in Q^*$ not both zero such that

$$(\alpha^T \nabla f(x^*) + \beta^T \nabla g(x^*))(x - x^*) \geqq 0, \quad \text{for all } x \in C$$
$$\beta^T g(x^*) = 0,$$

equivalently, there exist $\alpha \in K^*$, $\beta \in Q^*$ and $\beta_1 \in C^*$, $(\alpha, \beta, \beta_1) \neq 0$ such that

$$\alpha^T \nabla f(x^*) + \beta^T \nabla g(x^*) - \beta_1^T I = 0,$$
$$\beta^T g(x^*) = 0,$$
$$\beta_1^T x^* = 0.$$

Proof. (Sufficiency) Substituting $x = 0$ and $x = 2x^*$,
we get $(\alpha^T \nabla f(x^*) + \beta^T \nabla g(x^*))x^* = 0$. Since $\alpha^T \nabla f(x^*) + \beta^T \nabla g(x^*) \in C^*$,
Let $\beta_1 = \alpha^T \nabla f(x^*) + \beta^T \nabla g(x^*)$. Then

$$\alpha^T \nabla f(x^*) + \beta^T \nabla g(x^*) - \beta_1^T I = 0,$$
$$\beta^T g(x^*) = 0,$$
$$\beta_1^T x^* = 0.$$

(Necessity) Since $\alpha^T\nabla f(x^*)+\beta^T\nabla g(x^*) = \beta_1 \in C^*$. we get $(\alpha^T\nabla f(x^*)+\beta^T\nabla g(x^*))x \geqq 0$, for all $x \in C$, and $\beta_1^T x^* = (\alpha^T\nabla f(x^*) + \beta^T\nabla g(x^*))x^* = 0$. Therefore

$$(\alpha^T\nabla f(x^*) + \beta^T\nabla g(x^*))(x - x^*) \geqq 0, \quad \text{for all } x \in C$$

$$\beta^T g(x^*) = 0.$$

□

Theorem 4.3.2 (Strong Duality). Let $(\bar{x}, \bar{y}, \bar{\lambda})$ be a weakly efficient solution for $\mathbf{(MSP)}_M$. Assume that

(I) $\nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})$ is positive definite.

(II) $\{\nabla_y f_i(\bar{x}, \bar{y}),\ i = 1, 2, \cdots, p\}$ is linearly independent.

Then $(\bar{x}, \bar{y}, \bar{\lambda})$ is feasible for $\mathbf{(MSD)}_M$, and the objective values of $\mathbf{(MSP)}_M$ and $\mathbf{(MSD)}_M$ are equal. Furthermore, under the assumptions of Theorem 4.3.1, $(\bar{x}, \bar{y}, \bar{\lambda})$ is a weakly efficient solution for $\mathbf{(MSD)}_M$.

Proof. Since $(\bar{x}, \bar{y}, \bar{\lambda})$ is a weakly efficient solution of $\mathbf{(MSP)}_M$, from Lemma 4.3.1, there exist $\alpha \in K^*$, $\beta_1 \in C_2$, $\beta_2 \in \mathbb{R}_+$, $\beta_3 \in C_1^*$, $\beta_4 \in K$, $(\alpha, \beta_1, \beta_2, \beta_3, \beta_4) \neq 0$ such that for each $(x, y) \in C_1 \times C_2$ and $\lambda \in K^*$,

$$\alpha^T\nabla_x f(\bar{x}, \bar{y}) + (\beta_1 - \beta_2\bar{y})\nabla_{yx}(\bar{\lambda}^T f)(\bar{x}, \bar{y}) - \beta_3^T = 0, \tag{4.9}$$

$$(\alpha - \beta_2\bar{\lambda})^T\nabla_y f(\bar{x}, \bar{y}) + (\beta_1 - \beta_2\bar{y})^T\nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y}) = 0, \tag{4.10}$$

$$\nabla_y f(\bar{x}, \bar{y})(\beta_1 - \beta_2\bar{y}) - \beta_4 = 0, \tag{4.11}$$

$$\beta_1^T\nabla_y(\bar{\lambda}^T f)(\bar{x}, \bar{y}) = 0, \tag{4.12}$$

$$\beta_2^T\bar{y}^T\nabla_y(\bar{\lambda}^T f)(\bar{x}, \bar{y}) = 0, \tag{4.13}$$

$$\beta_3^T\bar{x} = 0, \tag{4.14}$$

$$\beta_4^T\bar{\lambda} = 0. \tag{4.15}$$

Multiplying (4.10) by $(\beta_1 - \beta_2\bar{y})$ and applying (4.12) and (4.13) gives

$$\alpha^T \nabla_y f(\bar{x}, \bar{y})(\beta_1 - \beta_2\bar{y}) + (\beta_1 - \beta_2\bar{y})^T \nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})(\beta_1 - \beta_2\bar{y}) = 0.$$

Using the result in equality (4.11), we get

$$\alpha^T \beta_4 + (\beta_1 - \beta_2\bar{y})^T \nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})(\beta_1 - \beta_2\bar{y}) = 0.$$

Since $\alpha \in K^*$ and $\beta_4 \in K$, $\alpha^T \beta_4 \geqq 0$ and hence

$$(\beta_1 - \beta_2\bar{y})^T \nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})(\beta_1 - \beta_2\bar{y}) \leqq 0.$$

Since $\nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})$ is positive definite, then $\beta_1 = \beta_2\bar{y}$. By (4.11), $\beta_4 = 0$. From (4.10) and the fact that $\beta_1 = \beta_2\bar{y}$, $(\alpha - \beta_2\bar{\lambda})^T \nabla_y f(\bar{x}, \bar{y}) = 0$. Since $\nabla_y f(\bar{x}, \bar{y})$ is linearly independent, $\alpha = \beta_2\bar{\lambda}$. If $\alpha = 0$, then $\beta_2 = 0$, $\beta_1 = 0$, $\beta_3 = 0$, $\beta_4 = 0$. This is not possible since $\alpha \neq 0$, $\beta_2 > 0$. By (4.9) and the fact that $\beta_1 = \beta_2\bar{y}$, and $\alpha = \beta_2\bar{\lambda}$,

$$\beta_2 \nabla_x(\bar{\lambda}^T f)(\bar{x}, \bar{y}) = \beta_3 \in C_1^*. \tag{4.16}$$

Since $\beta_2 > 0$, $\nabla_x(\bar{\lambda}^T f)(\bar{x}, \bar{y}) \in C_1^*$. Multiplying (4.16) by $\bar{x}$ and using equation (4.14), we get $\beta_2 \bar{x}^T \nabla_x(\bar{\lambda}^T f)(\bar{x}, \bar{y}) = 0$. Since $\beta_2 > 0$, $\bar{x}^T \nabla_x(\bar{\lambda}^T f)(\bar{x}, \bar{y}) = 0$. Thus $(\bar{x}, \bar{y}, \bar{\lambda})$ is feasible for $\mathbf{(MSD)}_M$ and the objective functions are equal.

By Theorem 4.3.1, $f(\bar{x}, \bar{y}) - f(u, v) \notin -intK$ for any $\mathbf{(MSD)}_M$-feasible solution (u, v, λ). Since $(\bar{x}, \bar{y}, \bar{\lambda})$ is a $\mathbf{(MSD)}_M$-feasible solution, $(\bar{x}, \bar{y}, \bar{\lambda})$ is a weakly efficient solution of $\mathbf{(MSD)}_M$. Hence the result holds. □

Theorem 4.3.3 (Converse Duality). Let $(\bar{u}, \bar{v}, \bar{\lambda})$ be a weakly efficient solution for $(\mathbf{MSD})_M$. Assume that $\nabla_{xx}(\bar{\lambda}^T f)(\bar{u}, \bar{v})$ is positive definite and the set $\{\nabla_x f_i(\bar{u}, \bar{v}),\ i = 1, 2, \cdots, p\}$ is linearly independent, then $(\bar{u}, \bar{v}, \bar{\lambda})$ is feasible for $(\mathbf{MSP})_M$ and the objective values of $(\mathbf{MSP})_M$ and $(\mathbf{MSD})_M$ are equal. Furthermore, under the assumptions of Theorem 4.3.1, $(\bar{u}, \bar{v}, \bar{\lambda})$ is an weakly efficient solution for $(\mathbf{MSP})_M$.

Proof. It is analogus to the proof of the lines of Theorem 4.3.2. □

Assume that $m = n$, $f(x, y) = -f(y, x)$, that is, f is skew-symmetric and $C_1 = C_2$. It follows that $(\mathbf{MSD})_M$ may be rewritten as a minimization problem:

$$\begin{array}{lll} (\mathbf{MSD'})_M & K - \text{minimize} & -f(u, v) \\ & \text{subject to} & v \in C_2 \\ & & \nabla_x(\lambda^T f)(u, v) \in {C_1}^* \\ & & u^T \nabla_x(\lambda^T f)(u, v) \leqq 0 \\ & & \lambda \in K^*,\ e \in intK,\ \lambda^T e = 1. \end{array}$$

Since $\nabla_x f(u, v) = -\nabla_y f(v, u)$, the problem $(\mathbf{MSD'})_M$ reduces to

$$\begin{array}{ll} K - \text{minimize} & f(v, u) \\ \text{subject to} & v \in C_2 \\ & -\nabla_y(\lambda^T f)(v, u) \in {C_1}^* \\ & u^T \nabla_y(\lambda^T f)(v, u) \geqq 0 \\ & \lambda \in K^*,\ e \in intK,\ \lambda^T e = 1. \end{array}$$

An optimization problem is said to be self-dual if, when the dual is written in the form of the primal, the new program so obtained is the same as the primal problem.

Now we establish the self-duality of $\mathbf{(MSP)}_M$.

Theorem 4.3.4 (Self Duality). Assume that $m = n$, $f(x, y) = -f(y, x)$, $C_1 = C_2$ and the conditions of Theorem 4.3.1 are satisfied, and if $(\bar{x}, \bar{y}, \bar{\lambda})$ is a weakly efficient solution for $\mathbf{(MSP)}_M$ and $\nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})$ is positive definite and the set $\{\nabla_y f_i(\bar{x}, \bar{y}) : i = 1, 2, \cdots, p\}$ is linearly independent, then $(\bar{y}, \bar{x}, \bar{\lambda})$ is a weakly efficient solution for both $\mathbf{(MSP)}_M$ and $\mathbf{(MSD)}_M$, and the common optimal value is 0.

Proof. By Theorem 4.3.2, $(\bar{x}, \bar{y}, \bar{\lambda})$ is a weakly efficient solution for $\mathbf{(MSD)}_M$ and the optimal values of $\mathbf{(MSP)}_M$ and $\mathbf{(MSD)}_M$ are equal to $f(\bar{x}, \bar{y})$. From self-duality, $(\bar{y}, \bar{x}, \bar{\lambda})$ is feasible for both $\mathbf{(MSP)}_M$ and $\mathbf{(MSD)}_M$ and using Theorem 4.3.1 and Theorem 4.3.2, we get that it is optimal for both problems. Since f is skew-symmetric, we have $f(\bar{x}, \bar{y}) = -f(\bar{y}, \bar{x})$. Hence

$$f(\bar{x}, \bar{y}) = f(\bar{y}, \bar{x}) = -f(\bar{x}, \bar{y}),$$

and so

$$f(\bar{x}, \bar{y}) = f(\bar{y}, \bar{x}) = 0.$$

□

4.4. Wolfe Type Duality Theorems

Consider the following Wolfe type multiobjective symmetric dual problems:

$$\begin{aligned}
\textbf{(MSP)}_W \qquad K-\text{minimize} \quad & f(x,y)-[y^T\nabla_y(\lambda^T f)(x,y)]e \\
\text{subject to} \quad & x \in C_1 \\
& -\nabla_y(\lambda^T f)(x,y) \in {C_2}^*, \qquad (4.17) \\
& \lambda \in K^*,\ e \in int\ K,\ \lambda^T e = 1, \quad (4.18)
\end{aligned}$$

and

$$\begin{aligned}
\textbf{(MSD)}_W \qquad K-\text{maximize} \quad & f(u,v)-[u^T\nabla_x(\lambda^T f)(u,v)]e \\
\text{subject to} \quad & v \in C_2 \\
& \nabla_x(\lambda^T f)(u,v) \in {C_1}^*, \qquad (4.19) \\
& \lambda \in K^*,\ e \in int\ K,\ \lambda^T e = 1,
\end{aligned}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ be a twice differentiable function, C_1 and C_2 be closed convex cones in $\mathbb{R}^n$, $\mathbb{R}^m$ with nonempty interiors, respectively. C_1^* and C_2^* are positive polar cones of C_1 and C_2 respectively, K is a closed convex cone in $\mathbb{R}^p$ such that $intK \neq \emptyset$.

We establish the symmetric duality theorems for $\textbf{(MSP)}_W$ and $\textbf{(MSD)}_W$.

Theorem 4.4.1 (Weak Duality). Let (x, y, λ) be feasible for $\textbf{(MSP)}_W$ and let (u, v, λ) be feasible to $\textbf{(MSD)}_W$ respectively. Let $f(\cdot, y)$ be K-preinvex with respect to η_1 for fixed y and $-f(x, \cdot)$ be K-preinvex with respect to η_2 for fixed x. If $\eta_1(x, u) + u \in C_1$ and $\eta_2(v, y) + y \in C_2$, then

$$(f(x,y)-[y^T\nabla_y(\lambda^T f)(x,y)]e)-(f(u,v)-[u^T\nabla_y(\lambda^T f)(u,v)]e) \notin -intK.$$

Proof. Since $f(\cdot, y)$ is K-preinvex with respect to η_1 for fixed y, by Remark 4.2.1,

$$(\lambda^T f)(x, v) - (\lambda^T f)(u, v) \geqq \eta_1(x, u)^T \nabla_x (\lambda^T f)(u, v).$$

From (4.19) and $\eta_1(x, u) + u \in C_1$,

$$(\eta_1(x, u) + u)^T \nabla_x (\lambda^T f)(u, v) \geqq 0.$$

Hence

$$(\lambda^T f)(x, v) - (\lambda^T f)(u, v) \geqq -u^T \nabla_x (\lambda^T f)(u, v). \tag{4.20}$$

Since $-f(x, \cdot)$ is K-preinvex with respect to η_1 for fixed y, by Remark 4.2.1,

$$(\lambda^T f)(x, y) - (\lambda^T f)(x, v) \geqq -\eta_1(v, y)^T \nabla_y (\lambda^T f)(x, y).$$

From (4.17) and $\eta_2(v, y) + y \in C_2$,

$$(\eta_2(v, y) + y)^T \nabla_y (\lambda^T f)(x, y) \leqq 0.$$

Hence

$$(\lambda^T f)(x, y) - (\lambda^T f)(x, v) \geqq y^T \nabla_y (\lambda^T f)(x, y). \tag{4.21}$$

From (4.20) and (4.21), we have

$$(\lambda^T f)(x, y) - (\lambda^T f)(u, v) + u^T \nabla_x (\lambda^T f)(u, v) - y^T \nabla_y (\lambda^T f)(x, y) \geqq 0. \tag{4.22}$$

Suppose to the contrary that

$$(f(x, y) - [y^T \nabla_y (\lambda^T f)(x, y)]e) - (f(u, v) - [u^T \nabla_x (\lambda^T f)(u, v)]e) \in -intK.$$

Then $\lambda \in K^*$ and $\lambda \neq 0$,

$$(\lambda^T f)(x, y) - y^T \nabla_y (\lambda^T f)(x, y) - (\lambda^T f)(u, v) + u^T \nabla_x (\lambda^T f)(u, v) < 0,$$

which contradicts (4.22). $\square$

Theorem 4.4.2 (Strong Duality). Let $(\bar{x}, \bar{y}, \bar{\lambda})$ be a weakly efficient solution for $(\mathbf{MSP})_W$. Assume that

(I) $\nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})$ is positive definite.

(II) $\{\nabla_y f_i(\bar{x}, \bar{y}),\ i = 1, 2, \cdots, p\}$ is linearly independent.

Then $(\bar{x}, \bar{y}, \bar{\lambda})$ is feasible for $(\mathbf{MSD})_W$, and the objective values of $(\mathbf{MSP})_W$ and $(\mathbf{MSD})_W$ are equal. Furthermore, under the assumptions of Theorem 4.4.1, $(\bar{x}, \bar{y}, \bar{\lambda})$ is a weakly efficient solution for $(\mathbf{MSD})_W$.

Proof. Since $(\bar{x}, \bar{y}, \bar{\lambda})$ is a weakly efficient solution of $(\mathbf{MSP})_W$, from Lemma 4.3.1, there exist $\alpha \in K^*$, $\beta_1 \in C_2$, $\beta_2 \in C_1^*$, $\beta_3 \in K$, $(\alpha, \beta_1, \beta_2, \beta_3) \neq 0$ such that for each $(x, y) \in C_1 \times C_2$ and $\lambda \in K^*$,

$$\alpha^T \nabla_x f(\bar{x}, \bar{y}) + (\beta_1 - (\alpha^T e)\bar{y})^T \nabla_{yx}(\bar{\lambda}^T f)(\bar{x}, \bar{y}) - \beta_2^T = 0, \tag{4.23}$$

$$(\alpha - (\alpha^T e)\bar{\lambda})^T \nabla_y f(\bar{x}, \bar{y}) + (\beta_1 - (\alpha^T e)\bar{y})^T \nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y}) = 0, \tag{4.24}$$

$$\nabla_y f(\bar{x}, \bar{y})((\alpha^T e)\bar{y} - \beta_1) + \beta_3 = 0, \tag{4.25}$$

$$\beta_1^T \nabla_y(\bar{\lambda}^T f)(\bar{x}, \bar{y}) = 0, \tag{4.26}$$

$$\beta_2^T \bar{x} = 0, \tag{4.27}$$

$$\beta_3^T \bar{\lambda} = 0. \tag{4.28}$$

Multiplying (4.24) by $(\beta_1 - (\alpha^T e)\bar{y})$,

$$(\alpha - (\alpha^T e)\bar{\lambda})^T \nabla_y f(\bar{x}, \bar{y})(\beta_1 - (\alpha^T e)\bar{y})$$
$$+(\beta_1 - (\alpha^T e)\bar{y})^T \nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})(\beta_1 - (\alpha^T e)\bar{y}) = 0.$$

Using the result in equality (4.25) and (4.28), we get

$$\alpha^T\beta_3 + (\beta_1 - (\alpha^T e)\bar{y})^T \nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})(\beta_1 - (\alpha^T e)\bar{y}) = 0.$$

Since $\alpha \in K^*$ and $\beta_3 \in K$, $\alpha^T\beta_3 \geqq 0$ and hence

$$(\beta_1 - (\alpha^T e)\bar{y})^T \nabla_{yy}(\bar{\lambda}^T f)(\beta_1 - (\alpha^T e)\bar{y}) \leqq 0.$$

Since $\nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})$ is positive definite, then

$$\beta_1 = (\alpha^T e)\bar{y}. \tag{4.29}$$

By (4.25), $\beta_3 = 0$. From (4.24), $(\alpha - (\alpha^T e)\bar{\lambda})^T \nabla_y f(\bar{x}, \bar{y}) = 0$. Since $\nabla_y f(\bar{x}, \bar{y})$ is linearly independent, $\alpha = (\alpha^T e)\bar{\lambda}$. This gives $\alpha \neq 0$, since if $\alpha = 0$, then by (4.23) and (4.29), $\beta_1 = \beta_2 = \beta_3 = 0$. This is not possible since $\alpha \neq 0$. By (4.23) and the fact that $\beta_1 = (\alpha^T e)\bar{y}$ and $\alpha = (\alpha^T e)\bar{\lambda}$,

$$(\alpha^T e)\nabla_x(\bar{\lambda}^T f)(\bar{x}, \bar{y}) = \beta_2 \in C_1^*. \tag{4.30}$$

Since $(\alpha^T e) = 1$, $\nabla_x(\bar{\lambda}^T f)(\bar{x}, \bar{y}) \in C_1^*$. From (4.27) and (4.30), $\bar{x}^T \nabla_x(\bar{\lambda}^T f)(\bar{x}, \bar{y}) = 0$. By (4.26) and the fact that $\beta_1 = (\alpha^T e)\bar{y}$,

$$(\alpha^T e)\bar{y}^T \nabla_y(\bar{\lambda}^T f)(\bar{x}, \bar{y}) = 0.$$

Since $(\alpha^T e) = 1$, $\bar{y}^T \nabla_y(\bar{\lambda}^T f)(\bar{x}, \bar{y}) = 0$. Thus $f(\bar{x}, \bar{y}) - [\bar{y}^T \nabla_y(\bar{\lambda}^T f)(\bar{x}, \bar{y})]e = f(\bar{x}, \bar{y}) - [\bar{x}^T \nabla_x(\bar{\lambda}^T f)(\bar{x}, \bar{y})]e$.

By Theorem 4.4.1, for any feasible solution (u, v, λ) of $\mathbf{(MSD)}_W$,

$$(f(\bar{x}, \bar{y}) - [\bar{y}^T \nabla_y(\bar{\lambda}^T f)(\bar{x}, \bar{y})]e) - (f(u, v) - [u^T \nabla_x(\lambda^T f)(u, v)]e) \notin -intK.$$

Since $\bar{x}^T\nabla_x(\bar{\lambda}^T f)(\bar{x},\bar{y}) = \bar{y}^T\nabla_y(\bar{\lambda}^T f)(\bar{x},\bar{y}) = 0$, we have for any feasible solution (u, v, λ) of $(\mathbf{MSD})_W$,

$$(f(\bar{x},\bar{y}) - [\bar{x}^T\nabla_x(\bar{\lambda}^T f)(\bar{x},\bar{y})]e) - (f(u,v) - [u^T\nabla_x(\lambda^T f)(u,v)]e) \notin -intK.$$

Since $(\bar{x},\bar{y},\bar{\lambda})$ is a feasible solution of $(\mathbf{MSD})_W$, $(\bar{x},\bar{y},\bar{\lambda})$ is a weakly efficient solution of $(\mathbf{MSD})_W$. Hence the result holds. □

Theorem 4.4.3 (Converse Duality). Let $(\bar{u},\bar{v},\bar{\lambda})$ be a weakly efficient solution for $(\mathbf{MSD})_W$. Assume that $\nabla_{xx}(\bar{\lambda}^T f)(\bar{u},\bar{v})$ is negative definite and the set $\{\nabla_x f_i(\bar{u},\bar{v}),\ i = 1, 2, \cdots, p\}$ is linearly independent, then $(\bar{u},\bar{v},\bar{\lambda})$ is feasible for $(\mathbf{MSP})_W$ and the objective values of $(\mathbf{MSP})_W$ and $(\mathbf{MSD})_W$ are equal. Furthermore, under the assumptions of Theorem 4.4.1, $(\bar{u},\bar{v},\bar{\lambda})$ is an weakly efficient solution for $(\mathbf{MSP})_W$.

Proof. It is analogus to the proof of the lines of Theorem 4.4.2. □

Now we establish the self-duality of $(\mathbf{MSP})_W$.

Theorem 4.4.4 (Self Duality). Assume that $m = n$, $f(x,y) = -f(y,x)$, $C_1 = C_2$ and the conditions of Theorem 4.4.1 are satisfied, and if $(\bar{x},\bar{y},\bar{\lambda})$ is a weakly efficient solution for $(\mathbf{MSP})_W$ and $\nabla_{yy}(\bar{\lambda}^T f)(\bar{x},\bar{y})$ is positive definite and the set $\{\nabla_y f_i(\bar{x},\bar{y}) : i = 1, 2, \cdots, p\}$ is linearly independent, then $(\bar{y},\bar{x},\bar{\lambda})$ is a weakly efficient solution for both $(\mathbf{MSP})_W$ and $(\mathbf{MSD})_W$, and the common optimal value is 0.

Proof. By Theorem 4.4.2, $(\bar{x},\bar{y},\bar{\lambda})$ is a weakly efficient solution of $(\mathbf{MSD})_W$ and the optimal values of $(\mathbf{MSP})_W$ and $(\mathbf{MSD})_W$ are equal to $f(\bar{x},\bar{y})$. From self-duality, $(\bar{y},\bar{x},\bar{\lambda})$ is feasible for both $(\mathbf{MSP})_W$ and $(\mathbf{MSD})_W$, and using

Theorem 4.4.1 and Theorem 4.4.2, we get that it is optimal for both problems. Since f is skew-symmetric, we have $f(\bar{x},\bar{y}) = -f(\bar{y},\bar{x})$. Hence

$$f(\bar{x},\bar{y}) = f(\bar{y},\bar{x}) = -f(\bar{x},\bar{y}),$$

and so

$$f(\bar{x},\bar{y}) = f(\bar{y},\bar{x}) = 0.$$

□

Chapter 5

Optimality and Duality for Multiobjective Programming Problems Involving Generalized Arcwise Connected Functions

5.1. Introduction

Ortega and Rheinboldt ([62]) introduced arcwise connected functions defined on arcwise connected sets by replacing a line segment joining two points by a continuous arc. Singh ([65]) investigated some elementary basic properties of arcwise connected sets and functions. Arcwise connected functions were further extended to arcwise Q-connected and arcwise P-connected functions by Avriel and Zang ([3]). Bhatia and Mehra ([10]) investigated some arcwise connected functions in terms of their directional derivatives and related them with invex functions. Recently Davar and Mehera ([19]) studied fractional optimization problems involving arcwise connected and generalized arewise connected functions. Very recently Fu and Wang ([22]) introduced the concept of arcwise connected cone-convex functions in topological vector spaces. In this chapter, we introduce multiobjective programming problem **(MOP)** and its geneal dual **(MOD)** involving generalized arcwise connected functions defined on arcwise connected sets by replacing a line segment joining two points by a continuous arc. We obtain the necessary and sufficient optimality conditions for **(MOP)** and prove the weak duality and strong duality theorems for **(MOP)** based on the weakly efficiency. Also we obtain the multiobjective fractional programming problem **(MFP)** and its general dual **(MFD)**. We introduce parametric multiobjective optimization problem **(MFP)**$_\lambda$ to obtain the optimality conditions and duality theorems by establishing equivalent relationship between **(MFP)** and **(MFP)**$_\lambda$.

Now we present some elemently basic proterties of arcwise connected sets and functions, and some extended connected functions.

Definition 5.1.1 ([3]). A set $X \subset \mathbb{R}^n$ is said to be an arcwise connected set (in short, AC set) if for every pair of points, $x_1 \in X, x_2 \in X$, there exists a continuous vector-valued function H_{x_1,x_2}, called an arc, defined on the unit interval $[0,1]$ and with values in X, such that

$$H_{x_1,x_2}(0) = x_1, \quad H_{x_1,x_2}(1) = x_2.$$

Definition 5.1.2 ([3]). A real-valued function f, defined on an AC set $X \subset \mathbb{R}^n$, is called an arcwise connected function (in short, CN function) if, for every $x_1 \in X, x_2 \in X$, there exists an arc $H_{x_1,x_2} \subset X$ satisfying

$$f(H_{x_1,x_2}(\theta)) \leqq (1-\theta)f(x_1) + \theta f(x_2), \quad \text{for} \quad 0 \leq \theta \leq 1.$$

If the above inequality is satisfied as a strict inequality for $0 \leq \theta \leq 1$ then the function f is called a strictly arcwise connected (STCN) function.

Definition 5.1.3 ([3]). A real-valued function f, defined on an AC set $X \subset \mathbb{R}^n$, is called a Q-connected function (in short, QCN function) if, for every $x_1 \in X, x_2 \in X$ such that $f(x_2) \leq f(x_1)$, there exists an arc $H_{x_1,x_2} \subset X$ satisfying

$$f(H_{x_1,x_2}(\theta)) \leq f(x_1), \quad \text{for} \quad 0 \leq \theta \leq 1.$$

Definition 5.1.4 ([3]). A real-valued function f, defined on an AC set $X \subset \mathbb{R}^n$ is said to be a P-connected function (in short, PCN function) if, for every $x_1 \in X, x_2 \in X$ such that $f(x_2) < f(x_1)$, there exists an arc $H_{x_1,x_2} \subset X$ and a positive real number β_{x_1,x_2} satisfying

$$f(H_{x_1,x_2}(\theta)) \leqq f(x_1) - \theta\beta_{x_1,x_2}, \quad \text{for} \quad 0 < \theta < 1.$$

Definition 5.1.5 ([19]). Let f be a real-valued function defined on an AC set X. For $x_0 \in X, x \in X$, the right differential of f with respect to $H_{x_0,x}(\theta)$ at $\theta = 0$ is given by

$$\lim_{\theta \to 0+} \frac{f(H_{x_0,x}(\theta)) - f(x_0)}{\theta},$$

provided the limit exists. This limit is denoted by $df^+(x_0, H_{x_0,x}(0+))$.

The following theorem of alternatives for CN functions is proved by Jeyakumar [30].

Theorem 5.1.1. Let $h : X \to \mathbb{R}^k$ be a CN function defined on an AC set $X \subset \mathbb{R}^n$. Then, exactly one of the following systems is solvable.

(i) There exists $x \in X$ such that $h(x) < 0$.

(ii) There exists $\lambda \in \mathbb{R}^k, \lambda \geq 0$ such that $\lambda^T h(x) \geq 0$, for all $x \in X$.

Theorem 5.1.2 ([10]). Let f be a real-valued function defined on an AC set $X \subset R^n$ and for all $x_1 \in X, x_2 \in X$, let $H_{x_1,x_2} \subset X$ be the arc with respect to which f possesses a right differential at $\theta = 0$.

(i) If f is CN, then $f(x_2) - f(x_1) \geq df^+(x_1, H_{x_1,x_2}(0+))$.

(ii) If f is QCN, then $f(x_2) \leqq f(x_1) \Longrightarrow df^+(x_1, H_{x_1,x_2}(0+)) \leqq 0$.

(iii) If f is PCN, then $f(x_2) < f(x_1) \Longrightarrow df^+(x_1, H_{x_1,x_2}(0+)) < 0$.

5.2. Optimality Conditions

Consider the following multiobjective optimization problem **(MOP)**:

$$\textbf{(MOP)} \quad \text{Minimize} \quad f(x) := (f_1(x), \cdots, f_p(x))$$

$$\text{subject to} \quad g(x) \leqq 0, \; x \in X, \tag{5.1}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \cdots, p$, $g_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, \cdots, m$, and $X \subset \mathbb{R}^n$ is an AC set. Let $X^0 = \{x \in X : g(x) \leqq 0\}$ be the set of feasible solutions of **(MOP)**.

Theorem 5.2.1 (Fritz John Type Necessary Optimality Condition). Let x^* be a weakly efficient solution of **(MOP)** and let f, g be CN functions with respect to the same arc. Then there exists $\lambda_i, i = 1, \cdots, p$, and $\mu_j, j = 1, \cdots, m$ such that

$$\sum_{i=1}^{p} \lambda_i df_i^+(x^*, H_{x^*,x}(0+)) + \sum_{j=1}^{m} \mu_j dg_j^+(x^*, H_{x^*,x}(0+)) \geqq 0, \quad \text{for all } x \in X,$$

$$\sum_{j=1}^{m} \mu_j g_j(x^*) = 0,$$

$$(\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m) \geq 0.$$

Proof. Let x^* be a weakly efficient solution of **(MOP)**. Then the system

$$\begin{pmatrix} f_i(x) < f_i(x^*), \; i = 1, \cdots, p \\ g_j(x) < 0, \; j = 1, \cdots, m \end{pmatrix}$$

has no solution $x \in X$. Since f and g are CN functions with respect to the same arc, hence by Theorem 5.1.1, there exists $\lambda_i,\ i = 1, \cdots, p$ and $\mu_j,\ j = 1, \cdots, m$ such that

$$\sum_{i=1}^{p} \lambda_i \big[f_i(x) - f_i(x^*)\big] + \sum_{j=1}^{m} \mu_j g_j(x) \geqq 0, \quad \text{for all } x \in X, \tag{5.2}$$

$$(\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m) \geq 0.$$

Letting $x = x^*$ in (5.2), $\sum_{j=1}^{m} \mu_j g_j(x^*) \geqq 0$. But since $\mu_j \geqq 0,\ g_j(x^*) \leqq 0$, we also have $\sum_{j=1}^{m} \mu_j g_j(x^*) \leqq 0$. Therefore $\sum_{j=1}^{m} \mu_j g_j(x^*) = 0$. Since X is an AC set, $H_{x^*,x}(\theta) \in X$, for all $x \in X$ and for all $\theta \in (0, 1]$; in particular, from (5.2), we get

$$\sum_{i=1}^{p} \lambda_i \big[f_i(H_{x^*,x}(\theta)) - f_i(x^*)\big] + \sum_{j=1}^{m} \mu_j g_j(H_{x^*,x}(\theta)) \geqq 0,$$

for all $\theta \in (0, 1]$ and for all $x \in X$, which can be rewritten as

$$\sum_{i=1}^{p} \lambda_i \big[f_i(H_{x^*,x}(\theta)) - f_i(x^*)\big] + \sum_{j=1}^{m} \mu_j \big[g_j(H_{x^*,x}(\theta)) - g_j(x^*)\big] \geqq 0, \tag{5.3}$$

for all $\theta \in (0, 1]$ and for all $x \in X$. Dividing by $\theta > 0$ and taking limits as $\theta \to 0+$ in (5.3), we get

$$\sum_{i=1}^{p} \lambda_i df_i^+(x^*, H_{x^*,x}(0+)) + \sum_{j=1}^{m} \mu_j dg_j^+(x^*, H_{x^*,x}(0+)) \geq 0, \text{ for all } x \in X. \quad \square$$

Theorem 5.2.2 (Kuhn-Tucker Type Necessary Optimality Condition). Suppose that f, g be CN functions with respect to same arc and assume that there exists $\hat{x} \in X$ such that $g_j(\hat{x}) < 0,\ j = 1, \cdots, m$. If x^* is a weakly efficient solution of **(MOP)**, then there exist $\lambda_i, i = 1, \cdots, p,\ \mu_j, j = 1, \cdots, m$, such that

$$\sum_{i=1}^{p} \lambda_i df_i^+(x^*, H_{x^*,x}(0+)) + \sum_{j=1}^{m} \mu_j dg_j^+(x^*, H_{x^*,x}(0+)) \geqq 0, \text{for all } x \in X,$$

$$\sum_{j=1}^{m} \mu_j g_j(x^*) = 0,$$

$$(\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m) \geq 0,\ (\lambda_1, \cdots, \lambda_p) \neq 0.$$

Proof. Since x^* is a weakly efficient solution of **(MOP)**, by Theorem 5.2.1, there exists $(\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m) \geq 0$. Suppose that $(\lambda_1, \cdots, \lambda_p) = 0$. Then $(\mu_1, \cdots, \mu_m) \geq 0$, by Theorem 5.2.1,

$$\sum_{j=1}^{m} \mu_j dg_j^+(x^*, H_{x^*,x}(0+)) \geqq 0, \text{ for all } x \in X. \tag{5.4}$$

Since g is a CN function,

$$\sum_{j=1}^{m} \mu_j g_j(x) - \sum_{j=1}^{m} \mu_j g_j(x^*) \geqq \sum_{j=1}^{m} \mu_j dg_j^+(x^*, H_{x^*,x}(0+)), \text{ for all } x \in X.$$

Since $\sum_{j=1}^{m} \mu_j g_j(x^*) = 0$ and by (5.4), we get $\sum_{j=1}^{m} \mu_j g_j(x) \geqq 0$ for all $x \in X$. Since there exists $\hat{x} \in X$ such that $g_j(\hat{x}) < 0, j = 1, \cdots, m$. Therefore

$\sum_{j=1}^{m} \mu_j g_j(\hat{x}) < 0$ for all $\hat{x} \in X$, which is a contradiction. Thus $(\lambda_1, \cdots, \lambda_p) \neq 0$. □

Now we shall obtain sufficient optimality conditions for a point to be a weakly efficient solution of **(MOP)**.

Theorem 5.2.3 (Kuhn-Tucker Type Sufficient Optimality Conditions). Let $\lambda_i,\ i = 1, \cdots, p,\ \mu_j, j = 1, \cdots, m$, and $x^* \in X$, satisfy the following conditions;

$$\sum_{i=1}^{p} \lambda_i df_i^+(x^*, H_{x^*,x}(0+)) + \sum_{j=1}^{m} \mu_j dg_j^+(x^*, H_{x^*,x}(0+)) \geqq 0,\ \forall\, x \in X, \quad (5.5)$$

$$\sum_{j=1}^{m} \mu_j g_j(x^*) = 0,$$

$$(\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m) \geq 0,\ (\lambda_1, \cdots, \lambda_p) \neq 0.$$

Assume that

(a) f and $\sum_{j=1}^{m} \mu_j g_j$ are CN functions; or

(b) f is PCN and $\sum_{j=1}^{m} \mu_j g_j$ is QCN; or

(c) $\sum_{i=1}^{p} \lambda_i f_i$ is PCN and $\sum_{j=1}^{m} \mu_j g_j$ is QCN.

Then x^* is a weakly efficient solution of **(MOP)**.

Proof. (a) Suppose that x^* is not a weakly efficient solution of **(MOP)**. Then there exists $\bar{x} \in X^0$ such that $f_i(\bar{x}) < f_i(x^*),\ i = 1, \cdots, p,\ g_j(\bar{x}) \leqq 0,\ j = 1, \cdots, m$. Since $\mu_j \geqq 0$, $\sum_{j=1}^{m} \mu_j g_j(\bar{x}) \leqq 0$, we have $\sum_{j=1}^{m} \mu_j g_j(\bar{x}) \leqq \sum_{j=1}^{m} \mu_j g_j(x^*)$. Thus

$$f_i(\bar{x}) - f_i(x^*) < 0,\ i = 1, \cdots, p, \quad (5.6)$$

$$\sum_{j=1}^{m} \mu_j g_j(\bar{x}) - \sum_{j=1}^{m} \mu_j g_j(x^*) \leqq 0. \tag{5.7}$$

By the fact that f and $\sum_{j=1}^{m} \mu_j g_j$ are CN functions with respect to $H_{x^*,\bar{x}} \subset X$, we have

$$f_i(\bar{x}) - f_i(x^*) \geq df_i^+(x^*, H_{x^*,\bar{x}}(0+)),\ i = 1, \cdots, p,$$

$$\sum_{j=1}^{m} \mu_j g_j(\bar{x}) - \sum_{j=1}^{m} \mu_j g_j(x^*) \geqq \sum_{j=1}^{m} \mu_j dg_j^+(x^*, H_{x^*,\bar{x}}(0+)).$$

Using (5.6) and (5.7), we get

$$df_i^+(x^*, H_{x^*,\bar{x}}(0+)) < 0,\ i = 1, \cdots, p, \quad \sum_{j=1}^{m} \mu_j dg_j^+(x^*, H_{x^*,x}(0+)) \leqq 0.$$

Since $(\lambda_1, \cdots, \lambda_p) \geq 0$ and $(\lambda_1, \cdots, \lambda_p) \neq 0$,

$$\sum_{i=1}^{p} \lambda_i df_i^+(x^*, H_{x^*,\bar{x}}(0+)) + \sum_{j=1}^{m} \mu_j dg_j^+(x^*, H_{x^*,\bar{x}}(0+)) < 0, \text{ for all } x \in X,$$

which contradicts (5.5).
Hence x^* is a weakly efficient solution of **(MOP)**. □

(b) Suppose that x^* is not a weakly efficient solution of **(MOP)**. Then there exists $\bar{x} \in X$ such that $f_i(\bar{x}) < f_i(x^*)$, $i = 1, \cdots, p$, $g_j(\bar{x}) \leqq 0$, $j = 1, \cdots, m$. Since $\mu_j \geqq 0$, $\sum_{j=1}^{m} \mu_j g_j(\bar{x}) \leqq 0$, we have $\sum_{j=1}^{m} \mu_j g_j(\bar{x}) \leqq \sum_{j=1}^{m} \mu_j g_j(x^*)$. Since f is PCN, and $\sum_{j=1}^{m} \mu_j g_j$ are QCN functions with respect to $H_{x^*,\bar{x}} \subset X$, we have

$$df_i^+(x^*, H_{x^*,\bar{x}}(0+)) < 0, \; i = 1, \cdots, p, \quad \sum_{j=1}^{m} \mu_j dg_j^+(x^*, H_{x^*,\bar{x}}(0+)) \leqq 0.$$

Since $(\lambda_1, \cdots, \lambda_p) \geq 0$ and $(\lambda_1, \cdots, \lambda_p) \neq 0$,

$$\sum_{i=1}^{p} \lambda_i df_i^+(x^*, H_{x^*,\bar{x}}(0+)) + \sum_{j=1}^{m} \mu_j dg_j^+(x^*, H_{x^*,\bar{x}}(0+)) < 0, \text{ for all } x \in X,$$

which contradicts (5.5).

Hence x^* is a weakly efficient solution of **(MOP)**. □

(c) Suppose that x^* is not a weakly efficient solution of **(MOP)**. Then there exists $\bar{x} \in X$ such that $f_i(\bar{x}) < f_i(x^*)$, $i = 1, \cdots, p$, $g_j(\bar{x}) \leqq 0$, $j = 1, \cdots, m$. Since $\lambda_i \geqq 0$, $\lambda_i \neq 0$, $\mu_j \geqq 0$, we have $\sum_{i=1}^{p} \lambda_i f_i(\bar{x}) < \sum_{i=1}^{p} \lambda_i f_i(x^*)$, $\sum_{j=1}^{m} \mu_j g_j(\bar{x}) \leqq 0 = \sum_{j=1}^{m} \mu_j g_j(x^*)$. Since $\sum_{i=1}^{p} \lambda_i f_i$ is PCN, and $\sum_{j=1}^{m} \mu_j g_j$ are QCN functions with respect to $H_{x^*,\bar{x}} \subset X$, we have

$$\sum_{i=1}^{p} \lambda_i df_i^+(x^*, H_{x^*,\bar{x}}(0+)) < 0, \quad \sum_{j=1}^{m} \mu_j dg_j^+(x^*, H_{x^*,\bar{x}}(0+)) \leqq 0.$$

Therefore

$$\sum_{i=1}^{p} \lambda_i df_i^+(x^*, H_{x^*,\bar{x}}(0+)) + \sum_{j=1}^{m} \mu_j dg_j^+(x^*, H_{x^*,\bar{x}}(0+)) < 0, \quad \text{for all } x \in X,$$

which contradicts (5.5).

Hence x^* is a weakly efficient solution of **(MOP)**. □

5.3. Duality Theorems

We propose the following general dual **(MOD)** to **(MOP)**,

(MOD) Maximize $(f_1(u) + \sum_{j\in I_0} \bar{\mu}_j g_j(u), \cdots, f_p(u) + \sum_{j\in I_0} \bar{\mu}_j g_j(u))$

subject to

$$\sum_{i=1}^{p} \bar{\lambda}_i df_i^+(u, H_{u,x}(0+)) + \sum_{j=1}^{m} \bar{\mu}_j dg_j^+(u, H_{u,x}(0+)) \geqq 0, \tag{5.8}$$

$$\sum_{j\in I_\alpha} \bar{\mu}_j g_j(u) \geqq 0, \ \alpha = 1, 2, \cdots, r,$$

$$(\bar{\lambda}_1, \cdots, \bar{\lambda}_p, \bar{\mu}_1, \cdots, \bar{\mu}_m) \geq 0, \ \bar{\lambda} = (\bar{\lambda}_1, \cdots, \bar{\lambda}_p) \in \Lambda^+,$$

where $I_\alpha \subset M = \{1, 2, \cdots, m\}$, $\alpha = 0, 1, 2, \cdots, r$ with $\cup_{\alpha=0}^{r} I_\alpha = M$ and $I_\alpha \cap I_\beta = \emptyset$ if $\alpha \neq \beta$. Let $\Lambda^+ = \{\lambda \in \mathbb{R}^p : \lambda > 0, \lambda^t e = 1, e = (1, \cdots, 1) \in \mathbb{R}^p\}$.

Theorem 5.3.1 (Weak Duality). Assume that for all feasible x of **(MOP)** and all feasible $(u, \bar{\lambda}, \bar{\mu})$ of **(MOD)**, if $\sum_{j\in I_\alpha} \bar{\mu}_j g_j(\cdot)(\alpha = 1, 2, \cdots, r)$ is QCN function at u and assume that one of the following conditions holds:

(a) $f_i(\cdot) + \sum_{j\in I_0} \bar{\mu}_j g_j(\cdot)$ is PCN functions at u with respect to $H_{u,x}$;

(b) $\sum_{i=1}^{p} \bar{\lambda}_i f_i(\cdot) + \sum_{j\in I_0} \bar{\mu}_j g_j(\cdot)$ is PCN function at u with respect to $H_{u,x}$,

then the following cannot hold,

$$f(x) < f(u) + \sum_{j\in I_0} \bar{\mu}_j g_j(u)e.$$

Proof. As x is feasible for **(MOP)** and $(u, \bar{\lambda}, \bar{\mu})$ is feasible for **(MOD)**, we have

$$\sum_{j\in I_\alpha} \bar{\mu}_j g_j(x) \leqq 0 \leqq \sum_{j\in I_\alpha} \bar{\mu}_j g_j(u), \alpha = 1, 2, \cdots, r.$$

Since $\sum_{j\in I_\alpha} \bar{\mu}_j g_j(u)$ is QCN, $\alpha = 1, 2, \cdots, r$,

$$\sum_{j\in I_\alpha} \bar{\mu}_j dg_j^+(u, H_{u,x}(0+)) \leqq 0, \alpha = 1, 2, \cdots, r.$$

From (5.8), we have

$$\sum_{i=1}^{p} \bar{\lambda}_i df_i^+(u, H_{u,x}(0+)) + \sum_{j\in I_0} \bar{\mu}_j dg_j^+(u, H_{u,x}(0+)) \geqq 0. \tag{5.9}$$

Now suppose, contrary to the result, $f(x) < f(u) + \sum_{j\in I_0} \bar{\mu}_j g_j(u)e$. Since x is feasible for **(MOP)** and $\bar{\mu} \geqq 0$,

$$f(x) + \sum_{j\in I_0} \bar{\mu}_j g_j(x)e < f(u) + \sum_{j\in I_0} \bar{\mu}_j g_j(u)e. \tag{5.10}$$

(a) Since $f + \sum_{j\in I_0} \bar{\mu}_j g_j e$ is PCN functions at u,

$$df_i^+(u, H_{u,x}(0+)) + \sum_{j\in I_0} \bar{\mu}_j dg_j^+(u, H_{u,x}(0+)) < 0.$$

From $\bar{\lambda} \in \Lambda^+$, we have

$$\sum_{i=1}^{p} \bar{\lambda}_i df_i^+(u, H_{u,x}(0+)) + \sum_{j\in I_0} \bar{\mu}_j dg_j^+(u, H_{u,x}(0+)) < 0,$$

which contradicts (5.9).

(b) Suppose now that (b) is satisfied. From $\bar{\lambda} \in \Lambda^+$ and (5.10), it follows,

$$\sum_{i=1}^{p} \bar{\lambda}_i f_i(x) + \sum_{j \in I_0} \bar{\mu}_j g_j(x) < \sum_{i=1}^{p} \bar{\lambda}_i f_i(u) + \sum_{j \in I_0} \bar{\mu}_j g_j(u).$$

Then, by the PCN property of $\sum_{i=1}^{p} \bar{\lambda}_i f_i(\cdot) + \sum_{j \in I_0} \bar{\mu}_j g_j(\cdot)$ at u,

$$\sum_{i=1}^{p} \bar{\lambda}_i df_i^+(u, H_{u,x}(0+)) + \sum_{j \in I_0} \bar{\mu}_j dg_j^+(u, H_{u,x}(0+)) < 0,$$

which contradics (5.9). □

Theorem 5.3.2 (Strong Duality). Let x^* be a weakly efficient solution of **(MOP)**, and let f, g posess right differentials with respect to $H_{x^*,x}$ at $\theta = 0$, for all $x \in X$. Further, assume that f and g are CN functions and there exists $\hat{x} \in X$ such that $g_j(\hat{x}) < 0$, $j = 1, \cdots, m$. Then there exists (x^*, λ^*, μ^*), which is feasible for **(MOD)**. Further, if any one of the conditions of weak duality holds then (x^*, λ^*, μ^*) is a weakly efficient solution of **(MOD)**.

Proof. By Theorem 5.2.2, there exist $\lambda^* \in \mathbb{R}^p$, $\mu^* \in \mathbb{R}^m$ such that (x^*, λ^*, μ^*) is a feasible solution of **(MOD)**. By weak duality, $f(x^*) \not< f(u) + \sum_{j \in I_0} \mu_j g_j(u)e$, where (u, λ, μ) is any feasible solution of **(MOD)**. Since $\mu_j^* g_j(x^*) = 0$, we have $f(x^*) + \sum_{j \in I_0} \mu_j^* g_j(x^*)e \not< f(u) + \sum_{j \in I_0} \mu_j g_j(u)e$. Since (u^*, λ^*, μ^*) is a feasible solution of **(MOD)**, (u^*, λ^*, μ^*) is a weakly efficient solution of **(MOD)**. Hence the result holds. □

5.4. Special Cases

As special cases of our duality results between **(MOP)** and **(MOD)**, we give Mond-Weir type and Wolfe type duality theorems.

If $I_0 = \emptyset,\ I_\alpha = M$, then **(MOD)** reduced to the Mond-Weir type dual $\textbf{(MOD)}_M$.

$$\textbf{(MOD)}_M \quad \text{Maximize} \quad (f_1(u), \cdots, f_p(u))$$

subject to

$$\sum_{i=1}^{p} \bar{\lambda}_i df_i^+(u, H_{u,x}(0+)) + \sum_{j=1}^{m} \bar{\mu}_j dg_j^+(u, H_{u,x}(0+)) \geqq 0, \text{for all } x \in X,$$

$$\sum_{j=1}^{m} \bar{\mu}_j g_j(u) \geqq 0,$$

$$(\bar{\lambda}_1, \cdots, \bar{\lambda}_p, \bar{\mu}_1, \cdots, \bar{\mu}_m) \geq 0,\ (\bar{\lambda}_1, \cdots, \bar{\lambda}_p) \in \Lambda^+,$$

where $\Lambda^+ = \{\lambda \in \mathbb{R}^p : \lambda > 0, \lambda^t e = 1, e = (1, \cdots, 1) \in \mathbb{R}^p\}$.

Theorem 5.4.1 (Weak Duality). Let x be feasible for **(MOP)** and $(u, \bar{\lambda}, \bar{\mu})$ be feasible for $\textbf{(MOD)}_M$. And let $\sum_{j=1}^{m} \bar{\mu}_j g_j(\cdot)$ be QCN function at u. Assume that one of the following conditions holds :

(a) f is PCN functions at u with respect to $H_{u,x}$;

(b) $\bar{\lambda}^t f$ is PCN function at u with respect to $H_{u,x}$.

Then the following cannot hold.

$$f(x) < f(u).$$

Theorem 5.4.2 (Strong Duality). Let x^* be a weakly efficient solution of **(MOP)**, and let f, g possess right differentials with respect to $H_{x^*,x}$ at $\theta = 0$, for all $x \in X$. Further, assume that f and g are CN functions and there exists $\hat{x} \in X$ such that $g_j(\hat{x}) < 0$, $j = 1, \cdots, m$. Then there exists (x^*, λ^*, μ^*) is feasible for $\mathbf{(MOD)}_M$. Further, if any one of the conditions of weak duality holds then (x^*, λ^*, μ^*) is a weakly efficient solution of $\mathbf{(MOD)}_M$.

If $I_0 = M$, $I_\alpha = \emptyset$, then **(MOD)** reduced to the Wolfe type dual $\mathbf{(MOD)}_W$.

$\mathbf{(MOD)}_W$ Maximize $\quad (f_1(u) + \sum_{j=1}^{m} \bar{\mu}_j g_j(u), \cdots, f_p(u) + \sum_{j=1}^{m} \bar{\mu}_j g_j(u))$

subject to

$$\sum_{i=1}^{p} \bar{\lambda}_i df_i^+(u, H_{u,x}(0+)) + \sum_{j=1}^{m} \bar{\mu}_j dg_j^+(u, H_{u,x}(0+)) \geq 0,$$

$$(\bar{\lambda}_1, \cdots, \bar{\lambda}_p, \bar{\mu}_1, \cdots, \bar{\mu}_m) \geq 0, \bar{\lambda} = (\bar{\lambda}_1, \cdots, \bar{\lambda}_p) \in \Lambda^+,$$

where $\Lambda^+ = \{\lambda \in \mathbb{R}^p : \lambda > 0, \lambda^t e = 1, e = (1, \cdots, 1) \in \mathbb{R}^p\}$.

Theorem 5.4.3 (Weak Duality). Let x be feasible for **(MOP)** and $(u, \bar{\lambda}, \bar{\mu})$ be feasible for $\mathbf{(MOD)}_W$. Assume that one of the following conditions holds :

(a) $f_i(\cdot) + \sum_{j=1}^{m} \bar{\mu}_j g_j(\cdot)$ is PCN functions at u with respect to $H_{u,x}$; (b) $\sum_{i=1}^{p} \bar{\lambda}_i f_i(\cdot) + \sum_{j=1}^{m} \bar{\mu}_j g_j(\cdot)$ is PCN function at u , with respect to $H_{u,x}$, then the following cannot hold.

$$f(x) < f(u) + \sum_{j=1}^{m} \bar{\mu}_j g_j(u)e.$$

Theorem 5.4.4 (Strong Duality). Let x^* be a weakly efficient solution of **(MOP)**, and let f, g possess right differentials with respect to $H_{x^*,x}$ at $\theta = 0$, for all $x \in X$. Further, assume that f and g are CN functions and there exists $\hat{x} \in X$ such that $g_j(\hat{x}) < 0$, $j = 1, \cdots, m$. Then there exists (x^*, λ^*, μ^*) which is feasible for $\mathbf{(MOD)}_W$. Further, if any one of the conditions of weak duality holds then (x^*, λ^*, μ^*) is a weakly efficient solution of $\mathbf{(MOD)}_W$.

5.5. Multiobjective Fractional Programming Problems

We shall obtain necessary and sufficient optimality conditions for a point to be a weakly efficient solution of a multiobjective fractional programming problem. We consider the problem,

$$\textbf{(MFP)} \quad \text{Minimize} \quad \left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \cdots, \frac{f_p(x)}{g_p(x)}\right)$$

$$\text{subject to} \quad h(x) \leqq 0, \ x \in X,$$

where $f_i, g_i : X \to \mathbb{R}$, $i = 1, \cdots, p$, $h : X \to \mathbb{R}^m$, and $X \subset \mathbb{R}^n$ is an AC set. Further, $f_i(x) \geqq 0$, $g_i(x) > 0$, $i = 1, \cdots, p$ for all $x \in X^0$, where $X^0 = \{x \in X : h_j(x) \leqq 0, \ j = 1, \cdots, m\}$.

We associate the following parametric multiobjective optimization problem $\mathbf{(MFP)}_\lambda$, for $\lambda \in \mathbb{R}^p_+$, with **(MFP)**. ([28])

$(\mathbf{MFP})_\lambda$ Minimize $(f_1(x) - \lambda_1 g_1(x), f_2(x) - \lambda_2 g_2(x), \cdots, f_p(x) - \lambda_p g_p(x))$

subject to $x \in X^0$.

We establish an equivalent relationship between **(MFP)** and $(\mathbf{MFP})_\lambda$.

Lemma 5.5.1. Let x^* be a weakly efficient solution of **(MFP)**. Then there exist $\lambda^* \in \mathbb{R}^p_+$ such that x^* is a weakly efficient solution of $(\mathbf{MFP})_{\lambda^*}$. Conversely, if x^* is a weakly efficient solution of $(\mathbf{MFP})_{\lambda^*}$, where $\lambda^* = \frac{f(x^*)}{g(x^*)}$, then x^* is a weakly efficient solution of **(MFP)**.

We now establish the following necessary optimality theorem.

Theorem 5.5.1 (Fritz John Type Necessary Optimality Condition). Let x^* be a weakly efficient solution of **(MFP)** and let $f, -g$, and h be CN functions with respect to the same arc. Then there exists $\alpha_i^* \geqq 0$, $i = 1, \cdots, p$, $\beta_j^* \geqq 0$, $j = 1, \cdots, m$, such that the following conditions hold:

$$\sum_{i=1}^{p} \alpha_i^* \left[df_i^+(x^*, H_{x^*,x}(0+)) - \lambda_i^* dg_i^+(x^*, H_{x^*,x}(0+)) \right]$$

$$+ \sum_{j=1}^{m} \beta_j^* dh_j^*(x^*, H_{x^*,x}(0+)) \geqq 0, \text{ for all } x \in X,$$

$$\sum_{j=1}^{m} \beta_j^* h_j(x^*) = 0,$$

$$(\alpha^*, \beta^*) \neq 0,$$

where $\lambda^* = \frac{f(x^*)}{g(x^*)}$.

Proof. Since x^* is a weakly efficient solution of **(MFP)**, it follows from Lemma 5.5.1 that x^* is a weakly efficient solution of $(\mathbf{MFP})_{\lambda^*}$, where $\lambda^* = \frac{f(x^*)}{g(x^*)}$. Then the system

$$\left(\begin{array}{c} f_i(x) - \lambda_i^* g_i(x) < f_i(x^*) - \lambda_i^* g_i(x^*),\ i = 1, \cdots, p \\ h_j(x) < 0,\ j = 1, \cdots, m \end{array} \right)$$

has no solution $x \in X$. Since $f, -g$ and h are CN functions with respect to the same arc, hence, by Theorem 5.1.1, there exist $\alpha_i^* \geqq 0,\ \beta_j^* \geqq 0$, such that

$$\sum_{i=1}^{p} \alpha_i^* \left[f_i(x) - \lambda_i^* g_i(x) - f_i(x^*) + \lambda_i^* g_i(x^*) \right] + \sum_{j=1}^{m} \beta_j^* h_j(x) \geqq 0, \qquad (5.11)$$

$(\alpha^*, \beta^*) \neq 0.$

Taking $x = x^*$ in (5.11), we get

$$\sum_{j=1}^{m} \beta_j^* h_j(x^*) \geqq 0.$$

But since $\beta_j^* \geqq 0,\ h_j(x^*) \leqq 0$, we also have $\sum_{j=1}^{m} \beta_j^* h_j(x^*) \leqq 0$. Therefore $\sum_{j=1}^{m} \beta_j^* h_j(x^*) = 0$. Moreover, X is an AC set, $H_{x^*,x}(\theta) \in X$, for all $x \in X$, and for all $\theta \in (0, 1]$; in particular, from (5.11), we get

$$\sum_{i=1}^{p} \alpha_i^* \left[f_i(H_{x^*,x}(\theta)) - f_i(x^*) - \lambda_i^* (g_i(H_{x^*,x}(\theta)) - g_i(x^*)) \right]$$

$$+ \sum_{j=1}^{m} \beta_j^* (h_j(H_{x^*,x}(\theta)) - h_j(x^*)) \geqq 0. \qquad (5.12)$$

Dividing by $\theta > 0$ and taking limits as $\theta \to 0+$ in (5.12), we get

$$\sum_{i=1}^{p} \alpha_i^* \left[df^+(x^*, H_{x^*,x}(0+)) - \lambda_i^* dg_i^+(x^*, H_{x^*,x}(0+)) \right]$$

$$+ \textstyle\sum_{j=1}^{m} \beta_j^* dh_j^+(x^*, H_{x^*,x}(0+)) \geqq 0, \text{ for all } x \in X. \qquad \square$$

Theorem 5.5.2 (Kuhn-Tucker Type Necessary Optimality Condition). Let x^* be a weakly efficient solution of **(MFP)**, and let $f, -g$, and h be CN functions with respect to the same arc. Further, assume that there exists $\hat{x} \in X$ such that $h(\hat{x}) < 0$. Then there exist $\alpha^* \in \mathbb{R}^p$ and $\beta^* \in \mathbb{R}^m$ such that the following conditions hold:

$$\sum_{i=1}^{p} \alpha_i^* \left[df_i^+(x^*, H_{x^*,x}(0+)) - \lambda_i^* dg_i^+(x^*, H_{x^*,x}(0+)) \right]$$

$$+ \sum_{j=1}^{m} \beta_j^* dh_j^+(x^*, H_{x^*,x}(0+)) \geqq 0, \text{ for all } x \in X,$$

$$\sum_{j=1}^{m} \beta_j^* h_j(x^*) = 0,$$

$$\alpha^* > 0, \beta^* \geqq 0,$$

where $\lambda^* = \frac{f(x^*)}{g(x^*)}$.

Proof. Since x^* is a weakly efficient solution of **(MFP)**, by Theorem 5.5.1, there exist $(\alpha_1^*, \cdots, \alpha_p^*, \beta_1^*, \cdots, \beta_m^*) \geq 0$. Suppose that $(\alpha_1^*, \cdots, \alpha_p^*) = 0$. Then $(\beta_1^*, \cdots, \beta_m^*) \geq 0$, by Theorem 5.5.1,

$$\sum_{j=1}^{m} \beta_j^* dh_j^+(x^*, H_{x^*,x}(0+)) \geqq 0, \text{ for all } x \in X.$$

Since h is a CN function, we get

$$\sum_{j=1}^{m} \beta_j^* h_j(x) - \sum_{j=1}^{m} \beta_j^* h_j(x^*) \geqq \sum_{j=1}^{m} \beta_j^* dh_j^+(x^*, H_{x^*,x}(0+)), \text{ for all } x \in X.$$

Since $\sum_{j=1}^{m} \beta_j^* h_j(x^*) = 0$, we get $\sum_{j=1}^{m} \beta_j^* h_j(x) \geqq 0$, for all $x \in X$. Since there exists $\hat{x} \in X$ such that $h_j(\hat{x}) < 0, j = 1, \cdots, m$. Therefore $\sum_{j=1}^{m} \beta_j^* h_j(\hat{x}) < 0$ for all $\hat{x} \in X$, which is a contradiction. Thus $(\alpha_1^*, \cdots, \alpha_p^*) \neq 0$. □

Now we shall obtain sufficient optimality conditions for a point to be a weakly efficient solution of **(MFP)**.

Theorem 5.5.3 (Kuhn-Tucker Type Sufficient Optimality Condition). Let $x^* \in X^0$, and assume that there exist $\alpha_i^* \geqq 0$, $i = 1, \cdots, p$ $\beta_j^* \geqq 0$, $j = 1, \cdots, m$, such that

$$\sum_{i=1}^{p} \alpha_i^* \left[df_i^+(x^*, H_{x^*,x}(0+)) - \lambda_i^* dg_i^+(x^*, H_{x^*,x}(0+))\right]$$

$$+ \sum_{j \in I(x^*)} \beta_j^* dh_j^+(x^*, H_{x^*,x}(0+)) \geqq 0, \text{ for all } x \in X, \quad (5.13)$$

$$\sum_{j=1}^{m} \beta_j^* h_j(x^*) = 0,$$

$$(\alpha_1^*, \cdots, \alpha_p^*) \neq 0$$

where $\lambda^* = \frac{f(x^*)}{g(x^*)}$, $I(x^*) = \{i \mid h_i(x^*) = 0\}$

Assume that

(a) $f, -g$, and h_I are CN functions ; or

(b) $\sum_{i=1}^{p} \alpha_i^*(f_i - \lambda^* g_i)$ is PCN and $\sum_{j \in I(x^*)} \beta_j h_j$ is QCN.

Then x^* is a weakly efficient solution of **(MFP)**.

Proof. (a) Suppose that x^* is not a weakly efficient solution of **(MFP)**. Then there exists $x \in X^0$ such that $\frac{f_i(x)}{g_i(x)} < \frac{f_i(x^*)}{g_i(x^*)}$, $i = 1, 2 \cdots, p$. Since $g_i(x) > 0$ for $i = 1, \cdots, p$, we have $f_i(x) - \lambda_i^* g_i(x) < f_i(x^*) - \lambda_i^* g_i(x^*)$, $i = 1, 2, \cdots, p$. Since $f, -g$, and h_I are CN functions, we have

$$f_i(x) - f_i(x^*) \geqq df_i^+(x^*, H_{x^*,x}(0+)), \tag{5.14}$$

$$(-g_i)(x) - (-g_i)(x^*) \geqq d(-g_i)^+(x^*, H_{x^*,x}(0+)), \tag{5.15}$$

$$h_I(x) - h_I(x^*) \geqq dh_I^+(x^*, H_{x^*,x}(0+)). \tag{5.16}$$

Multiplying (5.15) by $\lambda_i^* \geqq 0$ and adding it to (5.14), we get

$$\begin{aligned}(f_i(x) - \lambda_i^* g_i(x)) &- (f_i(x^*) - \lambda_i^* g_i(x^*)) \\ &\geqq df_i^+(x^*, H_{x^*,x}(0+)) - \lambda_i^* dg_i^+(x^*, H_{x^*,x}(0+)).\end{aligned}$$

Thus, we have

$$df_i^+(x^*, H_{x^*,x}(0+)) - \lambda_i^* dg_i^+(x^*, H_{x^*,x}(0+)) < 0, \; i = 1, 2 \cdots, p. \tag{5.17}$$

Since x is feasible for **(MFP)**, it follows from (5.16) that

$$dh_I^+(x^*, H_{x^*,x}(0+)) \leqq 0. \tag{5.18}$$

Now, as $\alpha^* > 0$ and $\beta^* \geqq 0$, multiplying (5.17) by α_i^* and (5.18) by $\beta_j^*, j \in I(x^*)$ and adding, we get

$$\sum_{i=1}^{p} \alpha_i^* \left[df_i^+(x^*, H_{x^*,x}(0+)) - \lambda_i^* dg_i^+(x^*, H_{x^*,x}(0+)) \right]$$

$$+ \sum_{j \in I(x^*)} \beta_j^* dh_j^+(x^*, H_{x^*,x}(0+)) < 0,$$

which contradicts (5.13). □

(b) Let x^* be not a weakly efficient solution of **(MFP)**. Then there exists $x \in X^0$ such that

$$\frac{f_i(x)}{g_i(x)} < \frac{f_i(x^*)}{g_i(x^*)}, \ i = 1, 2 \cdots, p.$$

Since $g_i(x) > 0$ for $i = 1, \cdots, p$, we have $f_i(x) - \lambda_i^* g_i(x) < f_i(x^*) - \lambda_i^* g_i(x^*)$, $i = 1, 2, \cdots, p$. Moreover, $\alpha > 0$; hence $\sum_{i=1}^{p} \alpha_i^*(f_i(x) - \lambda_i^* g_i(x)) < \sum_{i=1}^{p} \alpha_i^*(f_i(x^*) - \lambda_i^* g_i(x^*))$. Since $\sum_{i=1}^{p} \alpha_i^*(f_i - \lambda_i^* g_i)$ is PCN, we get $d(\sum_{i=1}^{p} \alpha_i^* (f_i - \lambda_i^* g_i))^+(x^*, H_{x^*,x}(0+)) < 0$, which gives

$$\sum_{i=1}^{p} \alpha_i^* \left[df_i^+(x^*, H_{x^*,x}(0+)) - \lambda_i^* dg_i^+(x^*, H_{x^*,x}(0+)) \right] < 0. \tag{5.19}$$

Also, as $x \in X^0$, we have $\sum_{j \in I(x^*)} \beta_j^* h_j(x) \leqq 0 = \sum_{j \in I(x^*)} \beta_j^* h_j(x^*)$. Since $\sum_{j \in I(x^*)} \beta_j^* h_j$ is QCN, we get $d(\sum_{j \in I(x^*)} \beta_j^* h_j)^+(x^*, H_{x^*,x}(0+)) \leqq 0$, which gives that

$$\sum_{j \in I(x^*)} \beta_j^* dh_j^+(x^*, H_{x^*,x}(0+)) \leqq 0. \tag{5.20}$$

Adding (5.19) and (5.20), we get

$$\sum_{i=1}^{p} \alpha_i^* \big[df_i^+(x^*, H_{x^*,x}(0+)) - \lambda_i^* dg_i^+(x^*, H_{x^*,x}(0+))\big]$$

$$+ \sum_{j\in I(x^*)} \beta_j^* dh_j^+(x^*, H_{x^*,x}(0+)) < 0,$$

which contradicts (5.13). □

Now we formulate a Mond-Weir type dual problem for **(MFP)**, and show that duality theorems hold.

(MFD) Maximize $(\lambda_1, \lambda_2, \cdots, \lambda_p)$

subject to

$$\sum_{i=1}^{p} \alpha_i \big[df_i^+(u, H_{u,x}(0+)) - \lambda_i dg_i^+(u, H_{u,x}(0+))\big]$$

$$+ \sum_{j=1}^{m} \beta_j dh_j^+(u, H_{u,x}(0+)) \geqq 0, \text{ for all } x \in X \quad (5.21)$$

$$f_i(u) - \lambda_i g_i(u) \geqq 0, \ i = 1, 2, \cdots, p, \quad (5.22)$$

$$\sum_{j=1}^{m} \beta_j h_j(u) \geqq 0, \quad (5.23)$$

$$\alpha \in \mathbb{R}^p, \beta \in \mathbb{R}^m, \lambda \in \mathbb{R}^p, \alpha_i \geqq 0, \alpha \neq 0, \beta \geqq 0, \lambda \geqq 0.$$

We establish weak and strong duality theorems between **(MFP)** and **(MFD)**.

Theorem 5.5.4 (Weak Duality). Let x be feasible for **(MFP)** and let $(u, \alpha, \beta, \lambda)$ be feasible for **(MFD)**. Assume that any one of following conditions holds:

(i) $f - \lambda g$ and h are CN functions with respect to $H_{u,x}$; or

(ii) $\sum_{i=1}^{p} \alpha_i(f_i - \lambda_i g_i)$ is PCN and $\sum_{j=1}^{m} \beta_j h_j$ is QCN with respect to $H_{u,x}$.

Then the following cannot hold:

$$\frac{f_i(x)}{g_i(x)} < \lambda_i, \text{ for all } i = 1, \cdots, p.$$

Proof. (i) Suppose contrary to the result of the theorem that for some feasible x for **(MFP)** and $(u, \alpha, \beta, \lambda)$ for **(MFD)**, $\frac{f_i(x)}{g_i(x)} < \lambda_i$ for all $i = 1, \cdots, p$. Since $g_i(x) > 0,\ i = 1, \cdots, p$, we have

$$f_i(x) - \lambda_i g_i(x) < 0,\ i = 1, 2, \cdots, p. \tag{5.24}$$

Since x is feasible for **(MFP)** and $\beta_j \geqq 0, j = 1, 2, \cdots, m$, we have

$$\beta_j h_j(x) \leqq 0,\ j = 1, 2, \cdots, m. \tag{5.25}$$

Further, as h is CN, we have, using Theorem 5.1.2,

$$h_j(x) - h_j(u) \geqq dh_j^+(u, H_{u,x}(0+)).$$

Multiplying by $\beta_j \geqq 0$ and using (5.23) and (5.25), we get

$$\beta_j dh_j^+(u, H_{u,x}(0+)) \leqq 0,\ j = 1, 2, \cdots, m. \tag{5.26}$$

Moreover, $f_i - \lambda_i g_i$ is CN functions; hence we have

$$(f_i(x) - \lambda_i g_i(x)) - (f_i(u) - \lambda_i g_i(u)) \geqq df_i^+(u, H_{u,x}(0+)) - \lambda_i dg_i^+(u, H_{u,x}(0+)).$$

Multiplying by $\alpha_i > 0$ and adding, we get

$$\sum_{i=1}^{p} \alpha_i(f_i(x) - \lambda_i g_i(x)) - \sum_{i=1}^{p} \alpha_i(f_i(u) - \lambda_i g_i(u))$$

$$\geqq \sum_{i=1}^{p} \alpha_i \big[df_i^+(u, H_{u,x}(0+)) - \lambda_i dg_i^+(u, H_{u,x}(0+))\big].$$

This along with (5.22) and (5.24) gives

$$\sum_{i=1}^{p} \alpha_i \big[df_i^+(u, H_{u,x}(0+)) - \lambda_i dg_i^+(u, H_{u,x}(0+))\big] < 0. \qquad (5.27)$$

Adding (5.26) and (5.27), we get

$$\sum_{i=1}^{p} \alpha_i \big[df_i^+(u, H_{u,x}(0+)) - \lambda_i dg_i^+(u, H_{u,x}(0+))\big] + \sum_{j=1}^{m} \beta_j dh_j^+(u, H_{u,x}(0+)) < 0,$$

which contradicts the feasibility of $(u, \alpha, \beta, \lambda)$ for **(MFD)**. □

(ii) Suppose contrary to the result of the theorem that for some feasible x for **(MFP)** and $(u, \alpha, \beta, \lambda)$ for **(MFD)**, $\frac{f_i(x)}{g_i(x)} < \lambda_i$ for all $i = 1, \cdots, p$. Since $g_i(x) > 0,\ i = 1, \cdots, p$, and from (5.22) we have

$$f_i(x) - \lambda_i g_i(x) < 0 \leqq f_i(u) - \lambda_i g_i(u),\ i = 1, 2, \cdots, p.$$

As $\alpha > 0$, we obtain

$$\sum_{i=1}^{p} \alpha_i(f_i(x) - \lambda_i g_i(x)) < \sum_{i=1}^{p} \alpha_i(f_i(u) - \lambda_i g_i(u)).$$

Since $\sum_{i=1}^{p} \alpha_i(f_i - \lambda_i g_i)$ is PCN, we have

$$d(\sum_{i=1}^{p} \alpha_i(f_i - \lambda_i g_i))^+(u, H_{u,x}(0+)) < 0;$$

that is,

$$\sum_{i=1}^{p} \alpha_i \left[df_i^+(u, H_{u,x}(0+)) - \lambda_i dg_i^+(u, H_{u,x}(0+))\right] < 0. \tag{5.28}$$

Also, from primal feasibility of x and the fact that $\beta_j \geqq 0,\ j = 1, 2, \cdots, m$, it follows that

$$\sum_{j=1}^{m} \beta_j h_j(x) \leqq 0 \leqq \sum_{j=1}^{m} \beta_j h_j(u).$$

Since $\sum_{j=1}^{m} \beta_j h_j$ is QCN, we get

$$d(\sum_{j=1}^{m} \beta_j h_j)^+(u, H_{u,x}(0+)) \leqq 0;$$

that is,

$$\sum_{j=1}^{m} \beta_j dh_j^+(u, H_{u,x}(0+)) \leqq 0. \tag{5.29}$$

Adding (5.28) and (5.29), we get

$$\sum_{i=1}^{p} \alpha_i \left[df_i^+(u, H_{u,x}(0+)) - \lambda_i dg_i^+(u, H_{u,x}(0+))\right] + \sum_{j=1}^{m} \beta_j dh_j^+(u, H_{u,x}(0+)) < 0,$$

which contradicts the feasibility of $(u, \alpha, \beta, \lambda)$ for **(MFD)**. □

Theorem 5.5.5 (Strong Duality). Let x^* be a weakly efficient solution of **(MFP)**, and let f, g possess right differentials with respect to $H_{x^*,x}$ at $\theta = 0$, for all $x \in X$. Further, assume that $f, -g$ and h are CN functions and there exist $\hat{x} \in X$ such that $h(\hat{x}) < 0$. Then there exists $\alpha^* = (\alpha_1^*, \cdots, \alpha_p^*)$, $\alpha^* > 0$, $\beta^* = (\beta_1^*, \cdots, \beta_m^*)$ such that $(x^*, \alpha^*, \beta^*, \lambda^*)$ is a feasible for **(MFD)**. Further, if any one of the conditions of weak duality holds, then $(x^*, \alpha^*, \beta^*, \lambda^*)$ is a weakly efficient solution of **(MFD)**.

Proof. By Theorem 5.4.2, there exists $\alpha^* = (\alpha_1^*, \cdots, \alpha_p^*)$, $\alpha^* > 0$, $\beta^* = (\beta_1^*, \cdots, \beta_m^*)$ such that $(x^*, \alpha^*, \beta^*, \lambda^*)$ is a feasible for **(MFD)**, where $\lambda^* = \frac{f(x^*)}{g(x^*)}$. If $(x^*, \alpha^*, \beta^*, \lambda^*)$ is not an optimal solution of **(MFD)**, then there exists a feasible solution $(x, \alpha, \beta, \lambda)$ of **(MFD)** such that

$$\lambda^* < \lambda. \tag{5.30}$$

Since $\lambda^* = \frac{f(x^*)}{g(x^*)}$, from (5.30), it follows that

$$\frac{f(x^*)}{g(x^*)} < \lambda,$$

which contradicts weak duality theorem. Hence the result follows. □

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