Optimality and Duality for Multiobjective Variational and Control Problems with Generalized Invexity

일반화된 인벡시티를 갖는 다목적 변분문제와 제어문제에 관한 최적성과 쌍대성

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Optimality and Duality for Multiobjective Variational and Control Problems with Generalized Invexity

A Dissertation

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CONTENTS

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요 약

본 논문에서는 일반화된 인벡스함수를 정의하고 이들 함수를 이용하여 다목적 최적화 문제인 변분문제와 제어문제의 최적성과 쌍대성을 정립하였다. V - type I 인벡스 함수 조건아래에서 다목적 변분문제의 효율해 와 진성 효율해에 대한 충분 최적 정리들을 얻었고 다목적 변분문제에 대한 여러 쌍대문제들을 만들고 그 문제들 사이의 쌍대정리가 성립함을 보였다. 또한, 매개변수를 사용하여 V - type I 인벡스 함수 조건하에서 다목적 분수 변분 문제의 효율해에 대한 Kuhn-Tucker 충분 최적 정리를 얻었고 마찬가지로 쌍대문제를 만들어 두 문제사이의 쌍대 정리가 성립함을 보였다. Kuhn-Tucker 필요 최적조건을 이용하여 $V-\rho$인벡스 함수조건아래에서 다목적 제어문제에관한 Kuhn-Tucker 충분 최적 정리를 얻었고 효율해에 대한 쌍대문제를 만들어 두 문제사이의 쌍대 정리가 성립함을 보였다. 마지막으로, 다목적 분수제어 문제에 대하여 이것과 동치인 다목적 비분수 제어문제를 만들어 그에 관한 Kuhn-Tucker 충분 최적 정리를 얻었고 다목적 제어문제와 분수 제어문제의 효율해에 대한 쌍대문제를 만들어 두 문제사이의 쌍대 정리가 성립함을 보였다.

Chapter 1

Introduction and Preliminaries

Several authors have been interested in duality theorems for multiobjective variational problems. Bector and Husain [2] proved duality results for a multiobjective variational problem with convexity functions. Various generalizations of convexity have been made in the literature. Bector and Husain [2] proved duality results for a multiobjective variational problem with convexity functions. In [25], Mishra and Mukherjee discussed duality for multiobjective variational problems involving generalized (F, ρ)-convex functions. Also, Nahak and Nanda [33] proved Wolfe type and Mond-Weir type duality results for a multiobjective variational problem with pseudo-invexity functions.

Parallel to the above development in multiobjective programming there has been a very popular growth and application of invexity theory which was originated by Hanson [15] but so named by Craven [9]. Later Hanson and Mond [14] introduced type-I and type-II invexities which have been further generalized by many researchers and applied to nonlinear programming problems in different settings.

In Kaul, Suneja and Srivastava [18] considered a multiobjective nonlinear programming problem involving type I functions to obtain some duality results, where Wolfe and Mond-Weir duals are considered. Recently, Hanson, Pini and Singh [16] extended a (scalarized) generalized type I invexity into a vector invexity (V-type I) and they provided some duality results.

Introducing the concept of proper efficiency of solutions of multiobjective programs, Geoffrion [12] proved an equivalence between a multiobjective program with convex functions and a related parametric (scalar) objective program. Using this equivalence, Weir [38] formulated a dual program for a multiobjective

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX program having differentiable convex functions. Subsequently, Egudo [10] and Weir [38] proved duality results for a differentiable multiobjective program with pseudo convex/ quasi-convex functions. Kim, Lee and Kuk [20] proved duality results for a multiobjective fractional variational problem with generalized invexity functions.

A number of duality theorems for the single-objective control problem have appeared in the literature; see [13,19,26,31,35]. In general, these references give conditions under which an extremal solution of the control problem yields a solution of the corresponding dual. Mond and Hanson [28] established the converse duality theorem which gives conditions under which a solution of the dual problem yields a solution of the control problem. Mond and Smart [29] extended the results of Mond and Hanson [28] for duality in control problems to invex functions. It is also shown in Mond and Smart [29] that, for invex functions, the necessary conditions for optimality in the control problem are also sufficient. Also, Lee et al. [22] proved a sufficient optimality theorem for a multiobjective control problem and established the weak duality theorem and the strict converse duality theorem for Mond-Weir type dual problem under invexity conditions.

Bhatia and Kumar [5] extended the work of Mond and Hanson [28] to the content of multiobjective control problems and established duality results for Wolfe as well as Mond-Weir type duals under ρ-invexity assumptions and their generalization. Recently, Mishra and Mukherjee [27] obtained a pair of multiobjective control problems for Mond-Weir type duals under V-invexity assumptions and their generalization. Very recently, Zhian and Qingkai [40] discussed some duality results for multiobjective control problems with generalized invexity due to Mond and Smart [29].

Optimality conditions and duality in multiobjective fractional programs have been of much interest in the recent past [4, 7, 11, 23, 24, 32, 36, 37]. Following the approaches of Bector et al. [3], Liu [23, 24] obtained necessary and sufficient conditions and derived duality theorems for a class of nonsmooth multiobjective fractional programming problems involving either pseudoinvex functions or (F, ρ)-convex functions. In [17], Jeyakumar and Mond have defined generalized invex functions, called V-invex functions, and have applied them to the fractional programming problem.

Kuk et al. [21] obtained the generalized Karush-Kuhn-Tucker necessary and sufficient optimality theorems and proved weak, strong and strict converse duality theorems for nonsmooth multiobjective fractional programs involving V-ρ-invex functions.

Mond and Hanson [28] established the converse duality theorem which gives conditions under which a solution of the dual problem yields a solution of the control problem. Mond and Smart [29] extended the results of Mond and Hanson [28] for duality in control problems to invex functions. It is also shown in Mond and Smart [29] that, for invex functions, the necessary conditions for optimality in control problem are also sufficient. Recently, Mishra and Mukherjee [27] obtained duality results for multiobjective control problems under V-invexity assumptions and their generalizations. In 2001, Zhian and Qingkai [40] discussed some duality results for multiobjective control problems with generalized invexity due to Mond and Smart [29].

We recall some basic definitions and results pertaining to multiobjective problems. First, we need a consistent notation for vector inequalities.

For $x, y \in R^n$, the following order notation will be used:

$$x < y \Leftrightarrow x_i < y_i, \quad i = 1, 2, \cdots, n;$$

$$x \leqq y \Leftrightarrow x_i \leqq y_i, \quad i = 1, 2, \cdots, n;$$

$$x \leq y \Leftrightarrow x_i \leqq y_i, \quad i = 1, 2, \cdots, n \text{ but } x \neq y.$$

The following problems is called a multiobjective variational problem (MVP):

(MVP):

$$\begin{aligned} \text{Minimize} \quad & \int_a^b f(t,x,\dot{x})dt \\ & = \Big(\int_a^b f^1(t,x,\dot{x})dt, \cdots, \int_a^b f^p(t,x,\dot{x})dt \Big) \\ \text{subject to} \quad & x(a) = t_0, \;\; x(b) = t_f, \\ & g(t,x,\dot{x}) \leq 0, \;\; t \in I. \end{aligned}$$

Let $I = [a,b]$ be a real interval, $f : I \times R^n \times R^n \rightarrow R^p$ and $g : I \times R^n \times R^n \rightarrow R^m$ are continuously differentiable functions.

Let $X_0 := \{x \in C(I,R^n) | \; x(a) = t_0, \;\; x(b) = t_f, \;\; g(t,x,\dot{x}) \leq 0\}$ be the set of feasible solutions of problem (MVP).
Optimization of (MVP) is finding efficient solutions defined as follows;

Definition 1.1. A point $x^* \in X_0$ is said to be an efficient solution of the problem (MVP) if there exists no other $x \in X_0$ such that

$$\int_a^b f^i(t,x,\dot{x})dt \quad \leq \quad \int_a^b f^i(t,x^*,\dot{x}^*)dt, \;\; \forall i = 1, \cdots, p$$

and

$$\int_a^b f^{i_0}(t,x,\dot{x})dt \quad < \quad \int_a^b f^{i_0}(t,x^*,\dot{x}^*)dt, \;\; \text{for some } i_0 = 1, \cdots, p.$$

Definition 1.2. A point $x^* \in X_0$ is said to be a weak efficient solution of the problem (MVP) if there does not exist $x \in X_0$ such that

$$\int_a^b f^i(t,x,\dot{x})dt \quad < \quad \int_a^b f^i(t,x^*,\dot{x}^*)dt, \;\; \forall i = 1, \cdots, p.$$

Definition 1.3 (Geoffrion [12]). A point $x^* \in X_0$ is said to be a properly efficient solution of (MVP) if there exists a scalar $M > 0$ such that $\forall i = 1, \cdots, p$

$$\int_a^b f^i(t, x^*, \dot{x}^*)dt - \int_a^b f^i(t, x, \dot{x})dt \leq M\left\{\int_a^b f^j(t, x, \dot{x})dt - \int_a^b f^j(t, x^*, \dot{x}^*)dt\right\}$$

for some j, satisfying $\int_a^b f^j(t, x, \dot{x})dt > \int_a^b f^j(t, x^*, \dot{x}^*)dt$ whenever $x \in X_0$ and $\int_a^b f^i(t, x, \dot{x})dt < \int_a^b f^i(t, x^*, \dot{x}^*)dt$.

The multiobjective dual variational problem (MMVD)(: Mond-Weir multiobjective dual variational problem) for (MVP) can be expressed as the following form:

(MMVD) Maximize

$$\int_a^b f(t, y, \dot{y})\, dt = \left(\int_a^b f^1(t, y, \dot{y})\, dt, \cdots, \int_a^b f^p(t, y, \dot{y})\, dt\right)$$

subject to
$$y(a) = t_0, \quad y(b) = t_f,$$
$$\sum_{i=1}^{p} \tau_i \{f_x^i(t, y, \dot{y}) - \frac{d}{dt} f_{\dot{x}}^i(t, y, \dot{y})\} + \sum_{j=1}^{m} \lambda_j(t)\{g_x^j(t, y, \dot{y}) - \frac{d}{dt} g_{\dot{x}}^j(t, y, \dot{y})\} = 0,$$
$$\int_a^b \lambda_j(t) g^j(t, y, \dot{y})dt \geq 0, \quad \forall j = 1, \cdots, m,$$
$$\tau \in R^p, \tau \geq 0,$$
$$\lambda(t) \in R^m, \lambda(t) \geq 0, \quad t \in I,$$

where $\lambda(t)$ is a function from I into R^m.

Efficient solutions of (MMVD) can be defined analogously in definition 1.1 as follows;

Definition 1.4. A feasible solution (x^*, τ^*, λ^*) of (MMVD) is said to be an efficient solution of the problem (MMVD) if there does not exist a feasible solution (x, τ, λ) of (MMVD) such that

$$\int_a^b f^i(t, x^*, \dot{x}^*)dt \quad \leq \quad \int_a^b f^i(t, x, \dot{x})dt, \;\; \forall i = 1, \cdots, p$$

and

$$\int_a^b f^{i_0}(t, x^*, \dot{x}^*)dt \quad < \quad \int_a^b f^{i_0}(t, x, \dot{x})dt, \;\; \text{for some } i_0 = 1, \cdots, p.$$

The control problem is to choose, under given conditions, a control vector $u(t)$, such that the state vector $x(t)$ is brought from some specified initial state $x(a) = t_0$ to some specified final state $x(b) = t_0$ in such a way as to minimize a given functional.

The following problem is called a multiobjective control problem (MCP):

(MCP)

$$\begin{aligned}
\text{Minimize} \quad & \left(\int_a^b f^1(t, x, u)\, dt, \cdots, \int_a^b f^p(t, x, u)\, dt \right) \\
\text{subject to} \quad & x(a) = t_0, \;\; x(b) = t_f, \\
& g(t, x, u) \leq 0, \qquad t \in I, \\
& h(t, x, u) = \dot{x}, \qquad t \in I.
\end{aligned}$$

Each $f^i : I \times R^n \times R^m \to R$ for $i = 1, \cdots, p$, $g^j : I \times R^n \times R^m \to R$ $(j = 1, \cdots, k)$, and $h^r : I \times R^n \times R^m \to R$ $(r = 1, \cdots, n)$ is a continuously differentiable function.

Let $X := \{x \in C(I, R^n) | \; x(a) = t_0, \; x(b) = t_f, \; g(t, x, u) \leq 0, \; h(t, x, u) = \dot{x}\}$ be the set of feasible solutions of problem (MCP).
Optimization of (MCP) is finding efficient solutions defined as follows;

Definition 1.5. A feasible solution (x^*, u^*) of (MCP) is said to be an efficient solution of (MCP) if there does not exist a feasible solution (x, u) of (MCP) such that

$$\int_a^b f^i(t, x, u)dt \leq \int_a^b f^i(t, x^*, u^*)dt, \text{ for all } i = 1, \cdots, p$$

and

$$\int_a^b f^{i_0}(t, x, u)dt < \int_a^b f^{i_0}(t, x^*, u^*)dt, \text{ for some } i_0 = 1, \cdots, p.$$

The multiobjective dual control problem (MMCD)(: Mond-Weir multiobjective control dual problem) for (MCP) can be expressed as the following form:

(MMCD)

$$\begin{aligned} \text{Maximize} \quad & \left(\int_a^b f^1(t, x, u)dt, \cdots, \int_a^b f^p(t, x, u)dt \right) \\ \text{subject to} \quad & x(a) = t_0, \; x(b) = t_f, \\ & \sum_{i=1}^{p} \tau_i f_x^i(t, x, u) + \sum_{j=1}^{k} \lambda_j(t) g_x^j(t, x, u) \\ & + \sum_{r=1}^{n} \mu_r(t) h_x^r(t, x, u) + \dot{\mu}(t) = 0, \; t \in I, \end{aligned}$$

$$\sum_{i=1}^{p} \tau_i f_u^i(t,x,u) + \sum_{j=1}^{k} \lambda_j(t) g_u^j(t,x,u) + \sum_{r=1}^{n} \mu_r(t) h_u^r(t,x,u) = 0, \quad t \in I,$$

$$\int_a^b \sum_{r=1}^{n} \mu_r(t)\{h^r(t,x,u) - \dot{x}(t)\}dt \geq 0, \;\; t \in I,$$

$$\int_a^b \sum_{j=1}^{k} \lambda_j(t) g^j(t,x,u)dt \geq 0, \;\; t \in I,$$

$$\lambda(t) \geq 0, \quad t \in I,$$

$$\tau_i \geq 0, \;\; \sum_{i=1}^{p} \tau^i = 1,$$

where $\lambda(t)$ is a function from I into R^m and $\mu(t)$ is a function from I into R^r. Here $\lambda(t)$ and $\mu(t)$ are required to be continuous except perhaps at points of discontinuity of $u(t)$.

We can define efficient solutions of (MMCD) is ways similar to the case of (MCP):

Definition 1.6. A feasible solution $(x^*, u^*, \tau^*, \lambda^*, \mu^*)$ of (MMCD) is said to be an efficient solution of (MMCD) if there does not exist a feasible solution $(x, u, \tau, \lambda, \mu)$ of (MMCD) such that

$$\int_a^b f^i(t,x,u)dt \geq \int_a^b f^i(t,x^*,u^*)dt, \;\; \text{for all } \; i = 1, \cdots, p$$

and

$$\int_a^b f^{i_0}(t,x,u)dt > \int_a^b f^{i_0}(t,x^*,u^*)dt, \;\; \text{for some } \; i_0 = 1, \cdots, p.$$

In this paper, we formulate the multiobjective variational problem and control problem for generalized invex functions. We obtain sufficient optimality theorems and duality theorems for multiobjective variational problem involving generalized type I invex functions. Also, we obtain sufficient optimality theorems and duality theorems for multiobjective control problem involving generalized V-ρ- invex functions.

This thesis consists of five chapters.

In Chapter 2, a multiobjective variational problem with equality and inequality constrained are considered. We introduce vector type invexity along the lines of Jeyakumar and Mond [17] extending the pseudo, quasi, quasi-pseudo, pseudo-quasi type-I invexity of Kaul et al. [18]. Some sufficiency results are established. We formulate the Mond-Weir type dual and general Mond-Weir type dual problems and prove the duality theorems under generalized V-type I assumptions. As special case of our duality results, we obtain the Wolfe type duality theorems.

In Chapter 3, we consider a multiobjective fractional variational programming problem. For sufficient conditions, we define the generalized V-type I invex functions. We obtain the generalized Kuhn-Tucker sufficient optimality theorem and prove weak and strong duality theorems for the multiobjective fractional variational problem.

In Chapter 4, we obtain duality results for multiobjective control problems under V-ρ-invexity (V-ρ-pseudo invexity, V-ρ-quasi invexity) assumptions. The results of the present section extend the work of Mishra and Mukherjee [31] to more generalized V-ρ-invex functions. It is also shown that for V-ρ-invex functions, the necessary conditions for optimality in the control problem are also sufficient. Moreover, we formulate Wolfe type dual (WMCD) and Mond-Weir type dual (MMCD) for (MCP), and then establish their duality relations.

In Chapter 5, we consider a multiobjective fractional control problem. Using parametric approach Wolfe type and Mond-Weir type duality theorems are established under V-ρ-invexity (V-ρ-pseudo invexity, V-ρ-quasi invexity) on the functions involved. It is also shown that for V-ρ-invex functions, the necessary conditions for optimality in the control problem are also sufficient. The concept of efficiency is used to state sufficient optimality theorems and some duality results.

Chapter 2

Multiobjective Variational Problem with Generalized Type I Invexity

2.1. Introduction

The following problems is called a multiobjective variational problem with equality and inequality constraints:

(MVP):

$$\begin{aligned} \text{Minimize} \quad & \int_a^b f(t,x,\dot{x})dt \\ & = \Big(\int_a^b f^1(t,x,\dot{x})dt, \cdots, \int_a^b f^p(t,x,\dot{x})dt \Big) \\ \text{subject to} \quad & x(a) = t_0, \ \ x(b) = t_f, \\ & g(t,x,\dot{x}) \leq 0, \ \ t \in I, \end{aligned}$$

(MVPE):

$$\begin{aligned} \text{Minimize} \quad & \int_a^b f(t,x,\dot{x})dt \\ & = \Big(\int_a^b f^1(t,x,\dot{x})dt, \cdots, \int_a^b f^p(t,x,\dot{x})dt \Big) \\ \text{subject to} \quad & x(a) = t_0, \ \ x(b) = t_f, \\ & g(t,x,\dot{x}) \leq 0, \\ & h(t,x,\dot{x}) = 0, \ \ \forall t \in I, \end{aligned}$$

where $f : I \times R^n \times R^n \to R^p$, $g : I \times R^n \times R^n \to R^m$ and $h : I \times R^n \times R^n \to R^q$, are assumed to be continuously differentiable functions. Let $I = [a,b]$ be a real interval. In order to consider $f(t, x, \dot{x})$, where $x : I \rightarrow R^n$ with derivative $\dot{x}$, denote the partial derivative of f with respect to t, x, and $\dot{x}$, respectively, by f_t, f_x, and $f_{\dot{x}}$, such that

$$f_x = [\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}], \quad f_{\dot{x}} = [\frac{\partial f}{\partial \dot{x}_1}, \cdots, \frac{\partial f}{\partial \dot{x}_n}].$$

The partial derivatives of other functions used will be written similarly.

Let $C(I, R^n)$ denote the space of piecewise smooth functions x with norm $||x|| = ||x||_\infty + ||Dx||_\infty$, where the differentiation operator D is given by

$$u = Dx \Longleftrightarrow x(t) = t_0 + \int_a^t u(s)ds,$$

in which α is a given boundary value. Therefore, $D = \frac{d}{dt}$ except at discontinuities.

In this chapter, we are concerned with the multiobjective variational problem with equality and inequality constraints. We introduce new classes of generalized V-type I vector valued functions for variational problems and consider multiobjective variational problems (MVP) and (MVPE). A number of sufficiency results are established using Lagrange multiplier conditions under various types of generalized V-type I requirements. Duality theorems are proved for Mond-Weir and general Mond-Weir type duality under the above generalized V-type I assumptions and their generalizations. As special case of our duality results, we obtain the Wolfe type duality theorems.

2.2. Definitions and Preliminaries

Let us now denote by X_0 be the set of all feasible solutions of the problem (MVP) given by

$$X_0 := \{x \in C(I, R^n) |\ x(a) = t_0,\ x(b) = t_f,\ g(t, x, \dot{x}) \leq 0\}$$

and X_1 be the set of all feasible solutions of the problem (MVPE) given by

$$X_1 := \{x \in C(I, R^n) |\ x(a) = t_0,\ x(b) = t_f,\ g(t, x, \dot{x}) \leq 0,\ h(t, x, \dot{x}) = 0\}.$$

Following Aghezzaf and Hachimi [1] we define generalized type I invex functions for variational problems as follows.

Definition 2.2.1. (f, g) is said to be V-type I invex with respect to η, α_i and β_j at x^* if for all $i = 1, \cdots, p$ and $j = 1, \cdots, m$ there exist a differentiable vector function $\eta \in R^n$, and real-valued functions $\alpha_i \in R_+ \backslash \{0\}$ and $\beta_j \in R_+ \backslash \{0\}$ such that

$$\int_a^b f^i(t, x, \dot{x}) dt - \int_a^b f^i(t, x^*, \dot{x}^*) dt$$
$$\geq \int_a^b \alpha_i(x, x^*, \dot{x}, \dot{x}^*) \eta(t, x, x^*, \dot{x}, \dot{x}^*) \{f^i_x(t, x^*, \dot{x}^*) - \frac{d}{dt} f^i_{\dot{x}}(t, x^*, \dot{x}^*)\} dt \tag{2.1}$$

and

$$- \int_a^b g^j(t, x^*, \dot{x}^*)\, dt$$
$$\geq \int_a^b \beta_j(x, x^*, \dot{x}, \dot{x}^*) \eta(t, x, x^*, \dot{x}, \dot{x}^*) \{g^j_x(t, x^*, \dot{x}^*) - \frac{d}{dt} g^j_{\dot{x}}(t, x^*, \dot{x}^*)\}\, dt, \tag{2.2}$$

for every x.

If in the above definition, (2.2) is a strict inequality, then we say that (f,g) is semistrictly V-type I invex at x^*.

We now define and introduce the notions of weak strictly-pseudoquasi V-type I invexity, weak quasistrictly-pseudo V-type I invexity and weak strictly-pseudo V-type I invexity for (MVP).

Definition 2.2.2. (f,g) is said to be weak strictly-pseudoquasi V-type I invex with respect to η, α_i and β_j at x^* if there exist differentiable vector functions $\eta \in R^n$ and $\alpha_i \in R_+\backslash\{0\}$ and $\beta_j \in R_+\backslash\{0\}$, such that for some vector $\tau \in R^p, \tau \geq 0$ and piecewise smooth function $\lambda : I \to R^m, \lambda(t) \geq 0$,

$$\int_a^b \sum_{i=1}^p \tau_i \alpha_i(x,x^*,\dot{x},\dot{x}^*) f^i(t,x,\dot{x})dt < \int_a^b \sum_{i=1}^p \tau_i \alpha_i(x,x^*,\dot{x},\dot{x}^*) f^i(t,x^*,\dot{x}^*)dt$$

$$\Longrightarrow \int_a^b \sum_{i=1}^p \tau_i \eta(t,x,x^*,\dot{x},\dot{x}^*)\{f_x^i(t,x^*,\dot{x}^*) - \frac{d}{dt} f_{\dot{x}}^i(t,x^*,\dot{x}^*)\}dt < 0$$

and

$$-\int_a^b \sum_{j=1}^m \lambda_j(t)\beta_j(x,x^*,\dot{x},\dot{x}^*) g^j(t,x^*,\dot{x}^*)dt \leq 0$$

$$\Longrightarrow \int_a^b \sum_{j=1}^m \lambda_j(t)\eta(t,x,x^*,\dot{x},\dot{x}^*)\{g_x^j(t,x^*,\dot{x}^*) - \frac{d}{dt} g_{\dot{x}}^j(t,x^*,\dot{x}^*)\}dt \leq 0.$$

This definition is a slight extension of that of the weak strictly-pseudoquasi-type I functions [1].

Definition 2.2.3. (f,g) is said to be weak quasi strictly-pseudo V-type I invex with respect to η, α_i and β_j at x^* if there exist differentiable vector functions $\eta \in R^n$ and $\alpha_i \in R_+\backslash\{0\}$ and $\beta_j \in R_+\backslash\{0\}$, such that for some vector $\tau \in R^p, \tau \geq 0$ and piecewise smooth function $\lambda : I \to R^m, \lambda(t) \geq 0$,

$$\int_a^b \sum_{i=1}^p \tau_i \alpha_i(x, x^*, \dot{x}, \dot{x}^*) f^i(t, x, \dot{x}) dt < \int_a^b \sum_{i=1}^p \tau_i \alpha_i(x, x^*, \dot{x}, \dot{x}^*) f^i(t, x^*, \dot{x}^*) dt$$

$$\Longrightarrow \int_a^b \sum_{i=1}^p \tau_i \eta(t, x, x^*, \dot{x}, \dot{x}^*) \{ f_x^i(t, x^*, \dot{x}^*) - \frac{d}{dt} f_{\dot{x}}^i(t, x^*, \dot{x}^*) \} dt \le 0$$

and

$$-\int_a^b \sum_{j=1}^m \lambda_j(t) \beta_j(x, x^*, \dot{x}, \dot{x}^*) g^j(t, x^*, \dot{x}^*) dt \le 0$$

$$\Longrightarrow \int_a^b \sum_{j=1}^m \lambda_j(t) \eta(t, x, x^*, \dot{x}, \dot{x}^*) \{ g_x^j(t, x^*, \dot{x}^*) - \frac{d}{dt} g_{\dot{x}}^j(t, x^*, \dot{x}^*) \} dt < 0.$$

Definition 2.2.4. (f, g) is said to be weak strictly pseudo V-type I invex with respect to η, α_i and β_j at x^* if there exist differentiable vector functions $\eta \in R^n$ and $\alpha_i \in R_+ \backslash \{0\}$ and $\beta_j \in R_+ \backslash \{0\}$, such that for some vector $\tau \in R^p, \tau \ge 0$ and piecewise smooth function $\lambda : I \to R^m, \lambda(t) \ge 0$,

$$\int_a^b \sum_{i=1}^p \tau_i \alpha_i(x, x^*, \dot{x}, \dot{x}^*) f^i(t, x, \dot{x}) dt < \int_a^b \sum_{i=1}^p \tau_i \alpha_i(x, x^*, \dot{x}, \dot{x}^*) f^i(t, x^*, \dot{x}^*) dt$$

$$\Longrightarrow \int_a^b \sum_{i=1}^p \tau_i \eta(t, x, x^*, \dot{x}, \dot{x}^*) \{ f_x^i(t, x^*, \dot{x}^*) - \frac{d}{dt} f_{\dot{x}}^i(t, x^*, \dot{x}^*) \} dt < 0$$

and

$$-\int_a^b \sum_{j=1}^m \lambda_j(t) \beta_j(x, x^*, \dot{x}, \dot{x}^*) g^j(t, x^*, \dot{x}^*) dt \le 0$$

$$\Longrightarrow \int_a^b \sum_{j=1}^m \lambda_j(t) \eta(t, x, x^*, \dot{x}, \dot{x}^*) \{ g_x^j(t, x^*, \dot{x}^*) - \frac{d}{dt} g_{\dot{x}}^j(t, x^*, \dot{x}^*) \} dt < 0.$$

Following Hanson, Pini and Singh [16] we define vector type I invexity for variational problems as follows.

Definition 2.2.5. (f, g) is said to be quasi V-type I invex at x^* with respect to η, α_i and β_j at x^* if there exist differentiable vector functions $\eta \in R^n$ and $\alpha_i \in R_+\backslash\{0\}$ and $\beta_j \in R_+\backslash\{0\}$, such that for some vector $\tau \in R^p, \tau \geq 0$ and piecewise smooth function $\lambda : I \to R^m, \lambda(t) \geq 0$,

$$\int_a^b \sum_{i=1}^p \tau_i \alpha_i(x, x^*, \dot{x}, \dot{x}^*) f^i(t, x, \dot{x}) dt \leq \int_a^b \sum_{i=1}^p \tau_i \alpha_i(x, x^*, \dot{x}, \dot{x}^*) f^i(t, x^*, \dot{x}^*) dt$$

$$\Longrightarrow \int_a^b \sum_{i=1}^p \tau_i \eta(t, x, x^*, \dot{x}, \dot{x}^*) \{f_x^i(t, x^*, \dot{x}^*) - \frac{d}{dt} f_{\dot{x}}^i(t, x^*, \dot{x}^*)\} dt \leq 0 \tag{2.3}$$

and

$$\int_a^b \sum_{j=1}^m \lambda_j(t) \beta_j(x, x^*, \dot{x}, \dot{x}^*) g^j(t, x^*, \dot{x}^*) dt \geq 0$$

$$\Longrightarrow \int_a^b \sum_{j=1}^m \lambda_j(t) \eta(t, x, x^*, \dot{x}, \dot{x}^*) \{g_x^j(t, x^*, \dot{x}^*) - \frac{d}{dt} g_{\dot{x}}^j(t, x^*, \dot{x}^*)\} dt \leq 0. \tag{2.4}$$

If (f, g) is quasi V-type I invex at each x^*, we say (f, g) is quasi V-type I invex on $I \times R^n \times R^n$. If the second (implied) inequality in (2.3) is strict ($x \neq x^*$) (f, g) is semi strictly quasi V-type I invex at x^* or on $I \times R^n \times R^n$ as the case may be.

Definition 2.2.6. (f, g) is said to be pseudo V-type I invex at x^* with respect to η, α_i and β_j at x^* if there exist differentiable vector functions $\eta \in R^n$ and $\alpha_i \in R_+\backslash\{0\}$ and $\beta_j \in R_+\backslash\{0\}$, such that for some vector $\tau \in R^p, \tau \geq 0$ and piecewise smooth function $\lambda : I \to R^m, \lambda(t) \geq 0$, the implications

$$\int_a^b \sum_{i=1}^p \tau_i \eta(t, x, x^*, \dot{x}, \dot{x}^*) \{f_x^i(t, x^*, \dot{x}^*) - \frac{d}{dt} f_{\dot{x}}^i(t, x^*, \dot{x}^*)\} dt \geq 0$$

$$\Longrightarrow \int_a^b \sum_{i=1}^p \tau_i \alpha_i(x, x^*, \dot{x}, \dot{x}^*) f^i(t, x, \dot{x}) dt$$

$$\geq \int_a^b \sum_{i=1}^{p} \tau_i \alpha_i(x, x^*, \dot{x}, \dot{x}^*) f^i(t, x^*, \dot{x}^*) dt, \tag{2.5}$$

and

$$\int_a^b \sum_{j=1}^{m} \lambda_j(t) \eta(t, x, x^*, \dot{x}, \dot{x}^*) \{g_x^j(t, x^*, \dot{x}^*) - \frac{d}{dt} g_{\dot{x}}^j(t, x^*, \dot{x}^*)\} dt \geq 0$$

$$\Longrightarrow \int_a^b \sum_{j=1}^{m} \lambda_j(t) \beta_j(x, x^*, \dot{x}, \dot{x}^*) g^j(t, x^*, \dot{x}^*) dt \leq 0 \tag{2.6}$$

hold. If (f, g) is pseudo V-type I invex at each x^*, we say (f, g) is pseudo V-type I invex on $I \times R^n \times R^n$. If the second (implied) inequality in (2.5) (Eq. (2.6)) is strict, (f, g) is semi strictly pseudo V-type I invex in f (in g) at x or on $I \times R^n \times R^n$ as the case may be. If the second (implied) inequalities in (2.5) and (2.6) are both strict we say that (f, g) is strictly pseudo V-type I invex at x^* or on $I \times R^n \times R^n$ as the case may be.

Definition 2.2.7. (f, g) is said to be quasi pseudo V-type I invex at x^* with respect to η, α_i and β_j at x^* if there exist differentiable vector functions $\eta \in R^n$ and $\alpha_i \in R_+ \backslash \{0\}$ and $\beta_j \in R_+ \backslash \{0\}$, such that for some vector $\tau \in R^p, \tau \geq 0$ and piecewise smooth function $\lambda : I \to R^m, \lambda(t) \geq 0$, the implications

$$\int_a^b \sum_{i=1}^{p} \tau_i \alpha_i(x, x^*, \dot{x}, \dot{x}^*) f^i(t, x, \dot{x}) dt \leq \int_a^b \sum_{i=1}^{p} \tau_i \alpha_i(x, x^*, \dot{x}, \dot{x}^*) f^i(t, x, \dot{x}) dt$$

$$\Longrightarrow \int_a^b \sum_{i=1}^{p} \tau_i \eta(t, x, x^*, \dot{x}, \dot{x}^*) \{f_x^i(t, x^*, \dot{x}^*) - \frac{d}{dt} f_{\dot{x}}^i(t, x^*, \dot{x}^*)\} dt \leq 0 \tag{2.7}$$

and

$$\int_a^b \sum_{j=1}^{m} \lambda_j(t) \eta(t, x, x^*, \dot{x}, \dot{x}^*) \{g_x^j(t, x^*, \dot{x}^*) - \frac{d}{dt} g_{\dot{x}}^j(t, x^*, \dot{x}^*)\} dt \geq 0$$

$$\Longrightarrow \int_a^b \sum_{j=1}^{m} \lambda_j(t) \beta_j(x, x^*, \dot{x}, \dot{x}^*) g^j(t, x^*, \dot{x}^*) dt \leq 0 \tag{2.8}$$

hold. If (f, g) is quasi pseudo V-type I invex at each x^*, we say (f, g) is quasi pseudo V-type I invex on $I \times R^n \times R^n$. If the second (implied) inequality in (2.8) is strict, we say that (f, g) is quasi strictly pseudo V-type I invex at x^* or on $I \times R^n \times R^n$ as the case may be.

In order to prove the strong duality theorem we will invoke the following lemmas due to Changkong and Haimes [8].

Lemma 2.2.1. A point $x^* \in X_0$ is an efficient solution for (MVP) if and only if x^* solves $\forall k = 1, \cdots, p$,
$MVP_k(x^*)$:

$$\begin{aligned}
\text{Minimize} \quad & \int_a^b f^k(t, x, \dot{x})\, dt \\
\text{subject to} \quad & x(a) = t_0, \;\; x(b) = t_f, \\
& g(t, x, \dot{x}) \leq 0, \\
& \int_a^b f^j(t, x, \dot{x})\, dt \leq \int_a^b f^j(t, x^*, \dot{x}^*)\, dt, \\
& \qquad\qquad \forall \; j \in \{1, \cdots, p\}, j \neq k.
\end{aligned}$$

Lemma 2.2.2. A point $x^* \in X_1$ is an efficient solution for (MVPE) if and only if x^* solves $\forall k = 1, \cdots, p$,

$MVPE_k(x^*)$:

$$\begin{aligned}
\text{Minimize} \quad & \int_a^b f^k(t, x, \dot{x})\, dt \\
\text{subject to} \quad & x(a) = \alpha_0, \;\; x(b) = \beta_0, \\
& g(t, x, \dot{x}) \leq 0, \;\; h(t, x, \dot{x}) = 0, \\
& \int_a^b f^j(t, x, \dot{x})\, dt \leq \int_a^b f^j(t, x^*, \dot{x}^*)\, dt, \\
& \qquad\qquad \forall \; j \in \{1, \cdots, p\}, j \neq k.
\end{aligned}$$

2.3. Sufficient Optimality Theorems for (MVP)

We establish some sufficient conditions for an $x^* \in X_0$ to be an efficient solution of problem (MVP) under various generalized type I invexity conditions specified in the definitions given above.

Theorem 2.3.1 (Sufficiency). Suppose that

(i) $x^* \in X_0$;

(ii) there exist $\tau^* \in R^p, \tau^* \geq 0$, and a piecewise smooth function $\lambda^* : I \to R^m, \lambda^*(t) \geq 0$ such that

$$(a) \sum_{i=1}^{p} \tau_i^* \{f_x^i(t, x^*, \dot{x}^*) - \frac{d}{dt} f_{\dot{x}}^i(t, x^*, \dot{x}^*)\}$$
$$+ \sum_{j=1}^{m} \lambda_j^*(t)\{g_x^j(t, x^*, \dot{x}^*) - \frac{d}{dt} g_{\dot{x}}^j(t, x^*, \dot{x}^*)\} = 0,$$
$$(b) \int_a^b \sum_{j=1}^{m} \lambda_j^*(t) g^j(t, x^*, \dot{x}^*)\, dt = 0;$$

(iii) (f, g) is quasi strictly pseudo V-type I invex at x^* with respect to η, τ^*, λ^* and for some positive functions α_i, β_j, for $i = 1, \cdots, p, j = 1, \cdots, m$.

Then x^* is an efficient solution for (MVP).

Proof. Suppose x^* is not an efficient solution of (MVP). Then there exists a $x \in X_0$ such that

$$\int_a^b f^i(t, x, \dot{x})dt \quad \leq \quad \int_a^b f^i(t, x^*, \dot{x}^*)dt, \ \forall i = 1, \cdots, p$$

and

$$\int_a^b f^{i_0}(t, x, \dot{x})dt \quad < \quad \int_a^b f^{i_0}(t, x^*, \dot{x}^*)dt, \text{ for some } i_0 = 1, \cdots, p$$

which implies that

$$\int_a^b \sum_{i=1}^{p} \tau_i^* \alpha_i(x, x^*, \dot{x}, \dot{x}^*) f^i(t, x, \dot{x}) dt \;<\; \int_a^b \sum_{i=1}^{p} \tau_i^* \alpha_i(x, x^*, \dot{x}, \dot{x}^*) f^i(t, x^*, \dot{x}^*) dt.$$

From the above inequality and the hypothesis (iii), it follows that

$$\int_a^b \sum_{i=1}^{p} \tau_i^* \eta(t, x, x^*, \dot{x}, \dot{x}^*) \{ f_x^i(t, x^*, \dot{x}^*) - \frac{d}{dt} f_{\dot{x}}^i(t, x^*, \dot{x}^*) \} dt \leq 0. \qquad (2.9)$$

By the inequality (2.9) and hypothesis (ii)(a) we have

$$\int_a^b \sum_{j=1}^{m} \lambda_j^*(t) \eta(t, x, x^*, \dot{x}, \dot{x}^*) \{ g_x^j(t, x^*, \dot{x}^*) - \frac{d}{dt} g_{\dot{x}}^j(t, x^*, \dot{x}^*) \} dt \geq 0.$$

From the above inequality and hypothesis (iii) it follows that

$$\int_a^b \sum_{j=1}^{m} \lambda_j^*(t) \beta_j(x, x^*, \dot{x}, \dot{x}^*) g^j(t, x^*, \dot{x}^*) \, dt < 0. \qquad (2.10)$$

Now from hypotheses (i) and (ii)(b) it follows that $\int_a^b \lambda_j^*(t) g^j(t, x^*, \dot{x}^*) \, dt = 0$, for every j, which further implies that

$$\int_a^b \sum_{j=1}^{m} \lambda_j^*(t) \beta_j(x, x^*, \dot{x}, \dot{x}^*) g^j(t, x^*, \dot{x}^*) \, dt = 0.$$

The last equation contradicts the inequality (2.10) and hence x^* is an efficient solution of (MVP). □

Theorem 2.3.2 (Sufficiency). Suppose that

(i) $x^* \in X_0$;

(ii) there exist $\tau^* \in R^p, \tau^* > 0$, and a piecewise smooth function $\lambda^* : I \to R^m, \lambda^*(t) \geq 0$ such that

$$(a) \sum_{i=1}^{p} \tau_i^* \{f_x^i(t, x^*, \dot{x}^*) - \frac{d}{dt} f_{\dot{x}}^i(t, x^*, \dot{x}^*)\}$$

$$+ \sum_{j=1}^{m} \lambda_j^*(t) \{g_x^j(t, x^*, \dot{x}^*) - \frac{d}{dt} g_{\dot{x}}^j(t, x^*, \dot{x}^*)\} = 0,$$

$$(b) \int_a^b \sum_{j=1}^{m} \lambda_j^*(t) g_j(t, x^*, \dot{x}^*)\, dt = 0;$$

(iii) (f, g) is pseudo V-type I invex at x^* with respect to η, τ^*, λ^* and for some positive functions α_i, β_j, for $i = 1, \cdots, p, j = 1, \cdots, m$.

Then x^* is an efficient solution for (MVP). If, further, there exist positive real numbers n_i, m_i such that $n_i < \alpha_i(x, x^*, \dot{x}, \dot{x}^*) < m_i$, for all $x \in X_0$ and for all $i = 1, \cdots, p$, then x^* is properly efficient for (MVP).

Proof. Suppose x^* is not an efficient solution of (MVP). Then there exists a $x(\neq x^*) \in X_0$ such that

$$\int_a^b f^i(t, x, \dot{x})dt \leq \int_a^b f^i(t, x^*, \dot{x}^*)dt, \quad \forall i = 1, \cdots, p$$

and

$$\int_a^b f^{i_0}(t, x, \dot{x})dt < \int_a^b f^{i_0}(t, x^*, \dot{x}^*)dt, \quad \text{for some } i_0 = 1, \cdots, p$$

which implies that

$$\int_a^b \sum_{i=1}^{p} \tau_i^* \alpha_i(x, x^*, \dot{x}, \dot{x}^*) f^i(t, x, \dot{x})dt$$

$$< \int_a^b \sum_{i=1}^{p} \tau_i^* \alpha_i(x, x^*, \dot{x}, \dot{x}^*) f^i(t, x^*, \dot{x}^*)dt. \tag{2.11}$$

Next, by the hypotheses (i) and (ii)(b), it follows that $\int_a^b \lambda_j^*(t)g^j(t, x^*, \dot{x}^*)\, dt = 0$, for every j, which further implies that

$$\int_a^b \sum_{j=1}^m \lambda_j^*(t)\beta_j(x, x^*, \dot{x}, \dot{x}^*)g^j(t, x^*, \dot{x}^*)\, dt = 0.$$

From the above equality and the hypothesis (iii), it follows that

$$\int_a^b \sum_{j=1}^m \lambda_j^*(t)\eta(t, x, x^*, \dot{x}, \dot{x}^*)\{g_x^j(t, x^*, \dot{x}^*) - \frac{d}{dt}g_{\dot{x}}^j(t, x^*, \dot{x}^*)\}\, dt \leq 0. \qquad (2.12)$$

Now by (2.12) and the hypothesis (ii)(a), we have

$$\int_a^b \sum_{i=1}^p \tau_i^*\eta(t, x, x^*, \dot{x}, \dot{x}^*)\{f_x^i(t, x^*, \dot{x}^*) - \frac{d}{dt}f_{\dot{x}}^i(t, x^*, \dot{x}^*)\}\, dt \geq 0. \qquad (2.13)$$

Finally, by (2.13) and the hypothesis (iii), we have

$$\begin{aligned}&\int_a^b \sum_{i=1}^p \tau_i^*\alpha_i(x, x^*, \dot{x}, \dot{x}^*)f^i(t, x, \dot{x})dt \\ &\geq \int_a^b \sum_{i=1}^p \tau_i^*\alpha_i(x, x^*, \dot{x}, \dot{x}^*)f^i(t, x^*, \dot{x}^*)dt. \qquad (2.14)\end{aligned}$$

Since (2.11) and (2.14) contradict each other, we have the conclusion that x^* is an efficient solution of (MVP).

Next let for $p \geqq 2$

$$M = (p-1)\ max_{i,j}\Big(\frac{(m_j\tau_j^*)}{(n_i\tau_i^*)}\Big)\ \ \forall i \neq j;\ \ 1 \leqq i, j \leqq p.$$

Suppose x^* is not properly efficient for (MVP). Then there exists an $x_0 \in X_0$ such that for some i with $\int_a^b f^i(t, x^*, \dot{x}^*)dt > \int_a^b f^i(t, x_0, \dot{x}_0)dt$,

$$\int_a^b f^i(t, x^*, \dot{x}^*)dt - \int_a^b f^i(t, x_0, \dot{x}_0)dt$$
$$> M\left\{\int_a^b f^j(t, x_0, \dot{x}_0)dt - \int_a^b f^j(t, x^*, \dot{x}^*)dt\right\}$$
$$\forall j \text{ such that } \int_a^b f^j(t, x_0, \dot{x}_0)dt > \int_a^b f^j(t, x^*, \dot{x}^*)dt. \tag{2.15}$$

From (2.15) it follows that

$$\int_a^b f^i(t, x^*, \dot{x}^*)dt - \int_a^b f^i(t, x_0, \dot{x}_0)dt$$
$$> \ (p-1)\ \left(\frac{(m_j \tau_j^*)}{(n_i \tau_i^*)}\right)\ \left\{\int_a^b f^j(t, x_0, \dot{x}_0)dt - \int_a^b f^j(t, x^*, \dot{x}^*)dt\right\}\ \forall i \neq j,$$

which implies that

$$\int_a^b f^i(t, x^*, \dot{x}^*)dt - \int_a^b f^i(t, x_0, \dot{x}_0)dt$$
$$> \ \left(\frac{(p-1)(\alpha_j \tau_j^*)}{(\alpha_i \tau_i^*)}\right)\ \left\{\int_a^b f^j(t, x_0, \dot{x}_0)dt - \int_a^b f^j(t, x^*, \dot{x}^*)dt\right\}\ \forall i \neq j,$$

which further implies that

$$\frac{1}{p-1}\ \alpha_i \tau_i^*\ \left\{\int_a^b f^i(t, x^*, \dot{x}^*)dt - \int_a^b f^i(t, x_0, \dot{x}_0)dt\right\}$$
$$> \ \alpha_j \tau_j^*\ \left\{\int_a^b f^j(t, x_0, \dot{x}_0)dt - \int_a^b f^j(t, x^*, \dot{x}^*)dt\right\}. \tag{2.16}$$

Summing (2.16) with respect to j, we have that

$$\alpha_i \tau_i^* \left\{ \int_a^b f^i(t, x^*, \dot{x}^*) dt - \int_a^b f^i(t, x_0, \dot{x}_0) dt \right\}$$
$$> \sum_{j \neq i} \alpha_j \tau_j^* \left\{ \int_a^b f^j(t, x_0, \dot{x}_0) dt - \int_a^b f^j(t, x^*, \dot{x}^*) dt \right\},$$

that is,

$$\int_a^b \sum_j \alpha_j \tau_j^* f^j(t, x_0, \dot{x}_0) dt < \int_a^b \sum_j \alpha_j \tau_j^* f^j(t, x^*, \dot{x}^*) dt. \qquad (2.17)$$

Now (2.17) contradicts (2.14) and hence x^* is a properly efficient solution for (MVP). □

Theorem 2.3.3 (Sufficiency). Suppose that

(i) $x^* \in X_0$;

(ii) there exist $\tau^* \in R^p, \tau^* \geq 0$, and a piecewise smooth function $\lambda^* : I \to R^m, \lambda^*(t) \geq 0$ such that

$$(a) \sum_{i=1}^{p} \tau_i^* \{ f_x^i(t, x^*, \dot{x}^*) - \frac{d}{dt} f_{\dot{x}}^i(t, x^*, \dot{x}^*) \}$$
$$+ \sum_{j=1}^{m} \lambda_j^*(t) \{ g_x^j(t, x^*, \dot{x}^*) - \frac{d}{dt} g_{\dot{x}}^j(t, x^*, \dot{x}^*) \} = 0,$$
$$(b) \int_a^b \sum_{j=1}^{m} \lambda_j^*(t) g^j(t, x^*, \dot{x}^*)\, dt = 0;$$

(iii) (f, g) is semi strictly quasi V-type I invex at x^* with respect to η, τ^*, λ^* and for some positive functions α_i, β_j, for $i = 1, \cdots, p, j = 1, \cdots, m$.

Then x^* is an efficient solution for (MVP).

Proof. Suppose that there exists a $x(\neq x^*) \in X_0$, such that

$$\int_a^b f^i(t,x,\dot{x})dt \leq \int_a^b f^i(t,x^*,\dot{x}^*)dt, \ \ \forall i = 1,\cdots,p$$

and

$$\int_a^b f^{i_0}(t,x,\dot{x})dt < \int_a^b f^{i_0}(t,x^*,\dot{x}^*)dt, \ \ \text{for some } i_0 = 1,\cdots,p,$$

this implies that

$$\begin{aligned}&\int_a^b \sum_{i=1}^p \tau_i^* \alpha_i(x,x^*,\dot{x},\dot{x}^*) f^i(t,x,\dot{x})dt \\ &\quad < \int_a^b \sum_{i=1}^p \tau_i^* \alpha_i(x,x^*,\dot{x},\dot{x}^*) f^i(t,x^*,\dot{x}^*)dt. \end{aligned} \tag{2.18}$$

From inequality (2.18) and the hypothesis (iii), it follows that

$$\int_a^b \sum_{i=1}^p \tau_i^* \eta(t,x,x^*,\dot{x},\dot{x}^*)\{f_x^i(t,x^*,\dot{x}^*) - \frac{d}{dt} f_{\dot{x}}^i(t,x^*,\dot{x}^*)\}dt < 0. \tag{2.19}$$

Since $\int_a^b \sum_{j=1}^m \lambda_j^*(t) g^j(t,x^*,\dot{x}^*)\, dt = 0$ implies that $\int_a^b \lambda_j^*(t) g^j(t,x^*,\dot{x}^*)\, dt = 0$ for all j and $\beta_j > 0$ for all j, we have

$$\int_a^b \sum_{j=1}^m \lambda_j^*(t) \beta_j(x,x^*,\dot{x},\dot{x}^*) g^j(t,x^*,\dot{x}^*)\, dt = 0. \tag{2.20}$$

Now (2.20) and the hypothesis (iii) imply that

$$\int_a^b \sum_{j=1}^m \lambda_j^*(t)\eta(t,x,x^*,\dot{x},\dot{x}^*)\{g_x^j(t,x^*,\dot{x}^*) - \frac{d}{dt}g_{\dot{x}}^j(t,x^*,\dot{x}^*)\}dt \leq 0. \tag{2.21}$$

Adding (2.19) and (2.21) we see that the hypothesis (ii)(a) is contradict. Hence x^* is an efficient solution of (MVP). □

Theorem 2.3.4 (Sufficiency). Suppose that

(i) $x^* \in X_0$;

(ii) there exist $\tau^* \in R^p, \tau^* \geq 0$, and a piecewise smooth function $\lambda^* : I \to R^m, \lambda^*(t) \geq 0$ such that

$$\begin{aligned}(a) &\sum_{i=1}^p \tau_i^*\{f_x^i(t,x^*,\dot{x}^*) - \frac{d}{dt}f_{\dot{x}}^i(t,x^*,\dot{x}^*)\} \\ &+ \sum_{j=1}^m \lambda_j^*(t)\{g_x^j(t,x^*,\dot{x}^*) - \frac{d}{dt}g_{\dot{x}}^j(t,x^*,\dot{x}^*)\} = 0, \\ (b) &\int_a^b \sum_{j=1}^m \lambda_j^*(t)g^j(t,x^*,\dot{x}^*)\,dt = 0;\end{aligned}$$

(iii) (f,g) is strictly pseudo V-type I invex at x^* with respect to η, τ^*, λ^* and for some positive functions α_i, β_j, for $i = 1, \cdots, p, j = 1, \cdots, m$.

Then x^* is an efficient solution of (MVP). If, further $\tau^* > 0$ and there exist positive real numbers n_i, m_i such that $n_i < \alpha_i(x,x^*,\dot{x},\dot{x}^*) < m_i$, for all $x \in X_0$ and for all $i = 1, \cdots, p$, then x^* is properly efficient for (MVP).

Proof. By hypothesis (ii)(b) it follows that

$$\int_a^b \sum_{j=1}^m \lambda_j^*(t)\beta_j(x,x^*,\dot{x},\dot{x}^*)g^j(t,x^*,\dot{x}^*)\,dt = 0,$$

which implies by the hypothesis (iii) that

$$\int_a^b \sum_{j=1}^m \lambda_j^*(t)\eta(t,x,x^*,\dot{x},\dot{x}^*)\{g_x^j(t,x^*,\dot{x}^*) - \frac{d}{dt}g_{\dot{x}}^j(t,x^*,\dot{x}^*)\}dt < 0,$$

which in turn implies by the hypothesis (ii)(a) that

$$\int_a^b \sum_{i=1}^p \tau_i^*\eta(t,x,x^*,\dot{x},\dot{x}^*)\{f_x^i(t,x^*,\dot{x}^*) - \frac{d}{dt}f_{\dot{x}}^i(t,x^*,\dot{x}^*)\}dt > 0. \tag{2.22}$$

Now from (2.22) and hypothesis (iii), we have

$$\begin{aligned}&\int_a^b \sum_{i=1}^p \tau_i^*\alpha_i(x,x^*,\dot{x},\dot{x}^*)f^i(t,x,\dot{x})dt\\ &\quad > \int_a^b \sum_{i=1}^p \tau_i^*\alpha_i(x,x^*,\dot{x},\dot{x}^*)f^i(t,x^*,\dot{x}^*)dt.\end{aligned} \tag{2.23}$$

Next if x^* is not an efficient solution of (MVP), then there exiats an $x \in X_0$ such that

$$\int_a^b f^i(t,x,\dot{x})dt \leq \int_a^b f^i(t,x^*,\dot{x}^*)dt, \ \ \forall i = 1,\cdots,p$$

and

$$\int_a^b f^{i_0}(t,x,\dot{x})dt < \int_a^b f^{i_0}(t,x^*,\dot{x}^*)dt, \ \ \text{for some } i_0 = 1,\cdots,p,$$

which implies that

$$\begin{aligned}&\int_a^b \sum_{i=1}^p \tau_i^*\alpha_i(x,x^*,\dot{x},\dot{x}^*)f^i(t,x,\dot{x})dt\\ &\quad < \int_a^b \sum_{i=1}^p \tau_i^*\alpha_i(x,x^*,\dot{x},\dot{x}^*)f^i(t,x^*,\dot{x}^*)dt.\end{aligned} \tag{2.24}$$

Since (2.23) and (2.24) contradict each other, the conclusion follows.

To establish the proper efficiency of x^* (MVP), we follow the same argument as in the proof of Theorem 2.3.2 except in the end we appeal to the inequality (2.24) for a contradiction. □

2.4. Formulations of Four Pairs of Variational Dual Problem

We formulate four pairs of the following multiobjective variational dual problems.

(MVD): Maximize

$$
\begin{aligned}
&\int_a^b \{f(t,y,\dot{y}) + \sum \lambda_A(t) g_A(t,y,\dot{y}) e\}\, dt \\
&= \Big(\int_a^b \{f^1(t,y,\dot{y}) + \sum \lambda_A(t) g_A(t,y,\dot{y})\}\, dt, \\
&\cdots, \int_a^b \{f^p(t,y,\dot{y}) + \sum \lambda_A(t) g_A(t,y,\dot{y})\}\, dt \Big)
\end{aligned}
$$

subject to

$$
\begin{aligned}
& y(a) = t_0, \;\; y(b) = t_f, \\
& \sum_{i=1}^{p} \tau_i \{ f_x^i(t,y,\dot{y}) - \frac{d}{dt} f_{\dot{x}}^i(t,y,\dot{y}) \} \\
& + \sum_{j=1}^{m} \lambda_j(t) \{ g_x^j(t,y,\dot{y}) - \frac{d}{dt} g_{\dot{x}}^j(t,y,\dot{y}) \} = 0, \\
& \int_a^b \lambda_B(t) g_B(t,y,\dot{y}) dt \geq 0, \\
& \tau \in R^p, \tau \geq 0, \\
& \lambda(t) \in R^m, \lambda(t) \geq 0, \;\; t \in I,
\end{aligned}
$$

where $e = (1,1,\cdots,1)^t \in R^p$ and $A \cup B = \{1,\cdots,m\}$.

When $A = \emptyset$ and $B = \{1, \cdots, m\}$, our dual problem (MVD) is reduced as follows:

(MMVD): Maximize

$$
\begin{aligned}
& \int_a^b f(t, y, \dot{y})\, dt \\
& = \Big(\int_a^b f^1(t, y, \dot{y})\, dt, \cdots, \int_a^b f^p(t, y, \dot{y})\, dt \Big) \\
\text{subject to} \quad & y(a) = t_0, \ \ y(b) = t_f, \\
& \sum_{i=1}^{p} \tau_i \{ f_x^i(t, y, \dot{y}) - \frac{d}{dt} f_{\dot{x}}^i(t, y, \dot{y}) \} \\
& + \sum_{j=1}^{m} \lambda_j(t) \{ g_x^j(t, y, \dot{y}) - \frac{d}{dt} g_{\dot{x}}^j(t, y, \dot{y}) \} = 0, \\
& \int_a^b \lambda_j(t) g^j(t, y, \dot{y}) dt \geq 0, \ \ \forall j = 1, \cdots, m, \\
& \tau \in R^p, \tau \geq 0, \\
& \lambda(t) \in R^m, \lambda(t) \geq 0, \ \ t \in I.
\end{aligned}
$$

We let Y_0 be the set of feasible solutions of problem (MMVD); i.e.,

$$
\begin{aligned}
& Y_0 = \Big\{ (y, \tau, \lambda) \mid y(a) = t_0, \ \ y(b) = t_f, \\
& \sum_{i=1}^{p} \tau_i \{ f_x^i(t, y, \dot{y}) - \frac{d}{dt} f_{\dot{x}}^i(t, y, \dot{y}) \} + \sum_{j=1}^{m} \lambda_j(t) \{ g_x^j(t, y, \dot{y}) - \frac{d}{dt} g_{\dot{x}}^j(t, y, \dot{y}) \} = 0, \\
& \int_a^b \lambda_j(t) g^j(t, y, \dot{y}) dt \geq 0, \ \ \forall j = 1, \cdots, m, \\
& \tau \in R^p, \ \ \tau \geq 0, \ \ \lambda(t) \in R^m, \ \ \lambda(t) \geq 0, \ \ t \in I \Big\}.
\end{aligned}
$$

Analogous to (MMVD) and (MVP), the following problem (MMVDE) is a dual to (MVPE).

(MMVDE):

$$
\begin{aligned}
\text{Maximize} \quad & \int_a^b f(t,y,\dot{y})dt \\
& = \Big(\int_a^b f^1(t,y,\dot{y})dt, \cdots, \int_a^b f^p(t,y,\dot{y})dt \Big) \\
\text{subject to} \quad & y(a) = t_0, \;\; y(b) = t_f, && (2.25) \\
& \sum_{i=1}^{p} \tau_i \{ f_x^i(t,y,\dot{y}) - \frac{d}{dt} f_{\dot{x}}^i(t,y,\dot{y}) \} \\
& + \sum_{j=1}^{m} \lambda_j(t) \{ g_x^j(t,y,\dot{y}) - \frac{d}{dt} g_{\dot{x}}^j(t,y,\dot{y}) \} \\
& + \sum_{l=1}^{q} \mu_l(t) \{ h_x^l(t,y,\dot{y}) - \frac{d}{dt} h_{\dot{x}}^l(t,y,\dot{y}) \} = 0, && (2.26) \\
& \int_a^b \lambda_j(t) g^j(t,y,\dot{y})dt \geq 0, \;\; \forall j = 1, \cdots, m, && (2.27) \\
& \int_a^b \mu_l(t) h^l(t,y,\dot{y})dt = 0, \;\; \forall l = 1, \cdots, q, && (2.28) \\
& \lambda(t) \geq 0, && (2.29) \\
& \tau_i \geq 0, \;\; \sum_{i=1}^{p} \tau_i = 1, \;\; t \in I. && (2.30)
\end{aligned}
$$

We let Y_1 be the set of feasible solutions of problem (MMVDE); i.e.,

$$Y_1 = \Bigg\{(y,\tau,\lambda,\mu) \mid y(a)=t_0,\ y(b)=t_f,$$

$$\sum_{i=1}^{p}\tau_i\{f^i_x(t,y,\dot y) - \frac{d}{dt}f^i_{\dot x}(t,y,\dot y)\} \quad + \sum_{j=1}^{m}\lambda_j(t)\{g^j_x(t,y,\dot y)$$

$$- \frac{d}{dt}g^j_{\dot x}(t,y,\dot y)\Bigg\} + \sum_{l=1}^{q}\mu_l(t)\{h^l_x(t,y,\dot y) - \frac{d}{dt}h^l_{\dot x}(t,y,\dot y)\} = 0,$$

$$\int_a^b \lambda_j(t)g^j(t,y,\dot y)dt \geq 0,\ \forall j = 1,\cdots,m,$$

$$\int_a^b \mu_l(t)h^l(t,y,\dot y)dt = 0,\ \forall l = 1,\cdots,q,$$

$$\lambda(t) \geq 0,$$

$$\tau_i \geq 0,\ \sum_{i=1}^{p}\tau_i = 1,\ t \in I\Bigg\}.$$

We consider the following general Mond-Weir [30] type dual problem:

(GMMVDE): Maximize

$$\Big(\int_a^b \{f^1(t,y,\dot y) + \sum_{j\in J_0}\lambda_j(t)g^j(t,y,\dot y) + \sum_{l\in K_0}\mu_l(t)h^l(t,y,\dot y)\}\,dt,$$

$$\cdots,\int_a^b \{f^p(t,y,\dot y) + \sum_{j\in J_0}\lambda_j(t)g^j(t,y,\dot y) + \sum_{l\in K_0}\mu_l(t)h^l(t,y,\dot y)\}\,dt\Big)$$

subject to $y(a)=t_0,\ y(b)=t_f,$ (2.31)

$$\sum_{i=1}^{p}\tau_i\{f^i_x(t,y,\dot y) - \frac{d}{dt}f^i_{\dot x}(t,y,\dot y)\}$$

$$+ \sum_{j=1}^{m}\lambda_j(t)\{g^j_x(t,y,\dot y) - \frac{d}{dt}g^j_{\dot x}(t,y,\dot y)\}$$

$$+ \sum_{l=1}^{q}\mu_l(t)\{h^l_x(t,y,\dot y) - \frac{d}{dt}h^l_{\dot x}(t,y,\dot y)\} = 0, \tag{2.32}$$

$$\int_a^b \{\sum_{j\in J_\alpha} \lambda_j(t)g^j(t,y,\dot{y}) + \sum_{l\in K_\alpha} \mu_l(t)h^l(t,y,\dot{y})\}dt \geq 0, \quad \alpha = 1,\cdots,r, \tag{2.33}$$

$$\lambda(t) \geq 0, \tag{2.34}$$

$$\tau_i \geq 0, \quad \sum_{i=1}^{p} \tau_i = 1, \tag{2.35}$$

where $J_\alpha \subset \{1,\cdots,m\}, \alpha = 0,1,\cdots,r$ with $J_\alpha \cap J_\beta = \emptyset, \alpha \neq \beta$ and $\cup_{\alpha=0}^{r} J_\alpha = \{1,\cdots,m\}$ and $K_\alpha \subset \{1,\cdots,k\}$, $\alpha = 0,1,\cdots,r$ with $K_\alpha \cap K_\beta = \emptyset, \alpha \neq \beta$ and $\cup_{\alpha=0}^{r} K_\alpha = \{1,\cdots,k\}$.

2.5. Duality Theorems

Now we establish some duality theorems between the multiobjective variational problem (MVP) and its dual problem (MMVD).

Theorem 2.5.1 (Weak Duality). Suppose that

(i) $x \in X_0$;

(ii) $(y,\tau,\lambda) \in Y_0$ and $\tau > 0$;

(iii) (f,g) is pseudo V-type I invex at y with respect to η,τ,λ and for some positive functions α_i,β_j, for $i = 1,\cdots,p, j = 1,\cdots,m$.

Then the following cannot hold:

$$\int_a^b f^i(t,x,\dot{x})dt \leq \int_a^b f^i(t,y,\dot{y})dt, \quad \forall i = 1,\cdots,p, \tag{2.36}$$

and

$$\int_a^b f^{i_0}(t,x,\dot{x})dt < \int_a^b f^{i_0}(t,y,\dot{y})dt, \quad \text{for some } i_0 = 1,\cdots,p. \tag{2.37}$$

Proof. By hypothesis (ii) we have $\int_a^b \lambda_j(t) g^j(t, y, \dot{y}) dt \geq 0, \;\; \forall j = 1, \cdots, m,$ which implies that

$$\int_a^b \sum_{j=1}^{m} \lambda_j(t) \beta_j(x, y, \dot{x}, \dot{y}) g^j(t, y, \dot{y}) dt \geq 0. \tag{2.38}$$

By the hypothesis (iii) and (2.38) it follows that

$$\int_a^b \sum_{j=1}^{m} \lambda_j(t) \eta(t, x, y, \dot{x}, \dot{y}) \{g_x^j(t, y, \dot{y}) - \frac{d}{dt} g_{\dot{x}}^j(t, y, \dot{y})\} dt \leq 0. \tag{2.39}$$

Using the inequality (2.39) and the hypothesis (ii) we have

$$\int_a^b \sum_{i=1}^{p} \tau_i \eta(t, x, y, \dot{x}, \dot{y}) \{f_x^i(t, y, \dot{y}) - \frac{d}{dt} f_{\dot{x}}^i(t, y, \dot{y})\} dt \geq 0. \tag{2.40}$$

Hypothesis (iii) and (2.40) give

$$\int_a^b \sum_{i=1}^{p} \tau_i \alpha_i(x, y, \dot{x}, \dot{y}) f^i(t, x, \dot{x}) dt \;\; \geq \;\; \int_a^b \sum_{i=1}^{p} \tau_i \alpha_i(x, y, \dot{x}, \dot{y}) f^i(t, y, \dot{y}) dt. \tag{2.41}$$

Suppose contrary to the result that (2.36) and (2.37) hold. Then since each $\alpha_i > 0$ and $\tau > 0$, we have

$$\int_a^b \sum_{i=1}^{p} \tau_i \alpha_i(x, y, \dot{x}, \dot{y}) f^i(t, x, \dot{x}) dt \;\; < \;\; \int_a^b \sum_{i=1}^{p} \tau_i \alpha_i(x, y, \dot{x}, \dot{y}) f^i(t, y, \dot{y}) dt,$$

which contradicts (2.41). Hence the conclusion follows. □

Theorem 2.5.2 (Weak Duality). Suppose that

(i) $x \in X_0$;

(ii) $(y, \tau, \lambda) \in Y_0$;

(iii) (f, g) is semi strictly V-type I invex at y for some positive functions $\alpha_i, \ \beta_j, \ \text{ for } i = 1, \cdots, p, \ j = 1, \cdots, m$.

Then (2.36) and (2.37) cannot hold.

Proof. By the hypothesis (ii) we have $\int_a^b \lambda_j(t) g^j(t, y, \dot{y}) dt \geq 0, \ \forall j = 1, \cdots, m$, which implies that

$$\int_a^b \sum_{j=1}^m \lambda_j(t) \beta_j(x, y, \dot{x}, \dot{y}) g^j(t, y, \dot{y}) dt \geq 0. \tag{2.42}$$

By (2.42) and the hypothesis (iii) it follows that

$$\int_a^b \sum_{j=1}^m \lambda_j(t) \eta(t, x, y, \dot{x}, \dot{y}) \{ g_x^j(t, y, \dot{y}) - \frac{d}{dt} g_{\dot{x}}^j(t, y, \dot{y}) \} dt < 0. \tag{2.43}$$

Using the inequality (2.43) and the hypothesis (ii) we have

$$\int_a^b \sum_{i=1}^p \tau_i \eta(t, x, y, \dot{x}, \dot{y}) \{ f_x^i(t, y, \dot{y}) - \frac{d}{dt} f_{\dot{x}}^i(t, y, \dot{y}) \} dt \geq 0. \tag{2.44}$$

By (2.44) and the hypothesis (iii), we have

$$\int_a^b \sum_{i=1}^p \tau_i \alpha_i(x, y, \dot{x}, \dot{y})^{-1} f^i(t, x, \dot{x}) dt > \int_a^b \sum_{i=1}^p \tau_i \alpha_i(x, y, \dot{x}, \dot{y})^{-1} f^i(t, y, \dot{y}) dt. \tag{2.45}$$

Suppose contrary to the result that (2.36) and (2.37) hold. Then since each $\alpha_i > 0$ and $\tau \geq 0$, we have

$$\int_a^b \sum_{i=1}^p \tau_i \alpha_i(x, y, \dot{x}, \dot{y})^{-1} f^i(t, x, \dot{x}) dt \leq \int_a^b \sum_{i=1}^p \tau_i \alpha_i(x, y, \dot{x}, \dot{y})^{-1} f^i(t, y, \dot{y}) dt,$$

which contradicts (2.45).

□

Corollary 2.5.1. Assume that weak duality theorems (2.5.1, 2.5.2) hold between (MVP) and (MMVD). If (y^*, τ^*, λ^*) is feasible for (MMVD) such that y^* is feasible for (MVP), then y^* is an efficient solution for (MVP) and (y^*, τ^*, λ^*) is an efficient solution for (MMVD).

Proof. Suppose that y^* is not efficient for (MVP). Then there exists some feasible x for (MVP) such that

$$\int_a^b f^i(t, x, \dot{x})\, dt \leq \int_a^b f^i(t, y^*, \dot{y}^*)\, dt, \quad \forall\ i = 1, \cdots, p$$
and
$$\int_a^b f^{i_0}(t, x, \dot{x})\, dt < \int_a^b f^{i_0}(t, y^*, \dot{y}^*)\, dt, \quad \text{for some } i_0 = 1, \cdots, p.$$

This contradicts weak duality. Hence y^* is an efficient for (MVP). Now suppose (y^*, τ^*, λ^*) is not an efficient for (MMVD). Then there exist some (x, τ, λ) feasible for (MMVD) such that

$$\int_a^b f^i(t, x, \dot{x})\, dt \geq \int_a^b f^i(t, y^*, \dot{y}^*)\, dt, \quad \forall\ i = 1, \cdots, p$$
and
$$\int_a^b f^{i_0}(t, x, \dot{x})\, dt > \int_a^b f^{i_0}(t, y, \dot{y})\, dt, \quad \text{for some } i_0 = 1, \cdots, p.$$

This contradicts weak duality. Hence (y^*, τ^*, λ^*) is an efficient for (MMVD).

□

Theorem 2.5.3 (Strong Duality). Assume that

(i) x^* is an efficient solution for (MVP);

(ii) for all $k = 1, \cdots, p$, x^* a constraint qualification for $MVP_k(x^*)$ at x^* is satisfied.

Then there exists $\tau^* \in R^p, \tau^* > 0$, and piecewise smooth function $\lambda^* : I \to R^m, \lambda^*(t) \geq 0$ such that $(x^*, \tau^*, \lambda^*) \in Y_0$. Further, if the assumption of weak duality theorems (2.5.1 or 2.5.2) are satisfied, then (x^*, τ^*, λ^*) is an efficient for (MMVD).

Proof. Since x^* is an efficient solution of (MVP), then from Lemma 2.2.1, x^* solves $MVP_k(x^*)$ for each $k = 1, \cdots, p$. From Kuhn-Tucker necessary conditions for each $k = 1, \cdots, p$, we obtain $\tau_i^k \geq 0$ for all $i \neq k$, and $\lambda^i(t)(\geq 0) \in R^m$ such that

$$\{f_x^i(t, x^*, \dot{x}^*) - \frac{d}{dt} f_{\dot{x}}^i(t, x^*, \dot{x}^*)\} + \sum_{i \neq k} \tau_i^k \{f_x^k(t, x^*, \dot{x}^*) - \frac{d}{dt} f_{\dot{x}}^k(t, x^*, \dot{x}^*)\}$$

$$+ \sum_{j=1}^{m} \lambda_j^i(t)\{g_x^j(t, x^*, \dot{x}^*) - \frac{d}{dt} g_{\dot{x}}^j(t, x^*, \dot{x}^*) = 0, \tag{2.46}$$

$$\int_a^b \sum_{j=1}^{m} \lambda_j^i(t) g^j(t, x^*, \dot{x}^*) dt = 0. \tag{2.47}$$

Summing (2.46) over $i = 1, \cdots, p$, we have

$$(1 + \tau_2^1 + \cdots + \tau_p^1)\{f_x^1(t, x^*, \dot{x}^*) - \frac{d}{dt} f_{\dot{x}}^1(t, x^*, \dot{x}^*)\}$$

$$+ \sum_{j=1}^{m} \lambda_j^1(t)\{g_x^j(t, x^*, \dot{x}^*) - \frac{d}{dt} g_{\dot{x}}^j(t, x^*, \dot{x}^*)\}$$

$$+ (\tau_1^2 + 2 + \cdots + \tau_p^2)\{f_x^2(t, x^*, \dot{x}^*) - \frac{d}{dt} f_{\dot{x}}^2(t, x^*, \dot{x}^*)\}$$

$$+ \sum_{j=1}^{m} \lambda_j^2(t)\{g_x^j(t, x^*, \dot{x}^*) - \frac{d}{dt} g_{\dot{x}}^j(t, x^*, \dot{x}^*)\} + \cdots$$

$$+ (\tau_1^p + \tau_2^p + \cdots + 1)\{f_x^p(t, x^*, \dot{x}^*) - \frac{d}{dt} f_{\dot{x}}^p(t, x^*, \dot{x}^*)\}$$
$$+ \sum_{j=1}^{m} \lambda_j^p(t)\{g_x^j(t, x^*, \dot{x}^*) - \frac{d}{dt} g_{\dot{x}}^j(t, x^*, \dot{x}^*)\} = 0$$

Let $\tau_1^* = 1 + \tau_2^1 + \cdots + \tau_p^1$, $\tau_2^* = \tau_1^2 + 1 + \cdots + \tau_p^2$, $\cdots, \tau_p^* = \tau_1^p + \tau_2^p + \cdots + 1$, $\sum_{k=1}^{p} \lambda_j^k(t) = \lambda_j^*(t)$, $j = 1, \cdots, m$, $\lambda_1^*(t) = (\lambda_1^*(t), \cdots, \lambda_m^*(t))$. Then we have

$$\sum_{i=1}^{p} \tau_i^*\{f_x^i(t, x^*, \dot{x}^*) - \frac{d}{dt} f_{\dot{x}}^i(t, x^*, \dot{x}^*)\}$$
$$+ \sum_{j=1}^{m} \lambda_j^*(t)\{g_x^j(t, x^*, \dot{x}^*) - \frac{d}{dt} g_{\dot{x}}^j(t, x^*, \dot{x}^*)\} = 0.$$

Summing (2.47) for $i = 1, \cdots, p$, we have $\int_a^b \sum_{j=1}^{m} \lambda_j^*(t) g^j(t, x^*, \dot{x}^*)dt = 0$. We conclude that (x^*, τ^*, λ^*) is feasible for (MMVD). Efficiency of (x^*, τ^*, λ^*) for (MMVD) now follows from Corollary 2.5.1. □

Theorem 2.5.4 (Converse Duality). Suppose that

(i) $(y^*, \tau^*, \lambda^*) \in Y_0$ with $\tau^* > 0$;

(ii) $y^* \in X_0$;

(iii) (f, g) is V-type I invex at y^* for some positive functions α_i, β_j for $i = 1, \cdots, p$, $j = 1, \cdots, m$.

Then y^* is an efficient solution of (MVP).

Proof. It follows by the hypothesis (i) that

$$\int_a^b \lambda_j^*(t) g^j(t, y^*, \dot{y}^*)dt \geq 0, \ \ \forall j = 1, \cdots, m. \tag{2.48}$$

By hypothesis (iii), for any $x \in X_0$, we have

$$\int_a^b f^i(t,x,\dot{x})dt - \int_a^b f^i(t,y^*,\dot{y}^*)dt$$
$$\geq \int_a^b \alpha_i \eta(t,x,y^*,\dot{x},\dot{y}^*)\{f_x^i(t,y^*,\dot{y}^*) - \frac{d}{dt}f_{\dot{x}}^i(t,y^*,\dot{y}^*)\}dt, \quad (2.49)$$
$$-\int_a^b g^j(t,y^*,\dot{y}^*)\,dt$$
$$\geq \int_a^b \beta_j \eta(t,x,y^*,\dot{x},\dot{y}^*)\{g_x^j(t,y^*,\dot{y}^*) - \frac{d}{dt}g_{\dot{x}}^j(t,y^*,\dot{y}^*)\}\,dt. \quad (2.50)$$

Now by the facts that $\alpha_i > 0$, $\beta_j > 0 \ \forall \ i,j$ and $\tau^* > 0, \lambda^*(t) \geq 0$, it follows by (2.49) and (2.50) that

$$\int_a^b \sum_{i=1}^p \frac{\tau_i^*}{\alpha_i} f^i(t,x,\dot{x})dt - \int_a^b \sum_{i=1}^p \frac{\tau_i^*}{\alpha_i} f^i(t,y^*,\dot{y}^*)dt - \int_a^b \sum_{j=1}^m \frac{\lambda_j^*(t)}{\beta_j} g^j(t,y^*,\dot{y}^*)dt$$
$$\geq \int_a^b \eta(t,x,y^*,\dot{x},\dot{y}^*)\Big(\sum_{i=1}^p \tau_i^*\{f_x^i(t,y^*,\dot{y}^*) - \frac{d}{dt}f_{\dot{x}}^i(t,y^*,\dot{y}^*)\}$$
$$+\sum_{j=1}^m \lambda_j^*(t)\{g_x^j(t,y^*,\dot{y}^*) - \frac{d}{dt}g_{\dot{x}}^j(t,y^*,\dot{y}^*)\}\Big)dt$$
$$= 0. \quad (2.51)$$

From (2.48) and (2.51) it follows that

$$\int_a^b \sum_{i=1}^p \frac{\tau_i^*}{\alpha_i(x,y^*,\dot{x},\dot{y}^*)} f^i(t,x,\dot{x})dt \geq \int_a^b \sum_{i=1}^p \frac{\tau_i^*}{\alpha_i(x,y^*,\dot{x},\dot{y}^*)} f^i(t,y^*,\dot{y}^*)dt. \quad (2.52)$$

Now suppose that y^* is not an efficient solution of (MVP). Then there exists an $x \in X_0$ such that

$$\int_a^b f^i(t,x,\dot{x})dt \leq \int_a^b f^i(t,y^*,\dot{y}^*)dt, \ \ \forall i = 1,\cdots,p$$

and

$$\int_a^b f^{i_0}(t,x,\dot{x})dt < \int_a^b f^{i_0}(t,y^*,\dot{y}^*)dt, \ \text{for some } i_0 = 1,\cdots,p$$

which implies that

$$\int_a^b \sum_{i=1}^p \frac{\tau_i^*}{\alpha_i(x,y^*,\dot{x},\dot{y}^*)} f^i(t,x,\dot{x})dt < \int_a^b \sum_{i=1}^p \frac{\tau_i^*}{\alpha_i(x,y^*,\dot{x},\dot{y}^*)} f^i(t,y^*,\dot{y}^*)dt. \tag{2.53}$$

Now (2.52) and (2.53) contradict each other. Hence the conclusion follows.

□

Theorem 2.5.5 (Converse Duality). Suppose that

(i) $(y^*,\tau^*,\lambda^*) \in Y_0$;

(ii) $y^* \in X_0$;

(iii) (f,g) is strictly pseudo quasi V-type I at y^* with respect to τ^*, λ^* and for some positive functions α_i, β_j for $i = 1,\cdots,p, j = 1,\cdots,m$.

Then y^* is an efficient solution of (MVP).

Proof. It follows by hypotheses (i) and (ii) that $\int_a^b \lambda_j^*(t)g^j(t,y^*,\dot{y}^*)dt \geq 0$, $\forall j = 1,\cdots,m$, which implies that $\int_a^b \sum_{j=1}^m \lambda_j^*(t)\beta_j(x,y^*,\dot{x},\dot{y}^*)g^j(t,y^*,\dot{y}^*)dt \geq 0$. From the above equality and Definition 2.2.6, we have

$$\int_a^b \sum_{j=1}^m \lambda_j^*(t)\eta(t,x,y^*,\dot{x},\dot{y}^*)\{g_x^j(t,y^*,\dot{y}^*) - \frac{d}{dt}g_{\dot{x}}^j(t,y^*,\dot{y}^*)\}dt \leq 0. \tag{2.54}$$

Combining (2.54) with hypothesis (i) and appealing to Definition 2.2.6 again we have

$$\int_a^b \sum_{i=1}^p \tau_i^* \alpha_i(x, y^*, \dot{x}, \dot{y}^*) f^i(t, x, \dot{x})dt > \int_a^b \sum_{i=1}^p \tau_i^* \alpha_i(x, y^*, \dot{x}, \dot{y}^*) f^i(t, y^*, \dot{y}^*)dt. \tag{2.55}$$

Next, if y^* is not an efficient solution of (MVP), then there exists an $x \in X_0$ such that

$$\int_a^b f^i(t, x, \dot{x})dt \leq \int_a^b f^i(t, y^*, \dot{y}^*)dt, \quad \forall i = 1, \cdots, p$$

and

$$\int_a^b f^{i_0}(t, x, \dot{x})dt < \int_a^b f^{i_0}(t, y^*, \dot{y}^*)dt, \quad \text{for some} \quad i_0 = 1, \cdots, p,$$

which implies that

$$\int_a^b \sum_{i=1}^p \tau_i^* \alpha_i(x, y^*, \dot{x}, \dot{y}^*) f^i(t, x, \dot{x})dt < \int_a^b \sum_{i=1}^p \tau_i^* \alpha_i(x, y^*, \dot{x}, \dot{y}^*) f^i(t, y^*, \dot{y}^*)dt. \tag{2.56}$$

Since (2.55) and (2.56) contradict each other, it follows that y^* is an efficient solution for (MVP). □

Now we establish some duality theorems between the multiobjective variational problem (MVPE) and its dual problem (MMVDE).

Theorem 2.5.6 (Weak Duality). Assume that for all feasible x for (MVPE) and all feasible (y, τ, λ, μ) for (MMVDE), any of the following holds:

(i) $(f, g + h)$ is weak strictly-pseudoquasi V-type I invex at y with respect to η and $\tau > 0$ and for some positive functions α_i, β_j for $i = 1, \cdots, p$, $j = 1, \cdots, m$;

(ii) $(f, g + h)$ is weak strictly pseudo V-type I invex at y with respect to η and $\tau > 0$ and for some positive functions α_i, β_j for $i = 1, \cdots, p, j = 1, \cdots, m$.

Then the following inequalities cannot hold:

$$\int_a^b f^i(t, x, \dot{x})dt \leq \int_a^b f^i(t, y, \dot{y})dt, \ \ \forall i = 1, \cdots, p \tag{2.57}$$

and

$$\int_a^b f^{i_0}(t, x, \dot{x})dt \quad < \quad \int_a^b f^{i_0}(t, y, \dot{y})dt, \ \text{ for some } i_0 = 1, \cdots, p. \tag{2.58}$$

Proof. Suppose contrary to the result that (2.57) and (2.58) hold. Then since each $\alpha_i > 0$ and $\tau > 0$, we have

$$\int_a^b \sum_{i=1}^p \tau_i \alpha_i(x, y, \dot{x}, \dot{y}) f^i(t, x, \dot{x})dt < \int_a^b \sum_{i=1}^p \tau_i \alpha_i(x, y, \dot{x}, \dot{y}) f^i(t, y, \dot{y})dt. \tag{2.59}$$

Since (y, τ, λ, μ) is feasible for (MMVDE), it follows that

$$\int_a^b \sum_{j=1}^m \lambda_j(t)\beta_j(x, y, \dot{x}, \dot{y}) g^j(t, y, \dot{y})dt \geq 0$$

and

$$\int_a^b \sum_{l=1}^q \mu_l(t)\gamma_l(x, y, \dot{x}, \dot{y}) h^l(t, y, \dot{y})dt = 0.$$

Hence

$$-\int_a^b \Big\{\sum_{j=1}^m \lambda_j(t)\beta_j(x,y,\dot{x},\dot{y})g^j(t,y,\dot{y}) + \sum_{l=1}^q \mu_l(t)\gamma_l(x,y,\dot{x},\dot{y})h^l(t,y,\dot{y})\Big\}dt \leq 0. \tag{2.60}$$

By the hypothesis (i) i.e; $(f, g+h)$ is weak strictly-pseudoquasi V-type I invex, (2.59) and (2.60) imply,

$$\int_a^b \sum_{i=1}^p \tau_i \eta(t,x,y,\dot{x},\dot{y})\{f_x^i(t,y,\dot{y}) - \frac{d}{dt}f_{\dot{x}}^i(t,y,\dot{y})\}dt < 0,$$

$$\int_a^b \eta(t,x,y,\dot{x},\dot{y})\Big(\sum_{j=1}^m \lambda_j(t)\{g_x^j(t,y,\dot{y}) - \frac{d}{dt}g_{\dot{x}}^j(t,y,\dot{y})\}$$
$$+\sum_{l=1}^q \mu_l(t)\{h_x^l(t,y,\dot{y}) - \frac{d}{dt}h_{\dot{x}}^l(t,y,\dot{y})\}\Big)dt \leq 0.$$

The above inequalities give

$$\int_a^b \eta(t,x,y,\dot{x},\dot{y})\Big(\sum_{i=1}^p \tau_i\{f_x^i(t,y,\dot{y}) - \frac{d}{dt}f_{\dot{x}}^i(t,y,\dot{y})\}$$
$$+\sum_{j=1}^m \lambda_j(t)\{g_x^j(t,y,\dot{y}) - \frac{d}{dt}g_{\dot{x}}^j(t,y,\dot{y})\}$$
$$+\sum_{l=1}^q \mu_l(t)\{h_x^l(t,y,\dot{y}) - \frac{d}{dt}h_{\dot{x}}^l(t,y,\dot{y})\}\Big)dt < 0, \tag{2.61}$$

which contradicts (2.26).

By we have the hypothesis (ii) i.e; $(f, g+h)$ is weak strictly pseudo V-type I invex, (2.59) and (2.60) imply

$$\int_a^b \sum_{i=1}^p \tau_i \eta\{f_x^i(t,y,\dot{y}) - \frac{d}{dt} f_{\dot{x}}^i(t,y,\dot{y})\}dt < 0, \tag{2.62}$$

$$\int_a^b \eta\Big(\sum_{j=1}^m \lambda_j(t)\{g_x^j(t,y,\dot{y}) - \frac{d}{dt} g_{\dot{x}}^j(t,y,\dot{y})\}$$
$$+\sum_{l=1}^q \mu_l(t)\{h_x^l(t,y,\dot{y}) - \frac{d}{dt} h_{\dot{x}}^l(t,y,\dot{y})\}\Big)dt < 0. \tag{2.63}$$

(2.62) and (2.63) imply (2.61), again contradicting (2.26). □

Corollary 2.5.2. Assume that weak duality holds between (MVPE) and (MMVDE). If $(y^*, \tau^*, \lambda^*, \mu^*)$ is feasible for (MMVDE) such that y^* is feasible for (MVPE), then y^* is an efficient solution for (MVPE) and $(y^*, \tau^*, \lambda^*, \mu^*)$ is an efficient solution for (MMVDE).

Proof. Suppose that y^* is not an efficient for (MVPE); then there exists a feasible x for (MVPE) such that (2.57) and (2.59) hold. But $(y^*, \tau^*, \lambda^*, \mu^*)$ is feasuble for (MMVDE), hence the result of weak duality is contradicted. Therefore, y^* must be efficient for (MVPE). Now suppose $(y^*, \tau^*, \lambda^*, \mu^*)$ is not an efficient for (MMVDE). Then there exist some (x, τ, λ, μ) feasible for (MMVDE) such that

$$\int_a^b f^i(t,x,\dot{x})\,dt \geq \int_a^b f^i(t,y^*,\dot{y}^*)\,dt, \ \ \forall\ i = 1, \cdots, p$$
$$\text{and} \ \int_a^b f^{i_0}(t,x,\dot{x})\,dt > \int_a^b f^{i_0}(t,y^*,\dot{y}^*)\,dt, \ \ \text{for some } i_0 = 1, \cdots, p.$$

This contradicts weak duality. Hence $(y^*, \tau^*, \lambda^*, \mu^*)$ is an efficient for (MMVDE).

□

Theorem 2.5.7 (Strong Duality). Assume that

(i) x^* is an efficient solution for (MVPE);

(ii) for all $k = 1, \cdots, p$, x^* a constraint qualification for problem $MVPE_k(x^*)$ is satisfied at x^*.

Then there exist $\tau^* \in R^p, \tau^* > 0$, and piecewise smooth function $\lambda^* : I \to R^m, \lambda^*(t) \geq 0$ and $\mu^* : I \to R^q$ such that $(x^*, \tau^*, \lambda^*, \mu^*)$ is feasible for (MMVDE).

Further, if also weak duality holds between (MVPE) and (MMVDE), then $(x^*, \tau^*, \lambda^*, \mu^*)$ is an efficient solution for (MMVDE).

Proof. Since x^* is an efficient solution of (MVPE), then from Lemma 2.2.2, x^* solves $MVPE_k(x^*)$ for each $k = 1, \cdots, p$. From Kuhn-Tucker necessary conditions for each $i = 1, \cdots, p$, we obtain $\tau_i^k \geq 0$ for all $i \neq k$, $\lambda^i(t) \geq 0 \in R^m$ and $\mu^i(t) \in R^q$ such that

$$\{f_x^i(t, x^*, \dot{x}^*) - \frac{d}{dt} f_{\dot{x}}^i(t, x^*, \dot{x}^*)\} + \sum_{i \neq k} \tau_i^k \{f_x^k(t, x^*, \dot{x}^*) - \frac{d}{dt} f_{\dot{x}}^k(t, x^*, \dot{x}^*)\}$$

$$+ \sum_{j=1}^{m} \lambda_j^i(t) \{g_x^j(t, x^*, \dot{x}^*) - \frac{d}{dt} g_{\dot{x}}^j(t, x^*, \dot{x}^*)\}$$

$$+ \sum_{l=1}^{q} \mu_l^i(t) \{h_x^l(t, x^*, \dot{x}^*) - \frac{d}{dt} h_{\dot{x}}^l(t, x^*, \dot{x}^*)\} = 0, \tag{2.64}$$

$$\int_a^b \sum_{j=1}^{m} \lambda_j^i(t) g^j(t, x^*, \dot{x}^*) dt = 0. \tag{2.65}$$

Summing (2.64) over $i = 1, \cdots, p$, we have

$$(1+\tau_2^1+\cdots+\tau_p^1)\{f_x^1(t,x^*,\dot{x}^*)-\frac{d}{dt}f_{\dot{x}}^1(t,x^*,\dot{x}^*)\}$$
$$+\sum_{j=1}^{m}\lambda_j^1(t)\{g_x^j(t,x^*,\dot{x}^*)-\frac{d}{dt}g_{\dot{x}}^j(t,x^*,\dot{x}^*)\}$$
$$+\sum_{l=1}^{q}\mu_l^1(t)\{h_x^l(t,x^*,\dot{x}^*)-\frac{d}{dt}h_{\dot{x}}^l(t,x^*,\dot{x}^*)\}$$
$$+(\tau_1^2+1+\cdots+\tau_p^2)\{f_x^2(t,x^*,\dot{x}^*)-\frac{d}{dt}f_{\dot{x}}^2(t,x^*,\dot{x}^*)\}$$

$$+\sum_{j=1}^{m}\lambda_j^2(t)\{g_x^j(t,x^*,\dot{x}^*)-\frac{d}{dt}g_{\dot{x}}^j(t,x^*,\dot{x}^*)\}$$
$$+\sum_{l=1}^{q}\mu_l^2(t)\{h_x^l(t,x^*,\dot{x}^*)-\frac{d}{dt}h_{\dot{x}}^l(t,x^*,\dot{x}^*)\}+\cdots$$
$$+(\tau_1^p+\tau_2^p+\cdots+1)\{f_x^p(t,x^*,\dot{x}^*)-\frac{d}{dt}f_{\dot{x}}^p(t,x^*,\dot{x}^*)\}$$
$$+\sum_{j=1}^{m}\lambda_j^p(t)\{g_x^j(t,x^*,\dot{x}^*)-\frac{d}{dt}g_{\dot{x}}^j(t,x^*,\dot{x}^*)\}$$
$$+\sum_{l=1}^{q}\mu_l^p(t)\{h_x^l(t,x^*,\dot{x}^*)-\frac{d}{dt}h_{\dot{x}}^l(t,x^*,\dot{x}^*)\}=0$$

Let $\tau_1^*=1+\tau_2^1+\cdots+\tau_p^1,\ \tau_2^*=\tau_1^2+1+\cdots+\tau_p^2,\ \cdots,\tau_p^*=\tau_1^p+\tau_2^p+\cdots+1,\ \lambda_j^*(t)=\sum_{k=1}^{p}\lambda_j^k(t),\ j=1,\cdots,m,\ \lambda^*(t)=(\lambda_1^*(t),\cdots,\lambda_m^*(t)),\ \sum_{k=1}^{p}\mu_l^k(t)=\mu_l^*(t),\ l=1,\cdots,q$ and $\mu^*(t)=(\mu_1^*(t),\cdots,\mu_q^*(t))$.

Then we have

$$\sum_{i=1}^{p}\tau_i^*\{f_x^i(t,x^*,\dot{x}^*)-\frac{d}{dt}f_{\dot{x}}^i(t,x^*,\dot{x}^*)\}+\sum_{j=1}^{m}\lambda_j^*(t)\{g_x^j(t,x^*,\dot{x}^*)-\frac{d}{dt}g_{\dot{x}}^j(t,x^*,\dot{x}^*)\}$$
$$+\sum_{l=1}^{q}\mu_l^*(t)\{h_x^l(t,x^*,\dot{x}^*)-\frac{d}{dt}h_{\dot{x}}^l(t,x^*,\dot{x}^*)\}=0.$$

Summing (2.65) for $i = 1, \cdots, p$, we have $\int_a^b \sum_{j=1}^m \lambda_j^*(t) g^j(t, x^*, \dot{x}^*)dt = 0$. We conclude that $(x^*, \tau^*, \lambda^*, \mu^*)$ is feasible for (MMVDE). Efficiency of $(x^*, \tau^*, \lambda^*, \mu^*)$ for (MMVDE) now follows from Corollary 2.5.2. □

Theorem 2.5.8 (Converse Duality). Suppose that

(i) $(y^*, \tau^*, \lambda^*, \mu^*) \in Y_1$;

(ii) $y^* \in X_1$;

(iii) $(f + \sum_{l=1}^q \mu_l(t) h_l, g)$ is V-type I invex at y^* with respect to η and $\tau^* > 0$ and for some positive functions α_i, β_j for $i = 1, \cdots, p, j = 1, \cdots, m$;

Then y^* is an efficient solution of (MVPE).

Proof. It follows by the hypotheses (i) and (ii) that

$$\int_a^b \lambda_j^*(t) g^j(t, y^*, \dot{y}^*)dt = 0, \quad \forall j = 1, \cdots, m. \tag{2.66}$$

By hypothesis (iii), for any $x \in X_1$, we have

$$\begin{aligned}
&\int_a^b f^i(t, x, \dot{x})dt - \int_a^b \{f^i(t, y^*, \dot{y}^*) + \sum_{l=1}^q \mu_l^*(t) h^l(t, y^*, \dot{y}^*)\}dt \\
&\geq \int_a^b \alpha_i(x, y^*, \dot{x}, \dot{y}^*) \eta(t, x, y^*, \dot{x}, \dot{y}^*) \Big(\{f_x^i(t, y^*, \dot{y}^*) - \frac{d}{dt} f_{\dot{x}}^i(t, y^*, \dot{y}^*)\} \\
&+ \sum_{l=1}^q \mu_l^*(t) \{h^l(t, y^*, \dot{y}^*) - \frac{d}{dt} h_{\dot{x}}^l(t, y^*, \dot{y}^*)\} \Big) dt, \ \forall i = 1, \cdots, p,
\end{aligned} \tag{2.67}$$

$$\begin{aligned}
&- \int_a^b g^j(t, y^*, \dot{y}^*)dt \\
&\geq \int_a^b \beta_j(x, y^*, \dot{x}, \dot{y}^*) \eta(t, x, y^*, \dot{x}, \dot{y}^*) \{g_x^j(t, y^*, \dot{y}^*) - \frac{d}{dt} g_{\dot{x}}^j(t, y^*, \dot{y}^*)\}dt, \\
&\forall j = 1, \cdots, m.
\end{aligned} \tag{2.68}$$

Now by the facts that $\alpha_i > 0,\ \beta_j > 0\ \forall\ i,j$ and $\tau^* > 0, \lambda^*(t) \geq 0$, it follows by (2.67) and (2.68) that

$$\int_a^b \sum_{i=1}^p \frac{\tau_i^*}{\alpha_i} f^i(t,x,\dot{x})dt - \int_a^b \sum_{i=1}^p \frac{\tau_i^*}{\alpha_i}\{f^i(t,y^*,\dot{y}^*) + \sum_{l=1}^q \mu_l^*(t)h^l(t,y^*,\dot{y}^*)\}dt$$
$$- \int_a^b \sum_{j=1}^m \frac{\lambda_j^*(t)}{\beta_j} g^j(t,y^*,\dot{y}^*)dt$$
$$\geq \int_a^b \eta(t,x,y^*,\dot{x},\dot{y}^*)\Big(\sum_{i=1}^p \tau_i^*\{f_x^i(t,y^*,\dot{y}^*) - \frac{d}{dt} f_{\dot{x}}^i(t,y^*,\dot{y}^*)\}$$
$$+ \sum_{j=1}^m \lambda_j^*(t)\{g_x^j(t,y^*,\dot{y}^*) - \frac{d}{dt} g_{\dot{x}}^j(t,y^*,\dot{y}^*)\}$$
$$+ \sum_{l=1}^q \mu_l^*(t)\{h_x^l(t,y^*,\dot{y}^*) - \frac{d}{dt} h_{\dot{x}}^l(t,y^*,\dot{y}^*)\}\Big)dt = 0. \tag{2.69}$$

From (2.28), (2.66) and (2.69) it follows that

$$\int_a^b \sum_{i=1}^p \frac{\tau_i^*}{\alpha_i(x,y^*,\dot{x},\dot{y}^*)} f^i(t,x,\dot{x})dt \geq \int_a^b \sum_{i=1}^p \frac{\tau_i^*}{\alpha_i(x,y^*,\dot{x},\dot{y}^*)} f^i(t,y^*,\dot{y}^*)dt. \tag{2.70}$$

Now suppose that y^* is not an efficient solution of (MVPE). Then there exists an $x \in X_1$ such that

$$\int_a^b f^i(t,x,\dot{x})dt \leq \int_a^b f^i(t,y^*,\dot{y}^*)dt,\ \ \forall i = 1,\cdots,p$$

and

$$\int_a^b f^{i_0}(t,x,\dot{x})dt < \int_a^b f^{i_0}(t,y^*,\dot{y}^*)dt,\ \text{for some } i_0 = 1,\cdots,p$$

which implies that

$$\int_a^b \sum_{i=1}^p \frac{\tau_i^*}{\alpha_i(x,y^*,\dot{x},\dot{y}^*)} f^i(t,x,\dot{x})dt < \int_a^b \sum_{i=1}^p \frac{\tau_i^*}{\alpha_i(x,y^*,\dot{x},\dot{y}^*)} f^i(t,y^*,\dot{y}^*)dt. \tag{2.71}$$

Now (2.70) and (2.71) contradict each other. Hence the conclusion follows. □

Now we establish some duality theorems between the multiobjective variational problem (MVPE) and its generalized Mond-Weir dual problem (GMMVDE).

Theorem 2.5.9 (Weak Duality). Assume that for all feasible x for (MVPE) and all feasible (y, τ, λ, μ) for (GMMVDE), any of the following holds:

(i) $\tau > 0$, and $(f+\sum_{j\in J_0}\lambda_j(t)g^j+\sum_{l\in K_0}\mu_l(t)h^l,\ \sum_{j\in J_\alpha}\lambda_j(t)g^j+\sum_{l\in K_\alpha}\mu_l(t)h^l)$ is weak strictly-pseudoquasi V-type I invex at y with respect to η for any $\alpha, 1 \le \alpha \le r$ and for some positive functions α_i, for $i = 1, \cdots, p$ and β;

(ii) $(f+\sum_{j\in J_0}\lambda_j(t)g^j+\sum_{l\in K_0}\mu_l(t)h^l,\ \sum_{j\in J_\alpha}\lambda_j(t)g^j+\sum_{l\in K_\alpha}\mu_l(t)h^l)$ is weak strictly pseudo V-type I invex at y with respect to η and for some positive functions α_i, for $i = 1, \cdots, p$ and β;

(iii) $(f+\sum_{j\in J_0}\lambda_j(t)g^j+\sum_{l\in K_0}\mu_l(t)h^l,\ \sum_{j\in J_\alpha}\lambda_j(t)g^j+\sum_{l\in K_\alpha}\mu_l(t)h^l)$ is weak quasistrictly-pseudo V-type I invex at y with respect to η and for some positive functions α_i, for $i = 1, \cdots, p$ and β.

Then the following inequalities cannot hold:

$$\int_a^b f^i(t,x,\dot{x})\,dt$$
$$\leq \int_a^b \{f^i(t,y,\dot{y}) + \sum_{j\in J_0} \lambda_j(t) g^j(t,y,\dot{y}) + \sum_{l\in K_0} \mu_l(t) h^l(t,y,\dot{y})\}\,dt,$$
$$\forall\ i = 1,\cdots,p \tag{2.72}$$

and

$$\int_a^b f^{i_0}(t,x,\dot{x})\,dt$$
$$< \int_a^b \{f^{i_0}(t,y,\dot{y}) + \sum_{j\in J_0} \lambda_j(t) g^j(t,y,\dot{y}) + \sum_{l\in K_0} \mu_l(t) h^l(t,y,\dot{y})\}\,dt,$$
$$\text{for some} \quad i_0 = 1,\cdots,p. \tag{2.73}$$

Proof. Suppose contrary to the result that (2.72) and (2.73) hold. Since x is feasible for (MVPE), and $\lambda(t) \geq 0$, (2.72) and (2.73) imply

$$\int_a^b \{f^i(t,x,\dot{x}) + \sum_{j\in J_0} \lambda_j(t) g^j(t,x,\dot{x}) + \sum_{l\in K_0} \mu_l(t) h^l(t,x,\dot{x})\}\,dt$$
$$\leq \int_a^b \{f^i(t,y,\dot{y}) + \sum_{j\in J_0} \lambda_j(t) g^j(t,y,\dot{y}) + \sum_{l\in K_0} \mu_l(t) h^l(t,y,\dot{y})\}\,dt,$$
$$\forall\ i = 1,\cdots,p$$

and

$$\int_a^b \{f^{i_0}(t,x,\dot{x}) + \sum_{j\in J_0} \lambda_j(t) g^j(t,x,\dot{x}) + \sum_{l\in K_0} \mu_l(t) h^l(t,x,\dot{x})\}\,dt$$
$$< \int_a^b \{f^{i_0}(t,y,\dot{y}) + \sum_{j\in J_0} \lambda_j(t) g^j(t,y,\dot{y}) + \sum_{l\in K_0} \mu_l(t) h^l(t,y,\dot{y})\}\,dt,$$
$$\text{for some} \quad i_0 = 1,\cdots,p.$$

Then since $\alpha_i > 0$, (2.35) we have

$$\int_a^b \sum_{i=1}^p \tau_i \alpha_i(x,y,\dot{x},\dot{y})\{f^i(t,x,\dot{x}) + \sum_{j\in J_0} \lambda_j(t) g^j(t,x,\dot{x}) + \sum_{l\in K_0} \mu_l(t) h^l(t,x,\dot{x})\}\, dt$$
$$< \int_a^b \sum_{i=1}^p \tau_i \alpha_i(x,y,\dot{x},\dot{y})\{f^i(t,y,\dot{y}) + \sum_{j\in J_0} \lambda_j(t) g^j(t,y,\dot{y}) + \sum_{l\in K_0} \mu_l(t) h^l(t,y,\dot{y})\}\, dt. \tag{2.74}$$

Also, from (2.33) and $\beta > 0$ we have

$$-\int_a^b \{\sum_{j\in J_\alpha} \lambda_j(t)\beta(x,y,\dot{x},\dot{y}) g^j(t,y,\dot{y}) + \sum_{l\in K_\alpha} \mu_l(t)\beta(x,y,\dot{x},\dot{y}) h^l(t,y,\dot{y})\} dt \leq 0,$$

for all $1 \leq \alpha \leq r$. (2.75)

Using hypothesis (i), we see that (2.74) and (2.75) together give

$$\int_a^b \sum_{i=1}^p \tau_i \eta(t,x,y,\dot{x},\dot{y}) \Big(\{f^i_x(t,y,\dot{y}) - \frac{d}{dt} f^i_{\dot{x}}(t,y,\dot{y})\}$$
$$+ \sum_{j\in J_0} \lambda_j(t)\{g^j_x(t,y,\dot{y}) - \frac{d}{dt} g^j_{\dot{x}}(t,y,\dot{y})\}$$
$$+ \sum_{l\in K_0} \mu_l(t)\{h^l_x(t,y,\dot{y}) - \frac{d}{dt} h^l_{\dot{x}}(t,y,\dot{y})\} \Big)\, dt < 0,$$
$$\int_a^b \eta(t,x,y,\dot{x},\dot{y}) \Big(\sum_{j\in J_\alpha} \lambda_j(t)\{g^j_x(t,y,\dot{y}) - \frac{d}{dt} g^j_{\dot{x}}(t,y,\dot{y})\}$$
$$+ \sum_{l\in K_\alpha} \mu_l(t)\{h^l_x(t,y,\dot{y}) - \frac{d}{dt} h^l_{\dot{x}}(t,y,\dot{y})\} \Big)\, dt \leq 0, \ \ \forall 1 \leq \alpha \leq r.$$

Since (2.35), the above inequalities give

$$\int_a^b \eta(t,x,y,\dot{x},\dot{y})\Big(\sum_{i=1}^{p}\tau_i\{f_x^i(t,y,\dot{y})-\frac{d}{dt}f_{\dot{x}}^i(t,y,\dot{y})\}+\sum_{j=0}^{r}\lambda_j(t)\{g_x^j(t,y,\dot{y})$$

$$-\frac{d}{dt}g_{\dot{x}}^j(t,y,\dot{y})\}+\sum_{l=0}^{r}\mu_l(t)\{h_x^l(t,y,\dot{y})-\frac{d}{dt}h_{\dot{x}}^l(t,y,\dot{y})\}\Big)\,dt<0. \tag{2.76}$$

Since $J_0, J_1, \cdots, J_r$ are partitions of $\{1, \cdots, m\}$ and $K_0, K_1, \cdots, K_r$ are partitions of $\{1, \cdots, k\}$, (2.76) is equivalent to

$$\int_a^b \eta(t,x,y,\dot{x},\dot{y})\Big(\sum_{i=1}^{p}\tau_i\{f_x^i(t,y,\dot{y})-\frac{d}{dt}f_{\dot{x}}^i(t,y,\dot{y})\}+\sum_{j=1}^{m}\lambda_j(t)\{g_x^j(t,y,\dot{y})$$

$$-\frac{d}{dt}g_{\dot{x}}^j(t,y,\dot{y})\}+\sum_{l=1}^{q}\mu_l(t)\{h_x^l(t,y,\dot{y})-\frac{d}{dt}h_{\dot{x}}^l(t,y,\dot{y})\}\Big)\,dt<0, \tag{2.77}$$

which contradicts (2.32).

Suppose now that (ii) is satisfied. Again from (2.74) and (2.75) it follows that

$$\int_a^b \sum_{i=1}^{p}\tau_i\eta(t,x,y,\dot{x},\dot{y})\Big(\{f_x^i(t,y,\dot{y})-\frac{d}{dt}f_{\dot{x}}^i(t,y,\dot{y})\}$$

$$+\sum_{j\in J_0}\lambda_j(t)\{g_x^j(t,y,\dot{y})-\frac{d}{dt}g_{\dot{x}}^j(t,y,\dot{y})\}$$

$$+\sum_{l\in K_0}\mu_l(t)\{h_x^l(t,y,\dot{y})-\frac{d}{dt}h_{\dot{x}}^l(t,y,\dot{y})\}\Big)\,dt<0,$$

$$\int_a^b \eta(t,x,y,\dot{x},\dot{y})\Big(\sum_{j\in J_\alpha}\lambda_j(t)\{g_x^j(t,y,\dot{y})-\frac{d}{dt}g_{\dot{x}}^j(t,y,\dot{y})\}$$

$$+\sum_{l\in K_\alpha}\mu_l(t)\{h_x^l(t,y,\dot{y})-\frac{d}{dt}h_{\dot{x}}^l(t,y,\dot{y})\}\Big)\,dt<0,\ \ \forall 1\le\alpha\le r.$$

Since (2.35), the above inequalities give

$$\int_a^b \eta(t,x,y,\dot{x},\dot{y})\Big(\sum_{i=1}^{p}\tau_i\{f_x^i(t,y,\dot{y})-\frac{d}{dt}f_{\dot{x}}^i(t,y,\dot{y})\}+\sum_{j=0}^{r}\lambda_j(t)\{g_x^j(t,y,\dot{y})$$
$$-\frac{d}{dt}g_{\dot{x}}^j(t,y,\dot{y})\}+\sum_{l=0}^{r}\mu_l(t)\{h_x^l(t,y,\dot{y})-\frac{d}{dt}h_{\dot{x}}^l(t,y,\dot{y})\}\Big)\,dt<0, \tag{2.78}$$

and then again we have (2.77). Also we obtain a contradiction.

Using hypothesis (iii), we see that (2.74) and (2.75) together give

$$\int_a^b \sum_{i=1}^{p}\tau_i\eta(t,x,y,\dot{x},\dot{y})\Big(\{f_x^i(t,y,\dot{y})-\frac{d}{dt}f_{\dot{x}}^i(t,y,\dot{y})\}$$
$$+\sum_{j\in J_0}\lambda_j(t)\{g_x^j(t,y,\dot{y})-\frac{d}{dt}g_{\dot{x}}^j(t,y,\dot{y})\}$$
$$+\sum_{l\in K_0}\mu_l(t)\{h_x^l(t,y,\dot{y})-\frac{d}{dt}h_{\dot{x}}^l(t,y,\dot{y})\}\Big)\,dt\leq 0,$$
$$\int_a^b \eta(t,x,y,\dot{x},\dot{y})\Big(\sum_{j\in J_\alpha}\lambda_j(t)\{g_x^j(t,y,\dot{y})-\frac{d}{dt}g_{\dot{x}}^j(t,y,\dot{y})\}$$
$$+\sum_{l\in K_\alpha}\mu_l(t)\{h_x^l(t,y,\dot{y})-\frac{d}{dt}h_{\dot{x}}^l(t,y,\dot{y})\}\Big)\,dt<0,\ \ \forall 1\leq\alpha\leq r.$$

and then again we have (2.77). Also we obtain a contradiction. □

Corollary 2.5.3. Assume that weak duality holds between (MVPE) and (GMMVDE). If $(y^*,\tau^*,\lambda^*,\mu^*)$ is feasible for (GMMVDE) with $\sum_{j\in J_0}\lambda_j^*(t)g^j(t,y^*,\dot{y}^*)=0$ and y^* is feasible for (MVPE), then y^* is an efficient for (MVPE) and $(y^*,\tau^*,\lambda^*,\mu^*)$ is an efficient for (GMMVDE).

Proof. Suppose that y^* is not an efficient for (MVPE). Then there exists a feasible x for (MVPE) such that

$$\int_a^b f^i(t,x,\dot{x})\,dt \leq \int_a^b f^i(t,y^*,\dot{y}^*)\,dt, \quad \forall\ i=1,\cdots,p \tag{2.79}$$

and

$$\int_a^b f^{i_0}(t,x,\dot{x})\,dt < \int_a^b f^{i_0}(t,y^*,\dot{y}^*)\,dt, \quad \text{for some } i_0=1,\cdots,p. \tag{2.80}$$

By hypotheses $\sum_{j\in J_0}\lambda_j^*(t)g^j(t,y^*,\dot{y}^*)=0$ and $\sum_{l=1}^q \mu_l^*(t)h^l(t,y^*,\dot{y}^*)=0$, so (2.79) and (2.80) can be written as

$$\int_a^b f^i(t,x,\dot{x})\,dt \leq \int_a^b \{f_i(t,y^*,\dot{y}^*)+\sum_{j\in J_0}\lambda_j^*(t)g^j(t,y^*,\dot{y}^*)$$
$$+\sum_{l=1}^q \mu_l^*(t)h^l(t,y^*,\dot{y}^*)\}\,dt, \quad \forall\ i=1,\cdots,p$$

and

$$\int_a^b f^{i_0}(t,x,\dot{x})\,dt < \int_a^b \{f^{i_0}(t,y^*,\dot{y}^*)+\sum_{j\in J_0}\lambda_j^*(t)g^j(t,y^*,\dot{y}^*)$$
$$+\sum_{l=1}^q \mu_k^*(t)h^l(t,y^*,\dot{y}^*)\}\,dt, \quad \text{for some } i_0=1,\cdots,p.$$

Since $(y^*,\tau^*,\lambda^*,\mu^*)$ is feasible in (GMMVDE) and x is feasible for (MVPE), these inequalities contradict weak duality.

Also suppose that $(y^*,\tau^*,\lambda^*,\mu^*)$ is not an efficient for (GMMVDE). Then there exists a feasible (y,τ,λ,μ) for (GMMVDE) such that

$$\int_a^b \{f^i(t,y,\dot{y}) + \sum_{j\in J_0} \lambda_j(t) g^j(t,y,\dot{y}) + \sum_{l\in K_0} \mu_l(t) h^l(t,y,\dot{y})\}\, dt$$
$$\geq \int_a^b \{f^i(t,y^*,\dot{y}^*) + \sum_{j\in J_0} \lambda_j^*(t) g^j(t,y^*,\dot{y}^*) + \sum_{l\in K_0} \mu_l^*(t) h^l(t,y^*,\dot{y}^*)\}\, dt,$$
$$\forall\ i = 1,\cdots,p \tag{2.81}$$

and

$$\int_a^b \{f^{i_0}(t,y,\dot{y}) + \sum_{j\in J_0} \lambda_j(t) g^j(t,y,\dot{y}) + \sum_{l\in K_0} \mu_l(t) h^l(t,y,\dot{y})\}\, dt$$
$$> \int_a^b \{f^{i_0}(t,y^*,\dot{y}^*) + \sum_{j\in J_0} \lambda_j^*(t) g^j(t,y^*,\dot{y}^*) + \sum_{l\in K_0} \mu_l^*(t) h^l(t,y^*,\dot{y}^*)\}\, dt,$$
$$\text{for some } i_0 = 1,\cdots,p, \tag{2.82}$$

and since $\sum_{j\in J_0} \lambda_j^*(t) g^j(t,y^*,\dot{y}^*) = 0$, $\sum_{l\in K_0} \mu_l^*(t) h^l(t,y^*,\dot{y}^*) = 0$, (2.81) and (2.82) reduce to

$$\int_a^b \{f^i(t,y,\dot{y}) + \sum_{j\in J_0} \lambda_j(t) g^j(t,y,\dot{y}) + \sum_{l\in K_0} \mu_l(t) h^l(t,y,\dot{y})\}\, dt$$
$$\geq \int_a^b f^i(t,y^*,\dot{y}^*)\, dt, \quad \forall\ i = 1,\cdots,p$$

and

$$\int_a^b \{f^{i_0}(t,y,\dot{y}) + \sum_{j\in J_0} \lambda_j(t) g^j(t,y,\dot{y}) + \sum_{l\in K_0} \mu_l(t) h^l(t,y,\dot{y})\}\, dt$$
$$> \int_a^b f^{i_0}(t,y^*,\dot{y}^*)\, dt, \quad \text{for some } i_0 = 1,\cdots,p.$$

Since y^* is feasible for (MVPE), these inequalities contradict weak duality. Therefore y^* and $(y^*, \tau^*, \lambda^*, \mu^*)$ are an efficient for their respective problems.

□

Theorem 2.5.10 (Strong Duality). Assume that

(i) x^* is an efficient solution for (MVPE);

(ii) for all $k = 1, \cdots, p$, x^* a constraint qualification for problem $MVPE_k(x^*)$ is satisfied at x^*.

Then there exist $\tau^* \in R^p, \tau^* > 0$, and piecewise smooth functions $\lambda^* : I \to R^m, \lambda^*(t) \geq 0$ and $\mu^* : I \to R^q$ such that $(x^*, \tau^*, \lambda^*, \mu^*)$ is feasible for (GM-MVDE) and $\sum_{j \in J_0} \lambda_j^*(t) g^j(t, x^*, \dot{x}^*) + \sum_{l \in K_0} \mu_l^*(t) h^l(t, x^*, \dot{x}^*) = 0$.

Further, if also weak duality holds between (MVPE) and (GMMVDE), then $(x^*, \tau^*, \lambda^*, \mu^*)$ is an efficient solution for (GMMVDE). □

Proof. Similar to the proof of Theorem 2.5.7 and Corollary 2.5.3 above.

2.6. Special Case

As a special case of our duality results between (MVPE) and (GMMVDE), we give Wolfe type duality theorems.

If $J_0 = \{1, \cdots, m\}, J_\alpha = \emptyset, K_0 = \{1, \cdots, k\}, K_\alpha = \emptyset$, then (GMMVDE) reduced to the Wolfe type dual [2].

(WMVDE): Maximize

$$\Big(\int_a^b \{ f^1(t, y, \dot{y}) + \sum_{j=1}^m \lambda_j(t) g^j(t, y, \dot{y}) + \sum_{l=1}^q \mu_l(t) h^l(t, y, \dot{y}) \} \, dt,$$
$$\cdots, \int_a^b \{ f^p(t, y, \dot{y}) + \sum_{j=1}^m \lambda_j(t) g^j(t, y, \dot{y}) + \sum_{l=1}^q \mu_l(t) h^l(t, y, \dot{y}) \} \, dt \Big)$$

subject to
$$y(a) = t_0, \; y(b) = t_f,$$
$$\sum_{i=1}^{p} \tau_i\{f_x^i(t,y,\dot{y}) - \frac{d}{dt}f_{\dot{x}}^i(t,y,\dot{y})\}$$
$$+ \sum_{j=1}^{m} \lambda_j(t)\{g_x^j(t,y,\dot{y}) - \frac{d}{dt}g_{\dot{x}}^j(t,y,\dot{y})\}$$
$$+ \sum_{l=1}^{q} \mu_l(t)\{h_x^l(t,y,\dot{y}) - \frac{d}{dt}h_{\dot{x}}^l(t,y,\dot{y})\} = 0,$$
$$\lambda(t) \geq 0,$$
$$\tau_i \geq 0, \; \sum_{i=1}^{p} \tau_i = 1.$$

Now we establish some duality theorems between the multiobjective variational problem (MVPE) and its Wolfe type dual problem (WMVDE).

Theorem 2.6.1 (Weak Duality). Assume that for all feasible x for (MVPE) and all feasible (y, τ, λ, μ) for (WMVDE), any of the following holds:

(i) $(f + \sum_{j=1}^{m} \lambda_j(t)g^j + \sum_{l=1}^{q} \mu_l(t)h^l, \; 0)$ is weak strictly-pseudoquasi V-type I invex at y with respect to η and for some positive functions α_i, for $i = 1, \cdots, p$;

(ii) $(f + \sum_{j=1}^{m} \lambda_j(t)g^j + \sum_{l=1}^{q} \mu_l(t)h^l, \; 0)$ is weak strictly pseudo V-type I invex at y with respect to η and for some positive functions α_i, for $i = 1, \cdots, p$;

(iii) $(f + \sum_{j=1}^{m} \lambda_j(t)g^j + \sum_{l=1}^{q} \mu_l(t)h^l, \; 0)$ is weak quasistrictly-pseudo V-type I invex at y with respect to η and for some positive functions α_i, for $i = 1, \cdots, p$.

Then the following cannot hold:

$$\int_a^b f^i(t,x,\dot{x})\,dt$$
$$\leq \int_a^b \{f^i(t,y,\dot{y}) + \sum_{j=1}^m \lambda_j(t) g^j(t,y,\dot{y}) + \sum_{l=1}^q \mu_l(t) h^l(t,y,\dot{y})\}\,dt,$$
$$\forall\ i = 1,\cdots,p \tag{2.83}$$

and

$$\int_a^b f^{i_0}(t,x,\dot{x})\,dt$$
$$< \int_a^b \{f^{i_0}(t,y,\dot{y}) + \sum_{j=1}^m \lambda_j(t) g^j(t,y,\dot{y}) + \sum_{l=1}^q \mu_l(t) h^l(t,y,\dot{y})\}\,dt,$$
$$\text{for some} \quad i_0 = 1,\cdots,p. \tag{2.84}$$

Proof. Suppose contrary to the result that (2.83) and (2.84) hold. Since x is feasible for (MVPE), and $\lambda(t) \geq 0$, (2.83) and (2.84) imply

$$\int_a^b \{f^i(t,x,\dot{x}) + \sum_{j=1}^m \lambda_j(t) g^j(t,x,\dot{x}) + \sum_{l=1}^q \mu_l(t) h^l(t,x,\dot{x})\}\,dt$$
$$\leq \int_a^b \{f^i(t,y,\dot{y}) + \sum_{j=1}^m \lambda_j(t) g^j(t,y,\dot{y}) + \sum_{l=1}^q \mu_l(t) h^l(t,y,\dot{y})\}\,dt, \quad \forall\ i = 1,\cdots,p$$

and

$$\int_a^b \{f^{i_0}(t,x,\dot{x}) + \sum_{j=1}^m \lambda_j(t) g^j(t,x,\dot{x}) + \sum_{l=1}^q \mu_l(t) h^l(t,x,\dot{x})\}\,dt$$
$$< \int_a^b \{f^{i_0}(t,y,\dot{y}) + \sum_{j=1}^m \lambda_j(t) g^j(t,y,\dot{y}) + \sum_{l=1}^q \mu_l(t) h^l(t,y,\dot{y})\}\,dt,$$

for some $\quad i_0 = 1,\cdots,p.$

Then since $\alpha_i > 0$, (2.35) we have

$$\int_a^b \sum_{i=1}^p \tau_i\alpha_i(x,y,\dot{x},\dot{y})\{f^i(t,x,\dot{x}) + \sum_{j=1}^m \lambda_j(t)g^j(t,x,\dot{x}) + \sum_{l=1}^q \mu_l(t)h^l(t,x,\dot{x})\}\,dt$$
$$< \int_a^b \sum_{i=1}^p \tau_i\alpha_i(x,y,\dot{x},\dot{y})\{f^i(t,y,\dot{y}) + \sum_{j=1}^m \lambda_j(t)g^j(t,y,\dot{y}) + \sum_{l=1}^q \mu_l(t)h^l(t,y,\dot{y})\}\,dt. \tag{2.85}$$

Using hypothesis (i), we see that (2.85) give

$$\int_a^b \sum_{i=1}^p \tau_i\eta(t,x,y,\dot{x},\dot{y})\Big(\{f_x^i(t,y,\dot{y}) - \frac{d}{dt}f_{\dot{x}}^i(t,y,\dot{y})\} + \sum_{j=1}^m \lambda_j(t)\{g_x^j(t,y,\dot{y})$$
$$- \frac{d}{dt}g_{\dot{x}}^j(t,y,\dot{y})\} + \sum_{l=1}^q \mu_l(t)\{h_x^l(t,y,\dot{y}) - \frac{d}{dt}h_{\dot{x}}^l(t,y,\dot{y})\}\Big)\,dt < 0.$$

Since (2.35), the above inequalities give

$$\int_a^b \eta(t,x,y,\dot{x},\dot{y})\Big(\sum_{i=1}^p \tau_i\{f_x^i(t,y,\dot{y}) - \frac{d}{dt}f_{\dot{x}}^i(t,y,\dot{y})\} + \sum_{j=1}^m \lambda_j(t)\{g_x^j(t,y,\dot{y})$$
$$- \frac{d}{dt}g_{\dot{x}}^j(t,y,\dot{y})\} + \sum_{l=1}^q \mu_l(t)\{h_x^l(t,y,\dot{y}) - \frac{d}{dt}h_{\dot{x}}^l(t,y,\dot{y})\}\Big)\,dt < 0,$$

which contradicts (2.32).

(ii) and (iii) are similar to the proof of (i). □

Corollary 2.6.1. Assume that weak duality holds between (MVPE) and (WMVDE). If $(y^*, \tau^*, \lambda^*, \mu^*)$ is feasible for (WMVDE) with $\sum_{j=1}^m \lambda_j^*(t)g^j(t, y^*,$

$\dot{y}^*) = 0$ and y^* is feasible for (MVPE), then y^* is an efficient for (MVPE) and $(y^*, \tau^*, \lambda^*, \mu^*)$ is an efficient for (WMVDE). □

Theorem 2.6.2 (Strong Duality). Assume that

(i) x^* is an efficient solution for (MVPE) ;

(ii) for all $k = 1, \cdots, p$, x^* a constraint qualification for problem $MVPE_k(x^*)$ is satisfied at x^*.

Then there exist $\tau^* \in R^p, \tau^* > 0$, and piecewise smooth functions $\lambda^* : I \to R^m, \lambda^*(t) \geq 0$ and $\mu^* : I \to R^q$ such that $(x^*, \tau^*, \lambda^*, \mu^*)$ is feasible for (WMVDE) and $\sum_{j=1}^m \lambda_j^*(t) g^j(t, x^*, \dot{x}^*) + \sum_{l=1}^q \mu_l^*(t) h^l(t, x^*, \dot{x}^*) = 0$. Further, if also weak duality holds between (MVPE) and (WMVDE), then $(x^*, \tau^*, \lambda^*, \mu^*)$ is an efficient solution for (WMVDE). □

Chapter 3

Multiobjective Fractional Variational Problem with Generalized Type I Invexity

3.1. Introduction

The following problem is called a multiobjective fractional variational problem (MFVP):

(MFVP):

$$\begin{aligned}
\text{Minimize} \quad & \frac{\int_a^b f(t,x,\dot{x})dt}{\int_a^b g(t,x,\dot{x})dt} \\
& = \Big(\frac{\int_a^b f^1(t,x,\dot{x})dt}{\int_a^b g^1(t,x,\dot{x})dt}, \cdots, \frac{\int_a^b f^p(t,x,\dot{x})dt}{\int_a^b g^p(t,x,\dot{x})dt}\Big) \\
\text{subject to} \quad & x(a) = t_0, \;\; x(b) = t_f, \\
& h(t,x,\dot{x}) \leq 0, \quad t \in I,
\end{aligned}$$

where $f^i, g^i : I \times R^n \times R^n \to R,\; i = 1, \cdots, p$ and $h^j : I \times R^n \times R^n \to R,\; j = 1, \cdots, m$, are continuously differentiable functions. $I = [a, b]$ is a real interval. We assume in the sequel that $f^i(t,x,\dot{x}) \geq 0$ and $g^i(t,x,\dot{x}) > 0$ on $I \times R^n \times R^n$ for $i = 1, \cdots, p$.

In this chapter, we consider a multiobjective fractional variational problem. For sufficient conditions, we define the generalized V-type I invex functions. We obtain the generalized Kuhn-Tucker sufficient optimality theorem and prove weak and strong duality theorems for the multiobjective fractional variational problem.

3.2. Definitions and Preliminaries

Let us now denote by X_2 be the set of all feasible solutions of problem (MFVP) given by

$$X_2 := \{x \in C(I, R^n) |\ x(a) = t_0,\ x(b) = t_f,\ h(t, x, \dot{x}) \leq 0\}.$$

Definition 3.2.1. A point $u \in X_2$ is said to be an efficient solution of (MFVP) if there does not $x \in X_2$ such that

$$\frac{\int_a^b f^i(t, x, \dot{x})dt}{\int_a^b g^i(t, x, \dot{x})dt} \leq \frac{\int_a^b f^i(t, u, \dot{u})dt}{\int_a^b g^i(t, u, \dot{u})dt} \quad \text{for all } i = 1, \cdots, p$$

and

$$\frac{\int_a^b f^{i_0}(t, x, \dot{x})dt}{\int_a^b g^{i_0}(t, x, \dot{x})dt} < \frac{\int_a^b f^{i_0}(t, u, \dot{u})dt}{\int_a^b g^{i_0}(t, u, \dot{u})dt} \quad \text{for some } i_0 = 1, \cdots, p.$$

In order to prove the strong duality theorem we will invoke the following lemma due to Changkong and Haimes [8].

Lemma 3.2.1. u is an efficient solution of (MFVP) if and only if u solves $(\text{MFVP})_k, k = 1, \cdots, p$, where $(\text{MFVP})_k$, is the following problem:

$(MFVP)_k$:

$$\begin{aligned} \text{Minimize} \quad & \frac{\int_a^b f^k(t, x, \dot{x})dt}{\int_a^b g^k(t, x, \dot{x})dt} \\ \text{subject to} \quad & x(a) = t_0, x(b) = t_f, \\ & \frac{\int_a^b f^i(t, x, \dot{x})dt}{\int_a^b g^i(t, x, \dot{x})dt} \leq \frac{\int_a^b f^i(t, u, \dot{u})dt}{\int_a^b g^i(t, u, \dot{u})dt} \quad \text{for all } i \neq k, \\ & h^j(t, x, \dot{x}) \leq 0, \quad j = 1, \cdots, m. \end{aligned}$$

3.3. Sufficient Optimality Theorem for (MFVP)

Theorem 3.3.1 (Sufficient Optimality Conditions). Suppose that

(i) $u \in X_2$;

(ii) there exist $\tau \in R^p, \tau > 0$, and a piecewise smooth function $\lambda : I \to R^m, \lambda(t) \geq 0$ such that

$$(a) \sum_{i=1}^{p} \tau_i \Big(\{f_x^i(t,u,\dot{u}) - v^i g_x^i(t,u,\dot{u})\} - \frac{d}{dt}\{f_{\dot{x}}^i(t,u,\dot{u}) - v^i g_{\dot{x}}^i(t,u,\dot{u})\}$$
$$+ \sum_{j=1}^{m} \lambda_j(t)\{h_x^j(t,u,\dot{u}) - \frac{d}{dt} h_{\dot{x}}^j(t,u,\dot{u})\}\Big) = 0,$$

$$(b) \int_a^b \sum_{j=1}^{m} \lambda_j(t) h^j(t,u,\dot{u})\, dt = 0;$$

(iii) $(f - vg, h)$ is quasi strictly pseudo V-type I invex at u with respect to η, τ, λ and for some positive functions α_i, β_j, for $i = 1, \cdots, p, j = 1, \cdots, m$, where $v_i = \frac{\int_a^b f^i(t,u,\dot{u})dt}{\int_a^b g^i(t,u,\dot{u})dt}$, $i = 1, \cdots, p$.

Then u is an efficient solution for (MFVP).

Proof. Suppose that u is not an efficient solution of (MFVP). Then there exists $x \in X_2$ such that

$$\frac{\int_a^b f^i(t,x,\dot{x})dt}{\int_a^b g^i(t,x,\dot{x})dt} \leq \frac{\int_a^b f^i(t,u,\dot{u})dt}{\int_a^b g^i(t,u,\dot{u})dt}, \quad \text{for all } i = 1, \cdots, p$$

and

$$\frac{\int_a^b f^{i_0}(t,x,\dot{x})dt}{\int_a^b g^{i_0}(t,x,\dot{x})dt} < \frac{\int_a^b f^{i_0}(t,u,\dot{u})dt}{\int_a^b g^{i_0}(t,u,\dot{u})dt}, \quad \text{for some } \quad i_0 = 1, \cdots, p.$$

Since $g^i(t,x,\dot{x}) > 0$ for all $i = 1,\cdots,p$, we have

$$\int_a^b \{f^i(t,x,\dot{x}) - v_i g^i(t,x,\dot{x})\}dt \;\le\; \int_a^b \{f^i(t,u,\dot{u}) - v_i g^i(t,u,\dot{u})\}dt,$$
for all $i = 1,\cdots,p$

and
$$\int_a^b \{f^{i_0}(t,x,\dot{x}) - v_{i_0} g^{i_0}(t,x,\dot{x})\}dt \;<\; \int_a^b \{f^{i_0}(t,u,\dot{u}) - v_{i_0} g^{i_0}(t,u,\dot{u})\}dt,$$
for some $i_0 = 1,\cdots,p$.

Since $\tau_i > 0$ and $\alpha_i > 0$ for all $i = 1,\cdots,p$, we have

$$\begin{aligned}&\int_a^b \sum_{i=1}^p \tau_i\alpha_i(x,u,\dot{x},\dot{u})\{f^i(t,x,\dot{x}) - v_i g^i(t,x,\dot{x})\}dt \\ &\quad < \int_a^b \sum_{i=1}^p \tau_i\alpha_i(x,u,\dot{x},\dot{u})\{f^i(t,u,\dot{u}) - v_i g^i(t,u,\dot{u})\}dt.\end{aligned}$$

From the above inequality and the hypothesis (iii), it follows that

$$\int_a^b \sum_{i=1}^p \tau_i\eta\Big(\{f_x^i(t,u,\dot{u}) - v_i g_x^i(t,u,\dot{u})\} - \frac{d}{dt}\{f_{\dot{x}}^i(t,u,\dot{u}) - v_i g_{\dot{x}}^i(t,u,\dot{u})\}\Big)dt \le 0. \tag{3.1}$$

By the inequality (3.1) and hypothesis (ii)(a) we have

$$\int_a^b \sum_{j=1}^m \lambda_j(t)\eta(t,x,u,\dot{x},\dot{u})\{h_x^j(t,u,\dot{u}) - \frac{d}{dt}h_{\dot{x}}^j(t,u,\dot{u})\}dt \ge 0.$$

From the above inequality and hypothesis (iii) it follows that

$$\int_a^b \sum_{j=1}^m \lambda_j(t)\beta_j(x,u,\dot{x},\dot{u})h^j(t,u,\dot{u})\,dt < 0. \tag{3.2}$$

Now by hypotheses (i) and (ii)(b) it follows that $\int_a^b \lambda_j(t)h^j(t,u,\dot{u})\,dt = 0$, for every j, which further implies that

$$\int_a^b \sum_{j=1}^{m} \lambda_j(t)\beta_j(x,u,\dot{x},\dot{u})h^j(t,u,\dot{u})\,dt = 0.$$

The last equation contradicts the inequality (3.2) and hence u is an efficient solution of (MFVP). □

3.4. Formulation of Fractional Variational Dual Problem

Following the parametric approch of Bector et al. [4], we formulate the following multiobjective variational dual problem for (MFVP).

(MFVD):

$$\begin{aligned}
\text{Maximize} \quad & (v_1, \cdots, v_p) \\
\text{subject to} \quad & x(a) = t_0, \ \ x(b) = t_f, && (3.3) \\
& \sum_{i=1}^{p} \tau_i \Big(\{f_x^i(t,u,\dot{u}) - v^i g_x^i(t,u,\dot{u})\} - \frac{d}{dt}\{f_{\dot{x}}^i(t,u,\dot{u}) - v^i g_{\dot{x}}^i(t,u,\dot{u})\} \Big) \\
& + \sum_{j=1}^{m} \lambda_j(t)\{h_x^j(t,u,\dot{u}) - \frac{d}{dt}\{h_{\dot{x}}^j(t,u,\dot{u})\} = 0, && (3.4) \\
& \int_a^b \{f^i(t,u,\dot{u}) - v_i g^i(t,u,\dot{u})\}dt \geq 0, \ \ i = 1, \cdots, p, && (3.5) \\
& \int_a^b \lambda_j(t)h^j(t,u,\dot{u})dt \geq 0, \ \ j = 1, \cdots, m, && (3.6) \\
& \tau \in R^p, \lambda(t) \in R^m, v \in R^p, \tau_i \geq 0, \ \ \lambda(t) \geq 0, \ \ t \in I. && (3.7)
\end{aligned}$$

3.5. Duality Theorems

We establish weak and strong duality theorems between (MFVP) and (MFVD).

Theorem 3.5.1 (Weak Duality). Assume that for all feasible x for (MFVP) and all feasible (u, τ, λ, v) for (MFVD), any of the following holds:

(i) $(f - vg, h)$ is weak strictly-pseudoquasi V-type I invex at u with respect to η and for some positive functions α_i, β_j, for $i = 1, \cdots, p, j = 1, \cdots, m$:

(ii) $(f - vg, h)$ is weak strictly pseudo V-type I invex at u with respect to η and for some positive functions α_i, β_j, for $i = 1, \cdots, p, j = 1, \cdots, m$.

Then the following inequalities cannot hold:

$$\frac{\int_a^b f^i(t, x, \dot{x})dt}{\int_a^b g^i(t, x, \dot{x})dt} \leq v_i, \ \forall \ i = 1, \cdots, p \tag{3.8}$$

and

$$\frac{\int_a^b f^{i_0}(t, x, \dot{x})dt}{\int_a^b g^{i_0}(t, x, \dot{x})dt} < v_{i_0}, \ \text{for some} \quad i_0 = 1, \cdots, p. \tag{3.9}$$

Proof. Suppose, contrary to the result of the theorem, that for some feasible x for (MFVP) and (u, τ, λ, v) for (MFVD),

$$\frac{\int_a^b f^i(t, x, \dot{x})dt}{\int_a^b g^i(t, x, \dot{x})dt} \leq v_i, \ \forall \ i = 1, \cdots, p$$

and

$$\frac{\int_a^b f^{i_0}(t, x, \dot{x})dt}{\int_a^b g^{i_0}(t, x, \dot{x})dt} < v_{i_0}, \ \text{for some} \quad i_0 = 1, \cdots, p.$$

Then, we have

$$\int_a^b \{f^i(t,x,\dot{x}) - v_i g^i(t,x,\dot{x})\}dt \leq 0, \quad \forall i = 1,\cdots,p \tag{3.10}$$

and

$$\int_a^b \{f^{i_0}(t,x,\dot{x}) - v_{i_0} g^{i_0}(t,x,\dot{x})\}dt < 0, \quad \text{for some } i_0 = 1,\cdots,p. \tag{3.11}$$

Hence, from (3.5) and (3.7), we obtain

$$\begin{aligned}\int_a^b \sum_{i=1}^p \tau_i \alpha_i(x,u,\dot{x},\dot{u})\{f^i(t,x,\dot{x}) - v_i g^i(t,x,\dot{x})\}dt \\ < \int_a^b \sum_{i=1}^p \tau_i \alpha_i(x,u,\dot{x},\dot{u})\{f^i(t,u,\dot{u}) - v_i g^i(t,u,\dot{u})\}dt.\end{aligned} \tag{3.12}$$

Since (u,τ,λ,v) is feasible for (MFVD) and each $\beta_j > 0$, it follows that

$$-\int_a^b \sum_{j=1}^m \lambda_j(t)\beta_j(x,u,\dot{x},\dot{u})h^j(t,u,\dot{u})dt \leq 0. \tag{3.13}$$

By the hypothesis (i) i.e $(f - vg, h)$ is weak strictly-pseudoquasi V-type I invex, (3.12) and (3.13) imply

$$\int_a^b \sum_{i=1}^p \tau_i \eta\Big(\{f_x^i(t,u,\dot{u}) - v_i g_x^i(t,u,\dot{u})\} - \frac{d}{dt}\{f_{\dot{x}}^i(t,u,\dot{u}) - v_i g_{\dot{x}}^i(t,u,\dot{u})\}\Big)dt < 0,$$

$$\int_a^b \lambda_j(t)\eta\{h_x^j(t,u,\dot{u}) - \frac{d}{dt}h_{\dot{x}}^j(t,u,\dot{u})\}dt \leq 0.$$

The above inequalities give

$$\begin{aligned}\int_a^b \eta\Big[\sum_{i=1}^p \tau_i\Big(\{f_x^i(t,u,\dot{u}) - v_i g_x^i(t,u,\dot{u})\} - \frac{d}{dt}\{f_{\dot{x}}^i(t,u,\dot{u}) - v_i g_{\dot{x}}^i(t,u,\dot{u})\}\Big) \\ + \sum_{j=1}^m \lambda_j(t)\{h_x^j(t,u,\dot{u}) - \frac{d}{dt}h_{\dot{x}}^j(t,u,\dot{u})\}\Big]dt < 0,\end{aligned} \tag{3.14}$$

which contradicts (3.4).

By the hypothesis (ii) i.e $(f - vg, h)$ is weak strictly pseudo V-type I invex, (3.12) and (3.13) imply

$$\int_a^b \sum_{i=1}^{p} \tau_i \eta\Big(\{f_x^i(t,u,\dot{u}) - v_i g_x^i(t,u,\dot{u})\}$$
$$- \frac{d}{dt}\{f_{\dot{x}}^i(t,u,\dot{u}) - v_i g_{\dot{x}}^i(t,u,\dot{u})\}\Big)dt < 0, \tag{3.15}$$
$$\sum_{j=1}^{m} \lambda_j(t)\eta\{h_x^j(t,u,\dot{u}) - \frac{d}{dt}h_{\dot{x}}^j(t,u,\dot{u})\}dt < 0. \tag{3.16}$$

(3.15) and (3.16) imply (3.14), again contradicting (3.4). □

Corollary 3.5.1 may be merely stated since its proof would run analogously to that of Corollary 2.5.1.

Corollary 3.5.1. Assume that the condition of weak duality theorem (3.5.1) hold. If x^* is feasible for (MFVP) and $(u^*, \tau^*, \lambda^*, v^*)$ is feasible for (MFVD), with $v_i^* = \frac{\int_a^b f^i(t,x^*,\dot{x}^*)dt}{\int_a^b g^i(t,x^*,\dot{x}^*)dt}$, $i = 1, \cdots, p$, then x^* and $(u^*, \tau^*, \lambda^*, v^*)$ are an efficient solutions of problems (MFVP) and (MFVD), respectively. □

Theorem 3.5.3 (Strong Duality). Let x^* be an efficient solution of (MFVP) and assume that a constraint qualification for problem $(\text{MFVP})_k$ is satisfiedat x^*. Then there exist $\tau^* \in R^p$ and piecewise smooth function $\lambda^* : I \to R^m, \lambda^*(t) \geq 0$, and $v \in R^p$ such that $(x^*, \tau^*, \lambda^*, v^*)$ is a feasible solution for (MFVD). If the assumptions of weak duality theorem (3.5.1) also hold, then $(x^*, \tau^*, \lambda^*, v^*)$ is an efficient solution for (MFVD).

Proof. Since x^* is an efficient solution of (MFVP), then from Lemma 3.2.1, x^* solves $(MFVP)_k$ for each $k = 1, \cdots, p$. From Kuhn-Tucker necessary conditions for each $i = 1, \cdots, p$, we obtain $\tau_i^k \geqq 0$ for all $i \neq k$, $\lambda^i(t)(\geq 0) \in R^m$ and $v_i(t) \in R^p$ such that

$$\Big(\{f_x^i(t,x^*,\dot{x}^*) - v_i g_x^i(t,x^*,\dot{x}^*)\} - \frac{d}{dt}\{f_{\dot{x}}^i(t,x^*,\dot{x}^*) - v_i g_{\dot{x}}^i(t,x^*,\dot{x}^*)\}\Big)$$
$$+ \sum_{i\neq k} \tau_i^k \Big(\{f_x^k(t,x^*,\dot{x}^*) - v_i g_x^k(t,x^*,\dot{x}^*)\} - \frac{d}{dt}\{f_{\dot{x}}^k(t,x^*,\dot{x}^*) - v_i g_{\dot{x}}^k(t,x^*,\dot{x}^*)\}\Big)$$
$$+ \sum_{j=1}^m \lambda_j^i(t)\{h_x^j(t,x^*,\dot{x}^*) - \frac{d}{dt}h_{\dot{x}}^j(t,x^*,\dot{x}^*)\} = 0, \tag{3.17}$$
$$\int_a^b \sum_{j=1}^m \lambda_j^i(t) h^j(t,x^*,\dot{x}^*)dt = 0. \tag{3.18}$$

Summing (3.17) over $i = 1, \cdots, p$, we have

$$(1 + \tau_2^1 + \cdots + \tau_p^1)\Big(\{f_x^1(t,x^*,\dot{x}^*) - v_1 g_x^1(t,x^*,\dot{x}^*)\}$$
$$- \frac{d}{dt}\{f_{\dot{x}}^1(t,x^*,\dot{x}^*) - v_1 g_{\dot{x}}^1(t,x^*,\dot{x}^*)\}\Big) + \sum_{j=1}^m \lambda_j^1(t)\{h_x^j(t,x^*,\dot{x}^*)$$
$$- \frac{d}{dt}h_{\dot{x}}^j(t,x^*,\dot{x}^*)\} + (\tau_1^2 + 1 + \cdots + \tau_p^2)\Big(\{f_x^2(t,x^*,\dot{x}^*) - v_2 g_x^2(t,x^*,\dot{x}^*)\}$$
$$- \frac{d}{dt}\{f_{\dot{x}}^2(t,x^*,\dot{x}^*) - v_2 g_{\dot{x}}^2(t,x^*,\dot{x}^*)\}\Big) + \sum_{j=1}^m \lambda_j^2(t)\{h_x^j(t,x^*,\dot{x}^*)$$
$$- \frac{d}{dt}h_{\dot{x}}^j(t,x^*,\dot{x}^*)\} + \cdots + (\tau_1^p + \tau_2^p + \cdots + 1)\Big(\{f_x^p(t,x^*,\dot{x}^*) - v_p g_x^p(t,x^*,\dot{x}^*)\}$$
$$- \frac{d}{dt}\{f_{\dot{x}}^p(t,x^*,\dot{x}^*) - v_p g_{\dot{x}}^p(t,x^*,\dot{x}^*)\}\Big) + \sum_{j=1}^m \lambda_j^p(t)\{h_x^j(t,x^*,\dot{x}^*)$$
$$- \frac{d}{dt}h_{\dot{x}}^j(t,x^*,\dot{x}^*)\} = 0.$$

Let $\tau_1^* = 1 + \tau_2^1 + \cdots + \tau_p^1$, $\tau_2^* = \tau_1^2 + 1 + \cdots + \tau_p^2$, $\cdots, \tau_p^* = \tau_1^p + \tau_2^p + \cdots + 1$, $\lambda_j^*(t) = \sum_{k=1}^p \lambda_j^k(t)$, $j = 1, \cdots, m$, $\lambda^*(t) = (\lambda_1^*(t), \cdots, \lambda_m^*(t))$. Then we have

$$\sum_{i=1}^{p} \tau_i^* \Big(\{ f_x^i(t, x^*, \dot{x}^*) - v_i g_x^i(t, x^*, \dot{x}^*) \} - \frac{d}{dt} \{ f_{\dot{x}}^i(t, x^*, \dot{x}^*) - v_i g_{\dot{x}}^i(t, x^*, \dot{x}^*) \} \Big)$$
$$+ \sum_{j=1}^{m} \lambda_j^*(t) \{ h_x^j(t, x^*, \dot{x}^*) - \frac{d}{dt} h_{\dot{x}}^j(t, x^*, \dot{x}^*) \} = 0.$$

Summing (3.18) for $i = 1, \cdots, p$, we have $\int_a^b \sum_{j=1}^m \lambda_j^*(t) h^j(t, x^*, \dot{x}^*) dt = 0$. We conclude that $(x^*, \tau^*, \lambda^*, v^*)$ is feasible for (MFVD) and $v_i^* = \frac{\int_a^b f^i(t, x^*, \dot{x}^*) dt}{\int_a^b g^i(t, x^*, \dot{x}^*) dt}$, $i = 1, \cdots, p$. Efficiency of $(x^*, \tau^*, \lambda^*, v^*)$ for (MFVD) now follows from Corollary 3.5.1. $\square$

Chapter 4

Multiobjective Control Problem with Generalized V-ρ Invexity

4.1. Introducton

The following problem is called a multiobjective control problem (MCP):

(MCP):

$$\begin{aligned} \text{Minimize} \quad & \left(\int_a^b f^1(t,x,u)\,dt, \cdots, \int_a^b f^p(t,x,u)\,dt \right) \\ \text{subject to} \quad & x(a) = t_0, \;\; x(b) = t_f, & (4.1) \\ & g(t,x,u) \leq 0, \qquad t \in I, & (4.2) \\ & h(t,x,u) = \dot{x}, \qquad t \in I. & (4.3) \end{aligned}$$

Here R^n denotes an n-dimensional Euclidean space and $I = [a,b]$ is a real interval. Each $f^i : I \times R^n \times R^m \longmapsto R$ for $i = 1, \cdots, p$, $g = (g^1, \cdots, g^k), g^j : I \times R^n \times R^m \to R$ $(j = 1, \cdots, k)$, and $h = (h^1, \cdots, h^n), h^r : I \times R^n \times R^m \to R$ $(r = 1, \cdots, n)$ is a continuously differentiable function.

Let $x : I \to R^n$ be differentiable with its derivative $\dot{x}$, and let $u : I \to R^m$ be a differentiable function. Denote the partial derivatives of f by f_t, f_x, and f_u, that is,

$$f_t = \frac{\partial f}{\partial t}, \;\; f_x = [\frac{\partial f}{\partial x^1}, \cdots, \frac{\partial f}{\partial x^n}], \;\; f_u = [\frac{\partial f}{\partial u^1}, \cdots, \frac{\partial f}{\partial u^m}],$$

where the superscripts denote the vector components.

Similarly, we have g_t, g_x, g_u, and h_t, h_x, h_u. X is the space of continuously differentiable state functions $x : I \to R^n$ such that $x(a) = t_0$ and $x(b) = t_f$

and is equipped with the norm $||x|| = ||x||_\infty + ||D_x||_\infty$; and Y is the space of piecewise continuous control functions $u : I \to R^m$, and has the uniform norm $||\cdot||_\infty$. The differential equation (4.3) with initial conditions expressed as $x(t) = x(a) + \int_a^t h^r(s, x(s), u(s))\, ds,\ t \in I$ may be written as $\dot{x} = H^r(x, u)$, where $H^r : X \times Y \to C(I, R^n)$, $C(I, R^n)$ being the space of continuous functions from I to R^n defined as $H^r(x, u)(t) = h^r(t, x(t), u(t))$.

In this chapter, we will define generalized V-ρ-invex functions for optimal control problems and consider a multiobjective control problem (MCP). The sufficient optimality conditions of the Kuhn-Tucker type for (MCP) are given under generalized invexity condition. Moreover, we formulate Wolfe type dual (WMCD) and Mond-Weir type dual (MMCD) for (MCP), and then establish their duality relations.

4.2. Definitions and Preliminaries

Definition 4.2.1. Let h^i be a function from $I \times R^n \times R^n \times R^m$ into R and let $H^i(x, u) = \int_a^b h^i(t, x, \dot{x}, u)\, dt$. Let there exist differentiable vector functions $\eta(t, x, x^*, \dot{x}, \dot{x}^*, u, u^*) \in R^n$ with $\eta = 0$ at t if $x(t) = x^*(t)$, and $\xi(t, x, x^*, \dot{x}, \dot{x}^*, u, u^*) \in R^m$, $\zeta(x, x^*, \dot{x}, \dot{x}^*, u, u^*) \in R^n$. Let $||\zeta(x, x^*, \dot{x}, \dot{x}^*, u, u^*)|| = \sup_{t \in I} ||\zeta(x, x^*, \dot{x}, \dot{x}^*, u, u^*)||$ and ρ_i real numbers.

(1) A vector function $H = (H^1, \cdots, H^n)$ is said to be V-ρ-invex in $x^*, \dot{x}^*$, and u^* on I with respect to η, ξ, ζ and α if there exist differentiable vector functions $\eta \in R^n$, $\xi \in R^m$, $\zeta \in R^n$, $\alpha_i(x, x^*, \dot{x}, \dot{x}^*, u, u^*) \in R_+ \backslash \{0\}$, and $\rho_i \in R$, $i = 1, \cdots, n$ such that, for each $x, x^* \in X$ and $u, u^* \in Y$,

$$H^i(x, u) - H^i(x^*, u^*) \geq \int_a^b \{\eta^T \alpha_i h_x^i(t, x^*, \dot{x}^*, u^*) + \frac{d\eta^T}{dt} \alpha_i h_{\dot{x}}^i(t, x^*, \dot{x}^*, u^*)$$
$$+ \xi^T \alpha_i h_u^i(t, x^*, \dot{x}^*, u^*)\}\, dt + \rho_i ||\zeta||^2.$$

(2) The vector function $H = (H^1, \cdots, H^n)$ is said to be V-ρ-pseudo-invex in $x^*, \dot{x}^*$, and u^* on I with respect to η, ξ, ζ and β if there exist η, ξ, ζ as above, $\beta_i(x, x^*, \dot{x}, \dot{x}^*, u, u^*) \in R_+ \backslash \{0\}$ and $\rho_i \in R$, $i = 1, \cdots, n$ such that, for each $x, x^* \in X$ and $u, u^* \in Y$,

$$\int_a^b \sum_{i=1}^n \{\eta^T h_x^i(t, x^*, \dot{x}^*, u^*) + \frac{d\eta^T}{dt} h_{\dot{x}}^i(t, x^*, \dot{x}^*, u^*)$$

$$+ \xi^T h_u^i(t, x^*, \dot{x}^*, u^*)\} \, dt + \sum_{i=1}^n \rho_i ||\zeta||^2 \geq 0$$

$$\Longrightarrow \int_a^b \sum_{i=1}^n \beta_i h^i(t, x, \dot{x}, u) \, dt \geq \int_a^b \sum_{i=1}^n \beta_i h^i(t, x^*, \dot{x}^*, u^*) \, dt$$

(3) The vector function $H = (H^1, \cdots, H^n)$ is said to be V-ρ-quasi-invex in $x^*, \dot{x}^*$, and u^* on I with respect to η, ξ, ζ and γ if there exist η, ξ, ζ as above, the vector $\gamma_i(x, x^*, \dot{x}, \dot{x}^*, u, u^*) \in R_+ \backslash \{0\}$, and $\rho_i \in R$, $i = 1, \cdots, n$ such that, for each $x, x^* \in X$ and $u, u^* \in Y$,

$$\int_a^b \sum_{i=1}^n \gamma_i h^i(t, x, \dot{x}, u) dt \leq \int_a^b \sum_{i=1}^n \gamma_i h^i(t, x^*, \dot{x}^*, u^*) dt$$

$$\Longrightarrow \int_a^b \sum_{i=1}^n \{\eta^T h_x^i(t, x^*, \dot{x}^*, u^*) + \frac{d\eta^T}{dt} h_{\dot{x}}^i(t, x^*, \dot{x}^*, u^*)$$

$$+ \xi^T h_u^i(t, x^*, \dot{x}^*, u^*)\} dt + \sum_{i=1}^n \rho_i ||\zeta||^2 \leq 0$$

Lemma 1 of [34] states that (x^*, u^*) is an efficient solution for (MCP) if and only if (x^*, u^*) solves

$MCP_k(x^*, u^*)$:

$$\begin{aligned}
\text{Minimize} \quad & \int_a^b f^k(t,x,u)\,dt \\
\text{subject to} \quad & x(a) = t_0, x(b) = t_f, \\
& g(t,x,u) \le 0, \\
& h(t,x,u) = \dot{x}, \\
& \int_a^b f^j(t,x,u)dt \le \int_a^b f^j(t,x^*,u^*)dt, \\
& \qquad\qquad \forall\ j \in \{1,\cdots,p\}, j \ne k.
\end{aligned}$$

Chandra, Craven, and Husain [6] gave the Fritz John necessary optimality conditions for the existence of an extremal solution for the single objective control problem (CP):

(CP):

$$\begin{aligned}
\text{Minimize} \quad & \int_a^b f(t,x,u)\,dt \\
\text{subject to} \quad & x(a) = t_0, x(b) = t_f, \\
& g(t,x,u) \le 0, \\
& h(t,x,u) = \dot{x},
\end{aligned}$$

where f, g, h are as defined earlier.

Mond and Hanson [28] pointed out that if the optimal solution for (CP) is normal, then Fritz John conditions reduce to Kuhn-Tucker conditions.

Lemma 4.2.1 (Kuhn-Tucker Necessary Optimality Condition). Let $(x^*, u^*) \in X \times Y$ be an efficient for (MCP). If the Fréchet derivatives $[D - H_x^i(x^*, u^*)]$ is surjective and (x^*, u^*) is normal for $MCP_k(x^*, u^*)$ at least one $k \in \{1, \cdots, p\}$, then there exist $\tau^* \in R^p$, piecewise smooth functions $\lambda^* : I \to R^k$, *and* $\mu^* : I \to R^n$ satisfying the following equalities; for all $t \in I$,

$$\sum_{i=1}^{p} \tau_i^* f_x^i(t, x^*, u^*) + \sum_{j=1}^{k} \lambda_j^*(t) g_x^j(t, x^*, u^*) + \sum_{r=1}^{n} \mu_r^*(t) h_x^r(t, x^*, u^*) + \dot{\mu}^*(t) = 0,$$

$$\sum_{i=1}^{p} \tau_i^* f_u^i(t, x^*, u^*) + \sum_{j=1}^{k} \lambda_j^*(t) g_u^j(t, x^*, u^*) + \sum_{r=1}^{n} \mu_r^*(t) h_u^r(t, x^*, u^*) = 0,$$

$$\sum_{j=1}^{k} \lambda_j^*(t) g^j(t, x^*, u^*) = 0,$$

$$\tau_i^* \geq 0, \quad \sum_{i=1}^{p} \tau_i^* = 1, \quad \lambda^*(t) \geq 0.$$

4.3. Formulation of Control Dual Problem

We formulate two pairs of the following multiobjective dual control problems.

The Wolfe type dual [39]:

(WMCD): Maximize

$$\left(\int_a^b \{ f^1(t, x, u) + \sum_{j=1}^{k} \lambda_j(t) g^j(t, x, u) + \sum_{r=1}^{n} \mu_r(t)(h^r(t, x, u) - \dot{x}) \} \, dt, \right.$$
$$\ldots,$$
$$\left. \int_a^b \{ f^p(t, x, u) + \sum_{j=1}^{k} \lambda_j(t) g^j(t, x, u) + \sum_{r=1}^{n} \mu_r(t)(h^r(t, x, u) - \dot{x}) \} \, dt \right)$$

subject to $x(a) = t_0, \;\; x(b) = t_f,$ (4.4)

$$\sum_{i=1}^{p} \tau_i f_x^i(t,x,u) + \sum_{j=1}^{k} \lambda_j(t) g_x^j(t,x,u)$$

$$+ \sum_{r=1}^{n} \mu_r(t) h_x^r(t,x,u) + \dot{\mu}(t) = 0, \;\; t \in I, \tag{4.5}$$

$$\sum_{i=1}^{p} \tau_i f_u^i(t,x,u) + \sum_{j=1}^{k} \lambda_j(t) g_u^j(t,x,u)$$

$$+ \sum_{r=1}^{n} \mu_r(t) h_u^r(t,x,u) = 0, \;\; t \in I, \tag{4.6}$$

$$\int_a^b \sum_{r=1}^{n} \mu_r(t)(h^r(t,x,u) - \dot{x}(t))\, dt \geq 0, \;\; t \in I, \tag{4.7}$$

$$\lambda(t) \geq 0, \quad t \in I, \tag{4.8}$$

$$\tau_i \geq 0, \;\; \sum_{i=1}^{p} \tau_i = 1. \tag{4.9}$$

The Mond-Weir type dual [30]:

(MMCD):

$$\text{Maximize} \quad \left(\int_a^b f^1(t,x,u)dt, \cdots, \int_a^b f^p(t,x,u)dt \right)$$

subject to $x(a) = t_0, \;\; x(b) = t_f,$ (4.10)

$$\sum_{i=1}^{p} \tau_i f_x^i(t,x,u) + \sum_{j=1}^{k} \lambda_j(t) g_x^j(t,x,u)$$

$$+ \sum_{r=1}^{n} \mu_r(t) h_x^r(t,x,u) + \dot{\mu}(t) = 0, \;\; t \in I, \tag{4.11}$$

$$\sum_{i=1}^{p} \tau_i f_u^i(t,x,u) + \sum_{j=1}^{k} \lambda_j(t) g_u^j(t,x,u) + \sum_{r=1}^{n} \mu_r(t) h_u^r(t,x,u) = 0, \quad t \in I, \tag{4.12}$$

$$\int_a^b \sum_{r=1}^{n} \mu_r(t)(h^r(t,x,u) - \dot{x}(t))dt \geq 0, \ \ t \in I, \tag{4.13}$$

$$\int_a^b \sum_{j=1}^{k} \lambda_j(t) g^j(t,x,u)dt \geq 0, \ \ t \in I, \tag{4.14}$$

$$\lambda(t) \geq 0, \quad t \in I, \tag{4.15}$$

$$\tau_i \geq 0, \ \sum_{i=1}^{p} \tau_i = 1. \tag{4.16}$$

4.4. Sufficient Optimality Theorem for (MCP)

We obtain a Kuhn-Tucker type sufficient optimality theorem of (MCP) as follows:

Theorem 4.4.1. Suppose that (x^*, u^*) is feasible for (MCP) such that there exist $\tau^* > 0$, $\lambda^*(t)$ and $\mu^*(t)$ such that

$$\sum_{i=1}^{p} \tau_i{}^* f_x^i(t,x^*,u^*) + \sum_{j=1}^{k} \lambda_j{}^*(t) g_x^j(t,x^*,u^*) + \sum_{r=1}^{n} \mu_r{}^*(t) h_x^r(t,x^*,u^*) + \dot{\mu}^*(t) = 0, \tag{4.17}$$

$$\sum_{i=1}^{p} \tau_i{}^* f_u^i(t,x^*,u^*) + \sum_{j=1}^{k} \lambda_j{}^*(t) g_u^j(t,x^*,u^*) + \sum_{r=1}^{n} \mu_r{}^*(t) h_u^r(t,x^*,u^*) = 0, \tag{4.18}$$

$$\sum_{j=1}^{k} \lambda_j{}^*(t) g^j(t,x^*,u^*) = 0, \tag{4.19}$$

$$\sum_{r=1}^{n} \mu_r{}^*(t)(h^r(t,x^*,u^*) - \dot{x}^*(t)) = 0, \tag{4.21}$$

$$\lambda^*(t) \geq 0, \tag{4.21}$$

$$\tau_i{}^* \geq 0, \ \sum_{i=1}^{p} \tau_i^* = 1 \tag{4.22}$$

hold through $a \leq t \leq b$ (except that at t corresponding to discontinuities of $u^*(t)$, (4.17) holds for right and left limits). If $\int_a^b f^i\,dt,\ i = 1,\cdots,p,\ \int_a^b \lambda_j^* g^j\,dt,$ $j = 1,\cdots,k,$ and $\int_a^b \mu_r^*(h^r - \dot{x}^*)\,dt,\ r = 1,\cdots,n,$ are all V-ρ-invex with respect to $\eta,\ \xi,\ \zeta,\ \alpha$ and $\sum \tau_i\rho_i + \sum \rho_j + \sum \rho_r \geq 0$, then (x^*, u^*) is an efficient solution of (MCP).

Proof. Suppose that (x^*, u^*) is not an efficient solution of (MCP). Then there exists $(x, u) \neq (x^*, u^*)$ such that (x, u) is feasible for (MCP), and

$$\int_a^b f^i(t,x,u)\,dt \leq \int_a^b f^i(t,x^*,u^*)\,dt, \ \text{for all} \quad i = 1,\cdots,p$$
and
$$\int_a^b f^{i_0}(t,x,u)\,dt < \int_a^b f^{i_0}(t,x^*,u^*)\,dt, \ \text{for some} \quad i_0 = 1,\cdots,p.$$

Since $\int_a^b f^i\,dt$ is V-ρ-inxex,

$$\int_a^b \{\eta^T\alpha_i f_x^i(t,x^*,u^*) + \xi^T\alpha_i f_u^i(t,x^*,u^*)\}\,dt + \rho_i||\zeta||^2 \leq 0, \text{for all} \quad i = 1,\cdots,p,$$
and
$$\int_a^b \{\eta^T\alpha_i f_x^{i_0}(t,x^*,u^*) + \xi^T\alpha_i f_u^{i_0}(t,x^*,u^*)\}\,dt + \rho_i||\zeta||^2 < 0, \text{for some} \quad i_0 = 1,$$

$\cdots,p$. Since $\tau_i^* > 0$ for all i,

$$\int_a^b \sum_{i=1}^{p} \alpha_i \{\eta^T \tau_i^* f_x^i(t, x^*, u^*) + \xi^T \tau_i^* f_u^i(t, x^*, u^*)\}\, dt + \sum_{i=1}^{p} \tau_i^* \rho_i ||\zeta||^2 < 0. \tag{4.23}$$

From the feasibility conditions, $\sum_{j=1}^{k} \lambda_j^*(t) g^j(t, x, u) \le 0 = \sum_{j=1}^{k} \lambda_j^*(t) g^j(t, x^*, u^*)$. By the V-$\rho$-invexity of $\int_a^b \lambda_j^*(t) g^j\, dt$, we have

$$\int_a^b \sum_{j=1}^{k} \beta_j \{\eta^T \lambda_j^*(t) g_x^j(t, x^*, u^*) + \xi^T \lambda_j^*(t) g_u^j(t, x^*, u^*)\}\, dt + \sum_{j=1}^{k} \rho_j ||\zeta||^2 \le 0. \tag{4.24}$$

From the feasibility conditions, $\sum_{r=1}^{n} \mu_r^*(t)(h^r(t, x, u) - \dot{x}) - \sum_{r=1}^{n} \mu_r^*(t)(h^r(t, x^*, u^*) - \dot{x}^*) = 0$.

By the V-ρ-invexity of $\int_a^b \mu_r^*(h^r - \dot{x}^*)\, dt$, we have

$$\int_a^b \sum_{r=1}^{n} \gamma_r \{\eta^T \mu_r^*(t) h_x^r(t, x^*, u^*) - \frac{d\eta^T}{dt} \mu^*(t) + \xi^T \mu_r^*(t) h_u^r(t, x^*, u^*)\}\, dt$$
$$+ \sum_{r=1}^{n} \rho_r ||\zeta||^2 \le 0. \tag{4.25}$$

By integrating $\frac{d\eta^T}{dt} \mu^*(t)$ from a to b and applying the boundary condition, we have

$$\int_a^b \frac{d\eta^T}{dt} \mu^*(t)\, dt = -\int_a^b \eta^T \dot{\mu}^*(t)\, dt. \tag{4.26}$$

Using (4.26) in (4.25), we have

$$\int_a^b \sum_{r=1}^{n} \gamma_r \{\eta^T \mu_r^*(t) h_x^r(t,x^*,u^*) + \eta^T \dot{\mu}^*(t) + \xi^T \mu_r^*(t) h_u^r(t,x^*,u^*)\}\, dt$$
$$+ \sum_{r=1}^{n} \rho_r ||\zeta||^2 \leq 0. \tag{4.27}$$

Since (4.23), (4.24) and (4.27) hold the same α, we have

$$\int_a^b \Bigg(\eta^T \alpha \{\sum_{i=1}^{p} \tau_i^* f_x^i(t,x^*,u^*) + \sum_{j=1}^{k} \lambda_j^*(t) g_x^j(t,x^*,u^*) + \sum_{r=1}^{n} \mu_r^*(t) h_x^r(t,x^*,u^*)$$
$$+ \dot{\mu}^*(t)\} + \xi^T \alpha \{\sum_{i=1}^{p} \tau_i^* f_u^i(t,x^*,u^*) + \sum_{j=1}^{k} \lambda_j^*(t) g_u^j(t,x^*,u^*)$$
$$+ \sum_{r=1}^{n} \mu_r^*(t) h_u^r(t,x^*,u^*)\} \Bigg) dt + \sum_{i=1}^{p} \tau_i^* \rho_i ||\zeta||^2 + \sum_{j=1}^{k} \rho_j ||\zeta||^2 + \sum_{r=1}^{n} \rho_r ||\zeta||^2$$
$$< 0. \tag{4.28}$$

From (4.17), (4.18) and the fact that $\sum \tau_i^* \rho_i + \sum \rho_j + \sum \rho_r \geq 0$, we have

$$\int_a^b \Bigg(\eta^T \alpha \{\sum_{i=1}^{p} \tau_i^* f_x^i(t,x^*,u^*) + \sum_{j=1}^{k} \lambda_j^* g_x^j(t,x^*,u^*) + \sum_{r=1}^{n} \mu_r h_x^r(t,x^*,u^*)$$
$$+ \dot{\mu}^*(t)\} + \xi^T \alpha \{\sum_{i=1}^{p} \tau_i^* f_u^i(t,x^*,u^*) + \sum_{j=1}^{k} \lambda_j^* g_u^j(t,x^*,u^*)$$
$$+ \sum_{r=1}^{n} \mu_r^* h_u^r(t,x^*,u^*)\} \Bigg) dt + \sum_{i=1}^{p} \rho_i ||\zeta||^2 + \sum_{j=1}^{k} \rho_j ||\zeta||^2 + \sum_{r=1}^{n} \rho_r ||\zeta||^2 \geq 0,$$

which contradicts the inequality (4.28). Hence (x^*, u^*) is an efficient solution of (MCP). □

4.5. Duality Theorems

Now we establish some duality theorems between the multiobjective control problem (MCP) and its Wolfe type dual problem (WMCD).

Theorem 4.5.1 (Weak Duality). Assume that, for all feasible (x^*, u^*) for (MCP) and all feasible $(x, u, \tau, \lambda, \mu)$ for (WMCD),

(i) $$\left(\int_a^b f^1(\cdot,\cdot,\cdot)\,dt, \quad \cdots, \quad \int_a^b f^p(\cdot,\cdot,\cdot)\,dt\right),$$

(ii) $$\left(\int_a^b \lambda_1 g^1(\cdot,\cdot,\cdot)\,dt, \quad \cdots, \quad \int_a^b \lambda_k g^k(\cdot,\cdot,\cdot)\,dt\right),$$

and

(iii) $$\left(\int_a^b \mu_1(h^1(\cdot,\cdot,\cdot) - \dot{x})\,dt, \quad \cdots, \quad \int_a^b \mu_n(h^n(\cdot,\cdot,\cdot) - \dot{x})\,dt\right)$$

are all V-ρ-invex with respect to the same functions η, ξ, ζ and α and

(iv) $\sum_{i=1}^{p} \tau_i \rho_i + \sum_{j=1}^{k} \rho_j + \sum_{r=1}^{n} \rho_r \geq 0$,

then the following inequalities cannot hold:

$$\int_a^b f^i(t, x^*, u^*)\,dt$$
$$\leq \int_a^b \{f^i(t, x, u) + \sum_{j=1}^{k} \lambda_j(t) g^j(t, x, u) + \sum_{r=1}^{q} \mu_r(t)(h^r(t, x, u) - \dot{x})\}\,dt,$$

for all $i = 1, \cdots, p$ (4.29)

and

$$\int_a^b f^{i_0}(t, x^*, u^*)\,dt$$
$$< \int_a^b \{f^{i_0}(t, x, u) + \sum_{j=1}^{k} \lambda_j(t) g^j(t, x, u) + \sum_{r=1}^{q} \mu_r(t)(h^r(t, x, u) - \dot{x})\}\,dt,$$

for some $i_0 = 1, \cdots, p.$ (4.30)

Proof. Suppose contrary to the result that (4.29) and (4.30) hold. Then, since $\tau_i > 0$ and $\sum_{i=1}^{p} \tau_i = 1$,

$$\int_a^b \sum_{i=1}^{p} \tau_i f^i(t, x^*, u^*)\, dt < \int_a^b \{\sum_{i=1}^{p} \tau_i f^i(t, x, u) + \sum_{j=1}^{k} \lambda_j(t) g^j(t, x, u) + \sum_{r=1}^{n} \mu_r(t)(h^r(t, x, u) - \dot{x})\}\, dt. \quad (4.31)$$

By (i), we have

$$\int_a^b f^i(t, x^*, u^*)\, dt - \int_a^b f^i(t, x, u)\, dt \geq \int_a^b \{\eta^T \alpha_i f_x^i(t, x, u) + \xi^T \alpha_i f_u^i(t, x, u)\}\, dt + \rho_i ||\zeta||^2.$$

Since $\tau_i > 0$ and $\sum_{i=1}^{p} \tau_i = 1$ we can get

$$\int_a^b \sum_{i=1}^{p} \tau_i f^i(t, x^*, u^*)\, dt - \int_a^b \sum_{i=1}^{p} \tau_i f^i(t, x, u)\, dt \geq \int_a^b \sum_{i=1}^{p} \alpha_i \{\eta^T \tau_i f_x^i(t, x, u) + \xi^T \tau_i f_u^i(t, x, u)\}\, dt + \sum_{i=1}^{p} \tau_i \rho_i ||\zeta||^2. \quad (4.32)$$

By (ii), we have

$$\int_a^b \lambda_j(t) g^j(t, x^*, u^*)\, dt - \int_a^b \lambda_j(t) g^j(t, x, u)\, dt \geq \int_a^b \{\eta^T \alpha_j \lambda_j(t) g_x^j(t, x, u) + \xi^T \alpha_j \lambda_j(t) g_u^j(t, x, u)\}\, dt + \rho_j ||\zeta||^2. \quad (4.33)$$

Using (4.2) and (4.8), from (4.33), we have

$$
\begin{aligned}
&-\int_a^b \lambda_j(t) g^j(t,x,u)\,dt \\
&\geq \int_a^b \{\eta^T \alpha_j \lambda_j(t) g_x^j(t,x,u) + \xi^T \alpha_j \lambda_j(t) g_u^j(t,x,u)\}\,dt + \rho_j ||\zeta||^2,
\end{aligned}
$$

which implies,

$$
\begin{aligned}
&-\int_a^b \sum_{j=1}^{k} \lambda_j(t) g^j(t,x,u)\,dt \\
&\geq \int_a^b \sum_{j=1}^{k} \alpha_j \{\eta^T \lambda_j(t) g_x^j(t,x,u) + \xi^T \lambda_j(t) g_u^j(t,x,u)\}\,dt + \sum_{j=1}^{k} \rho_j ||\zeta||^2.
\end{aligned}
\tag{4.34}
$$

By (iii), we have

$$
\begin{aligned}
&\int_a^b \mu_r(t)(h^r(t,x^*,u^*) - \dot{x}^*)\,dt - \int_a^b \mu_r(t)(h^r(t,x,u) - \dot{x})\,dt \\
&\geq \int_a^b \{\eta^T \alpha_r \mu_r(t) h_x^r(t,x,u) + \frac{d\eta^T}{dt} \alpha_r(-\mu(t)) + \xi^T \alpha_r \mu_r(t) h_u^r(t,x,u)\}\,dt \\
&\quad + \rho_r ||\zeta||^2.
\end{aligned}
\tag{4.35}
$$

Using (4.2), we have

$$
\begin{aligned}
-\int_a^b \mu_r(t)(h^r(t,x,u) - \dot{x})\,dt \geq \int_a^b &\{\eta^T \alpha_r \mu_r(t) h_x^r(t,x,u) + \frac{d\eta^T}{dt} \alpha_r(-\mu(t)) \\
&+ \xi^T \alpha_r \mu_r(t) h_u^r(t,x,u)\}\,dt + \rho_r ||\zeta||^2,
\end{aligned}
$$

which implies,

$$-\int_a^b \sum_{r=1}^n \mu_r(t)(h^r(t,x,u)-\dot{x})\,dt \geq \int_a^b \sum_{r=1}^n \alpha_r\{\eta^T\mu_r(t)h_x^r(t,x,u) - \frac{d\eta^T}{dt}\mu(t) + \xi^T\mu_r(t)h_u^r(t,x,u)\}\,dt + \sum_{r=1}^n \rho_r||\zeta||^2. \tag{4.36}$$

By integration $\frac{d\eta^T}{dt}\mu(t)$ from a to b and applying the boundary condition, we have

$$\int_a^b \frac{d\eta^T}{dt}\mu(t)\,dt = \eta^T\mu(t)|_a^b - \int_a^b \eta^T\dot{\mu}(t)\,dt = -\int_a^b \eta^T\dot{\mu}(t)\,dt. \tag{4.37}$$

Using (4.37) in (4.36), we have

$$-\int_a^b \sum_{r=1}^n \mu_r(t)(h^r(t,x,u)-\dot{x})\,dt \geq \int_a^b \sum_{r=1}^n \alpha_r\{\eta^T\mu_r(t)h_x^r(t,x,u) + \eta^T\dot{\mu}(t) + \xi^T\mu_r(t)h_u^r(t,x,u)\}\,dt + \sum_{r=1}^n \rho_r||\zeta||^2. \tag{4.38}$$

Since (4.32), (4.34) and (4.38) hold the same α, we have

$$\begin{aligned}
&\int_a^b \sum_{i=1}^p \tau_i f^i(t,x^*,u^*)\,dt - \int_a^b \sum_{i=1}^p \tau_i f^i(t,x,u)\,dt \\
&- \int_a^b \sum_{j=1}^k \lambda_j(t)g^j(t,x,u)\,dt - \int_a^b \sum_{r=1}^n \mu_r(t)(h^r(t,x,u)-\dot{x})\,dt \\
&\geq \int_a^b \Bigg(\eta^T\alpha\{\sum_{i=1}^p \tau_i f_x^i(t,x,u) + \sum_{j=1}^k \lambda_j(t)g_x^j(t,x,u) + \sum_{r=1}^n \mu_r(t)h_x^r(t,x,u) + \dot{\mu}(t)\} \\
&\quad + \xi^T\alpha\{\sum_{i=1}^p \tau_i f_u^i(t,x,u) + \sum_{j=1}^k \lambda_j(t)g_u^j(t,x,u) + \sum_{r=1}^n \mu_r(t)h_u^r(t,x,u)\}\Bigg)\,dt
\end{aligned}$$

$+\sum_{i=1}^{p}\tau_i\rho_i||\zeta||^2+\sum_{j=1}^{k}\rho_j||\zeta||^2+\sum_{r=1}^{n}\rho_r||\zeta||^2 \geq 0$ by (4.5),(4.6) and (iv). Hence

$$\int_a^b \sum_{i=1}^{p}\tau_i f^i(t,x^*,u^*)\,dt$$
$$\geq \int_a^b \{\sum_{i=1}^{p}\tau_i f^i(t,x,u)+\sum_{j=1}^{k}\lambda_j(t)g^j(t,x,u)+\sum_{r=1}^{q}\mu_r(t)(h^r(t,x,u)-\dot{x})\}\,dt,$$

which is a contradiction to (4.31). □

Corollary 4.5.1. Assume that weak duality (Theorem 4.5.1) holds between (MCP) and (WMCD). If (x,u) is feasible for (MCP), (x,u,τ,λ,μ) is feasible for (WMCD) with $\sum_{j=1}^{k}\lambda_j(t)g^j(t,x,u)=0$, then (x,u) is an efficient for (MCP) and (x,u,τ,λ,μ) is an efficient for (WMCD).

Proof. Suppose (x,u) is not an efficient for (MCP). Then there exists some feasible (x^*,u^*) for (MCP) such that

$$\int_a^b f^i(t,x^*,u^*)\,dt \leq \int_a^b f^i(t,x,u)\,dt, \quad \text{for all } i=1,\cdots,p$$
and
$$\int_a^b f^{i_0}(t,x^*,u^*)\,dt < \int_a^b f^{i_0}(t,x,u)\,dt, \quad \text{for some } i_0=1,\cdots,p.$$

Since $\sum_{j=1}^{k}\lambda_j(t)g^j(t,x,u)=0$ and $\sum_{r=1}^{n}\mu_r(t)(h^r(t,x,u)-\dot{x})=0$, we get

$$\int_a^b f^i(t,x^*,u^*)\,dt \leq \int_a^b \{f^i(t,x,u)+\sum_{j=1}^{k}\lambda_j(t)g^j(t,x,u)$$
$$+\sum_{r=1}^{n}\mu_r(t)(h^r(t,x,u)-\dot{x})\}\,dt,$$

for all $i = 1, \cdots, p$
and

$$\int_a^b f^{i_0}(t, x^*, u^*)\, dt < \int_a^b \{f^{i_0}(t, x, u) + \sum_{j=1}^{k} \lambda_j(t) g^j(t, x, u) + \sum_{r=1}^{n} \mu_r(t)(h^r(t, x, u) - \dot{x})\}\, dt,$$

for some $i_0 = 1, \cdots, p$.

This contradicts the weak duality. Hence (x, u) is an efficient for (MCP). Now suppose $(x, u, \tau, \lambda, \mu)$ is not an efficient for (WMCD). Then there exists some $(x^*, u^*, \tau^*, \lambda^*, \mu^*)$ feasible for (WMCD) such that

$$\int_a^b \{f^i(t, x^*, u^*) + \sum_{j=1}^{k} \lambda_j^*(t) g^j(t, x^*, u^*) + \sum_{r=1}^{n} \mu_r^*(t)(h^r(t, x^*, u^*) - \dot{x}^*)\}\, dt$$
$$\geq \int_a^b \{f^i(t, x, u) + \sum_{j=1}^{k} \lambda_j(t) g^j(t, x, u) + \sum_{r=1}^{n} \mu_r(t)(h^r(t, x, u) - \dot{x})\}\, dt,$$

for all $i = 1, \cdots, p$
and

$$\int_a^b \{f^{i_0}(t, x^*, u^*) + \sum_{j=1}^{k} \lambda_j^*(t) g^j(t, x^*, u^*) + \sum_{r=1}^{n} \mu_r^*(t)(h^r(t, x^*, u^*) - \dot{x}^*)\}\, dt$$
$$> \int_a^b \{f^{i_0}(t, x, u) + \sum_{j=1}^{k} \lambda_j(t) g^j(t, x, u) + \sum_{r=1}^{n} \mu_r(t)(h^r(t, x, u) - \dot{x})\}\, dt,$$

for some $i_0 = 1, \cdots, p$.

Since $\sum_{j=1}^{k} \lambda_j(t) g^j(t,x,u) = 0$ and $\sum_{r=1}^{n} \mu_r(t)(h^r(t,x,u) - \dot{x}) = 0$,

$$\int_a^b \{f^i(t,x^*,u^*) + \sum_{j=1}^{k} \lambda_j^*(t) g^j(t,x^*,u^*) + \sum_{r=1}^{n} \mu_r^*(t)(h^r(t,x^*,u^*) - \dot{x}^*)\}\, dt$$
$$\geq \int_a^b f^i(t,x,u)\, dt, \quad \text{for all } i = 1,\cdots,p$$

and

$$\int_a^b \{f^{i_0}(t,x^*,u^*) + \sum_{j=1}^{k} \lambda_j^*(t) g^j(t,x^*,u^*) + \sum_{r=1}^{n} \mu_r^*(t)(h^r(t,x^*,u^*) - \dot{x})\}\, dt$$
$$> \int_a^b f^{i_0}(t,x,u)\, dt, \quad \text{for some } i_0 = 1,\cdots,p.$$

This contradicts the weak duality. Hence (x,u,τ,λ,μ) is an efficient for (WMCD).

□

Theorem 4.5.2 (Strong Duality). Let (x^*,u^*) be an efficient for (MCP) and assume that (x^*,u^*) satisfies the constraint qualification for $MCP_k(x^*,u^*)$ for at least one $k \in \{1,\cdots,p\}$. Then there exist $\tau^* \in R^p$ and piecewise smooth functions $\lambda^* : I \to R^k$ and $\mu^* : I \to R^n$ such that $(x^*,u^*,\tau^*,\lambda^*,\mu^*)$ is feasible for (WMCD) and $\sum_{j=1}^{k} \lambda_j^*(t) g^j(t,x^*,u^*) = 0$. If weak duality also holds between (MCP) and (WMCD), then $(x^*,u^*,\tau^*,\lambda^*,\mu^*)$ is an efficient for (WMCD).

Proof. It follows from Lemma 4.2.1 that there exist $\tau^* \in R^p$, and piecewise smooth functions $\lambda^* : I \to R^k$ and $\mu^* : I \to R^n$, satisfying the following relations, for all $t \in I$:

$$\sum_{i=1}^{p} \tau_i^* f_x^i(t, x^*, u^*) + \sum_{j=1}^{k} \lambda_j^*(t) g_x^j(t, x^*, u^*) + \sum_{r=1}^{n} \mu_r^*(t) h_x^r(t, x^*, u^*) + \dot{\mu}^* = 0,$$

$$\sum_{i=1}^{p} \tau_i^* f_u^i(t, x^*, u^*) + \sum_{j=1}^{k} \lambda_j^*(t) g_u^j(t, x^*, u^*) + \sum_{r=1}^{n} \mu_r^*(t) h_u^r(t, x^*, u^*) = 0,$$

$$\sum_{j=1}^{k} \lambda_j^*(t) g^j(t, x^*, u^*) = 0,$$

$$\tau_i^* \geq 0, \quad \sum_{i=1}^{p} \tau_i^* = 1, \quad \lambda^*(t) \geq 0.$$

As (x^*, u^*) is feasible for (MCP), $\dot{x}^* = h^r(t, x^*, u^*)$ and $\int_a^b \sum_{r=1}^{n} \mu_r(t)(h^r(t, x^*, u^*) - \dot{x}^*)dt \geq 0$. therefore $(x^*, u^*, \tau^*, \lambda^*, \mu^*)$ is feasible for (WMCD). The result now follows from Corollary 4.5.1. □

Now we establish some duality theorems between the multiobjective control problem (MCP) and its Mond-Weir type dual problem (MMCD).

Theorem 4.5.3 (Weak Duality). Assume that, for all feasible (x^*, u^*) for (MCP) and all feasible $(x, u, \tau, \lambda, \mu)$ for (MMCD),

(i) $\left(\int_a^b f^1(\cdot,\cdot,\cdot)\, dt, \quad \cdots, \quad \int_a^b f^p(\cdot,\cdot,\cdot)\, dt\right)$ is V-ρ-pseudo-invex with respect to η, ξ, ζ and α,

(ii) $\left(\int_a^b \lambda_1 g^1(\cdot,\cdot,\cdot)\, dt, \quad \cdots, \quad \int_a^b \lambda_k g^k(\cdot,\cdot,\cdot)\, dt\right)$ is V-ρ-quasi-invex with respect to η, ξ, ζ and β,

(iii) $\left(\int_a^b \mu_1(h^1(\cdot,\cdot,\cdot) - \dot{x})\, dt, \quad \cdots, \quad \int_a^b \mu_n(h^n(\cdot,\cdot,\cdot) - \dot{x})\, dt\right)$ is V-ρ-quasi-invex with respect to η, ξ, ζ and γ, and

(iv) $\sum \tau_i \rho_i + \sum \rho_j + \sum \rho_r \geq 0$.

Then the following relations cannot hold:

$$\int_a^b f^i(t, x^*, u^*)\,dt \le \int_a^b f^i(t, x, u)\,dt, \quad \text{for all} \quad i = 1, \cdots, p \tag{4.39}$$

and

$$\int_a^b f^{i_0}(t, x^*, u^*)\,dt < \int_a^b f^{i_0}(t, x, u)\,dt, \quad \text{for some} \quad i_0 = 1, \cdots, p. \tag{4.40}$$

Proof. Suppose contrary to the result that (4.39) and (4.40) hold. Since $\alpha_i(x, x^*, \dot{x}, \dot{x}^*, u, u^*) > 0$,

$$\int_a^b \sum_{i=1}^p \alpha_i f^i(t, x^*, u^*)\,dt < \int_a^b \sum_{i=1}^p \alpha_i f^i(t, x, u)\,dt.$$

Then (i) yields

$$\int_a^b \sum_{i=1}^p \{\eta^T f_x^i(t, x, u) + \xi^T f_u^i(t, x, u)\}\,dt + \sum_{i=1}^p \rho_i ||\zeta||^2 < 0.$$

Since $\tau_i \ge 0$, we have

$$\int_a^b \sum_{i=1}^p \{\eta^T \tau_i f_x^i(t, x, u) + \xi^T \tau_i f_u^i(t, x, u)\}\,dt + \sum_{i=1}^p \tau_i \rho_i ||\zeta||^2 < 0. \tag{4.41}$$

From the feasibility conditions,

$$\int_a^b \lambda_j(t) g^j(t, x^*, u^*)\,dt \le \int_a^b \lambda_j(t) g^j(t, x, u)\,dt, \ \text{for each} \quad j = 1, \cdots, k.$$

Since $\beta_j > 0, \ \forall j = 1, \cdots, k$, we have

$$\int_a^b \sum_{j=1}^k \beta_j \lambda_j(t) g^j(t, x^*, u^*)\, dt \leq \int_a^b \sum_{j=1}^k \beta_j \lambda_j(t) g^j(t, x, u)\, dt.$$

It now follows from (ii) that

$$\int_a^b \sum_{j=1}^k \{\eta^T \lambda_j(t) g_x^j(t, x, u) + \xi^T \lambda_j(t) g_u^j(t, x, u)\}\, dt + \sum_{j=1}^k \lambda_j \rho_j ||\zeta||^2 \leq 0. \tag{4.42}$$

From (4.3) and (4.13), we have

$$\int_a^b \sum_{r=1}^n \mu_r(t)(h^r(t, x^*, u^*) - \dot{x})\, dt \leq \int_a^b \sum_{r=1}^n \mu_r(t)(h^r(t, x, u) - \dot{x})\, dt.$$

Since $\gamma_r > 0, \ \forall r = 1, \cdots, n$, we have

$$\int_a^b \sum_{r=1}^n \gamma_r \mu_r(t)(h^r(t, x^*, u^*) - \dot{x})\, dt \leq \int_a^b \sum_{r=1}^n \gamma_r \mu_r(t)(h^r(t, x, u) - \dot{x})\, dt.$$

From (iii) it follows that

$$\int_a^b \sum_{r=1}^n \{\eta^T \mu_r(t) h_x^r(t, x, u) - \frac{d\eta^T}{dt}\mu(t) + \xi^T \mu_r h_u^r(t, x, u)\} dt + \sum_{r=1}^n \rho_r ||\zeta||^2 \leq 0. \tag{4.43}$$

By integrating $\frac{d\eta^T}{dt}\mu(t)$ from a to b and applying the boundary condition (4.1), we have

$$\int_a^b \frac{d\eta^T}{dt}\mu(t)\,dt = -\int_a^b \eta^T \dot{\mu}(t)\,dt. \tag{4.44}$$

Using (4.44) in (4.43), we have

$$\int_a^b \{\sum_{r=1}^{n} \eta^T \mu_r(t) h_x^r(t,x,u) + \eta^T \dot{\mu}(t) + \sum_{r=1}^{n} \xi^T \mu_r h_u^r(t,x,u)\}\,dt + \sum_{r=1}^{n} \rho_r ||\zeta||^2 \leq 0. \tag{4.45}$$

Adding (4.41), (4.42) and (4.45), we have

$$\int_a^b \Bigg(\eta^T \{\sum_{i=1}^{p} \tau_i f_x^i(t,x,u) + \sum_{j=1}^{k} \lambda_j g_x^j(t,x,u) + \sum_{r=1}^{n} \mu_r h_x^r(t,x,u) + \dot{\mu}(t)\}$$
$$+ \xi^T \{\sum_{i=1}^{p} \tau_i f_u^i(t,x,u) + \sum_{j=1}^{k} \lambda_j g_u^j(t,x,u) + \sum_{r=1}^{q} \mu_r h_u^r(t,x,u)\} \Bigg)\,dt$$
$$+ \sum_{i=1}^{p} \tau_i \rho_i ||\zeta||^2 + \sum_{j=1}^{k} \rho_j ||\zeta||^2 + \sum_{r=1}^{n} \rho_r ||\zeta||^2 < 0,$$

which is a contradiction to (4.11), (4.12) and (iv). □

Corollary 4.5.2. Assume that weak duality theorem (4.5.3) holds between (MCP) and (MMCD). If (x, u) is feasible for (MCP) and $(x, u, \tau, \lambda, \mu)$ is feasible for (MMCD), then (x, u) is an efficient for (MCP) and $(x, u, \tau, \lambda, \mu)$ is an efficient for (MMCD).

Proof. Suppose (x, u) is not an efficient for (MCP). Then there exists some feasible (x^*, u^*) for (MCP) such that

$$\int_a^b f^i(t,x^*,u^*)\,dt \leq \int_a^b f^i(t,x,u)\,dt, \text{ for all } i=1,\cdots,p$$

and

$$\int_a^b f^{i_0}(t,x^*,u^*)\,dt < \int_a^b f^{i_0}(t,x,u)\,dt, \text{ for some } i_0=1,\cdots,p.$$

But (x,u,τ,λ,μ) is feasible for (MMCD), hence the result of weak duality theorem is contradict. Therefore (x,u) is an efficient for (MCP). Now suppose (x,u,τ,λ,μ) is not an efficient for (MMCD). Then there exist some feasible $(x^*,u^*,\tau,\lambda,\mu)$ for (MMCD) such that

$$\int_a^b f^i(t,x^*,u^*)\,dt \geq \int_a^b f^i(t,x,u)\,dt, \ \forall\ i=1,\cdots,p$$

and

$$\int_a^b f^{i_0}(t,x^*,u^*)\,dt > \int_a^b f^{i_0}(t,x,u)\,dt, \text{ for some } i_0=1,\cdots,p.$$

This contradicts weak duality. Hence (x,u,τ,λ,μ) is an efficient for (MMCD).

□

Theorem 4.5.4 (Strong Duality). Let (x^*,u^*) be an efficient for (MCP) and assume that (x^*,u^*) satisfies the constraint qualification for $MCP_k(x^*,u^*)$ for at least one $k\in\{1,\cdots,p\}$. Then there exist $\tau^*\in R^p$ and piecewise smooth functions $\lambda^*: I\to R^k$ and $\mu^*: I\to R^n$ such that $(x^*,u^*,\tau^*,\lambda^*,\mu^*)$ is feasible for (MMCD). If also weak duality holds between (MCP) and (MMCD), then $(x^*,u^*,\tau^*,\lambda^*,\mu^*)$ is an efficient for (MMCD).

Proof. Proceeding on the same lines as in Theorem 4.5.2, it follows that there exist $\tau^*\in R^p$, and piecewise smooth functions $\lambda^*: I\to R^k$ and $\mu^*: I\to R^n$, satisfying for all $t\in I$ the following relations:

$$\sum_{i=1}^{p} \tau_i^* f_x^i(t, x^*, u^*) + \sum_{j=1}^{k} \lambda_j^*(t) g_x^j(t, x^*, u^*) + \sum_{r=1}^{n} \mu_r^*(t) h_x^r(t, x^*, u^*) + \dot{\mu}^* = 0,$$

$$\sum_{i=1}^{p} \tau_i^* f_u^i(t, x^*, u^*) + \sum_{j=1}^{k} \lambda_j^*(t) g_u^j(t, x^*, u^*) + \sum_{r=1}^{n} \mu_r^*(t) h_u^r(t, x^*, u^*) = 0,$$

$$\sum_{j=1}^{k} \lambda_j^*(t) g^j(t, x^*, u^*) = 0,$$

$$\tau_i^* \geq 0, \quad \sum_{i=1}^{p} \tau_i^* = 1, \quad \lambda^*(t) \geq 0.$$

The relations $\int_a^b \sum_{j=1}^k \lambda_j^*(t) g^j(t, x^*, u^*)\, dt = 0$, and $\int_a^b \sum_{r=1}^n \mu_r^*(t)(h^r(t, x^*, u^*) - \dot{x}^*)\, dt \geq 0$ are obvious.

The above relations imply that $(x^*, u^*, \tau^*, \lambda^*, \mu^*)$ is feasible for (MMCD). The result now follows from Corollary 4.5.2. □

Chapter 5

Multiobjective Fractional Control Problem with Generalized V-ρ Invexity

5.1. Introduction

The following problem is called a multiobjective fractional control problem:

(MFCP)

$$\text{Minimize} \quad \frac{\int_a^b f(t,x(t),u(t))dt}{\int_a^b g(t,x(t),u(t))dt}$$
$$=\Big(\frac{\int_a^b f^1(t,x(t),u(t))dt}{\int_a^b g^1(t,x(t),u(t))dt},\cdots,\frac{\int_a^b f^p(t,x(t),u(t))dt}{\int_a^b g^p(t,x(t),u(t))dt}\Big)$$

$$\text{subject to} \quad x(a)=t_0,\ \ x(b)=t_f, \tag{5.1}$$

$$h^j(t,x(t),u(t))\leq 0, \quad t\in I, \tag{5.2}$$

$$k^l(t,x(t),u(t))=\dot{x}, \quad t\in I, \tag{5.3}$$

where $f^i, g^i : I\times R^n\times R^m\to R,\ i=1,\cdots,p,\ h^j : I\times R^n\times R^m\to R,\ j=1,\cdots,m$, and $k^l : I\times R^n\times R^m\to R,\ l=1,\cdots,n$, are continuously differentiable functions. $I=[a,b]$ is a real interval. We assume that $f^i(t,x(t),u(t))\geq 0$ and $g^i(t,x(t),u(t))>0$ on $I\times R^n\times R^m$ for $i=1,\cdots,p$.

Let $x : I\to R^n$ be differentiable with its derivative $\dot{x}$, and let $u : I\to R^m$ be a differentiable function. Denote the partial derivatives of f by f_t, f_x, and f_u, that is

$$f_t=\frac{\partial f}{\partial t},\ \ f_x=[\frac{\partial f}{\partial x^1},\cdots,\frac{\partial f}{\partial x^n}],\ \ f_u=[\frac{\partial f}{\partial u^1},\cdots,\frac{\partial f}{\partial u^m}],$$

where the superscripts denote the vector components.

Similarly, we have $g_t, g_x, g_u, h_t, h_x, h_u$, and k_t, k_x, k_u. Denote by X the space of piecewise smooth functions $x : I \to R^n$, with the norm $||x|| = ||x||_\infty + ||Dx||_\infty$ and by Y the space of piecewise continuous control functions $u : I \to R^m$ with the norm $||u||_\infty$. The differential equation (5.3) with initial conditions expressed as $x(t) = x(a) + \int_a^t k^l(s, x(s), u(s))\, ds,\ t \in I$ may be written as $\dot{x} = K^l(x, u)$, where $K^l : X \times Y \longmapsto C(I, R^n)$, $C(I, R^n)$ being the space of continuous functions from I to R^n defined as $K^l(x, u)(t) = k^l(t, x(t), u(t))$.

In this chapter, we will define generalized V-ρ-invex functions for optimal control problems and consider a multiobjective fractional control problem (MFCP). We obtain the sufficient optimality conditions of the Kuhn-Tucker type for (MFCP) under generalized invexity conditions. Moreover, we formulate Wolfe type dual (WFCD) and Mond-Weir type dual (MFCD) for (MFCP), and then establish their duality relations.

5.2. Definitions and Preliminaries

Definition 5.2.1. A feasible solution (x^*, u^*) of (MFCP) is said to be an efficient solution of (MFCP) if there does not exist a feasible solution (x, u) of (MFCP) such that

$$\frac{\int_a^b f^i(t,x,u)dt}{\int_a^b g^i(t,x,u)dt} \leq \frac{\int_a^b f^i(t,x^*,u^*)dt}{\int_a^b g^i(t,x^*,u^*)dt} \quad \forall i = 1, \cdots, p$$

and

$$\frac{\int_a^b f^{i_0}(t,x,u)dt}{\int_a^b g^{i_0}(t,x,u)dt} < \frac{\int_a^b f^{i_0}(t,x^*,u^*)dt}{\int_a^b g^{i_0}(t,x^*,u^*)dt} \quad \text{for some } i_0 = 1, \cdots, p.$$

Following Bector et al. [3], the problem $(\text{MFCP})_v$ stated below is associated with the given problem (MFCP) for $v \in R^p_+$, where R^p_+ is the positive orthant of R^p.

(MFCP)$_v$

$$\begin{aligned}
\text{Minimize} \quad & \int_a^b \{f(t,x(t),u(t)) - v^T g(t,x(t),u(t))\}dt \\
& = \Big(\int_a^b \{f^1(t,x(t),u(t)) - v_1 g^1(t,x(t),u(t))\}dt, \\
& \cdots, \int_a^b \{f^p(t,x(t),u(t)) - v_p g^p(t,x(t),u(t))\}dt \Big) \\
\text{subject to} \quad & (5.1) - (5.3).
\end{aligned}$$

The following Lemma connecting (MFCP) and (MFCP)$_v$ has been proved in [4].

Lemma 5.2.1. (x^*, u^*) is an efficient solution of (MFCP) if and only if (x^*, u^*) is an efficient solution of (MFCP)$_v$, where $v_i = \frac{\int_a^b f^i(t,x^*,u^*)dt}{\int_a^b g^i(t,x^*,u^*)dt}$.

Lemma 1 of [34] states that (x^*, u^*) is an efficient solution for (MFCP)$_v$ if and only if (x^*, u^*) solves

$(MFCP_v)_k(x^*, u^*)$:

$$\begin{aligned}
\text{Minimize} \quad & \int_a^b \{f^k(t,x,u) - v_k g^k(t,x,u)\}\, dt \\
\text{subject to} \quad & x(a) = t_0, x(b) = t_f, \\
& h^j(t,x,u) \leq 0, \;\; j = 1, \cdots, m, \\
& k^l(t,x,u) = \dot{x}, \;\; l = 1, \cdots, n, \\
& \{f^j(t,x,u) - v_j g^j(t,x,u)\} \leq \{f^j(t,x^*,u^*) - v_j g^j(t,x^*,u^*)\}, \\
& \qquad\qquad \textit{for all } \; j \in \{1, \cdots, p\}, j \neq k,
\end{aligned}$$

for each $k = 1, \cdots, p$.

Chandra, Craven, and Husain [6] gave the Fritz John necessary optimality conditions for the existence of an extremal solution for the single objective control problem (CP):

(CP)

$$\begin{aligned} \text{Minimize} \quad & \int_a^b f(t, x, u)\, dt \\ \text{subject to} \quad & x(a) = t_0, x(b) = t_f, \\ & g(t, x, u) \leq 0, \\ & h(t, x, u) = \dot{x}, \end{aligned}$$

where $f : I \times R^n \times R^m \to R$, $g : I \times R^n \times R^m \to R^m$ and $h : I \times R^n \times R^m \to R^n$ are assumed to be continuously differentiable functions.

Mond and Hanson [28] pointed out that if the optimal solution for (CP) is normal, then Fritz John conditions reduce to Kuhn-Tucker conditions.

Lemma 5.2.2 (Kuhn-Tucker Necessary Optimality Condition). Let $(x^*, u^*) \in X \times Y$ be an efficient for $(\text{MFCP})_v$. If the Fréchet derivatives $[D - K_x^i(x^*, u^*)]$ is surjective and (x^*, u^*) is normal for at least one $P_k(x^*, u^*)$ then there exist $\tau^* \in R^p$, and piecewise smooth functions $\lambda^* : I \to R^m$ *and* $\mu^* : I \to R^n$, satisfying the following equalities, for all $t \in I$,

$$\begin{aligned} & \sum_{i=1}^p \tau_i^* \{f_x^i(t, x^*, u^*) - v_i g_x^i(t, x^*, u^*)\} + \sum_{j=1}^m \lambda_j^*(t) h_x^j(t, x^*, u^*) \\ & + \sum_{l=1}^n \mu_l^*(t) k_x^l(t, x^*, u^*) + \dot{\mu}^*(t) = 0, \end{aligned}$$

$$\sum_{i=1}^{p} \tau_i^* \{f_u^i(t,x^*,u^*) - v_i g_u^i(t,x^*,u^*)\} + \sum_{j=1}^{m} \lambda_j^*(t) h_u^j(t,x^*,u^*)$$
$$+ \sum_{l=1}^{n} \mu_l^*(t) k_u^l(t,x^*,u^*) = 0,$$
$$\sum_{j=1}^{m} \lambda_j^*(t) h^j(t,x^*,u^*) = 0,$$
$$\tau_i^* \geq 0, \quad \sum_{i=1}^{p} \tau_i^* = 1, \quad \lambda^*(t) \geq 0.$$

Definition 5.2.2. Let k^i be a function from $I \times R^n \times R^n \times R^m$ into R and let $K^i(x,u) = \int_a^b k^i(t,x,\dot{x},u)\,dt$. Let there exist differentiable vector functions $\eta(t,x,x^*,\dot{x},\dot{x}^*,u,u^*) \in R^n$ with $\eta = 0$ at t if $x(t) = x^*(t)$, and $\xi(t,x,x^*,\dot{x},\dot{x}^*,u,u^*) \in R^m$, $\zeta(x,x^*,\dot{x},\dot{x}^*,u,u^*) \in R^n$ Let $||\zeta(x,x^*,\dot{x},\dot{x}^*,u,u^*)$ $|| = \sup_{t \in I} ||\zeta(x,x^*,\dot{x},\dot{x}^*,u,u^*)||^2$ and ρ_i real numbers.

(1) A vector function $K = (K^1, \cdots, K^n)$ is said to be V-ρ-invex in $x^*, \dot{x}^*$, and u^* on I with respect to η, ξ, ζ, and α if there exist differentiable vector functions $\eta \in R^n, \xi \in R^m, \zeta \in R^n, \alpha_i(x,x^*,\dot{x},\dot{x}^*,u,u^*) \in R_+ \backslash \{0\}$ and $\rho_i \in R$ such that, for each $x, x^* \in X$, $u, u^* \in Y$ and for $i = 1, \cdots, n$,

$$K^i(x,u) - K^i(x^*,u^*)$$
$$\geq \int_a^b \{\eta^T \alpha_i k_x^i(t,x^*,\dot{x}^*,u^*) + \frac{d\eta^T}{dt} \alpha_i k_{\dot{x}}^i(t,x^*,\dot{x}^*,u^*) + \xi^T \alpha_i k_u^i(t,x^*,\dot{x}^*,u^*)\}\,dt$$
$$+ \rho_i ||\zeta||^2.$$

(2) The vector function $K = (K^1, \cdots, K^n)$ is said to be V-ρ-pseudo-invex (strictly V-ρ-pseudo-invex) in $x^*, \dot{x}^*$, and u^* on I with respect to η, ξ, ζ and

β if there exist $\eta,\ \xi,\ \zeta$ as above and $\beta_i(x, x^*, \dot{x}, \dot{x}^*, u, u^*) \in R_+\backslash\{0\}$ and $\rho_i \in R$ such that, for each $x, x^* \in X, u, u^* \in Y$ and for $i = 1, \cdots, n$,

$$\int_a^b \sum_{i=1}^n \{\eta^T k_x^i(t, x^*, \dot{x}^*, u^*) + \frac{d\eta^T}{dt} k_{\dot{x}}^i(t, x^*, \dot{x}^*, u^*) + \xi^T k_u^i(t, x^*, \dot{x}^*, u^*)\}\, dt$$
$$+ \sum \rho_i ||\zeta||^2 \geq 0$$
$$\Longrightarrow \int_a^b \sum_{i=1}^n \beta_i k^i(t, x, \dot{x}, u)\, dt \geq \int_a^b \sum_{i=1}^n \beta_i k^i(t, x^*, \dot{x}^*, u^*)\, dt \ \ (>)$$

(3) The vector function $K = (K^1, \cdots, K^n)$ is said to be V-ρ-quasi-invex in $x^*, \dot{x}^*$, and u^* on I with respect to $\eta,\ \xi,\ \zeta$, and γ if there exist $\eta,\ \xi,\ \zeta$, as above and the vector $\gamma_i(x, x^*, \dot{x}, \dot{x}^*, u, u^*) \in R_+\backslash\{0\}$ and $\rho_i \in R$ such that, for each $x, x^* \in X$, $u, u^* \in Y$ and for $i = 1, \cdots, n$,

$$\int_a^b \sum_{i=1}^n \gamma_i k^i(t, x, \dot{x}, u) dt \leq \int_a^b \sum_{i=1}^n \gamma_i k^i(t, x^*, \dot{x}^*, u^*) dt$$
$$\Longrightarrow \int_a^b \sum_{i=1}^n \{\eta^T k_x^i(t, x^*, \dot{x}^*, u^*) + \frac{d\eta^T}{dt} k_{\dot{x}}^i(t, x^*, \dot{x}^*, u^*) + \xi^T k_u^i(t, x^*, \dot{x}^*, u^*)\} dt$$
$$+ \sum \rho_i ||\zeta||^2 \leq 0$$

5.3. Formulation of Fractional Control Dual Problem

We formulate two pairs of the following multiobjective fractional dual control problems.

The Wolfe type dual [39]:

(WFCD)

$$\text{Maximize} \quad \Big[\int_a^b \Big(\{f^1(t,x,u) - v_1 g^1(t,x,u)\} + \sum_{j=1}^m \lambda_j(t) h^j(t,x,u) + \sum_{l=1}^n \mu_l(t)(k^l(t,x,u) - \dot{x})\Big)\, dt, \cdots, \int_a^b \Big(\{f^p(t,x,u) - v_p g^p(t,x,u)\} + \sum_{j=1}^m \lambda_j(t) h^j(t,x,u) + \sum_{l=1}^n \mu_l(t)(k^l(t,x,u) - \dot{x})\Big)\, dt\Big]$$

subject to

$$x(a) = t_0, \ \ x(b) = t_f, \tag{5.4}$$

$$\sum_{i=1}^p \tau_i\{f_x^i(t,x,u) - v_i g_x^i(t,x,u)\} + \sum_{j=1}^m \lambda_j(t) h_x^j(t,x,u) + \sum_{l=1}^n \mu_l(t) k_x^l(t,x,u) + \dot{\mu}(t) = 0, \ \ t \in I, \tag{5.5}$$

$$\sum_{i=1}^p \tau_i\{f_u^i(t,x,u) - v_i g_u^i(t,x,u)\} + \sum_{j=1}^m \lambda_j(t) h_u^j(t,x,u) + \sum_{l=1}^n \mu_l(t) k_u^l(t,x,u) = 0, \ \ t \in I, \tag{5.6}$$

$$\lambda(t) \geq 0, \quad t \in I, \tag{5.7}$$

$$v_i \geq 0, \quad i = 1, \cdots, p, \ \ \tau_i \geq 0, \ \ \sum_{i=1}^p \tau_i = 1. \tag{5.8}$$

The Mond-Weir type dual [30]:

(MFCD)

$$\text{Maximize} \quad \Big(\int_a^b \{f^1(t,x,u) - v_1 g^1(t,x,u)\} dt, \cdots, \int_a^b \{f^p(t,x,u) - v_p g^p(t,x,u)\} dt\Big)$$

subject to $\quad x(a) = t_0, \ \ x(b) = t_f,$ (5.9)

$$\sum_{i=1}^{p} \tau_i \{ f_x^i(t,x,u) - v_i g_x^i(t,x,u)\} + \sum_{j=1}^{m} \lambda_j(t) h_x^j(t,x,u)$$

$$+ \sum_{l=1}^{n} \mu_l(t) k_x^l(t,x,u) + \dot{\mu}(t) = 0, \ \ t \in I, \tag{5.10}$$

$$\sum_{i=1}^{p} \tau_i \{ f_u^i(t,x,u) - v_i g_u^i(t,x,u)\} + \sum_{j=1}^{m} \lambda_j(t) h_u^j(t,x,u)$$

$$+ \sum_{l=1}^{n} \mu_l(t) k_u^l(t,x,u) = 0, \ \ t \in I, \tag{5.11}$$

$$\int_a^b \sum_{j=1}^{m} \lambda_j(t) h^j(t,x,u) dt \geq 0, \ \ t \in I, \tag{5.12}$$

$$\int_a^b \sum_{l=1}^{n} \mu_l(t) (k^l(t,x,u) - \dot{x}) dt \geq 0, \ \ t \in I, \tag{5.13}$$

$$\lambda(t) \geq 0, \quad t \in I, \tag{5.14}$$

$$\tau_i \geq 0, \ \ \sum_{i=1}^{p} \tau_i = 1. \tag{5.15}$$

5.4. Sufficient Optimality Theorem for $(\mathrm{MFCP})_v$

We obtain a Kuhn-Tucker type sufficient optimality theorem of $(\mathrm{MFCP})_v$ as follows:

Theorem 5.4.1. Suppose that (x^*, u^*) is feasible for $(\mathrm{MFCP})_v$ such that there exist $\tau^* > 0, \ \ \lambda^*(t)$ and $\mu^*(t)$ such that

$$\sum_{i=1}^{p} \tau_i^* \{f_x^i(t, x^*, u^*) - v_i g_x^i(t, x^*, u^*)\} + \sum_{j=1}^{m} \lambda_j^*(t) h_x^j(t, x^*, u^*)$$

$$+ \sum_{l=1}^{n} \mu_l^*(t) k_x^l(t, x^*, u^*) + \dot{\mu}^*(t) = 0, \tag{5.16}$$

$$\sum_{i=1}^{p} \tau_i^* \{f_u^i(t, x^*, u^*) - v_i g_u^i(t, x^*, u^*)\} + \sum_{j=1}^{m} \lambda_j^*(t) h_u^j(t, x^*, u^*)$$

$$+ \sum_{l=1}^{n} \mu_l^*(t) k_u^l(t, x^*, u^*) = 0, \tag{5.17}$$

$$\sum_{j=1}^{m} \lambda_j^*(t) h^j(t, x^*, u^*) = 0, \tag{5.18}$$

$$\sum_{l=1}^{n} \mu_l^*(t)(k^l(t, x^*, u^*) - \dot{x}^*) = 0, \tag{5.19}$$

$$\tau_i^* \geq 0, \quad \sum_{i=1}^{p} \tau_i^* = 1, \tag{5.20}$$

$$\text{and } \lambda^*(t) \geq 0. \tag{5.21}$$

hold through $a \leq t \leq b$ (except that at t corresponding to discontinuities of $u^*(t)$, (5.16) holds for right and left limits). If $\int_a^b (f^i - v_i g^i)\, dt$, $i = 1, \cdots, p$, $\int_a^b \lambda_j^* h^j\, dt$, $j = 1, \cdots, m$, and $\int_a^b \mu_l^*(k^l - \dot{x}^*)\, dt$, $l = 1, \cdots, n$ are all V-ρ-invex with respect to η, ξ, ζ, and α, and $\sum \tau_i \rho_i + \sum \rho_j + \sum \rho_l \geq 0$, then (x^*, u^*) is an efficient solution of $(\text{MFCP})_v$.

Proof. Suppose that (x^*, u^*) is not an efficient solution of $(\text{MFCP})_v$. Then there exists $(x, u) \neq (x^*, u^*)$ such that (x, u) is feasible for $(\text{MFCP})_v$, and

$$\int_a^b \{f^i(t,x,u) - v_i g^i(t,x,u)\}\,dt \leq \int_a^b \{f^i(t,x^*,u^*) - v_i g^i(t,x^*,u^*)\}\,dt,$$
$\forall\ i = 1,\cdots,p$

and

$$\int_a^b \{f^{i_0}(t,x,u) - v_{i_0} g^{i_0}(t,x,u)\}\,dt < \int_a^b \{f^{i_0}(t,x^*,u^*) - v_{i_0} g^{i_0}(t,x^*,u^*)\}\,dt,$$
for some $i_0 = 1,\cdots,p$.

Since $\int_a^b (f^i - v_i g^i)\,dt$ is V-ρ-invex, we have

$$\int_a^b \Big(\eta^T \alpha_i \{f_x^i(t,x^*,u^*) - v_i g_x^i(t,x^*,u^*)\} + \xi^T \alpha_i \{f_u^i(t,x^*,u^*) - v_i g_u^i(t,x^*,u^*)\}\Big)\,dt + \rho_i ||\zeta||^2 \leq 0,\ \ \forall\ i = 1,\cdots,p,$$

and

$$\int_a^b \Big(\eta^T \alpha_i \{f_x^{i_0}(t,x^*,u^*) - v_{i_0} g_x^{i_0}(t,x^*,u^*)\} + \xi^T \alpha_i \{f_u^{i_0}(t,x^*,u^*) - v_{i_0} g_u^{i_0}(t,x^*,u^*)\}\Big)\,dt + \rho_i ||\zeta||^2 < 0,$$
for some $i_0 = 1,\cdots,p$.

Since $\tau_i^* > 0$ for all i, we get

$$\int_a^b \sum_{i=1}^{p} \alpha_i \Big(\eta^T \tau_i^* \{f_x^i(t,x^*,u^*) - v_i g_x^i(t,x^*,u^*)\} + \xi^T \tau_i^* \{f_u^i(t,x^*,u^*) - v_i g_u^i(t,x^*,u^*)\}\Big)\,dt + \sum_{i=1}^{p} \tau_i^* \rho_i ||\zeta||^2 < 0. \quad (5.22)$$

From the feasibility conditions, $\sum_{j=1}^m \lambda_j^*(t) h^j(t,x,u) < 0 = \sum_{j=1}^m \lambda_j^*(t) h^j(t,x^*,u^*)$. By the V-$\rho$-invexity of $\int_a^b \lambda_j^*(t) h^j\,dt$, we have

$$\int_a^b \sum_{j=1}^m \beta_j\{\eta^T\lambda_j^*(t)h_x^j(t,x^*,u^*)+\xi^T\lambda_j^*(t)h_u^j(t,x^*,u^*)\}\,dt+\sum_{j=1}^m \rho_j||\zeta||^2 \leq 0. \tag{5.23}$$

From the feasibility conditions, $\sum_{l=1}^n \mu_l^*(t)(k^l(t,x,u)-\dot{x})-\sum_{l=1}^n \mu_l^*(t)(k^l(t,x^*,u^*)-\dot{x}^*)=0$. By the V-$\rho$-invexity of $\int_a^b \mu_l^*(k^l-\dot{x}^*)\,dt$, we have

$$\int_a^b \sum_{l=1}^n \gamma_l\{\eta^T\mu_l^*(t)k_x^l(t,x^*,u^*)-\frac{d\eta^T}{dt}\mu^*(t)+\xi^T\mu_l^*(t)k_u^l(t,x^*,u^*)\}\,dt$$
$$+\sum_{l=1}^n \rho_l||\zeta||^2 \leq 0. \tag{5.24}$$

By integrating $\frac{d\eta^T}{dt}\mu^*(t)$ from a to b by parts and applying the boundary condition, we have

$$\int_a^b \frac{d\eta^T}{dt}\mu^*(t)\,dt=-\int_a^b \eta^T\dot{\mu}^*(t)\,dt. \tag{5.25}$$

Using (5.25) in (5.24), we have

$$\int_a^b \sum_{l=1}^n \gamma_l\{\eta^T\mu_l^*(t)k_x^l(t,x^*,u^*)+\eta^T\dot{\mu}^*(t)+\xi^T\mu_l^*(t)k_u^l(t,x^*,u^*)\}\,dt$$
$$+\sum_{l=1}^n \rho_l||\zeta||^2 \leq 0. \tag{5.26}$$

Since (5.22), (5.23) and (5.26) hold the same α, we have

$$\int_a^b \Big[\eta^T\alpha\Big(\sum_{i=1}^{p}\tau_i^*\{f_x^i(t,x^*,u^*)-v_ig_x^i(t,x^*,u^*)\}$$
$$+\sum_{j=1}^{m}\lambda_j^*(t)h_x^j(t,x^*,u^*)+\sum_{l=1}^{n}\mu_l^*(t)k_x^l(t,x^*,u^*)+\dot{\mu}^*(t)\Big)$$
$$+\xi^T\alpha\Big(\sum_{i=1}^{p}\tau_i^*\{f_u^i(t,x^*,u^*)-v_ig_u^i(t,x^*,u^*)\}$$
$$+\sum_{j=1}^{m}\lambda_j^*(t)h_u^j(t,x^*,u^*)+\sum_{l=1}^{n}\mu_l^*(t)k_u^l(t,x^*,u^*)\Big)\Big]\,dt$$
$$+\sum_{i=1}^{p}\tau_i^*\rho_i||\zeta||^2+\sum_{j=1}^{m}\rho_j||\zeta||^2+\sum_{l=1}^{n}\rho_l||\zeta||^2<0. \tag{5.27}$$

From (5.16), (5.17) and hypothesis, we have

$$\int_a^b \Big[\eta^T\alpha\Big(\sum_{i=1}^{p}\tau_i^*\{f_x^i(t,x^*,u^*)-v_ig_x^i(t,x^*,u^*)\}+\sum_{j=1}^{m}\lambda_j^*(t)h_x^j(t,x^*,u^*)$$
$$+\sum_{l=1}^{n}\mu_l^*(t)k_x^l(t,x^*,u^*)+\dot{\mu}^*(t)\Big)+\xi^T\alpha\Big(\sum_{i=1}^{p}\tau_i^*\{f_u^i(t,x^*,u^*)-v_ig_u^i(t,x^*,u^*)\}$$
$$+\sum_{j=1}^{m}\lambda_j^*(t)h_u^j(t,x^*,u^*)+\sum_{l=1}^{n}\mu_l^*(t)k_u^l(t,x^*,u^*)\Big)\Big]\,dt$$
$$+\sum_{i=1}^{p}\tau_i^*\rho_i||\zeta||^2+\sum_{j=1}^{m}\rho_j||\zeta||^2+\sum_{l=1}^{n}\rho_l||\zeta||^2\geq 0,$$

which contradicts the inequality (5.27). Hence (x^*,u^*) is an efficient solution of $(\mathrm{MFCP})_v$. □

5.5. Duality Theorems

Now we establish some duality theorems between the multiobjective fractional control problem $(\text{MFCP})_v$ and its Wolfe type dual problem (WFCD).

Theorem 5.5.1 (Weak Duality). Assume that for all feasible (x^*, u^*) for $(\text{MFCP})_v$ and all feasible $(x, u, \tau, v, \lambda, \mu)$ for (WFCD)

$$\text{(i)} \ \Big[\int_a^b \Big(\{f^1(t,\cdot,\cdot) - v_1 g^1(t,\cdot,\cdot)\} + \sum_{j=1}^m \lambda_j h^j(t,\cdot,\cdot) + \sum_{l=1}^n \mu_l (k^1(t,\cdot,\cdot) - \dot{x})\Big)\, dt,$$

$$\cdots, \int_a^b \Big(\{f^p(t,\cdot,\cdot) - v_p g^p(t,\cdot,\cdot)\} + \sum_{j=1}^m \lambda_j h^j(t,\cdot,\cdot) + \sum_{l=1}^n \mu_l (k^l(t,\cdot,\cdot) - \dot{x})\Big)\, dt\Big]$$

are V-ρ-invex with respect to the same η, ξ, ζ and α, and

(ii) $\tau_i > 0, \ \forall i = 1, \cdots, p,$ and $\rho \geq 0$, then

$$\int_a^b \{f^i(t, x^*, u^*) - v_i g^i(t, x^*, u^*)\}\, dt \leq \int_a^b \Big(\{f^i(t, x, u) - v_i g^i(t, x, u)\}$$
$$+ \sum_{j=1}^m \lambda_j(t) h^j(t, x, u) + \sum_{l=1}^n \mu_l(t)(k^l(t, x, u) - \dot{x})\Big)\, dt, \quad \forall\ i = 1, \cdots, p \qquad (5.28)$$

and

$$\int_a^b \{f^{i_0}(t, x^*, u^*) - v_{i_0} g^{i_0}(t, x^*, u^*)\}\, dt < \int_a^b \Big(\{f^{i_0}(t, x, u) - v_{i_0} g^{i_0}(t, x, u)\}$$
$$+ \sum_{j=1}^m \lambda_j(t) h^j(t, x, u) + \sum_{l=1}^n \mu_l(t)(k^l(t, x, u) - \dot{x})\Big)\, dt$$

for some $\quad i_0 = 1, \cdots, p \qquad (5.29)$

cannot hold.

Proof. Suppose contrary to the result, that (5.28) and (5.29) hold. As (x^*, u^*) is feasible for $(\text{MFCP})_v$ and $\lambda(t) \geq 0$, (5.28) and (5.29) imply

$$\int_a^b \Big(\{f^i(t, x^*, u^*) - v_i g^i(t, x^*, u^*)\} + \sum_{j=1}^m \lambda_j(t) h^j(t, x^*, u^*) + \sum_{l=1}^n \mu_l(t)(k^l(t, x^*, u^*) - \dot{x}^*) \Big)\, dt \leq \int_a^b \Big(\{f^i(t, x, u) - v_i g^i(t, x, u)\} + \sum_{j=1}^m \lambda_j(t) h^j(t, x, u) + \sum_{l=1}^n \mu_l(t)(k^l(t, x, u) - \dot{x}) \Big)\, dt, \quad \forall\ i = 1, \cdots, p$$

and

$$\int_a^b \Big(\{f^{i_0}(t, x^*, u^*) - v_{i_0} g^{i_0}(t, x^*, u^*)\} + \sum_{j=1}^m \lambda_j(t) h^j(t, x^*, u^*) + \sum_{l=1}^n \mu_l(t)(k^l(t, x^*, u^*) - \dot{x}^*) \Big)\, dt < \int_a^b \Big(\{f^{i_0}(t, x, u) - v_{i_0} g^{i_0}(t, x, u)\} + \sum_{j=1}^m \lambda_j(t) h^j(t, x, u) + \sum_{l=1}^n \mu_l(t)(k^l(t, x, u) - \dot{x}) \Big)\, dt,$$

for some $i_0 = 1, \cdots, p$.

Now hypothesis (ii) and $\sum_{i=1}^p \tau_i = 1$ imply

$$\int_a^b \Big(\sum_{i=1}^p \tau_i \{f^i(t, x^*, u^*) - v_i g^i(t, x^*, u^*)\} + \sum_{j=1}^m \lambda_j(t) h^j(t, x^*, u^*) + \sum_{l=1}^n \mu_l(t)(k^l(t, x^*, u^*) - \dot{x}^*) \Big)\, dt < \int_a^b \Big(\sum_{i=1}^p \tau_i \{f^i(t, x, u) - v_i g^i(t, x, u)\} + \sum_{j=1}^m \lambda_j(t) h^j(t, x, u) + \sum_{l=1}^n \mu_l(t)(k^l(t, x, u) - \dot{x}) \Big)\, dt. \tag{5.30}$$

Now according to (i),

$$\int_a^b \Big(\{f^i(t,x^*,u^*) - v_i g^i(t,x^*,u^*)\} + \sum_{j=1}^m \lambda_j(t) h^j(t,x^*,u^*)$$
$$+ \sum_{l=1}^n \mu_l(t)(k^l(t,x^*,u^*) - \dot{x}^*)\Big)\, dt - \int_a^b \Big(\{f^i(t,x,u) - v_i g^i(t,x,u)\}$$
$$+ \sum_{j=1}^m \lambda_j(t) h^j(t,x,u) + \sum_{l=1}^n \mu_l(t)(k^l(t,x,u) - \dot{x})\Big)\, dt$$
$$\geq \int_a^b \Big[\eta^T \alpha_i \Big(\{f_x^i(t,x,u) - v_i g_x^i(t,x,u)\} + \sum_{j=1}^m \lambda_j(t) h_x^j(t,x,u)$$
$$+ \sum_{l=1}^n \mu_l(t) k_x^l(t,x,u)\Big) + \frac{d\eta^T}{dt}\alpha_i(-\mu(t))$$
$$+ \xi^T \alpha_i \Big(\{f_u^i(t,x,u) - v_i g_u^i(t,x,u)\}$$
$$+ \sum_{j=1}^m \lambda_j(t) h_u^j(t,x,u) + \sum_{l=1}^n \mu_l(t) k_u^l(t,x,u)\Big)\Big]\, dt + \rho_i ||\zeta||^2. \tag{5.31}$$

By integrating $\frac{d\eta^T}{dt}\alpha_i\mu(t)$ from a to b by parts and applying the boundary conditions, we have

$$\int_a^b \frac{d\eta^T}{dt}\alpha_i \mu(t)\, dt = -\int_a^b \eta^T \alpha_i \dot{\mu}(t)\, dt. \tag{5.32}$$

Using (5.32) in (5.31), we have

$$
\int_a^b \Big(\{f^i(t,x^*,u^*) - v_i g^i(t,x^*,u^*)\} + \sum_{j=1}^m \lambda_j(t) h^j(t,x^*,u^*)
$$

$$
+ \sum_{l=1}^n \mu_l(t)(k^l(t,x^*,u^*) - \dot{x}^*)\Big)\, dt - \int_a^b \Big(\{f^i(t,x,u) - v_i g^i(t,x,u)\}
$$

$$
+ \sum_{j=1}^m \lambda_j(t) h^j(t,x,u) + \sum_{l=1}^n \mu_l(t)(k^l(t,x,u) - \dot{x})\Big)\, dt
$$

$$
\geq \int_a^b \Big[\eta^T \alpha_i \Big(\{f_x^i(t,x,u) - v_i g_x^i(t,x,u)\} + \sum_{j=1}^m \lambda_j(t) h_x^j(t,x,u)
$$

$$
+ \sum_{l=1}^n \mu_l(t) k_x^l(t,x,u) + \dot{\mu}(t)\Big) + \xi^T \alpha_i \Big(\{f_u^i(t,x,u) - v_i g_u^i(t,x,u)\}
$$

$$
+ \sum_{j=1}^m \lambda_j(t) h_u^j(t,x,u) + \sum_{l=1}^n \mu_l(t) k_u^l(t,x,u)\Big)\Big]\, dt + \rho_i ||\zeta||^2.
$$

Now $\tau_i > 0$ and $\sum_{i=1}^p \tau_i = 1$, imply

$$
\int_a^b \Big(\sum_{i=1}^p \tau_i \{f^i(t,x^*,u^*) - v_i g^i(t,x^*,u^*)\} + \sum_{j=1}^m \lambda_j(t) h^j(t,x^*,u^*)
$$

$$
+ \sum_{l=1}^n \mu_l(t)(k^l(t,x^*,u^*) - \dot{x}^*)\Big)\, dt - \int_a^b \Big(\sum_{i=1}^p \tau_i \{f^i(t,x,u) - v_i g^i(t,x,u)\}
$$

$$
+ \sum_{j=1}^m \lambda_j(t) h^j(t,x,u) + \sum_{l=1}^n \mu_l(t)(k^l(t,x,u) - \dot{x})\Big)\, dt
$$

$$
\geq \int_a^b \Big[\eta^T \alpha_i \Big(\sum_{i=1}^p \tau_i \{f_x^i(t,x,u) - v_i g_x^i(t,x,u)\} + \sum_{j=1}^m \lambda_j(t) h_x^j(t,x,u)
$$

$$
+ \sum_{l=1}^n \mu_l(t) k_x^l(t,x,u) + \dot{\mu}(t)\Big) + \xi^T \alpha_i \Big(\sum_{i=1}^p \tau_i \{f_u^i(t,x,u) - v_i g_u^i(t,x,u)\}
$$

$$+\sum_{j=1}^{m}\lambda_j(t)h_u^j(t,x,u)+\sum_{l=1}^{n}\mu_l(t)k_u^l(t,x,u)\Big)\Big]\,dt$$

$$+\sum_{i=1}^{p}\tau_i\rho_i||\zeta||^2\geq 0 \text{ by hypothesis (ii), (5.5) and (5.6).}$$

Hence

$$\int_a^b\Big(\sum_{i=1}^{p}\tau_i\{f_i(t,x^*,u^*)-v_ig_i(t,x^*,u^*)\}+\sum_{j=1}^{m}\lambda_j(t)h_j(t,x^*,u^*)$$
$$+\sum_{l=1}^{n}\mu_l(t)(k_l(t,x^*,u^*)-\dot{x}^*)\Big)\,dt\geq\int_a^b\Big(\sum_{i=1}^{p}\tau_i\{f_i(t,x,u)-v_ig_i(t,x,u)\}$$
$$+\sum_{j=1}^{m}\lambda_j(t)h_j(t,x,u)+\sum_{l=1}^{n}\mu_l(t)(k_l(t,x,u)-\dot{x})\Big)\,dt$$

which is a contradiction to (5.30). □

Theorem 5.5.2 (Weak Duality). Assume that for all feasible (x^*,u^*) for $(\text{MFCP})_v$ and all feasible (x,u,τ,v,λ,μ) for (WFCD)

(i) $\Big(\int_a^b\{f^1(t,\cdot,\cdot)-v_1g^1(t,\cdot,\cdot)\}\,dt,\cdots,\int_a^b\{f^p(t,\cdot,\cdot)-v_pg^p(t,\cdot,\cdot)\}\,dt\Big),$

(ii) $\Big(\int_a^b\lambda_1h^1(t,\cdot,\cdot)\,dt,\quad\cdots,\quad\int_a^b\lambda_mh^m(t,\cdot,\cdot)\,dt\Big)$ and

(iii) $\Big(\int_a^b\mu_1(k^1(t,\cdot,\cdot)-\dot{x})\,dt,\quad\cdots,\quad\int_a^b\mu_n(k^n(t,\cdot,\cdot)-\dot{x})\,dt\Big)$

are all V-ρ-invex with respect to the same η, ξ, ζ, and

(iv) $\tau_i > 0, \ \forall i = 1, \cdots, p,$ then (5.28) and (5.29) cannot hold.

Proof. The proof is on similar lines as that of Theorem 5.5.1. □

Corollary 5.5.1. Assume that weak dualities (Theorem 5.5.1, 5.5.2) hold between $(\text{MFCP})_v$ and (WFCD). If (x, u) is feasible for $(\text{MFCP})_v$ and $(x, u, \tau, v, \lambda, \mu)$ is feasible for (WFCD) with $\sum_{j=1}^{m} \lambda_j(t) h^j(t, x, u) = 0$. Then (x, u) is an efficient for $(\text{MFCP})_v$ and $(x, u, \tau, v, \lambda, \mu)$ is an efficient for (WFCD).

Proof. Suppose that (x, u) is not an efficient for $(\text{MFCP})_v$, then there exists some feasible (x^*, u^*) for $(\text{MFCP})_v$ such that

$$\int_a^b \{f^i(t, x^*, u^*) - v_i g^i(t, x^*, u^*)\}\, dt \leq \int_a^b \{f^i(t, x, u) - v_i g^i(t, x, u)\}\, dt, \quad \forall\ i = 1, \cdots, p \tag{5.33}$$

and

$$\int_a^b \{f^{i_0}(t, x^*, u^*) - v_{i_0} g^{i_0}(t, x^*, u^*)\}\, dt < \int_a^b \{f^{i_0}(t, x, u) - v_{i_0} g^{i_0}(t, x, u)\}\, dt, \quad \text{for some } i_0 = 1, \cdots, p. \tag{5.34}$$

Since $\sum_{j=1}^{m} \lambda_j(t) h^j(t, x, u) = 0$ and $\sum_{l=1}^{n} \mu_l(t)(k^l(t, x, u) - \dot{x}) = 0$, we get

$$\int_a^b \{f^i(t, x^*, u^*) - v_i g^i(t, x^*, u^*)\}\, dt \leq \int_a^b \Big(\{f^i(t, x, u) - v_i g^i(t, x, u)\} + \sum_{j=1}^{m} \lambda_j(t) h^j(t, x, u) + \sum_{l=1}^{n} \mu_l(t)(k^l(t, x, u) - \dot{x})\Big)\, dt, \quad \forall\ i = 1, \cdots, p$$

and

$$\int_a^b \{f^{i_0}(t, x^*, u^*) - v_{i_0} g^{i_0}(t, x^*, u^*)\}\, dt < \int_a^b \Big(\{f^{i_0}(t, x, u) - v_{i_0} g^{i_0}(t, x, u)\} + \sum_{j=1}^{m} \lambda_j(t) h^j(t, x, u) + \sum_{l=1}^{n} \mu_l(t)(k^l(t, x, u) - \dot{x})\Big)\, dt, \quad \text{for some } i_0 = 1, \cdots, p.$$

Since $(x, u, \tau, v, \lambda, \mu)$ is feasible for (WFCD) and (x^*, u^*) is feasible for $(\text{MFCP})_v$, these inequalities contradict weak duality. Hence (x, u) is efficient for $(\text{MFCP})_v$.

Next suppose $(x, u, \tau, v, \lambda, \mu)$ is not an efficient for (WFCD). Then there exists some feasible $(x^*, u^*, \tau^*, v^*, \lambda^*, \mu^*)$ for (WFCD) such that

$$\int_a^b \Big(\{f^i(t, x^*, u^*) - v_i g^i(t, x^*, u^*)\} + \sum_{j=1}^m \lambda_j^*(t) h^j(t, x^*, u^*)$$
$$+ \sum_{l=1}^n \mu_l^*(t)(k^l(t, x^*, u^*) - \dot{x}^*)\Big)\, dt \geq \int_a^b \Big(\{f^i(t, x, u) - v_i g^i(t, x, u)\}$$
$$+ \sum_{j=1}^m \lambda_j(t) h^j(t, x, u) + \sum_{l=1}^n \mu_l(t)(k^l(t, x, u) - \dot{x})\Big)\, dt, \;\; \forall\; i = 1, \cdots, p$$

and

$$\int_a^b \Big(\{f^{i_0}(t, x^*, u^*) - v_{i_0} g^{i_0}(t, x^*, u^*)\} + \sum_{j=1}^m \lambda_j^*(t) h^j(t, x^*, u^*)$$
$$+ \sum_{l=1}^n \mu_l^*(t)(k^l(t, x^*, u^*) - \dot{x}^*)\Big)\, dt > \int_a^b \Big(\{f^{i_0}(t, x, u) - v_{i_0} g^{i_0}(t, x, u)\}$$
$$+ \sum_{j=1}^m \lambda_j(t) h^j(t, x, u) + \sum_{l=1}^n \mu_l(t)(k^l(t, x, u) - \dot{x})\Big)\, dt, \;\; \text{for some } i_0 = 1, \cdots, p.$$

Since $\sum_{j=1}^m \lambda_j(t) h^j(t, x, u) = 0$ and $\sum_{l=1}^n \mu_l(t)(k^l(t, x, u) - \dot{x}) = 0$, we get

$$\int_a^b \Big(\{f^i(t, x^*, u^*) - v_i g^i(t, x^*, u^*)\} + \sum_{j=1}^m \lambda_j^*(t) h^j(t, x^*, u^*)$$
$$+ \sum_{l=1}^n \mu_l^*(t)(k^l(t, x^*, u^*) - \dot{x}^*)\Big)\, dt \geq \int_a^b \{f^i(t, x, u) - v_i g^i(t, x, u))\}\, dt,$$
$$\forall\; i = 1, \cdots, p$$

and

$$\int_a^b \Big(\{f^{io}(t,x^*,u^*) - v_{i_0} g^{io}(t,x^*,u^*)\} + \sum_{j=1}^m \lambda_j^*(t) h^j(t,x^*,u^*)$$

$$+ \sum_{l=1}^n \mu_l^*(t)(k^l(t,x^*,u^*) - \dot{x}^*)\Big)\, dt > \int_a^b \{f^{io}(t,x,u) - v_{i_0} g^{io}(t,x,u)\}\, dt,$$

for some $i_0 = 1, \cdots, p$.

Since (x^*, u^*) is feasible for $(\text{MFCP})_v$, these inequalities contradict weak duality. Hence $(x, u, \tau, v, \lambda, \mu)$ is an efficient for (WFCD). □

Theorem 5.5.3 (Strong Duality). Let (x^*, u^*) be an efficient for $(\text{MFCP})_v$ and assume that (x^*, u^*) satisfies the constraint qualification for $P_k(x^*, u^*)$ for at least one $k \in \{1, \cdots, p\}$. Then there exist $\tau^* \in R^p$ and piecewise smooth functions $\lambda^* : I \to R^m$ and $\mu^* : I \to R^n$ such that $(x^*, u^*, \tau^*, v^*,$ $\lambda^*, \mu^*)$ is feasible for (WFCD) and $\sum_{j=1}^m \lambda_j^*(t) h^j(t, x^*, u^*) = 0$.

Further, if weak duality also holds between $(\text{MFCP})_v$ and (WFCD), then $(x^*, u^*, \tau^*, v^*, \lambda^*, \mu^*)$ is an efficient for (WFCD).

Proof. It follows from Lemma 5.5.2 that there exist $\tau^* \in R^p$, and piecewise smooth functions $\lambda^* : I \to R^m$ and $\mu^* : I \to R^n$, satisfying for all $t \in I$ the following relations:

$$\sum_{i=1}^{p} \tau_i^* \{f_x^i(t, x^*, u^*) - v_i g_x^i(t, x^*, u^*)\} + \sum_{j=1}^{m} \lambda_j^*(t) h_x^j(t, x^*, u^*)$$

$$+ \sum_{l=1}^{n} \mu_l^*(t) k_x^l(t, x^*, u^*) + \dot{\mu}^*(t) = 0,$$

$$\sum_{i=1}^{p} \tau_i^* \{f_u^i(t, x^*, u^*) - v_i g_u^i(t, x^*, u^*)\} + \sum_{j=1}^{m} \lambda_j^*(t) h_u^j(t, x^*, u^*)$$

$$+ \sum_{l=1}^{n} \mu_l^*(t) k_u^l(t, x^*, u^*) = 0,$$

$$\sum_{j=1}^{m} \lambda_j^*(t) h^j(t, x^*, u^*) = 0,$$

$$\tau_i^* \geq 0, \quad \sum_{i=1}^{p} \tau_i^* = 1, \quad \lambda^*(t) \geq 0.$$

Hence $(x^*, u^*, \tau^*, v^*, \lambda^*, \mu^*)$ is feasible for (WFCD). The result now from Corollary 5.5.1. □

Now we establish some duality theorems between the multiobjective fractional control problem $(\text{MFCP})_v$ and its Mond-Weir type dual problem (MFCD).

Theorem 5.5.4 (Weak Duality). Assume that for all feasible (x^*, u^*) for $(\text{MFCP})_v$ and all feasible $(x, u, \tau, v, \lambda, \mu)$ for (MFCD)

(i) $\int_a^b \Big(\sum_{j=1}^{m} \lambda_j h^j(t, \cdot, \cdot) + \sum_{l=1}^{n} \mu_l (k^l(t, \cdot, \cdot) - \dot{x}) \Big) \, dt$ is V-ρ-quasi-invex at (x, u),

(ii) $\tau_i > 0, \ \forall i = 1, \cdots, p$ and

$$\Big(\int_a^b \{f^1(t, \cdot, \cdot) - v_1 g^1(t, \cdot, \cdot)\} \, dt, \cdots, \int_a^b \{f^p(t, \cdot, \cdot) - v_p g^p(t, \cdot, \cdot)\} \, dt \Big)$$

is V-ρ-pseudo-invex at(x, u) with respect to the sameη, ξ, ζ,

(iii) $\sum \tau_i \rho_i + \sum \rho_j \geq 0$, then the following cannot hold:

$$\int_a^b \{f^i(t,x^*,u^*) - v_i g^i(t,x^*,u^*)\}\,dt \le \int_a^b \{f^i(t,x,u) - v_i g^i(t,x,u)\}\,dt$$

$\forall i = 1,\cdots,p$ (5.35)

and

$$\int_a^b \{f^{i_0}(t,x^*,u^*) - v_{i_0} g^{i_0}(t,x^*,u^*)\}\,dt < \int_a^b \{f^{i_0}(t,x,u) - v_{i_0} g^{i_0}(t,x,u)\}\,dt$$

for some $i_0 = 1,\cdots,p.$ (5.36)

Proof. Let (x^*,u^*) be an arbitrary feasible solution of $(\text{MFCP})_v$ and (x,u,τ,v,λ,μ) be an arbitrary feasible solution of (MFCD). As $\lambda(t) \ge 0,$ we have that

$$\int_a^b \sum_{j=1}^m \lambda_j h^j(t,x^*,u^*)\,dt \le \int_a^b \sum_{j=1}^m \lambda_j h^j(t,x,u)\,dt$$

and

$$\int_a^b \sum_{l=1}^n \mu_l(k^l(t,x^*,u^*) - \dot{x}^*)\,dt \le \int_a^b \sum_{l=1}^n \mu_l(k^l(t,x,u) - \dot{x})\,dt.$$

Therefore

$$\int_a^b \Big(\sum_{j=1}^m \lambda_j h^j(t,x^*,u^*) + \sum_{l=1}^n \mu_l(k^l(t,x^*,u^*) - \dot{x}^*)\Big)\,dt$$
$$\le \int_a^b \Big(\sum_{j=1}^m \lambda_j h^j(t,x,u) + \sum_{l=1}^n \mu_l(k^l(t,x,u) - \dot{x})\Big)\,dt.$$

Since $\beta_j > 0, \ \forall j = 1,\cdots,m,$ we have

$$\int_a^b \sum_{j=1}^m \beta_j \Big(\lambda_j h^j(t, x^*, u^*) + \mu_l (k^l(t, x^*, u^*) - \dot{x}^*) \Big)\, dt$$
$$\leq \int_a^b \sum_{j=1}^m \beta_j \Big(\lambda_j h^j(t, x, u) + \mu_l (k^l(t, x, u) - \dot{x}) \Big)\, dt.$$

Then (i) yields

$$\int_a^b \sum_{j=1}^m \Big(\eta^T \{ \lambda_j h_x^j(t, x, u) + \mu_l k_x^l(t, x, u) \} + \frac{d\eta^T}{dt}(-\mu)$$
$$\xi^T \{ \lambda_j h_u^j(t, x, u) + \mu_l k_u^l(t, x, u) \} \Big)\, dt + \sum \rho_j ||\zeta||^2 \leq 0. \tag{5.37}$$

By integrating $\frac{d\eta^T}{dt}\mu$ from a to b by parts and applying the boundary conditions, we have

$$\int_a^b \frac{d\eta^T}{dt} \mu(t)\, dt = - \int_a^b \eta^T \dot{\mu}(t)\, dt. \tag{5.38}$$

Using (5.38) in (5.37), we have

$$\int_a^b \sum_{j=1}^m \Big(\eta^T \{ \lambda_j h_x^j(t, x, u) + \mu_l k_x^l(t, x, u) + \dot{\mu}(t) \}$$
$$+ \xi^T \{ \lambda_j h_u^j(t, x, u) + \mu_l k_u^l(t, x, u) \} \Big)\, dt + \sum \rho_j ||\zeta||^2 \leq 0. \tag{5.39}$$

On the other hand, suppose contrary to the result of the theorem that (5.35) and (5.36) hold.

Since $\alpha_i > 0$, we have

$$\int_a^b \Big(\sum_{i=1}^p \alpha_i\{f^i(t,x^*,u^*) - v_i g^i(t,x^*,u^*)\}\Big)\, dt$$
$$< \int_a^b \Big(\sum_{i=1}^p \alpha_i\{f^i(t,x,u) - v_i g^i(t,x,u)\}\Big)\, dt.$$

By the V-ρ-pseudo-invexity of $\int_a^b (f^i - v_i g^i)\, dt$,

$$\int_a^b \sum_{i=1}^p \Big(\eta^T\{f_x^i(t,x,u) - v_i g_x^i(t,x,u)\} + \xi^T\{f_u^i(t,x,u) - v_i g_u^i(t,x,u)\}\Big)\, dt$$
$$+ \sum \rho_i ||\zeta||^2 < 0.$$

Because $\tau_i > 0, \ \forall i = 1, \cdots, p$, we have

$$\int_a^b \sum_{i=1}^p \Big(\eta^T \tau_i\{f_x^i(t,x,u) - v_i g_x^i(t,x,u)\} + \xi^T \tau_i\{f_u^i(t,x,u) - v_i g_u^i(t,x,u)\}\Big)\, dt$$
$$+ \sum \tau_i \rho_i ||\zeta||^2 < 0. \qquad (5.40)$$

Adding (5.39) and (5.40), we have

$$\int_a^b \Big[\eta^T \Big(\sum_{i=1}^p \tau_i\{f_x^i(t,x,u) - v_i g_x^i(t,x,u)\} + \sum_{j=1}^m \lambda_j h_x^j(t,x,u)$$
$$+ \sum_{l=1}^n \mu_l k_x^l(t,x,u) + \dot{\mu}(t)\Big) + \xi^T \Big(\sum_{i=1}^p \tau_i\{f_u^i(t,x,u) - v_i g_u^i(t,x,u)\}$$
$$+ \sum_{j=1}^m \lambda_j h_u^j(t,x,u) + \sum_{l=1}^n \mu_l k_u^l(t,x,u)\Big)\Big]\, dt + \sum \tau_i \rho_i ||\zeta||^2 + \sum \rho_j ||\zeta||^2 < 0,$$

which is a contradiction to the hypothesis (iii), (5.10) and (5.11). □

Theorem 5.5.5 (Weak Duality). Assume that for all feasible (x^*, u^*) for $(\text{MFCP})_v$ and all feasible $(x, u, \tau, v, \lambda, \mu)$ for (MFCD)

(i) $\tau_i > 0, \ \forall i = 1, \cdots, p$

$$\Big(\int_a^b \{f^1(t,\cdot,\cdot) - v_1 g^1(t,\cdot,\cdot)\}\, dt, \cdots, \int_a^b \{f^p(t,\cdot,\cdot) - v_p g^p(t,\cdot,\cdot)\}\, dt\Big)$$

is V-ρ-pseudo-invex at (x, u),

(ii) $$\Big(\int_a^b \lambda_1 h^1(t,\cdot,\cdot)\, dt, \quad \cdots, \quad \int_a^b \lambda_m h^m(t,\cdot,\cdot)\, dt\Big) \text{ and}$$

(iii) $$\Big(\int_a^b \mu_1(k^1(t,\cdot,\cdot) - \dot{x})\, dt, \quad \cdots, \quad \int_a^b \mu_n(k^n(t,\cdot,\cdot) - \dot{x})\, dt\Big)$$

are V-ρ-quasi-invex with respect to the same η, ξ, ζ, γ

(iv) $\sum \tau_i \rho_i + \sum \rho_j + \sum \rho_l \geq 0$, then (5.35) and (5.36) cannot hold.

Proof. The proof is on similar lines as that of Theorem 5.5.2. □

The following result is very similar to Corollary 5.5.1.

Corollary 5.5.2. Assume that weak dualities (5.5.4, 5.5.5) hold between $(\text{MFCP})_v$ and (MFCD). If (x, u) is feasible for $(\text{MFCP})_v$ and $(x, u, \tau, v, \lambda, \mu)$ is feasible for (MFCD), then (x, u) is an efficient for $(\text{MFCP})_v$ and $(x, u, \tau, v, \lambda, \mu)$ is an efficient for (MFCD).

Proof. Suppose that (x, u) is not an efficient for $(\text{MFCP})_v$. Then there exists some feasible (x^*, u^*) for $(\text{MFCP})_v$ such that

$$\int_a^b \{f^i(t,x^*,u^*) - v_i g^i(t,x^*,u^*)\}\,dt \le \int_a^b \{f^i(t,x,u) - v_i g^i(t,x,u))\}\,dt,$$
$\forall\ i = 1,\cdots,p$

and

$$\int_a^b \{f^{i_0}(t,x^*,u^*) - v_{i_0} g^{i_0}(t,x^*,u^*)\}\,dt < \int_a^b \{f^{i_0}(t,x,u) - v_{i_0} g^{i_0}(t,x,u)\}\,dt,$$
for some $i_0 = 1,\cdots,p$.

But (x,u,τ,v,λ,μ) is feasible for (MFCD), hence the result of weak duality theorems (5.5.4, 5.5.5) is contradicted. So, (x,u) must be efficient for $(\text{MFCP})_v$. Similarly assuming (x,u,τ,v,λ,μ) is not an efficient for (MFCD), we get a contradiction and therefore (x,u,τ,v,λ,μ) is an efficient for (MFCD).

□

Theorem 5.5.6 (Strong Duality). Let (x^*,u^*) be an efficient for $(\text{MFCP})_v$ and assume that (x^*,u^*) satisfies the constraint qualification for $P_k(x^*,u^*)$ for at least one $k \in \{1,\cdots,p\}$. Then there exist $\tau^* \in R^p$ and piecewise smooth functions $\lambda^* : I \to R^m$ and $\mu^* : I \to R^n$ such that $(x^*,u^*,\tau^*,v^*,$ $\lambda^*,\mu^*)$ is feasible for (MFCD). If also weak duality holds between $(\text{MFCP})_v$ and (MFCD), then $(x^*,u^*,\tau^*,v^*,\lambda^*,\mu^*)$ is an efficient for (MFCD).

Proof. Proceeding on the same lines as in Theorem 4.5.3, it follows that there exist $\tau^* \in R^p$, and piecewise smooth functions $\lambda^* : I \to R^m$ and $\mu^* : I \to R^n$, satisfying for all $t \in I$ the following relations:

$$\sum_{i=1}^{p} \tau_i^* \{f_x^i(t,x^*,u^*) - v_i g_x^i(t,x^*,u^*)\} + \sum_{j=1}^{m} \lambda_j^*(t) h_x^j(t,x^*,u^*)$$
$$+ \sum_{l=1}^{n} \mu_l^*(t) k_x^l(t,x^*,u^*) + \dot{\mu}^*(t) = 0,$$

$$\sum_{i=1}^{p} \tau_i^* \{f_u^i(t,x^*,u^*) - v_i g_u^i(t,x^*,u^*)\} + \sum_{j=1}^{m} \lambda_j^*(t) h_u^j(t,x^*,u^*)$$
$$+ \sum_{l=1}^{n} \mu_l^*(t) k_u^l(t,x^*,u^*) = 0,$$
$$\sum_{j=1}^{m} \lambda_j^*(t) h^j(t,x^*,u^*) = 0,$$
$$\tau_i^* \geq 0, \quad \sum_{i=1}^{p} \tau_i^* = 1, \quad \lambda^*(t) \geq 0.$$

The relations $\int_a^b \sum_{j=1}^m \lambda_j^*(t) h^j(t,x^*,u^*)\,dt \geq 0$ and $\int_a^b \sum_{l=1}^n \mu_l^*(t)(k^l(t,x^*,u^*) - \dot{x}^*\}\,dt \geq 0$ are obvious.

The above relations imply that $(x^*, u^*, \tau^*, v^*, \lambda^*, \mu^*)$ is feasible for (MFCD). The result now follows from Corollary 4.5.2. □

Theorem 5.5.7 (Strict Converse Duality). Let (x^*, u^*) and $(x, u, \tau, v, \lambda, \mu)$ be efficient solutions of $(\text{MFCP})_v$ and (MFCD), respectively. If $\int_a^b (f^i - v_i g^i)dt,\ i = 1, \cdots, p$, are V-$\rho$-quasi-invex and $\int_a^b \Big(-\sum_{j=1}^m \lambda_j h^j - \sum_{l=1}^n \mu_l (k^l - \dot{x}) \Big) dt$ is strictly V-ρ-pseudo-invex at (x, u) with respect to the same η, ξ, ζ, then $(x^*, u^*) = (x, u)$, i.e., (x, u) is an efficient solution of $(\text{MFCP})_v$.

Proof. Suppose that $(x^*, u^*) \neq (x, u)$. Since (x^*, u^*) is an efficient solution of $(\text{MFCP})_v$ by strong duality, there exist $\tau^* \in R^p, v^* \in R_+^p$ and piecewise smooth functions $\lambda^* : I \to R^m$ and $\mu^* : I \to R^n$, such that $(x^*, u^*, \tau^*, v^*, \lambda^*, \mu^*)$ is an efficient solution of (MFCD). Since $(x, u, \tau, v, \lambda, \mu)$ is also an efficient solution of (MFCD), it follows that

$$\int_a^b \{f^i(t,x^*,u^*) - v_i g^i(t,x^*,u^*)\}dt = \int_a^b \{f^i(t,x,u) - v_i g^i(t,x,u)\}dt,$$
$\forall i = 1, \cdots, p.$

Since $\alpha_i > 0, \;\; i = 1, \cdots, p,$ we have

$$\int_a^b \sum_{i=1}^p \alpha_i \{f^i(t, x^*, u^*) - v_i g^i(t, x^*, u^*)\} dt = \int_a^b \sum_{i=1}^p \alpha_i \{f^i(t, x, u) - v_i g^i(t, x, u)\} dt.$$

By the V-ρ-quasi-invexity of $\int_a^b (f^i - v_i g^i) dt$ at (x, u), we have

$$\int_a^b \Big(\sum_{i=1}^p \eta^T \{f_x^i(t, x, u) - v_i g_x^i(t, x, u)\} + \xi^T \{f_u^i(t, x, u) - v_i g_u^i(t, x, u)\} \Big) dt$$
$$+ \sum \rho_i ||\zeta||^2 \leq 0.$$

Since $\tau_i \geq 0, \;\; i = 1, \cdots, p,$ we have

$$\int_a^b \Big(\sum_{i=1}^p \eta^T \tau_i \{f_x^i(t, x, u) - v_i g_x^i(t, x, u)\} + \xi^T \tau_i \{f_u^i(t, x, u) - v_i g_u^i(t, x, u)\} \Big) dt$$
$$+ \sum \tau_i \rho_i ||\zeta||^2 \leq 0.$$

Since $(x, u, \tau, v, \lambda, \mu)$ is an efficient solution of (MFCD), from (5.10) and (5.11), we have

$$\int_a^b -\Big(\eta^T \{ \sum_{j=1}^m \lambda_j h_x^j + \sum_{l=1}^n \mu_l k_x^l + \dot{\mu}(t) \} + \xi^T \{ \sum_{j=1}^m \lambda_j h_u^j + \sum_{l=1}^n \mu_l k_u^l \} \Big) dt$$
$$- \sum \tau_i \rho_i ||\zeta||^2 \geq 0.$$

Since $\int_a^b \eta^T \dot{\mu}(t) dt = \; -\int_a^b \frac{d\eta^T}{dt} \mu(t) dt,$ we have

$$\int_a^b -\Big(\eta^T \{ \sum_{j=1}^m \lambda_j h_x^j + \sum_{l=1}^n \mu_l k_x^l \} + \frac{d\eta^T}{dt} \mu(t) + \xi^T \{ \sum_{j=1}^m \lambda_j h_u^j + \sum_{l=1}^n \mu_l k_u^l \} \Big) dt$$
$$- \sum \tau_i \rho_i ||\zeta||^2 \geq 0.$$

By the strict V-ρ-pseudo-invexity of $\int_a^b \Big(-\sum_{j=1}^m \lambda_j h^j - \sum_{l=1}^n \mu_l (k^l - \dot{x}) \Big) dt$ at (x, u),

$$\int_a^b \Big(\sum_{j=1}^m \lambda_j h^j(t, x^*, u^*) + \sum_{l=1}^n \mu_l (k^l(t, x^*, u^*) - \dot{x}^*) \Big) dt$$
$$> \int_a^b \Big(\sum_{j=1}^m \lambda_j h^j(t, x, u) + \sum_{l=1}^n \mu_l (k^l(t, x, u) - \dot{x}) \Big) dt.$$

Since $\int_a^b \Big(\sum_{j=1}^m \lambda_j h^j(t, x, u) + \sum_{l=1}^n \mu_l (k^l(t, x, u) - \dot{x}) \Big) dt \geq 0,$ we obtain

$$\int_a^b \Big(\sum_{j=1}^m \lambda_j h^j(t, x^*, u^*) + \sum_{l=1}^n \mu_l (k^l(t, x^*, u^*) - \dot{x}^*) \Big) dt > 0. \qquad (5.41)$$

Since $h^j(t, x^*, u^*) \leq 0, \ \forall j = 1, \cdots, m, \quad \lambda(t) \geq 0$ and $k^l(t, x^*, u^*) = \dot{x}^*, \ \forall l = 1, \cdots, n$, we obtain

$$\int_a^b \Big(\sum_{j=1}^m \lambda_j h^j(t, x^*, u^*) + \sum_{l=1}^n \mu_l (k^l(t, x^*, u^*) - \dot{x}^*) \Big) dt \leq 0.$$

This contradicts the inequality (5.41). Hence $(x^*, u^*) = (x, u)$. □

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