Optimality and Duality for Nonsmooth Multiobjective Fractional Programs

비원활 다목적 분수 문제에 대한 최적성과 쌍대성

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CONTENTS

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요 약

본 논문에서는 국소 리프시즈 $V-\rho$-인벡스티 함수를 가지는 미분불가능한 다목적 분수 최적화 문제를 생각한다. $V-\rho$-인벡스티 함수는 볼록함수의 확장이다.

본 논문의 목적은 일반화된 Fritz-John 필요 최적 조건과 일반화된 Karush-Kuhn-Tucker 필요 충분 최적 조건을 정립하고, 매개변수를 사용하여 위의 다목적 분수최적화문제의 쌍대문제를 만들고 두 문제 사이에서 약쌍대정리와 강쌍대정리가 성립한다는 것을 보이는 것이다.

1. Introduction

Optimality conditions and duality in single objective or multiobjective fractional programs have been of much interest in the recent past [1, 2, 5, 9, 11, 12, 13, 15, 17, 18] (see also the references therein). In particular, using parametric approach, Bector *et al.* [1] derived Fritz John and Karush-Kuhn-Tucker necessary and sufficient optimality for a class of nondifferentiable convex multiobjective fractional programming problems, and they also established some duality theorems. Following the approaches of Bector *et al.* [1], Liu [12] obtained the necessary and sufficient conditions and derived duality theorems for a class of nonsmooth multiobjective fractional programming problems involving pseudoinvex functions.

Recently, Kuk *et al.* [8] defined the concept of V-ρ-invexity for vector-valued functions, which is a generalization of the V-invex function [7,14], and they proved the generalized Karush-Kuhn-Tucker sufficient optimality theorem, weak and strong duality for nonsmooth multiobjective programs under the V-ρ-invexity assumptions. Kuk, Lee and Tanino [9] extend their results to nonsmooth multiobjective fractional programs on the basis of efficiency.

The aim of this paper is to consider the results of Kuk, Lee and Tanino [9] for nonsmooth multiobjective fractional programs on the basis of weak efficiency.

In this thesis, we consider a nonsmooth multiobjective fractional programming problem. For sufficient conditions, we define the V-ρ-invex functions for locally Lipschitz functions. We obtain the generalized Karush-Kuhn-Tucker necessary and sufficient optimality theorems and prove weak and strong duality theorems for the multiobjective fractional programs. This thesis consists of four sections. In Section 2, we give notations, definitions and examples for next sections. In Section 3, we obtain generalized Fritz - John necessary optimality conditions, and the generalized Karush-Kuhn-Tucker necessary and sufficient optimality theorems. Finally, in Section 4, we prove duality theorems for the multiobjective fractional programs.

2. Notations, Definitions and Examples

Now we give mathematical notations definitions and examples for next sections. The real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be locally Lipschitz if for any $z \in \mathbb{R}^n$ there exists a positive constant K and a neighborhood N of z such that, for each $x,\ y \in N$,

$$|f(x) - f(y)| \leqq K\|x - y\|,$$

where $\|\cdot\|$ denotes any norm in $\mathbb{R}^n$.

In this paper, we consider the following multiobjective fractional programming problem:

$$\text{(FP)} \qquad \text{minimize} \quad \left(\frac{f_1(x)}{g_1(x)}, \cdots, \frac{f_p(x)}{g_p(x)}\right)$$
$$\text{subject to} \quad x \in X = \{x \in \mathbb{R}^n \mid h_j(x) \leqq 0, \quad j = 1, \cdots, m\}$$

where $f_i : \mathbb{R}^n \to \mathbb{R},\ g_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, \cdots, p$ and $h_j : \mathbb{R}^n \to \mathbb{R},\ j = 1, \cdots, m$ are locally Lipschitz functions.

We assume in the sequel that $f_i(x) \geqq 0$ and $g_i(x) > 0$ on $\mathbb{R}^n$ for $i = 1, \cdots, p$.

The Clarke [4] generalized directional derivative of a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ at x in the direction $d \in \mathbb{R}^n$ denoted by $f^\circ(x; d)$ is as follows:

$$f^\circ(x; d) = \limsup_{\substack{y \to x \\ t \downarrow 0}} t^{-1}(f(y + td) - f(y)).$$

Further the Clarke [4] generalized gradient of f at x is denoted by

$$\partial f(x) = \{\xi \mid f^\circ(x; d) \geqq \xi^T d, \quad \text{for all } d \in \mathbb{R}^n\}.$$

It is well-known that

$$f^\circ(x; d) = \max_{\xi \in \partial f(x)} \xi^t d.$$

Now we define the upper Dini directional derivative of the function f at x in the direction $d \in \mathbb{R}^n$:

$$f^+(x; d) = \limsup_{t \downarrow 0} \frac{f(x + td) - f(x)}{t}.$$

Then for any $x \in \mathbb{R}^n$ and any $d \in \mathbb{R}^n$,

$$f^+(x; d) \leqq f^0(x; d).$$

Example 2.1. Let

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Then f is not locally Lipschitz at 0.

Example 2.2. Let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Then f is locally Lipschitz on $\mathbb{R}$, $\partial f(0) = [-1, 1] \neq \{f'(0)\}$ and $f^o(0; d) = |d|$ for any $d \in \mathbb{R}$. But $f^+(0; d) = 0$ for any $d \in \mathbb{R}$, and hence $f^+(0; d) \leq f^o(0; d)$.

Example 2.3. Let n be a natural number with $n \geq 3$, and let

$$f(x) = \begin{cases} x^n \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Then f is locally Lipschitz on $\mathbb{R}$ (continuously differentiable on $\mathbb{R}$), $\partial f(0) = \{f'(0)\} = \{0\}$ and $f^0(0; d) = f^+(0; d) = 0$ for any $d \in \mathbb{R}$.

Lee [10] defined invexity of locally Lipschitz functions as follows:

Definition 2.1. A locally Lipschitz function f is said to be invex on $X_0 \subset \mathbb{R}^n$ if for $x,\ u \in X_0$ there exists a function $\eta(x, u) : X_0 \times X_0 \to \mathbb{R}^n$ such that

$$f(x) - f(u) \geq \xi^T \eta(x, u), \quad \text{for each} \;\; \xi \in \partial f(u).$$

Egudo and Hanson [6] generalized the V-invexity of Jeyakumar and Mond [7] to the nonsmooth case as follows:

Definition 2.2. A vector function $f : X_0 \to \mathbb{R}^n$ is said to be V-invex if there exist functions $\eta : X_0 \times X_0 \to \mathbb{R}^n$ and $\alpha_i : X_0 \times X_0 \to \mathbb{R}_+ \setminus \{0\}$ such that

$$f_i(x) - f_i(u) - \alpha_i(x,u)\xi_i^T \eta(x,u) \geqq 0, \quad \text{for each } \xi_i \in \partial f_i(u).$$

Kuk *et al.* [9] defined V-ρ-invexity to the nonsmooth case as follows:

Definition 2.3. Let $f_i : \mathbb{R}^n \to \mathbb{R}$, $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \cdots, p$ and $h_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, \cdots, m$ be locally Lipschitz functions, and let $v \in \mathbb{R}^p$.

(a) $f - vg := (f_1 - v_1 g_1, \cdots, f_p - v_p g_p)$ is said to be V-ρ-invex with respect to functions η and $\theta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ if there exist $\alpha_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \setminus \{0\}$ and $\rho_i \in \mathbb{R}$, $i = 1, \cdots, p$ such that for any x, $u \in \mathbb{R}^n$ and any $\xi_i \in \partial f_i(u)$ and $\zeta \in \partial g_i(u)$,

$$\alpha_i(x,u)\{f_i(x) - v_i g_i(x) - f_i(u) + v_i g_i(u)\} \geqq (\xi_i - v_i \zeta_i)\eta(x,u) + \rho_i \|\theta(x,u)\|^2. \quad (1)$$

If we have strict inequality in (1) for any $x, u \in \mathbb{R}^n$, with $x \neq u$, then $f - vg$ is said to be strictly V-ρ-invex with respect to functions η and $\theta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$.

(b) h is said to be V-σ-invex with respect to functions η and $\theta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ if there exist $\beta_j : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \setminus \{0\}$ and $\sigma_j \in \mathbb{R}$, $j = 1, \cdots, m$ such that for any x, $u \in \mathbb{R}^n$ and any $\mu_j \in \partial h_j(u)$,

$$\beta_j(x,u)\{h_j(x) - h_j(u)\} \geqq \mu_j \eta(x,u) + \sigma_j \|\theta(x,u)\|^2.$$

Remark 2.1. If in the above definition, (a) $\rho_i = 0$ and $g_i \equiv 0$ for all i, and $\sigma_j = 0$ in (b) of Definition 2.3, then the functions f and h are V-invex.

Remark 2.2 Let $h : \mathbb{R}^n \to \mathbb{R}$ be a convex function. The subdifferential of h at x is given by

$$\partial_c h(x) = \{\xi \in \mathbb{R}^n \mid h(y) - h(x) \geqq \xi^T(y - x) \text{ for all } y \in \mathbb{R}^n\}$$

Then $\partial h(x) = \partial_c h(x)$.

Let $\varphi; \mathbb{R}^n \to \mathbb{R}$ be a concave function: The subdifferential of φ at x is given by

$$\partial^c(\varphi)(x) = \{\zeta \in \mathbb{R}^n \mid \varphi(y) - \varphi(x) \leqq \zeta^T(y - x) \text{ for all } y \in \mathbb{R}^n\}$$

We can easily check that

$$\partial\varphi(x) = -\partial_c(-\varphi)(x) = \partial^c\varphi(x).$$

Hence if $f_i(x)$, $i = 1, \cdots, p$ are convex and g_i, $i = 1, \cdots, p$ are concave Then for any $\xi_i \in \partial_c f_i(u)$ and $\zeta_i \in \partial^c g_i(u)$,

$$f_i(x) - u_i g_i(x) - f_i(u) + u_i g_i(u) \geqq \xi_i^T(x - u) - \zeta_i(x - u).$$

Let $\alpha_i(x, u) = 1, \eta(x, u) = x - u, \rho_i = 0, \theta(x, u) = 0$. Then for such f_i and g_i, (1) in Definition 2.3 hold.

Definition 2.4.([3,16]) A point $\bar{x} \in X$ is said to be an efficient solution of (FP) if there exist no $x \in X$ such that

$$\frac{f_i(x)}{g_i(x)} \leqq \frac{f_i(\bar{x})}{g_i(\bar{x})} \qquad \text{for all } i = 1, \cdots, p,$$

and

$$\frac{f_k(x)}{g_k(x)} < \frac{f_k(\bar{x})}{g_k(\bar{x})} \qquad \text{for some } k.$$

Definition 2.5.([3,16]) A point $\bar{x} \in X$ is said to be a weakly efficient solution of (FP) if there exist no $x \in X$ such that

$$\frac{f_i(x)}{g_i(x)} < \frac{f_i(\bar{x})}{g_i(\bar{x})} \qquad \text{for all } i = 1, \cdots, p.$$

It is clear that an efficient solution of (FP) is a weakly efficient solution of (FP), but the converse does not hold.

3. Optimality Conditions

In this section, we establish generalized Fritz-John necessary theorems, and generalized Karush-Kuhn-Tucker necessary and sufficient optimality theorems for weakly efficient solutions of (FP).

Theorem 3.1. Let $\bar{x} \in X$ is a weakly efficient solution of (FP). Then the following statements hold.
(i) $\bar{x} \in X$ is an optimal solution of the following scalarizing optimization problem:

$$\begin{array}{lll} \text{(SP)} & \text{Minimize} & l(x) \\ & \text{subject to} & x \in X := \{x \mid h_j(x) \leq 0,\ j = 1, \cdots, m\} \end{array}$$

where $l(x) = \max\limits_{1 \leq i \leq p} \left[\frac{f_i(x)}{g_i(x)} - \frac{f_i(\bar{x})}{g_i(\bar{x})} \right]$.

(ii) If $I^* := \{j \mid h_j(\bar{x}) = 0\} = \emptyset$, $0 \in \partial l(\bar{x})$, where $I = \{1, \cdots, m\}$.

(iii) If $I^* \neq \emptyset$, then the system

$$\left\langle \begin{array}{l} l^\circ(\bar{x}; d) < 0, \\ h_j^\circ(\bar{x}; d) < 0, \ j \in I^* \end{array} \right\rangle$$

has no solution $d \in \mathbb{R}^n$.

(iv) If $I^* \neq \emptyset$, then $0 \in co\{\partial l(\bar{x}) \cup \bigcup_{j \in I^*} \partial h_j(\bar{x})\}$, where $co\ A$ is the convexhull of the set A.

(v) If $I^* \neq \emptyset$, there exists $\alpha \geqq 0, \ \mu_j \geqq 0, \ j \in I^*$ such that

$$(\alpha, \mu_j)_{j \in I^*} \neq 0, \text{ and}$$
$$0 \in \alpha \partial l(\bar{x}) + \sum_{j \in I^*} \mu_j \partial h_j(\bar{x}).$$

(vi) If $0 \notin co \bigcup_{j \in I^*} \partial h_j(\bar{x})$, then $\alpha > 0$, where α is a real number in (v).

(vii) If $\Omega_-^0 := \{d \in \mathbb{R}^n \mid h_j^0(\bar{x}; d) < 0, \ j \in I^*\} \neq \emptyset$, i.e., there exists $d \in \mathbb{R}^n$ such that $\forall j \in I^*, \ h_j^0(\bar{x}; d) < 0$, then $0 \notin co \bigcup_{j \in I^*} \partial h_j(\bar{x})$, equivalently, $0 \in \sum_{j \in I^*} \tilde{\mu}_j \partial h_j(\bar{x})$ and $\tilde{\mu}_j \geqq 0$ implies $\tilde{\mu}_j = 0, \ j \in I^*$.

(viii) If $0 \notin co \bigcup_{j \in I^*} \partial h_j(\bar{x})$ there exist $\tau_i \geqq 0, \ i = 1, \cdots, p, \ \lambda_j \geqq 0, \ j = 1, \cdots, m$ such that

$$0 \in \sum_{i=1}^{p} \tau_i \{\partial f_i(\bar{x}) - y_i \partial g_i(\bar{x})\} + \sum_{j=1}^{m} \lambda_j \partial h_j(\bar{x}), \tag{2}$$

$$\lambda_j h_j(\bar{x}) = 0, \quad j = 1, \cdots, m, \tag{3}$$

$$(\tau_1, \cdots, \tau_p) \neq 0, \tag{4}$$

where $y_i = \frac{f_i(\bar{x})}{g_i(\bar{x})},\ i = 1, \cdots, p$.

(viiii) there exist $\tau_i \geqq 0,\ i = 1, \cdots, p,\ \lambda_j \geqq 0,\ j = 1, \cdots, m$ such that

$$0 \in \sum_{i=1}^{p} \tau_i \{\partial f_i(\bar{x}) - y_i \partial g_i(\bar{x})\} + \sum_{j=1}^{m} \lambda_j \partial h_j(\bar{x}),$$

$$\lambda_j h_j(\bar{x}) = 0,$$

$$(\tau_1, \cdots, \tau_p, \lambda_1, \cdots, \lambda_m) \neq 0$$

Proof. (i) Since $\bar{x}$ is a weakly efficient solution of (FP),

$$\max_{1 \leqq i \leqq p} \left[\frac{f_i(x)}{g_i(x)} - \frac{f_i(\bar{x})}{g_i(\bar{x})} \right] \geqq 0 \quad \text{for} \quad \text{any } x \in D.$$

Since

$$0 = \max_{1 \leqq i \leqq p} \left[\frac{f_i(\bar{x})}{g_i(\bar{x})} - \frac{f_i(\bar{x})}{g_i(\bar{x})} \right],$$

the conclusion hold.

(ii) Since $I^* = \emptyset$, that is , for all $j = 1, \cdots, m, h_j(\bar{x}) < 0$ since h_j is continuous, there exists $\delta_1 > 0$ such that for any $x \in \bar{x} + B_\delta(0)$, for any $j \in I, h_j(\bar{x}) < 0$. So, $\bar{x} + B_{\delta_1}(0) \subset D$. Since $\bar{x}$ is a minimum of (SP), there exists $\delta_2 > 0$ such that for any $x \in D \cap (\bar{x} + B_{\delta_2}(0))$, $l(x) \geqq l(\bar{x})$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then $\bar{x} + B_\delta(0) \subset D$ and for any $x \in \bar{x} + B_\delta(0)$, $l(x) \geqq l(\bar{x})$ Thus $0 \in \partial l(\bar{x})$.

(iii) Suppose to the contrary that the system has a solution $d^* \in \mathbb{R}^n$. Notice that $l^+(\bar{x}; d^*) \leqq l^\circ(\bar{x}; d^*)$ and $h_j^+(\bar{x}; d^*) \leqq h_j^\circ(\bar{x}; d^*)$. So, $l^+(\bar{x}; d^*) < 0$ and

$h_j^+(\bar{x}; d^*) < 0, j \in I^*$. Since $l^+(\bar{x}; d^*) < 0$, there exists $\delta_1^* > 0$ such that for any $\lambda \in (0, \delta_1^*)$, $l(\bar{x} + \lambda d^*) < l(\bar{x})$. Similarly, for any $j \in I^*$ there exists $\delta_j^* > 0$ such that for any $\lambda \in (0, \delta_j^*)$, $h_j(\bar{x} + \lambda d^*) < h_j(\bar{x}) = 0$. Since h_j is continuous, and $h_j(\bar{x}) < 0$, for any $j \in I \setminus I^*$, there exists $\tilde{\delta}_j^* > 0$ such that for any $\lambda \in (0, \tilde{\delta}_j^*)$, $h_j(\bar{x} + \lambda d^*) < 0$. Let $\delta^* = \min\{\delta_1^*, \delta_j^*, \tilde{\delta}_{j'}^*\}$. Then for all $\lambda \in (0, \delta^*)$, $l(\bar{x} + \lambda d^*) < l(\bar{x})$ and $h_j(\bar{x} + \lambda d^*) < 0$ (and hence $\bar{x} + \lambda d^* \in D$). This contradicts the optimality of $\bar{x}$.

(iv) By (iii),

$$\left\langle \begin{array}{l} \max\limits_{\xi \in \partial l(\bar{x})} \xi^t d < 0 \\ \max\limits_{\xi \in \partial l(\bar{x})} \xi^t d < 0, \ j \in I^* \end{array} \right\rangle$$

has no solution. Suppose to the contrary that

$$0 \notin co\{\partial l(\bar{x}) \cup \bigcup_{j \in I^*} \partial h_j(\bar{x})\}.$$

By separation theorem, there exists $\tilde{d}(\neq 0) \in \mathbb{R}^n$ such that $\xi^t \tilde{d} < 0$, for any $\xi \in co\{\partial l(\bar{x}) \cup \bigcup_{j \in I^*} \partial h_j(\bar{x})\}$. Thus, $\xi^t \tilde{d} < 0$, for any $\xi \in \partial l(\bar{x})$, any $j \in I^*$ and any $\xi \in \partial h_j(\bar{x})$. Since $\partial l(\bar{x})$ and $\partial h_j(\bar{x})$ are compact, $\max\limits_{\xi \in \partial l(\bar{x})} \xi^t \tilde{d} < 0$ and $\max\limits_{\xi \in \partial h_j(\bar{x})} \xi^t \tilde{d} < 0$ for any $j \in I^*$. This is a contradiction. Thus

$$0 \in co\{\partial l(\bar{x}) \cup \bigcup_{j \in I^*} \partial h_j(\bar{x})\}.$$

(v) Since

$$
\begin{aligned}
&co\{\partial l(\bar{x}) \cup \bigcup_{j\in I^*} \partial h_j(\bar{x})\} \\
&= \{\alpha\xi_0 + \sum_{j\in I^*} \mu_j \xi_j \mid \alpha \geqq 0,\ \mu_j \geqq 0,\ j \in I^*, \alpha + \sum_{j\in I^*} \mu_j = 1, \\
&\quad \xi_0 \in \partial l(\bar{x}),\ \xi_j \in \partial h_j(\bar{x}),\ j \in I^*\}
\end{aligned}
$$

and $0 \in co\{\partial l(\bar{x}) \cup \bigcup_{j\in I^*} \partial h_j(\bar{x})\}$, there exist $\alpha \geqq 0,\ \mu_j \geqq 0,\ j \in I^*$ such that $(\alpha, \mu_j)_{j\in I^*} \neq 0$ and $0 \in \alpha\partial l(\bar{x}) + \sum_{j\in I^*} \mu_j \partial h_j(\bar{x})$.

(vi) Suppose to the contrary that $\alpha = 0$. From (v), $0 \in \sum_{j\in I^*} \mu_j \partial h_j(\bar{x}), (\mu_j)_{j\in I^*} \neq 0$, and $\mu_j \geqq 0,\ j \in I^*$. So, $0 \in co \bigcup_{j\in I^*} \partial h_j(\bar{x})$. This contradicts the assumption. Thus $\alpha > 0$.

(vii) Suppose to the contrary that $0 \in co \bigcup_{j\in I^*} \partial h_j(\bar{x})$. Then there exist $\mu_j^* \geqq 0$ and $\xi_j \in \partial h_j(\bar{x})$ such that $\sum_{j\in I^*} \mu_j^* = 1$ and $0 = \sum_{j\in I^*} \mu_j^* \xi_j^*$. So,

$$
\begin{aligned}
0 &= \left\langle \sum_{j\in I^*} \mu_j^* \xi_j^*, d \right\rangle \\
&= \sum_{j\in I^*} \mu_j^* \langle \xi_j^*, d \rangle \\
&\leq \sum_{j\in I^*} \mu_j^* h_j^0(\bar{x}; d) < 0 \quad \text{(by assumption)}.
\end{aligned}
$$

This is a contradiction.

If $0 \in \sum_{j \in I^*} \tilde{\mu}_j \partial h_j(\bar{x})$ and $\tilde{\mu}_j \geqq 0$ implies $\tilde{\mu}_j = 0,\ j \in I^*$, then $0 \notin co \bigcup_{j \in I^*} \partial h_j(\bar{x})$, and the converse holds.

(viii) Suppose that $0 \notin co \bigcup_{j \in I^*} \partial h_j(\bar{x})$. Then by (iv) and (v), there exist $\alpha > 0$ and $\mu_j \geqq 0,\ j \in I^*$ such that

$$0 \in \alpha \partial l(\bar{x}) + \sum_{j \in I^*} \mu_j \partial h_j(\bar{x}).$$

Letting $\lambda_j = \frac{1}{\alpha}\mu_j,\ j \in I^*$ and $\lambda_j = 0,\ j \in I \setminus I^*$, we have

$$0 \in \partial l(\bar{x}) + \sum_{j=1}^{m} \lambda_j \partial h_j(\bar{x})$$

$$\text{and } \lambda_j h_j(\bar{x}) = 0, j = 1, \cdots, m.$$

Since $l(x) = \max_{1 \leqq i \leqq p} \left[\frac{f_i(x)}{g_i(x)} - \frac{f_i(\bar{x})}{g_i(\bar{x})} \right]$ for any $x \in D$, we have

$$\begin{aligned} \partial l(\bar{x}) &\subset co\left\{ \partial(\frac{f_i}{g_i})(x) \mid i = 1, \cdots, p \right\} \quad \text{(see p. 47 in [4])} \\ &\subset co\left\{ \frac{1}{g_i(\bar{x})} \partial f_i(\bar{x}) - \frac{f_i(\bar{x})}{\{g_i(\bar{x})\}^2} \partial g_i(\bar{x}) \mid i = 1, \cdots, p \right\} \quad \text{(see p. 48 in [4])} \end{aligned}$$

Thus, we can get the conclusion.

(viiii) Using the argument in the proof of (viii), we can get the conclusion from (V).

Now we give sufficient optimality conditions for (FP) under generalized invexity assumptions.

Theorem 3.2. Let $(\bar{x}, \tau, \lambda) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m$ satisfy conditions (2)-(4). Assume that $f - yg := (f_1 - y_1 g_1, \cdots, f_p - y_p g_p)$ is V-ρ-invex and h is V-σ-invex with respect to the same η and θ and

$$\sum_{i=1}^{p} \tau_i \rho_i + \sum_{j=1}^{m} \lambda_j \sigma_j \geqq 0, \tag{5}$$

where $y_i = f_i(\bar{x})/g_i(\bar{x})$, $i = 1, \cdots, p$. Then $\bar{x}$ is a weakly efficient solution of (FP).

Proof. Suppose that $\bar{x}$ is not a weakly efficient solution of (FP). Then there exists $x \in X$ such that

$$\frac{f_i(x)}{g_i(x)} < \frac{f_i(\bar{x})}{g_i(\bar{x})}, \qquad \text{for } i = 1, \cdots, p.$$

Since $g_i(x) > 0$ for all $i = 1, \cdots, p$, we have

$$f_i(x) - y_i g_i(x) < f_i(\bar{x}) - y_i g_i(\bar{x}) \ \text{ for all } i = 1, \cdots, p.$$

Since $\tau > 0$ and $\alpha_i(x, \bar{x}) > 0$ for all $i = 1, \cdots, p$, we have

$$\sum_{i=1}^{p} \tau_i \alpha_i(x, \bar{x}) \{f_i(x) - f_i(\bar{x})\} < \sum_{i=1}^{p} \tau_i \alpha_i(x, \bar{x}) \{y_i g_i(x) - y_i g_i(\bar{x})\}.$$

Then, by the V-ρ-invexity of $f - yg$, we have

$$\sum_{i=1}^{p} \tau_i (\xi_i - y_i \zeta_i) \eta(x, \bar{x}) + \sum_{i=1}^{p} \tau_i \rho_i \|\theta(x, \bar{x})\|^2 < 0, \tag{6}$$

for each $\xi_i \in \partial f_i(\bar{x})$ and each $\zeta_i \in \partial g_i(\bar{x})$. From (2) and (5), (6) yields

$$\sum_{j=1}^{m} \lambda_j \mu_j \eta(x, \bar{x}) + \sum_{j=1}^{m} \lambda_j \sigma_j \|\theta(x, \bar{x})\|^2 > 0,$$

for some $\mu_j \in \partial h_j(\bar{x})$. Hence, by the V-σ-invexity of h, we obtain

$$\sum_{j=1}^{m} \lambda_j \beta_j(x, \bar{x})\{h_j(x) - h_j(\bar{x})\} > 0.$$

Since $\lambda_j h_j(\bar{x}) = 0$ for all $j = 1, \cdots, m$, we have

$$\sum_{j=1}^{m} \lambda_j \beta_j(x, \bar{x}) h_j(x) > 0,$$

which contradicts the conditions $\beta_j(x, \bar{x}) > 0,\ \lambda_j \geqq 0$, and $h_j(x) \leqq 0$ for all $j = 1, \cdots, m$. Thus $\bar{x}$ is a weakly efficient solution of (FP).

4. Duality Theorems

Following the parametric approach of Bector *et al.* [1], we formulate the following dual problem for (FP).

(FD) maximize $(v_1, \cdots, v_p)$

$$\text{subject to} \quad 0 \in \sum_{i=1}^{p} \tau_i\{\partial f_i(x) - v_i \partial g_i(x)\} + \sum_{j=1}^{m} \lambda_j \partial h_j(x), \tag{7}$$

$$f_i(x) - v_i g_i(x) \geqq 0, \quad i = 1, \cdots, p, \tag{8}$$

$$\lambda_j h_j(x) \geqq 0, \quad j = 1, \cdots, m, \tag{9}$$

$$\tau \in \mathbb{R}^p, \lambda \in \mathbb{R}^m, v \in \mathbb{R}^p, \tau_i \geqq 0, \tau \neq 0, \lambda \geqq 0. \tag{10}$$

We establish weak and strong duality theorems between (FP) and (FD).

Theorem 4.1 (Weak Duality). Let x be any feasible for (FP) and let (x, τ, λ, v) be any feasible for (FD). Assume that $f - vg := (f_1 - v_1 g_1, \cdots, f_p - v_p g_p)$ is V-ρ-invex and h is V-σ-invex with respect to the same η and θ and

$$\sum_{i=1}^{p} \tau_i \rho_i + \sum_{j=1}^{m} \lambda_j \sigma_j \geqq 0. \tag{11}$$

Then the following cannot hold:

$$\frac{f_i(x)}{g_i(x)} < v_i, \quad \text{for all } i = 1, \cdots, p. \tag{12}$$

Proof. Suppose contrary to the result of the theorem that for some feasible x for (FP) and (x, τ, λ, v) for (FD),

$$\frac{f_i(x)}{g_i(x)} < v_i, \quad \text{for all } i = 1, \cdots, p.$$

Then, we have

$$f_i(x) - v_i g_i(x) < 0, \quad \text{for all } i = 1, \cdots, p.$$

Hence, from (8) and (10), we obtain

$$f_i(x) - v_i g_i(x) < f_i(\bar{x}) - v_i g_i(\bar{x}). \tag{13}$$

By the $V - \rho$-invexity of $f - vge$, there exist $\alpha_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \setminus \{0\}$ and $\rho \in \mathbb{R}$, $i = 1, \cdots, p$ such that for any $\xi_i \in \partial f_i(\bar{x})$ and $\zeta_i \in \partial g_i(\bar{x})$,

$$\alpha_i(x, \bar{x})\{f_i(x) - v_i g_i(x) - f_i(\bar{x}) + v_i g_i(\bar{x})\} \geqq (\xi_i - v_i \zeta_i)\eta(x, \bar{x}) + \rho_i \|\theta(x, \bar{x})\|^2. \tag{14}$$

From (13) and (14)

$$0 > (\xi_i - v_i\zeta_i)\eta(x,\bar{x}) + \rho_i\|\theta(x,\bar{x})\|^2.$$

Since $\tau_i \geqq 0$ and $\tau \neq 0$, we have

$$0 > \sum_{i=1}^{p} \tau_i(\xi_i - v_i\zeta_i)\eta(x,\bar{x}) + \sum_{i=1}^{p} \tau_i\rho_i\|\theta(x,\bar{x})\|^2, \tag{15}$$

for any $\xi_i \in \partial f_i(\bar{x})$ and any $\xi_i \in \partial g_i(\bar{x})$. Hence

$$0 \in \sum_{i=1}^{p} \tau_i\{\partial f_i(\bar{x}) - v_i\partial g_i(\bar{x})\} + \sum_{j=1}^{m} \lambda_j\partial h_j.$$

This means that there exist $\bar{\xi_i} \in \partial f_i(\bar{x}),\ \bar{\zeta_i} \in \partial g_i(\bar{x}),\ w_j \in \partial h_j(\bar{x})$ such that $0 = \sum_{i=1}^{p} \tau_i(\bar{\xi_i} - w_i\bar{\zeta_i}) + \sum_{j=1}^{m} \lambda_j w_j$ and hence $\sum_{i=1}^{p} \tau_i(\bar{\xi_i} - w_i\bar{\zeta_i})\eta(x,\bar{x}) = -\sum_{j=1}^{m} \lambda_j w_j\eta(x,\bar{x})$. Thus from (15), $0 > -\sum_{j=1}^{m} \lambda_j w_j\eta(x,\bar{x}) + \sum_{i=1}^{m} \tau_i\rho_i\|\theta(x,\bar{x})\|^2$. By the assumption (11),

$$0 > -\sum_{j=1}^{m} \lambda_j w_j\eta(x,\bar{x}) - \sum_{j=1}^{m} \lambda_j\rho_j\|\theta(x,\bar{x})\|^2.$$

Thus we have

$$\sum_{j=1}^{m} \lambda_j w_j\eta(x,\bar{x}) + \sum_{j=1}^{m} \lambda_j\rho_j\|\theta(x,\bar{x})\|^2 > 0. \tag{16}$$

By the $V-\rho$-invexity of h, there exists $\beta_j : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \setminus \{0\}$ such that

$$\beta_j(x,\bar{x})\{h_j(x) - h_j(\bar{x})\} \geqq w_j\eta(x,\bar{x}) + \rho_j\|\theta(x,\bar{x})\|^2.$$

Thus we have

$$\sum_{j=1}^{m} \lambda_j \beta_j(x,\bar{x})\{h_j(x) - h_j(\bar{x})\} \geqq \sum_{j=1}^{m} \lambda_j w_j \eta(x,\bar{x}) + \sum_{j=1}^{m} \lambda_j \rho_j \|\theta(x,\bar{x})\|^2.$$

By (15), $\sum_{j=1}^{m} \lambda_j \beta_j(x,\bar{x})\{h_j(x) - h_j(\bar{x})\} > 0$. Since $\lambda_j h_j(\bar{x}) \geqq 0$, and $\beta_j(x,\bar{x}) > 0$, $\sum_{j=1}^{m} \lambda_j \beta_j(x,\bar{x}) h_j(x) > 0$. However, since $\lambda_j \geqq 0$, and $h_j(x) \leqq 0$,

$$\sum_{j=1}^{m} \lambda_j \beta_j(x,\bar{x}) h_j(x) \leqq 0.$$

This is a contradiction. So, the conclusion holds.

Theorem 4.2 (Strong Duality). Let $\bar{x}$ be a weakly efficient solution of (FP) and suppose that the condition in (vii) or (viii) holds at $\bar{x}$. Then there exist $\bar{\tau} \in \mathbb{R}^p$, $\bar{\lambda} \in \mathbb{R}^m$ and $\bar{v} \in \mathbb{R}^p$ such that $(\bar{x}, \bar{\tau}, \bar{\lambda}, \bar{v})$ is a feasible solution for (FD). If the assumptions of Theorem 4.1 hold, then $(\bar{x}, \bar{\tau}, \bar{\lambda}, \bar{v})$ is a weakly efficient solution for (FD).

Proof. By (viii) of Theorem 3.1, there exist $\bar{\tau} \in \mathbb{R}^p$ and $\bar{\lambda} \in \mathbb{R}^m$ such that

$$0 \in \sum_{i=1}^{p} \bar{\tau}\{\partial f_i(\bar{x}) - \bar{v}_i \partial g_i(\bar{x})\} + \sum_{j=1}^{m} \bar{\lambda}_j \partial h_j(\bar{x}),$$

$$\bar{\lambda}_j h_j(\bar{x}) = 0, \quad j = 1, \cdots, m,$$

$$\bar{\tau} \geqq 0,\ \bar{\tau} \neq 0,\ \bar{\lambda} \geqq 0,$$

where $\bar{v}_i = \frac{f_i(\bar{x})}{g_i(\bar{x})}$, $i = 1, \cdots, p$.

So, there exist $\bar{\tau} \in \mathbb{R}^p$, $\bar{\lambda} \in \mathbb{R}^m$, and $\bar{v} \in \mathbb{R}^p$ such that $(\bar{x}, \bar{\tau}, \bar{\lambda}, \bar{v})$ is a feasible solution of (FD). Since weak duality holds between (FP) and (FD), there does not exist a feasible solution $(v_1, \cdots, v_p)$ for (FD) such that

$$\bar{v}_i < v_i, \ i = 1, \cdots, p.$$

Thus $(\bar{x}, \bar{\tau}, \bar{\lambda}, \bar{v})$ is a weakly efficient solution of (FD).

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