# Relations And Applications of Short Exact Sequences Concerning Amalgamated Free Products

# 융합자유곱과 관련한 완전열들의 관계 및 응용

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by

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# Relations And Applications of Short Exact Sequences Concerning Amalgamated Free Products

A dissertation

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# 융합자유곱과 관련한 완전열들의 관계 및 응용

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요 약

융합자유곱(amalgamated free products)과 관련한 완전열들(short exact sequences) 사이에는 서로 밀접한 관계가 있으며, 구조관계를 규명하는 다이어그램(diagrams)을 활용하여 그 관계를 밝힐 수 있다. 제2장에서 완전열, 군(groups) 또는 모듈(modules)의 다이어그램, 융합자유곱의 개념을 도입하고 이 개념들을 제3장부터 시작되는 주요 연구에 적용한다.

본 논문에서는 첨가아이디알(augmentation ideal), 관계모듈(relation modules), 2차호모토피모듈(second homotopy modules)을 포함하는 완전열들 사이의 상호관계를 규명하며, 관계모듈의 군표시(group presentation)를 구하여 그 구조를 분석 파악하고 그들의 관계를 밝힌다. 특히 3행과 3열이 모두 완전열이 되는 견실한 구조의 교환가능한(commutative) 다이어그램을 발견하였다. 완전열을 구성하고 있는 관계모듈은 군표시에 대해 상당히 의존적이다. 즉 동형인 군이 서로 다른 군표시를 가지면 그 군에 의해 형성되는 관계모듈들은 서로 다를 수도 있는 반면에, 동형이 아닌 군에 의해 형성된 관계모듈들은 오히려 서로 동형일 수도 있다. 이 개념을 제3장에서 언급하며 제5장에서 몇 가지 응용을 조사하여 문제를 해결한다. 아울러 관계모듈의 구조를 파악하면 그 구조를 통하여 고차(코)호몰로지군(higher (co)homology groups)을 계산해내는 것이 가능하며, 특히 본 논문에서는 2차(코)호몰로지(second integral (co)homology)를 산출하였고 군표시의 효율성(efficiency) 여부를 판단하는 부분적인 해결을 얻었다. 제4장에서 2차호모토피모듈을 포함하는 완전열간의 관계를 밝히는 데 있어서는, 2차호모토피모듈이 위상적이고 대수적인 양면을 함유하고 있으므로 2차호모토피모듈에 대한 생성원들(generators)을 결정하는 조합기하적(combinatorial geometric)인 기술로서 그림(picture)이론을 도입하여 구면그림(spherical picture)의 동치류(equivalence class)로서 이루어지는 군을 생각한 후, 군작용(group action)에 의해 2차호모토피모듈을 정의하고 항등열(identity sequence)의 동치류 집합과 동일화(identify)하여 상호관계를 규명하였다. 또한 2차호모토피모듈과 관계모듈로서 구성된 특정한 완전열로 부터 관계모듈에 대한 군표시를 간편하게 구할 수 있으며, 이를 활용하여 관계모듈끼리의 관계를 밝히는 것은 대단히 유용하므로 그 응용을 제5장에서 보인다.

## Notations

Let $G$, $H$, and $K$ be groups.

$H \oplus K$ : the direct sum

$H * K$ : the free product

$H *_U K$ : the amalgamated free product

$G \cong H$ : $G$ is isomorphic to $H$

$G/H$ : the quotient of $G$ by $H$

$AutG$ : the automorphism group of $G$

$EndG$ : the endomorphism ring of $G$

$gp_H\{-\}$ : the subgroup generated by a subset of $H$

$[a, b]$ : the commutator of $a$ and $b$

$\mathbb{Z}G$ : the integral group ring

$IG$ : the augmentation ideal

$- \otimes_G -$ : the tensor product of $\mathbb{Z}G$-modules

$rk(G)$ : the rank of the torsion-free part (when $G$ is abelian)

$d(G)$ : the least number of generators

$\nu(G) = 1 - rk(H_1(G)) + d(H_2(G))$

$H_k(G, A)$ : the **k**-th homology group of G with coefficients in $A$

$H^k(G, B)$ : the **k**-th cohomology group of G with coefficients in $B$

$H_2(G)$ : the second integral homology of $G$

$H^2(G)$ : the second integral cohomology of $G$

$\rho_1, \rho_2$ : the standard surjections

$\mu_1, \mu_2$ : the standard injections

$A \xrightarrow{\iota} B$ : the inclusion of $A$ into $B$

$\mathbb{Z}$ : the integers

$ker\alpha$ : the kernel of $\alpha$

$im\alpha$ : the image of $\alpha$

*Notation concerning presentations.*

Let $\wp$ be a group presentation.

$\pi_2(\wp)$ : the second homotopy module

$M(\wp)$ : the relation module

$\chi(\wp)$ : the Euler characteristic

*Notation concerning pictures.*

Let $\mathbb{P}$ be a picture.

$W(\mathbb{P})$ : the label of $\mathbb{P}$

$\partial(\mathbb{P})$ : the boundary of $\mathbb{P}$

$-\mathbb{P}$ : the mirror image of $\mathbb{P}$

$\mathbb{P}^W$ : the spherical picture obtained from a spherical picture $\mathbb{P}$ by surrounding it by a collection of concentric closed arcs with total label $W$

$< \mathbb{P} >$ : the equivalence class containing $\mathbb{P}$

$W(\gamma)$ : the label of a path

$W(\Delta)$ : the label of a disc

$exp_R(\mathbb{P})$ : the exponent sum of $R$ in $\mathbb{P}$

$exp_x(W)$ : the exponent sum of $x$ in $W$

*Miscellaneous notation.*

Let $\sigma$ be a sequence of words.

$\Pi\sigma$ : the product of terms of $\sigma$

$< \sigma >$ : the equivalence class containing $\sigma$

$\sigma(\tilde{\gamma})$ : the sequence associated with $\tilde{\gamma}$

# Chapter 1

# Introduction

Let $\wp_1$ and $\wp_2$ be group presentations for $H$ and $K$, respectively and $\wp$ presentation for $G = H *_U K$, i.e., the amalgamated free product of $H$ and $K$ with subgroup $U$. It is known that short exact sequences of amalgamated free products are closely related. We can find out the relation among them by applying diagrams of groups(modules).

In this thesis, we investigate the mutual relation among short exact sequences of amalgamated free products which involve augmentation ideals, relation modules, and second homotopy modules. In particular, we find out commutative diagrams having a steady structure in the sense that all of their three columns and rows are short exact.

We can more easily obtain a group presentation of relation modules from a short exact sequence which is consist of second homotopy modules and relation modules. It follows that relation modules depend on their presentations heavily, that is amount to say that, if two isomorphic groups have two different presentations, then it is possible that their relation modules are different from each other. Moreover, even though two groups are not isomorphic, their relation modules can be isomorphic from each other.

In addition, if we know the structure of relation modules, then we are able to compute the higher (co)homology groups of $G$. In particular, we compute the second integral (co)homology of $G$ and investigate the efficiency of $G$ and Cockroft property.

There are five chapters, each of which is consist of several sections. The main themes with which we shall be principally concerned in this thesis are relations among short exact sequences which associate with amalgamated free products. We devote chapter 2 to a preliminary discussion of exact sequences, diagrams, and amalgamated free products.

In chapter 3, we deal with relations among short exact sequences involving augmentation ideal and relation modules. In addition, we study the relationship between two short exact sequences.

In chapter 4, we provide an overview of the theory of pictures from a homotopy theoretic perspective and deal with relations among short exact sequences concerning second homotopy module $\pi_2(\wp)$. We also describe combinatorial geometric techniques that determine explicit generators for the second homotopy modules.

The second homotopy modules arise from both topological and algebraic sources. The description focuses on the theory of pictures. Pictures have been used for many purposes. It is known that there is a formal expressing the second homotopy modules which is denoted by the group consisting of all elements $< \mathbb{P} >$ where $\mathbb{P}$ is a spherical picture. The set $\Sigma$ of all equivalence classes of all identity sequences forms a group. We identify $\pi_2(\wp)$ with $\Sigma$. We shall be concerned with their relation to short exact sequences of relation module $M(\wp)$ and second homotopy module $\pi_2(\wp)$.

In chapter 5, it will be presented how the presentation of relation modules can be built up from short exact sequences and the application about relation modules will be shown. We also compute the second integral (co)homology of $G$, and investigate the efficiency of $G$ and Cockroft property.

We now state the main results of the thesis ; the proofs of theorem 1.0.1 and corollary 1.0.2 will be given in chapter 3 and the proofs of theorem 1.0.3 and corollary 1.0.4 will be given in chapter 4. The following theorem and corollary give us the evident relation among short exact sequences.

**Theorem 1.0.1.** *For $G = H *_U K$, we have the following commutative dia-*

*grams :*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}G \otimes_U IU & \xrightarrow{\alpha_1} & (\mathbb{Z}G \otimes_H IH) \oplus (\mathbb{Z}G \otimes_K IK) & \xrightarrow{\beta_1} & IG & \longrightarrow & 0 \\
 & & \downarrow^{\iota'} & & \downarrow^{\iota^*} & & \downarrow^{\iota} & & \\
0 & \longrightarrow & \mathbb{Z}G \otimes_U \mathbb{Z}U & \xrightarrow{\alpha_2} & (\mathbb{Z}G \otimes_H \mathbb{Z}H) \oplus (\mathbb{Z}G \otimes_K \mathbb{Z}K) & \xrightarrow{\beta_2} & \mathbb{Z}G & \longrightarrow & 0 \\
 & & \downarrow^{\varepsilon'} & & \downarrow^{\varepsilon^*} & & \downarrow^{\varepsilon} & & \\
0 & \longrightarrow & \mathbb{Z}G \otimes_U \mathbb{Z} & \xrightarrow{\alpha_3} & (\mathbb{Z}G \otimes_H \mathbb{Z}) \oplus (\mathbb{Z}G \otimes_K \mathbb{Z}) & \xrightarrow{\beta_3} & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

*where* $IU$ , $IH$ , $IK$ , *and* $IG$ *are the augmentation ideals of* $\mathbb{Z}U$ , $\mathbb{Z}H$ , $\mathbb{Z}K$ , *and* $\mathbb{Z}G$ *respectively.*

**Corollary 1.0.2.** *(1-1) is short exact if and only if (1-2) is short exact.*

$(1-1)$ $0 \longrightarrow \mathbb{Z}G \otimes_U IU \xrightarrow{\alpha_1} (\mathbb{Z}G \otimes_H IH) \oplus (\mathbb{Z}G \otimes_K IK) \xrightarrow{\beta_1} IG \longrightarrow 0.$

$(1-2)$ $0 \longrightarrow \mathbb{Z}G \otimes_U \mathbb{Z} \xrightarrow{\alpha_3} (\mathbb{Z}G \otimes_H \mathbb{Z}) \oplus (\mathbb{Z}G \otimes_K \mathbb{Z}) \xrightarrow{\beta_3} \mathbb{Z} \longrightarrow 0.$

We also have the following theorem and corollary.

Let $P_2^H = \oplus_{R\in\mathbf{r_1}}\mathbb{Z}H\bar{t}_R$ , $P_2^K = \oplus_{S\in\mathbf{r_2}}\mathbb{Z}K\bar{t}_S$ , $P_1^U = \oplus_{i\in I}\mathbb{Z}U\bar{t}_i$ ,

$P_2 = (\oplus_{R\in\mathbf{r_1}}\mathbb{Z}Gt_R) \oplus (\oplus_{S\in\mathbf{r_2}}\mathbb{Z}Gt_S) \oplus (\oplus_{i\in I}\mathbb{Z}Gt_i)$ ,

$M(\wp_1) = N_1/N_1{}'$ , $M(\wp_2) = N_2/N_2{}'$ , and $M(\wp) = N/N'$ and let $N_* = ker\theta_*$ with $\theta_* : F_* \longmapsto U$ defined by $y_i \longmapsto a_iN_1$ $(i \in I)$ , where $a_i$ is non-empty freely reduced words such that $U = gp_H\{a_iN_1 : i \in I\}$ . Then we have:

**Theorem 1.0.3.** *The following diagram is commutative with exact rows and columns.*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & (\mathbb{Z}G \otimes_H \pi_2(\wp_1)) \oplus (\mathbb{Z}G \otimes_K \pi_2(\wp_2)) & \xrightarrow{\vartheta} & \pi_2(\wp) & \xrightarrow{\xi} & \mathbb{Z}G \otimes_U N_*/N_*{}' & \to & 0 \\
 & & \downarrow{\mu_2}' & & \downarrow{\mu_2} & & \downarrow{\mu_1}' & & \\
0 & \to & (\mathbb{Z}G \otimes_H P_2^H) \oplus (\mathbb{Z}G \otimes_K P_2^K) & \xrightarrow{\alpha} & P_2 & \xrightarrow{\beta} & \mathbb{Z}G \otimes_U P_1^U & \to & 0 \\
 & & \downarrow{\rho_2}' & & \downarrow{\rho_2} & & \downarrow{\rho_1}' & & \\
0 & \to & (\mathbb{Z}G \otimes_H M(\wp_1)) \oplus (\mathbb{Z}G \otimes_K M(\wp_2)) & \xrightarrow{\lambda} & M(\wp) & \xrightarrow{\mu} & \mathbb{Z}G \otimes_U IU & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

**Corollary 1.0.4.** *(1-3) is short exact if and only if (1-4) is short exact.*

$(1-3)$ $0 \to (\mathbb{Z}G \otimes_H \pi_2(\wp_1)) \oplus (\mathbb{Z}G \otimes_K \pi_2(\wp_2)) \xrightarrow{\vartheta} \pi_2(\wp) \xrightarrow{\xi} \mathbb{Z}G \otimes_U (N_*/N_*{}') \to 0.$

$(1-4)$ $0 \to (\mathbb{Z}G \otimes_H M(\wp_1)) \oplus (\mathbb{Z}G \otimes_K M(\wp_2)) \xrightarrow{\lambda} M(\wp) \xrightarrow{\mu} \mathbb{Z}G \otimes_U IU \to 0.$

We finally set up some general conventions. First, we use the left-handed convention, whereby the composite of the morphism $\alpha$ followed by the morphism $\beta$ is written $\beta\alpha$. Modules are understood to be left modules, unless the contrary is explicitly stated. Similarly, group actions are generally understood to be left actions. We allow ourselves to simplify notation once the strict notation has been introduced and established. The identity element of a multiplicative group $G$ is denoted by $1_G$ or $1$ and the same notation $1$ is also used for the trivial subgroup consisting of the identity element. The notation and terminology not defined in this thesis are standard and can be found in almost all standard books on related areas.

# Chapter 2

# Preliminaries

In this chapter, we present the exact sequences, the notions of diagrams and the amalgamated free products together with their properties. Some basic concepts and notation are also defined.

## 2.1 Exact sequences and diagrams of groups(modules

Suppose that we have a sequence $\{G_n\}$ of groups(modules) and a sequence of group(module) homomorphisms $f_i$ from $G_i$ into $G_{i+1}$ . We will express these homomorphisms by arrows between the groups(modules):

$$(2-1) \qquad \cdots \longrightarrow G_{n-1} \xrightarrow{f_{n-1}} G_n \xrightarrow{f_n} G_{n+1} \longrightarrow \cdots$$

The set of suffixes may be finite or infinite. The above sequence (2-1) is said to be *exact* if we have $im\ f_{n-1} = ker\ f_n$ for each $n$ . If $G_i = 0$ for $i \leq n-2$

and $G_i = 0$ for $i \geq n+2$, then

$$(2-2) \qquad 0 \longrightarrow G_{n-1} \longrightarrow G_n \longrightarrow G_{n+1} \longrightarrow 0.$$

The sequence (2-2) is called a *short exact sequence.*

**Remark 2.1.1.** *Suppose that* $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ *is short exact. Then* $f$ *is a monomorphism and* $g$ *is an epimorphism.*

Let $A, B, C,$ and $D$ be groups(modules) and let $\alpha, \beta, \gamma,$ and $\delta$ be group(module) homomorphisms. We say that the *diagram*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \downarrow^{\gamma} & & \downarrow^{\beta} \\ C & \xrightarrow{\delta} & D \end{array}$$

is *commutative* if $\beta\alpha = \delta\gamma : A \longrightarrow D$. This notion can be generalized to more complicated diagrams in an obvious way.

**Lemma 2.1.2.** *( Five lemma ) Let*

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow^{\alpha_1} & & \downarrow^{\alpha_2} & & \downarrow^{\alpha_3} & & \downarrow^{\alpha_4} & & \downarrow^{\alpha_5} \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

*be a commutative diagram with exact rows.*

*(a) If* $\alpha_1$ *is an epimorphism and* $\alpha_2, \alpha_4$ *are monomorphisms, then* $\alpha_3$ *is a monomorphism;*

*(b) If* $\alpha_5$ *is a monomorphism and* $\alpha_2, \alpha_4$ *are epimorphisms, then* $\alpha_3$ *is an epimorphism.*

**Lemma 2.1.3.** *( Snake lemma ) Given the commutative diagram with exact rows:*

$$\begin{array}{ccccccc} & & A & \xrightarrow{\lambda} & B & \xrightarrow{\mu} & C & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow & & \\ 0 & \longrightarrow & A' & \xrightarrow{\lambda'} & B' & \xrightarrow{\mu'} & C' & & \end{array}$$

*there exists a homomorphism* $\Delta : ker\gamma \longrightarrow coker\alpha$ *such that the sequence*

$$ker\alpha \xrightarrow{\lambda^*} ker\beta \xrightarrow{\mu^*} ker\gamma \xrightarrow{\Delta} coker\alpha \xrightarrow{\lambda'_*} coker\beta \xrightarrow{\mu'_*} coker\gamma$$

*is exact. Moreover, if* $\lambda$ *is monomorphic, then so is* $\lambda^*$ *and if* $\mu'$ *is epimorphic, then so is* $\mu'_*$ .

**Lemma 2.1.4.** *(* $3 \times 3$ *lemma )*

*Consider the following commutative diagram, where three columns are exact.*

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_1 & \longrightarrow & C_2 & \longrightarrow & C_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

*Suppose that the middle row is exact. Then the first row is exact if and only if the third row is exact.*

## 2.2 Amalgamated free products

Let $H$, $K$, and $U$ be groups and $\phi_1$ and $\phi_2$ homomorphisms:

$$\begin{array}{ccc} U & \xrightarrow{\phi_1} & H \\ {\scriptstyle\phi_2}\downarrow & & \\ K & & \end{array} \qquad (2-3)$$

A *solution* of the above diagram (2-3) is a group $G$ and homomorphisms $\psi_1$ and $\psi_2$ such that the following diagram commutes (i.e., $\psi_1\phi_1 = \psi_2\phi_2$ ):

$$\begin{array}{ccc} U & \xrightarrow{\phi_1} & H \\ {\scriptstyle\phi_2}\downarrow & & {\scriptstyle\psi_1}\downarrow \\ K & \xrightarrow{\psi_2} & G \end{array} \qquad (2-4)$$

A *push-out* of the diagram (2-3) is a solution $(G, \psi_1, \psi_2)$ such that, for any other solution $(L, \theta_1, \theta_2)$, there exists a unique homomorphism $\alpha : G \longrightarrow L$ such that $\theta_i = \alpha\psi_i$ $(i = 1,\ 2)$. As usual, the push-out is unique up to isomorphism.

Let $\wp = \langle\ \mathbf{x} : \mathbf{r}\ \rangle$ be a group presentation, where $\mathbf{x}$ is a set and $\mathbf{r}$ is a set of cyclically reduced words on $\mathbf{x} \cup \mathbf{x}^{-1}$. Let $N$ be the normal closure of $\mathbf{r}$ in $F$, where $F$ is the free group on $\mathbf{x}$. Then the quotient $G$ of $F$ by $N$ is called the *group defined by* $\wp$ .

**Theorem 2.2.1.** *A push-out exists for the diagram (2-3). Moreover, if $H$ and $K$ are defined by $\wp_1 = \langle\ \mathbf{x}_1 : \mathbf{r}_1\ \rangle$ and $\wp_2 = \langle\ \mathbf{x}_2 : \mathbf{r}_2\ \rangle$ respectively, then the push-out $G$ is defined by $\wp = \langle\ \mathbf{x}_1 \cup \mathbf{x}_2 : \mathbf{r}_1 \cup \mathbf{r}_2 \cup \{\phi_1(u)\phi_2(u)^{-1} : u \in U\}\ \rangle$ .*

A proof of this theorem can be found in [37] (Theorem 11.58). When both $\phi_1$ and $\phi_2$ are monomorphisms, the push-out $G$ is called the *amalgamated free product* of $H$ and $K$ with subgroup $U$. In this case we usually regard $U$ as a subgroup of $H$ and $K$, and regard $\phi_1$ and $\phi_2$ as inclusions. The usual notation for the amalgamated free product of $H$ and $K$ with subgroup $U$ is $H *_U K$. Sometimes it is more convienent to use the notation $H *_{U \cong V} K$ where $U \subseteq H$, $V \subseteq K$, and $U \cong V$. For more precision, we could mention the specific isomorphism from $U$ to $V$. For an amalgamated free product we see that $\psi_1$ and $\psi_2$ are monomorphisms, and we regard them as inclusions.

# Chapter 3

# The augmentation ideal and relation modules

In this chapter we will describe the basic concepts of the augmentation ideal and relation modules. We also present some short exact sequences concerned about the augmentation ideal and relation modules associated with amalgamated free products.

## 3.1 The augmentation ideal

In this section, we describe the relationship among short exact sequences involving augmentaion ideal with amalgamated free products.

Let $G$ be a group written multiplicatively. The *integral group ring* $\mathbb{Z}G$ of $G$ is defined as follows. Its underlying abelian group is the free abelian group on the set of elements of $G$ as basis ; the product of two basis elements is given by the product in $G$. Thus the elements of the group ring $\mathbb{Z}G$ are sums

$$\sum_{x \in G} m(x)x$$

where $m$ is a function from $G$ to $\mathbb{Z}$ which takes the value zero except on a finite number of elements of $G$. The multiplication is given by

$$(\sum_{x \in G} m(x)x) \cdot (\sum_{y \in G} m'(y)y) = \sum_{x,y \in G} (m(x) \cdot m'(y))xy.$$

The group ring is characterised by the following universal property. Let $i : G \longrightarrow \mathbb{Z}G$ be the obvious embedding.

**Proposition 3.1.1.** *Let $R$ be a ring. To each function $f : G \longrightarrow R$ such that $f(xy) = f(x) \cdot f(y)$ and $f(1) = 1_R$, there exists a unique ring homomorphism $f' : \mathbb{Z}G \longrightarrow R$ such that $f'i = f$.*

A (left) $G$-module is an abelian group $A$ together with a group homomorphism $\sigma : G \longrightarrow AutA$. In other words, each element of $G$ acts as an automorphism of $A$. Since $AutA \subseteq EndA$, the universal property of the group ring yields a ring homomorphism $\sigma' : \mathbb{Z}G \longrightarrow EndA$, making $A$ into a (left) module over $\mathbb{Z}G$. Conversely, if $A$ is a (left) module over $\mathbb{Z}G$ then $A$ is a (left) $G$-module, since any ring homomorphism takes invertible elements into invertible elements, and since the group elements in $\mathbb{Z}G$ are invertible. Thus we need not retain any distinction between the concepts of $G$-module and $\mathbb{Z}G$-module. A (left) $G$-module is called *trivial* if the structure map

$\sigma : G \longrightarrow AutA$ is trivial, i.e., if every element of $G$ acts as the identity in $A$. Every abelian group may be regarded as a trivial left or right $G$-module for each group $G$. We regard $\mathbb{Z}$ as a left $\mathbb{Z}G$-module with trivial $G$-action. The *augmentation map* $\varepsilon : \mathbb{Z}G \longrightarrow \mathbb{Z}$ is the homomorphism sending every $x \in G$ into $1 \in \mathbb{Z}$, that is

$$\varepsilon : \mathbb{Z}G \longrightarrow \mathbb{Z}$$

$$\sum_{x\in G} m(x)x \longmapsto \sum_{x\in G} m(x)$$

The kernel of $\varepsilon$ is denoted by $IG$ and is called the *augmentation ideal* of $\mathbb{Z}G$. Thus we have a short exact sequence

$$(3-1) \qquad 0 \longrightarrow IG \xrightarrow{\iota} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

Tensoring (3-1) with $IG$ over $\mathbb{Z}$, we obtain the short exact sequence

$$(3-2) \qquad 0 \longrightarrow IG \otimes_{\mathbb{Z}} IG \xrightarrow{\gamma} \mathbb{Z}G \otimes_{\mathbb{Z}} IG \xrightarrow{\delta} IG \longrightarrow 0$$

where $\gamma$ and $\delta$ are defined by

$$\gamma : (x-1) \otimes (x-1) \longmapsto (x-1) \otimes (x-1) \qquad (x \in G)$$

$$\delta : x \otimes (y-1) \longmapsto x(y-1) \qquad (x, y \in G).$$

**Lemma 3.1.2.** *(i) As an abelian group $IG$ is free on the set $\{x-1 \mid 1 \neq x \in G\}$.*
*(ii) If $S$ is a generating set for $G$, then the set $\{s-1 \mid s \in S\}$ generates $IG$ as a $\mathbb{Z}G$-module.*

**Lemma 3.1.3.** *Let $U$ be a subgroup of $G$. Then $\mathbb{Z}G$ is free as left(or right) $U$-module.*

Let $G = H *_U K$ be the amalgamated free product of $H$ and $K$ with subgroup $U$. Then we have:

**Proposition 3.1.4.** *There is a short exact sequence*

$$(3-3) \qquad 0 \longrightarrow \mathbb{Z}G \otimes_U IU \xrightarrow{\alpha_1} (\mathbb{Z}G \otimes_H IH) \oplus (\mathbb{Z}G \otimes_K IK) \xrightarrow{\beta_1} IG \longrightarrow 0$$

*where* $\alpha_1$ *and* $\beta_1$ *are defined by*

$$\alpha_1 : x \otimes (u-1) \longmapsto (x \otimes (u-1), -x \otimes (u-1)) \quad (x \in G,\ u \in U)$$

$$\beta_1 : (x \otimes (h-1), y \otimes (k-1)) \longmapsto x(h-1) + y(k-1) \quad (x, y \in G, h \in H, k \in K).$$

**Proposition 3.1.5.** *There is a short exact sequence*

$$(3-4) \qquad 0 \longrightarrow \mathbb{Z}G \otimes_U \mathbb{Z}U \xrightarrow{\alpha_2} (\mathbb{Z}G \otimes_H \mathbb{Z}H) \oplus (\mathbb{Z}G \otimes_K \mathbb{Z}K) \xrightarrow{\beta_2} \mathbb{Z}G \longrightarrow 0$$

*where* $\alpha_2$ *and* $\beta_2$ *are defined by*

$$\alpha_2 : x \otimes u \longmapsto (x \otimes u, -x \otimes u) \quad (x \in G,\ u \in U)$$

$$\beta_2 : (x \otimes h, y \otimes k) \longmapsto xh + yk \quad (x, y \in G,\ h \in H,\ k \in K).$$

**Proposition 3.1.6.** *There is a short exact sequence*

$$(3-5) \qquad 0 \longrightarrow \mathbb{Z}G \otimes_U \mathbb{Z} \xrightarrow{\alpha_3} (\mathbb{Z}G \otimes_H \mathbb{Z}) \oplus (\mathbb{Z}G \otimes_K \mathbb{Z}) \xrightarrow{\beta_3} \mathbb{Z} \longrightarrow 0$$

*where* $\alpha_3$ *and* $\beta_3$ *are defined by*

$$\alpha_3 : x \otimes a \longmapsto (x \otimes a, -x \otimes a) \quad (x \in G,\ a \in \mathbb{Z})$$

$$\beta_3 : (x \otimes a, y \otimes b) \longmapsto a + b \quad (x, y \in G,\ a, b \in \mathbb{Z}).$$

We now observe the relation among (3-3),(3-4), and (3-5) through the following theorem.

**Theorem 3.1.7.** *The following diagram is commutative :*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}G \otimes_U IU & \xrightarrow{\alpha_1} & (\mathbb{Z}G \otimes_H IH) \oplus (\mathbb{Z}G \otimes_K IK) & \xrightarrow{\beta_1} & IG & \longrightarrow & 0 \\
 & & \downarrow{\iota'} & & \downarrow{\iota^*} & & \downarrow{\iota} & & \\
0 & \longrightarrow & \mathbb{Z}G \otimes_U \mathbb{Z}U & \xrightarrow{\alpha_2} & (\mathbb{Z}G \otimes_H \mathbb{Z}H) \oplus (\mathbb{Z}G \otimes_K \mathbb{Z}K) & \xrightarrow{\beta_2} & \mathbb{Z}G & \longrightarrow & 0 \\
 & & \downarrow{\varepsilon'} & & \downarrow{\varepsilon^*} & & \downarrow{\varepsilon} & & \\
0 & \longrightarrow & \mathbb{Z}G \otimes_U \mathbb{Z} & \xrightarrow{\alpha_3} & (\mathbb{Z}G \otimes_H \mathbb{Z}) \oplus (\mathbb{Z}G \otimes_K \mathbb{Z}) & \xrightarrow{\beta_3} & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

*where*

$\iota' : x \otimes (u-1) \longmapsto x \otimes (u-1) \quad (x \in G,\ u \in U)$

$\varepsilon' : x \otimes u \longmapsto x \otimes 1 \quad (x \in G,\ u \in U)$

$\iota^* : (x \otimes (h-1), y \otimes (k-1)) \longmapsto (x \otimes (h-1), y \otimes (k-1)) \quad (x, y \in G,\ h \in H,\ k \in K)$

$\varepsilon^* : (x \otimes h, y \otimes k) \longmapsto (x \otimes 1, y \otimes 1) \quad (x, y \in G,\ h \in H,\ k \in K)$

*Proof.* (1) We consider the commutativity of the left upper hand square. Then

$$\iota^* \alpha_1(x \otimes (u-1)) = \iota^*(x \otimes (u-1), -x \otimes (u-1)) = (x \otimes (u-1), -x \otimes (u-1))$$
$$\alpha_2 \iota'(x \otimes (u-1)) = \alpha_2(x \otimes (u-1)) = (x \otimes (u-1), -x \otimes (u-1)).$$

Thus we have $\iota^*\alpha_1 = \alpha_2\iota'$. Hence the left upper hand square is commutative.

(2) We consider the commutativity of the right upper hand square. Then

$$\iota\beta_1(x\otimes(h-1), y\otimes(k-1)) = \iota(x(h-1)+y(k-1)) = x(h-1)+y(k-1)$$
$$\beta_2\iota^*(x\otimes(h-1), y\otimes(k-1)) = \beta_2(x\otimes(h-1), y\otimes(k-1)) = x(h-1)+y(k-1).$$

Thus we have $\iota\beta_1 = \beta_2\iota^*$. Hence the right upper hand square is commutative.

(3) We consider the commutativity of the left lower hand square. Then

$$\varepsilon^*\alpha_2(x\otimes u) = \varepsilon^*(x\otimes u, -x\otimes u) = (x\otimes 1, -x\otimes 1)$$
$$\alpha_3\varepsilon'(x\otimes u) = \alpha_3(x\otimes 1) = (x\otimes 1, -x\otimes 1).$$

Thus we have $\varepsilon^*\alpha_2 = \alpha_3\varepsilon'$. Hence the left lower hand square is commutative.

(4) We consider the commutativity of the right lower hand square. Then

$$\varepsilon\beta_2(x\otimes h, y\otimes k) = \varepsilon(xh+yk) = 1+1$$
$$\beta_3\varepsilon^*(x\otimes h, y\otimes k) = \beta_3(x\otimes 1, y\otimes 1) = 1+1.$$

Thus we have $\varepsilon\beta_2 = \beta_3\varepsilon^*$. Hence the right lower hand square is commutative.

Therefore we get the result by (1),(2),(3), and (4). □

As a consequence of the above theorem, we have the following corollary, which shows the evident relation between (3-3) and (3-5).

**Corollary 3.1.8.** *(3-3) is exact if and only if (3-5) is exact.*

*Proof.* The third column is given in (3-1). The first and second columns are given from (3-1) and by tensoring $\mathbb{Z}G\otimes_U -$ and $(\mathbb{Z}G\otimes_H -)\oplus(\mathbb{Z}G\otimes_K -)$ respectively. Then by $3\times 3$ Lemma and Proposition 3.1.5 we get the result. □

## 3.2 Relation modules

In this section, we describe the relation between two short exact sequences involving augmentaion ideal and relation modules.

Let $G$ be the group defined by a given presentation $\wp = \langle\, \mathbf{x} : \mathbf{r} \,\rangle$ and let $N$ be the normal closure of $\mathbf{r}$ in $F$, where $F$ is the free group on $\mathbf{x}$. Then we have a short exact sequence of groups

$$(3-6) \qquad 1 \longrightarrow N \longrightarrow F \stackrel{\pi}{\longrightarrow} G \longrightarrow 1.$$

The abelianization $N/N'$ of $N$ can be regarded as a left $\mathbb{Z}G$-module via $G$-action induced by conjugation in $F$ (if $U \in N$ and $W \in F$ then $(WN)(UN') = WUW^{-1}N'$). The $G$-module $N/N'$ is called the *relation module* determined by the short exact sequence (3-6).

Relation modules depend on their presentations heavily, that is amount to say that, if two isomorphic groups have two different presentations, then it is possible that their relation modules are different from each other. Moreover, even though two groups are not isomorphic, their relation modules can be isomorphic from each other. The application of these concepts is referred to Example 5.1.1 in chapter 5.

Next we consider the short sequences involving relation modules and augmentation ideals. Then we have the following short exact sequences :

$$(3-7) \qquad 0 \longrightarrow N/N' \stackrel{\iota}{\longrightarrow} F/N' \stackrel{\varphi}{\longrightarrow} G \longrightarrow 0$$

where

$$\iota : UN' \longmapsto UN' \quad (U \in N),$$

$$\varphi : WN' \longmapsto WN \quad (W \in F),$$

and

$$(3-8) \qquad 0 \longrightarrow N/N' \xrightarrow{\mu_1} \oplus_{x\in\mathbf{x}}\mathbb{Z}Gt_x \xrightarrow{\rho_1} IG \longrightarrow 0$$

where

$$WN' \longmapsto \sum_{x\in\mathbf{x}} \rho(\frac{\partial W}{\partial x})t_x \quad (W \in N)$$

$$t_x \longmapsto xN - 1 \qquad (x \in \mathbf{x})$$

Here $\frac{\partial}{\partial x} : \mathbb{Z}F \longrightarrow \mathbb{Z}F$ is the Fox derivation (See [29] Section II.3) and $\rho : \mathbb{Z}F \longrightarrow \mathbb{Z}G$ is induced by the natural epimorphism $F \longrightarrow G$. From (3-1) and (3-8), we get

$$(3-9) \qquad 0 \longrightarrow N/N' \xrightarrow{\mu_1} \oplus_{x\in\mathbf{x}}\mathbb{Z}Gt_x \xrightarrow{\rho_1} \mathbb{Z}G \longrightarrow 0.$$

**Lemma 3.2.1.** *Let*

$$1 \longrightarrow N \longrightarrow F \xrightarrow{\pi} G \longrightarrow 1$$

*be a short exact sequence of groups, where* $G$ *is a group defined by* $\wp = \langle\, \mathbf{x} : \mathbf{r} \,\rangle$ *and* $N$ *is the normal closure of* $\mathbf{r}$ *in* $F$ *free on* $\mathbf{x}$. *Then*

$$(3-10) \qquad 0 \longrightarrow N/N' \xrightarrow{\kappa} \mathbb{Z}G \otimes_F IF \xrightarrow{\nu} IG \longrightarrow 0$$

*is an exact sequence of* $G$*-modules where* $\kappa(UN') = 1_G \otimes (U-1)$ *and* $\nu(1_G \otimes (W-1)) = \pi(W) - 1$ $(U \in N,\ W \in F)$.

A proof of this lemma can be found in [23](Chapter VI, Theorem 6.3).

**Theorem 3.2.2.** *The two short exact sequences (3-2) and (3-10) are isomorphic.*

*Proof.* Consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N/N' & \xrightarrow{\kappa} & \mathbb{Z}G \otimes_F IF & \xrightarrow{\nu} & IG & \longrightarrow & 0 \\ & & \downarrow^{\alpha} & & \downarrow^{\beta} & & \downarrow^{\iota} & & \\ 0 & \longrightarrow & IG \otimes_{\mathbb{Z}} IG & \xrightarrow{\gamma} & \mathbb{Z}G \otimes_{\mathbb{Z}} IG & \xrightarrow{\delta} & IG & \longrightarrow & 0 \end{array}$$

where $\alpha, \beta, \gamma,$ and $\delta$ are defined by

$$\alpha : UN' \longmapsto 1_G \otimes (UN - 1) \quad (U \in N),$$

$$\beta : 1_G \otimes (W - 1) \longmapsto 1_G \otimes (WN - 1) \quad (W \in F),$$

$$\gamma : (WN - 1) \otimes (WN - 1) \longmapsto (WN - 1) \otimes (WN - 1) \quad (WN \in G),$$

$$\delta : 1_G \otimes (WN - 1) \longmapsto WN - 1 \quad (WN \in G).$$

(1) We consider the commutativity of the left hand square. Then

$$\beta\kappa(UN') = \beta(1_G \otimes (U - 1)) = 1_G \otimes (UN - 1)$$

$$\gamma\alpha(UN') = \gamma(1_G \otimes (UN - 1)) = 1_G \otimes (UN - 1).$$

Thus we have $\beta\kappa = \gamma\alpha$. Hence the left hand square is commutative.

(2) We consider the commutativity of the right hand square. Then

$$\iota\nu(1_G \otimes (W - 1)) = \iota(WN - 1) = WN - 1$$

$$\delta\beta(1_G \otimes (W - 1)) = \delta(1_G \otimes (WN - 1)) = WN - 1.$$

Thus we have $\iota\nu = \delta\beta$. Hence the right hand square is commutative. Now we want to show that $\alpha$ is an isomorphism. We show that $ker\alpha = 0$.

Let $UN' \in ker\alpha$. Then $0 = \gamma\alpha(UN') = \beta\kappa(UN')$. It is routine to show that $\beta$ is an isomorphism. Since $\beta$ is an isomorphism, $\kappa(UN') = 0$. Since $\kappa$ is injective, it follows that $UN' = 0$. Secondly, we shall show that $\alpha$ is surjective. Let $(UN - 1) \otimes (UN - 1) \in IG \otimes_{\mathbb{Z}} IG$. Then $\gamma((UN-1)\otimes(UN-1)) \in \mathbb{Z}G\otimes_{\mathbb{Z}}IG$. Since $\beta$ is an isomorphism, there exists $1_G\otimes(U-1) \in \mathbb{Z}G\otimes_F IF$ such that $\beta(1_G\otimes(U-1)) = \gamma((UN-1)\otimes(UN-1))$. Then $\iota\nu(1_G \otimes (U-1)) = \delta\beta(1_G \otimes (U-1)) = \delta\gamma((UN-1)\otimes(UN-1)) = 0$. Hence $\nu(1_G \otimes (U-1)) \in ker\iota$. Since $\iota$ is an isomorphism, it follows that $\nu(1_G \otimes (U-1)) = 0$. Then $1_G \otimes (U-1) \in ker\nu = im\kappa$. Hence there exists $UN' \in N/N'$ such that $\kappa(UN') = 1_G \otimes (U-1)$. This implies that

$$\begin{aligned}
&\gamma(\alpha(UN') - (UN-1)\otimes(UN-1)) \\
&= \gamma\alpha(UN') - \gamma((UN-1)\otimes(UN-1)) \\
&= \beta\kappa(UN') - \gamma((UN-1)\otimes(UN-1)) \\
&= \beta(1_G \otimes (U-1)) - \gamma((UN-1)\otimes(UN-1)) \\
&= 0.
\end{aligned}$$

Then $\alpha(UN') - ((UN-1)\otimes(UN-1)) \in ker\gamma$. Since $\gamma$ is injective, it follows that $\alpha(UN') - ((UN-1)\otimes(UN-1)) = 0$, i.e., $\alpha(UN') = (UN-1)\otimes(UN-1)$ Therefore $\alpha$ is surjective. Consequently, we obtain the result. □

Let $H$ and $K$ be the groups defined by $\wp_1 = \langle\, \mathbf{x}_1 : \mathbf{r}_1 \,\rangle$ and $\wp_2 = \langle\, \mathbf{x}_2 : \mathbf{r}_2 \,\rangle$, respectively and let $N_i$ be the normal closure of $\mathbf{r}_i$ in $F_i$, where $F_i$ is free group on $\mathbf{x}_i$ for $i = 1, 2$. Then we have short exact sequences

$$1 \longrightarrow N_1 \longrightarrow F_1 \xrightarrow{\pi_1} H \longrightarrow 1$$

and

$$1 \longrightarrow N_2 \longrightarrow F_2 \xrightarrow{\pi_2} K \longrightarrow 1.$$

Let $G = H *_U K$ be the amalgamated free product of groups $H$ and $K$ with subgroup $U$. Then there is a short exact sequence

$$(3-9) \qquad 1 \longrightarrow N \longrightarrow F_1 * F_2 \xrightarrow{\pi} H *_U K \longrightarrow 1$$

of $G$, where $F_1 * F_2$ is the free product of $F_1$ and $F_2$, and the epimorphism $\pi$ is defined by

$$\pi|_{F_1} = \pi_1, \ \pi|_{F_2} = \pi_2 \ .$$

**Theorem 3.2.3.** *There is a short exact sequence of $G$-modules*

$$0 \longrightarrow \mathbb{Z}G \otimes_H N_1/N_1{}' \xrightarrow{\kappa_1} \mathbb{Z}G \otimes_{F_1} IF_1 \xrightarrow{\nu_1} \mathbb{Z}G \otimes_H IH \longrightarrow 0$$

*where*

$$\kappa_1 : 1_G \otimes U_1 N_1{}' \longmapsto 1_G \otimes (U_1 - 1) \qquad (U_1 \in N_1)$$

$$\nu_1 : 1_G \otimes (W_1 - 1) \longmapsto 1_G \otimes (W_1 N_1 - 1) \qquad (W_1 \in F_1).$$

*Proof.* By Lemma 3.2.1,

$$0 \longrightarrow N_1/N_1{}' \longrightarrow \mathbb{Z}H \otimes_{F_1} IF_1 \longrightarrow IH \longrightarrow 0$$

is exact. Tensoring with $\mathbb{Z}G$ over $H$ yields

$$\mathbb{Z}G \otimes_H (\mathbb{Z}H \otimes_{F_1} IF_1) = (\mathbb{Z}G \otimes_H \mathbb{Z}H) \otimes_{F_1} IF_1 \cong \mathbb{Z}G \otimes_{F_1} IF_1$$

Hence the above sequence is exact. □

**Corollary 3.2.4.** *There is a short exact sequence of* $G$*-modules*

$$0 \to (\mathbb{Z}G \otimes_H N_1/N_1{}') \oplus (\mathbb{Z}G \otimes_K N_2/N_2{}') \xrightarrow{\kappa^*}$$
$$(\mathbb{Z}G \otimes_{F_1} IF_1) \oplus (\mathbb{Z}G \otimes_{F_2} IF_2) \xrightarrow{\nu^*} (\mathbb{Z}G \otimes_H IH) \oplus (\mathbb{Z}G \otimes_K IK) \to 0$$

*where*

$$\kappa^* : (1_G \otimes U_1 N_1{}', 1_G \otimes U_2 N_2{}') \longmapsto (1_G \otimes (U_1 - 1), 1_G \otimes (U_2 - 1))$$
$$U_1 \in N_1, \quad U_2 \in N_2$$
$$\nu^* : (1_G \otimes (W_1 - 1), 1_G \otimes (W_2 - 1)) \longmapsto (1_G \otimes (W_1 N_1 - 1), 1_G \otimes (W_2 N_2 - 1))$$
$$W_1 \in F_1, \quad W_2 \in F_2.$$

**Theorem 3.2.5.** *Let* $F = F_1 * F_2$ *and let* $G = H *_U K$. *Then the following diagram is commutative.*

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & (\mathbb{Z}G \otimes_H N_1/N_1') \oplus (\mathbb{Z}G \otimes_K N_2/N_2') & & N/N' & & \\
 & & & & \downarrow{\scriptstyle\kappa^*} & & \downarrow{\scriptstyle\kappa} & & \\
 & & & & (\mathbb{Z}G \otimes_{F_1} IF_1) \oplus (\mathbb{Z}G \otimes_{F_2} IF_2) & \xrightarrow{\zeta} & \mathbb{Z}G \otimes_F IF & & \\
 & & & & \downarrow{\scriptstyle\nu^*} & & \downarrow{\scriptstyle\nu} & & \\
0 & \to & \mathbb{Z}G \otimes_U IU & \xrightarrow{\alpha_1} & (\mathbb{Z}G \otimes_H IH) \oplus (\mathbb{Z}G \otimes_K IK) & \xrightarrow{\beta_1} & IG & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

*where*

$$\zeta : (1_G \otimes (W_1 - 1), 1_G \otimes (W_2 - 1)) \mapsto 1_G \otimes (W_1 - 1) + 1_G \otimes (W_2 - 1) \quad (W_1 \in F_1, W_2 \in F_2).$$

*Proof.* We want to show that $\nu\zeta = \beta_1\nu^*$.

$$\begin{aligned} &\nu\zeta(1_G \otimes (W_1 - 1), 1_G \otimes (W_2 - 1)) \\ &= \nu(1_G \otimes (W_1 - 1) + 1_G \otimes (W_2 - 1)) \\ &= (W_1N_1 - 1) + (W_2N_2 - 1). \end{aligned}$$

On the other hand

$$\begin{aligned} &\beta_1\nu^*(1_G \otimes (W_1 - 1), 1_G \otimes (W_2 - 1)) \\ &= \beta_1(1_G \otimes (W_1N_1 - 1), 1_G \otimes (W_2N_2 - 1)) \\ &= (W_1N_1 - 1) + (W_2N_2 - 1). \end{aligned}$$

Therefore we get the result. □

From the above theorem, we can obtain the short exact sequence

$$0 \longrightarrow (\mathbb{Z}G \otimes_H N_1/N_1') \oplus (\mathbb{Z}G \otimes_K N_2/N_2') \longrightarrow N/N' \longrightarrow \mathbb{Z}G \otimes_U IU \longrightarrow 0.$$

This can be found in [22].

# Chapter 4

# Second homotopy modules

In this chapter, we introduce the basic concepts of the pictures and the identity sequences. We also study some short exact sequences concerned about the second homotopy modules associated with amalgamated free products.

## 4.1 Pictures

A picture $\mathbb{P}$ is a geometric configuration consisting of the following:

(a) A disc $D^2$ with basepoint $O$ on $\partial D^2$.

(b) Disjoint discs $\Delta_1, \ldots, \Delta_n$ in the interior of $D^2$. Each disc $\Delta_\lambda$ ( $\lambda = 1, \ldots, n$ ) has a basepoint $O_\lambda$ on $\partial\Delta_\lambda$.

(c) A finite number of disjoint arcs $\alpha_1, \ldots, \alpha_m$. Each arc lies in the closure of $D^2 \setminus \bigcup_{\lambda=1}^{n} \Delta_\lambda$ and is either a simple closed curve having trivial intersection with $\partial D^2 \cup \partial\Delta_1 \cup \ldots \cup \partial\Delta_n$, or a simple non-closed curve which joins two points of $\partial D^2 \cup \partial\Delta_1 \cup \ldots \cup \partial\Delta_n$, neither point being a basepoint. Each arc has a normal orientation, indicated by a short arrow meeting the arc transversely.

A picture $\mathbb{P}$ is called to be *connected* if $\bigcup\{\Delta_1, \ldots, \Delta_n\} \cup \bigcup\{\alpha_1, \ldots, \alpha_n\}$ is connected.

For each disc $\Delta$, the *corners* of $\Delta$ are the closures of the connected components of $\partial\Delta \setminus \bigcup\{\alpha_1, \ldots, \alpha_m\}$, where $\alpha_1, \ldots, \alpha_m$ are arcs of $\Delta$. The *regions* of $\mathbb{P}$ are the closures of connected components of $D^2 \setminus (\bigcup\{discs\} \cup \bigcup\{arcs\})$. An *inner* region of $\mathbb{P}$ is a simply connected region of $\mathbb{P}$ that does not meet $\partial D^2$.

We remark that when we refer to the discs of $\mathbb{P}$ we mean the discs $\Delta_1, \ldots, \Delta_n$, but not the ambient disc $D^2$. We define $\partial\mathbb{P}$ to be $\partial D^2$.

We say that $\mathbb{P}$ is *spherical* if no arcs meet $\partial\mathbb{P}$. If $\mathbb{P}$ is spherical then we often omit $\partial\mathbb{P}$.

Let $\wp = \langle\, \mathbf{x} : \mathbf{r} \,\rangle$ be a group presentation, where $\mathbf{x}$ is a set and $\mathbf{r}$ is a set of cyclically reduced words on $\mathbf{x} \cup \mathbf{x}^{-1}$.

**Definition 4.1.1.** *A picture $\mathbb{P}$ is over $\wp$ if the following conditions hold:*

*(1) Each arc is labelled by an element of $\mathbf{x} \cup \mathbf{x}^{-1}$.*

*(2) If we travel around $\partial\Delta_\lambda$ once in the clockwise direction starting at $O_\lambda$ and read off the labels on the arcs encountered then we obtain a word which belongs to $\mathbf{r} \cup \mathbf{r}^{-1}$ and we call this word the label of $\Delta_\lambda$.*

**Example 4.1.1.** Let $\wp = \langle x, y, z : x^3, yzy^{-1}z^{-1}, xyx^{-2}y^{-1}, xzx^{-2}z^{-1} \rangle$ . Then the following picture is a spherical picture over $\wp$.

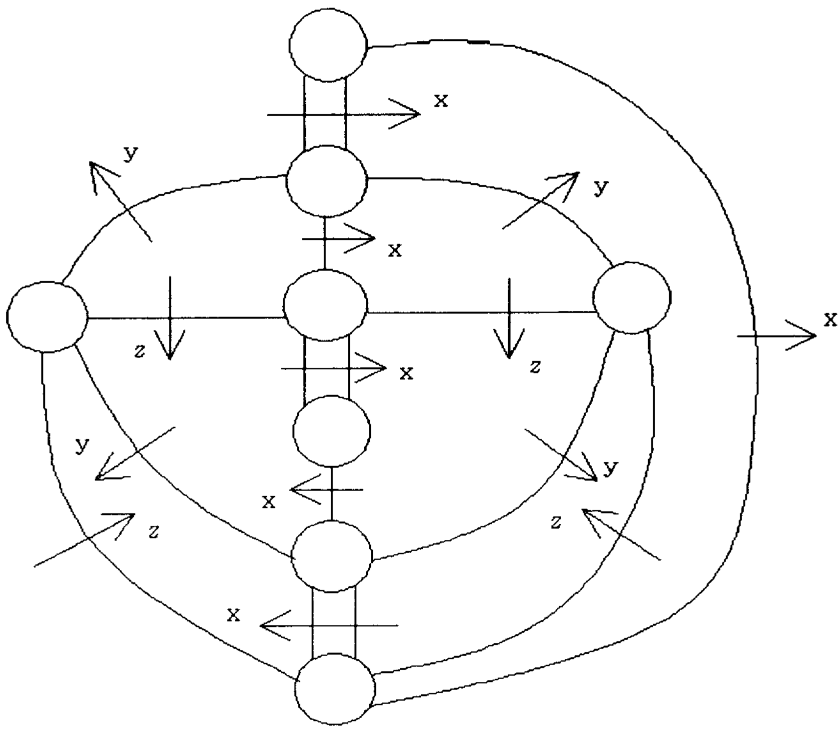


Figure 4.1.1. Spherical picture.

Let $\mathbf{y}$ and $\mathbf{s}$ be subsets of $\mathbf{x}$ and $\mathbf{r}$ respectively. An arc labelled by an element of $\mathbf{y} \cup \mathbf{y}^{-1}$ is called a $\mathbf{y}$-*arc* and a disc labelled by an element of $\mathbf{s} \cup \mathbf{s}^{-1}$ is called an $\mathbf{s}$-*disc*. The label on $\mathbb{P}$ (denoted $W(\mathbb{P})$ ) is the word read off by travelling around $\partial D^2$ once in the clockwise starting at O.

**Example 4.1.2.** Let $\wp = \langle\ a, b, c : a^2, (ab)^2, [b, c], [a, c]\ \rangle$.

Then $W(\mathbb{P}) = b^{-1}ac^{-1}b^{-1}ac$.

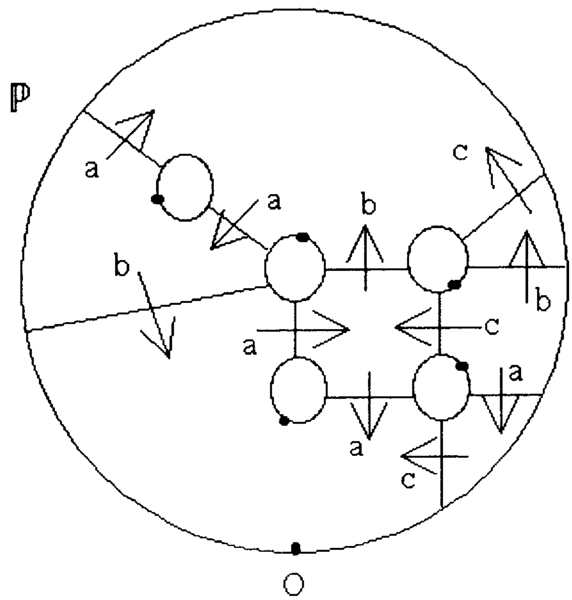


Figure 4.1.2. Label of $\mathbb{P}$.

A *(transverse)* path in $\mathbb{P}$ is a path in the closure of $D^2 \setminus \bigcup_{\lambda=1}^{n} \Delta_\lambda$ which intersects the arcs of $\mathbb{P}$ only finitely many times. If we travel along a path $\gamma$ from its initial point to its terminal point we will cross various arcs, and we can read off the labels on these arcs, giving a word $W(\gamma)$, the *label on* $\gamma$. A *spray* for $\mathbb{P}$ is a sequence $\tilde{\gamma} = (\gamma_1, \ldots, \gamma_n)$ of simple paths satisfying the following; for $\lambda = 1, \ldots, n$, $\gamma_\lambda$ starts at O and ends at the basepoint $O_{\theta(\lambda)}$ of $\Delta_{\theta(\lambda)}$, where $\theta$ is a permutation of $\{1, \ldots, n\}$ (depending on $\tilde{\gamma}$); for

$1 \leq \lambda < \mu \leq n$, $\gamma_\lambda$ and $\gamma_\mu$ intersect only at O; travelling around O clockwise in $\mathbb{P}$ we encounter the paths in the order $\gamma_1, \ldots, \gamma_n$.

Now we introduce the basic operations on pictures.

(A) Deletion of a closed arc which encircles no discs or arcs of $\mathbb{P}$ (such a closed arc is called a *floating circle*).

$(A)^{-1}$ Insertion of a floating circle.

A *cancelling pair* is a spherical picture with exactly two discs, and when their basepoint lie in the same region like Figure 4.1.3.

(B) If there is a simple closed path $\beta$ in $\mathbb{P}$ such that the part of $\mathbb{P}$ encircled by $\beta$ is a cancelling pair, then remove that part of $\mathbb{P}$ encircled by $\beta$.

$(B)^{-1}$ The opposite of (B).

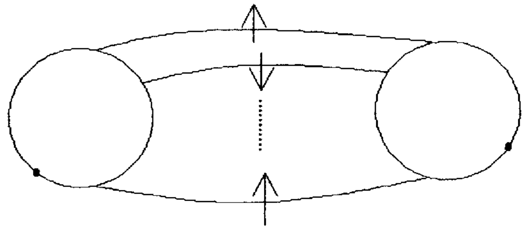

Figure 4.1.3. Cancelling pair.

(C) Bridge move.

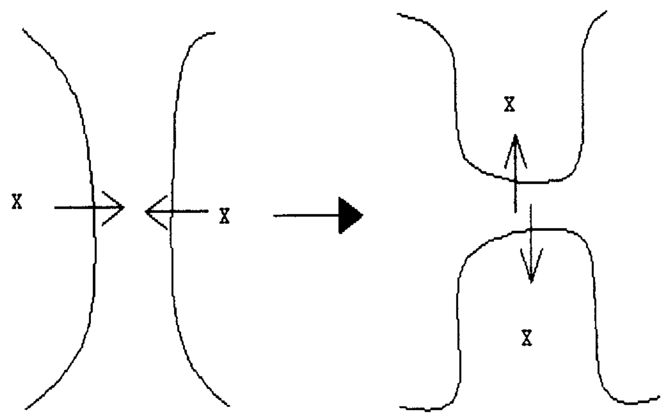


Figure 4.1.4. Bridge move.

Two pictures will be said to be *equivalent* if the pictures are both spherical and one can be transformed to the other by a finite number of operations (A), $(A)^{-1}$, (B), $(B)^{-1}$, and (C).

**Remark 4.1.2.** *Since we allow only one basepoint on each disc, when a relator is a proper power, we need more caution. That is to say,* $\mathbb{P}_1$ *and* $\mathbb{P}_2$ *are cancelling pair, whereas* $\mathbb{P}_3$ *is not. So we will only insert basepoints for discs* *whose labels are proper powers.*

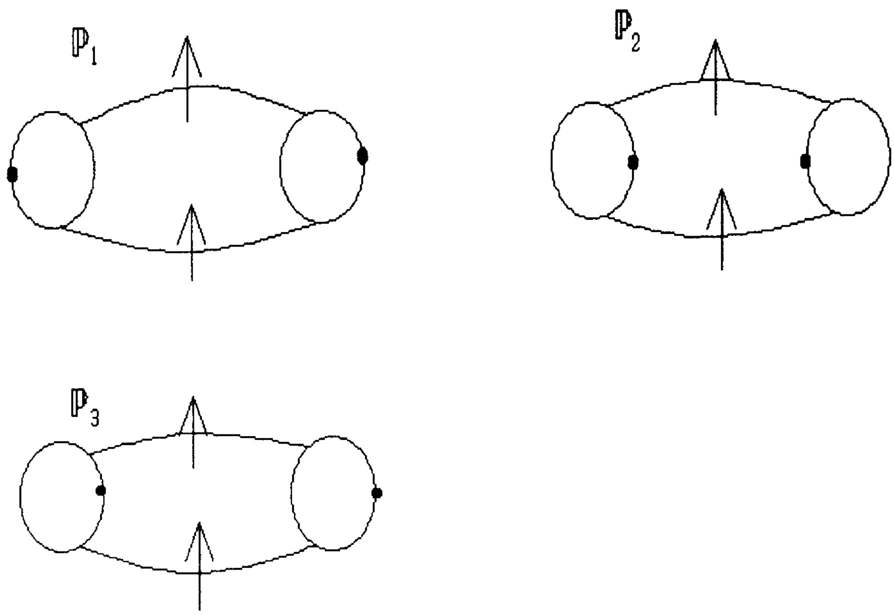


Figure 4.1.5. Cancelling pair and not cancelling pair.

The algebraic operations in second homotopy modules are easily visualized in terms of pictures. Let $\mathbb{P}, \mathbb{P}_1$, and $\mathbb{P}_2$ be based pictures over $\wp$. Two new pictures $\mathbb{P}_1 + \mathbb{P}_2$ and $-\mathbb{P}$ are constructed as in Figure 4.1.6 and Figure 4.1.7, respectively. Thus $\mathbb{P}_1 + \mathbb{P}_2$ is a certain sum of $\mathbb{P}_1$ and $\mathbb{P}_2$ and $-\mathbb{P}$ is a mirror image of $\mathbb{P}$ obtained by a planar reflection and by changing the signs on all discs of $\mathbb{P}$.

**Definition 4.1.3.** *(1)* $\mathbb{P}_1 + \mathbb{P}_2$ *is the sum of* $\mathbb{P}_1$ *and* $\mathbb{P}_2$ *.*

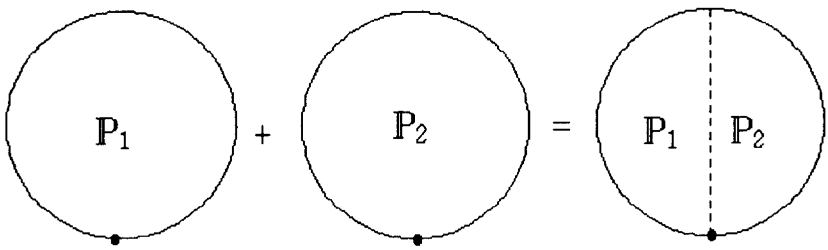


Figure 4.1.6. Sum of $\mathbb{P}_1$ *and* $\mathbb{P}_2$ .

*(2)* $-\mathbb{P}$ *is the mirror image of* $\mathbb{P}$ *.*

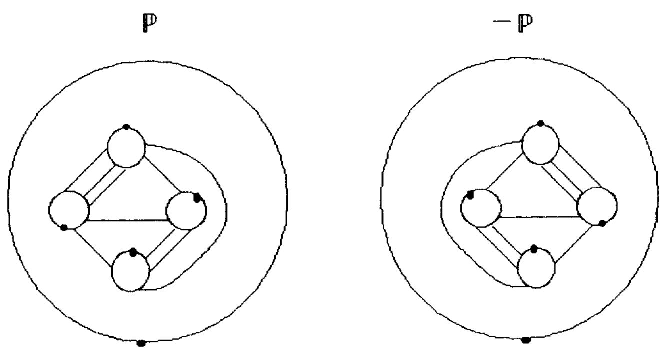


Figure 4.1.7. Mirror image of $\mathbb{P}$ .

We let $\langle\ \mathbb{P}\ \rangle$ denote the equivalence class containing $\mathbb{P}$. The set of all equivalence classes of all spherical pictures over $\wp$ forms a group under the following binary operation

$$\langle\ \mathbb{P}_1\ \rangle + \langle\ \mathbb{P}_2\ \rangle = \langle\ \mathbb{P}_1 + \mathbb{P}_2\ \rangle$$

where the inverse of $\langle\ \mathbb{P}\ \rangle$ is $\langle\ -\mathbb{P}\ \rangle$ and the identity is the equivalence class containing the empty picture. We let $\pi_2(\wp)$ denote the group consisting of all elements $\langle\ \mathbb{P}\ \rangle$ where $\mathbb{P}$ is a spherical picture. Let $\mathbb{P}^W$ be the spherical picture obtained from a spherical picture $\mathbb{P}$ by surrounding it by a collection of concentric closed arcs with total label $W$ like Figure 4.1.8.

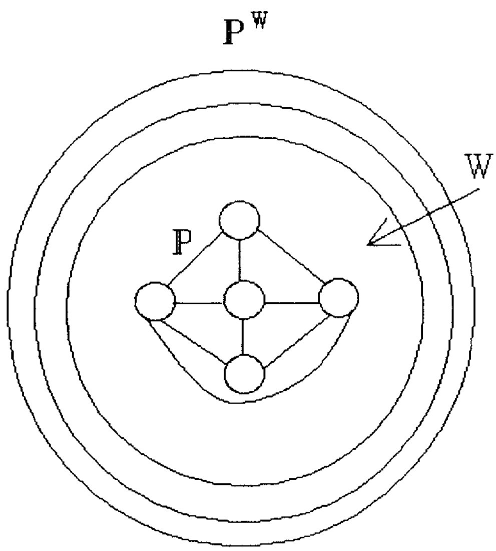


Figure 4.1.8. Spherical picture $\mathbb{P}^W$.

We can also consider $\pi_2(\wp)$ as a left $\mathbb{Z}G$-module by the $G$-action given by

$$WN.\langle \mathbb{P} \rangle = \langle \mathbb{P}^W \rangle \qquad (W \in F).$$

Then we call $\pi_2(\wp)$ the *second homotopy module of* $\wp$.

## 4.2 Identity sequences

Let $\wp = \langle \mathbf{x} : \mathbf{r} \rangle$ be a group presentation. Let $G$ be the group defined by $\wp$, that is, $G = F/N$, where $F$ is the free group on $\mathbf{x}$ and $N$ is the normal closure of $\mathbf{r}$ in $F$. We denote by $\mathbf{w}$ the set of all words on $\mathbf{x} \cup \mathbf{x}^{-1}$. If $\mathbf{s}$ is a subset of $\mathbf{r}$ then $\mathbf{s}^{\mathbf{w}}$ is the set of all words of the form $WS^{\varepsilon}W^{-1}(W \in \mathbf{w}, S \in \mathbf{s}, \varepsilon = \pm 1)$. Let $\sigma = (c_1, \cdots, c_n)$ where $c_i \in \mathbf{r}^{\mathbf{w}}(i = 1, \cdots, n)$. We define $\Pi\sigma$ to be the product $c_1 c_2 \cdots c_n$. If $\Pi\sigma$ is freely equal to 1 then $\sigma$ is called an *identity sequence.* We define the inverse $\sigma^{-1}$ of $\sigma$ to be $(c_n^{-1}, \cdots, c_1^{-1})$ and for $W \in \mathbf{w}$ we define the *conjugate* $W\sigma W^{-1}$ of $\sigma$ by $W$ to be $(Wc_1W^{-1}, \cdots, Wc_nW^{-1})$. We define operations on sequences as follows. Let $c_i = W_i R_i^{\varepsilon_i} W_i^{-1}(W_i \in \mathbf{w}, R_i \in \mathbf{r}, \varepsilon_i = \pm 1, i = 1, \ldots, n)$.

($\sharp$ 1) Replace each $W_i$ by a word freely equal to it.

($\sharp$ 2) Delete two consecutive terms if one is identically equal to the inverse of the other.

($\sharp$ 3) The opposite of ($\sharp$ 2).

($\sharp$ 4) Replace two consecutive terms $c_i, c_{i+1}$ by either $c_{i+1}, c_{i+1}^{-1} c_i c_{i+1}$ or by $c_i c_{i+1} c_i^{-1}, c_i$ .

Two sequences $\sigma$ and $\sigma'$ will be said to be *(Peiffer) equivalent* if one can be obtained from the other by a finite number of applications of the operations $(\sharp 1), (\sharp 2), (\sharp 3)$, and $(\sharp 4)$.

The equivalence class containing $\sigma$ will be denoted by $\langle\sigma\rangle$. The set $\Sigma$ of all equivalence classes of all identity sequences forms a group under the following binary operation $\langle\sigma_1\rangle + \langle\sigma_2\rangle = \langle\sigma_1\sigma_2\rangle$ where $\sigma_1\sigma_2$ is the juxtaposition of the two sequences $\sigma_1$ and $\sigma_2$.

We can also consider $\Sigma$ as a left $\mathbb{Z}G$-module via the $G$-action given by

$$WN.\langle\sigma\rangle = \langle W\sigma W^{-1}\rangle \qquad (W \in F).$$

We now define a map

$$\psi : \pi_2(\wp) \to \Sigma, \quad \langle\ \mathbb{P}\ \rangle \mapsto \langle\ \sigma\ \rangle$$

where $\sigma$ is an identity sequence. From now on, we will identify $\pi_2(\wp)$ with $\Sigma$. We can think of an identity sequence as *a relation (an identity) among relators*. So $\Sigma$ gives us a description of all relations among relators. Thus computing generators of $\Sigma$ amounts to determining a collection of identities among the relators of $\wp$ from which all other identities are derivable. The *sequence* $\sigma(\tilde{\gamma})$ *associated with* spray $\tilde{\gamma} = (\gamma_1, \ldots, \gamma_n)$ is

$$(W(\gamma_1)W(\Delta_{\theta(1)})W(\gamma_1)^{-1}, \ldots, W(\gamma_n)W(\Delta_{\theta(n)})W(\gamma_n)^{-1}).$$

A picture will be said to *represent* a sequence $\sigma$ if there is a spray for the picture whose associated sequence is $\sigma$.

**Example 4.2.1.** We consider $\wp = < x, y : x^3, xyxy^{-1} >$. Then we get the following picture like Figure 4.2.1. Let $R_1 = x^3$ and $R_2 = xyxy^{-1}$. Then we obtain the identity sequence $\sigma = (R_1, x^{-1}R_2^{-1}x, yxy^{-1}x^{-1}R_2^{-1}xyx^{-1}y^{-1}, yR_1y^{-1}, R_2^{-1})$. Hence $\mathbb{P}$ represents $\sigma$.

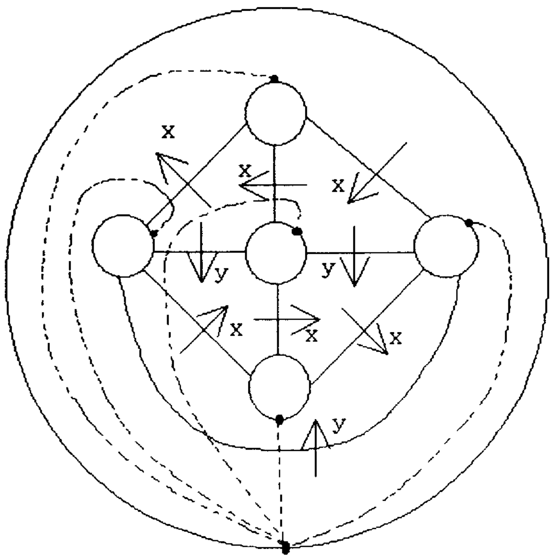


Figure 4.2.1. $\mathbb{P}$ representing $\sigma$.

Note that if $\mathbb{P}$ represents $\sigma$ then $-\mathbb{P}$ represents $\sigma^{-1}$. Note also that if $\mathbb{P}_1$ and $\mathbb{P}_2$ represent $\sigma_1$ and $\sigma_2$ respectively then $\mathbb{P}_1 + \mathbb{P}_2$ represents $\sigma_1\sigma_2$. Consider a collection $\mathbf{X}$ of spherical pictures over $\wp$. We introduce two further operations on $\pi_2(\wp)$ as follows.

(D) If there is a simple closed path $\beta$ in a picture such that the part of the picture enclosed by $\beta$ is a copy of $\mathbb{P}$ or $-\mathbb{P}$ ($\mathbb{P} \in \mathbf{X}$), then delete that part of the picture enclosed by $\beta$.

$(D)^{-1}$ The opposite of (D).

Two spherical picture will be said to be *equivalent* (rel $\mathbf{X}$) if one can be transformed to the other by a finite number of operations (A), $(A)^{-1}$, (B), $(B)^{-1}$, (C), (D), and $(D)^{-1}$.

**Theorem 4.2.1.** *[33] (Theorem 2.6 Corollary 1) The elements $\langle \mathbb{P} \rangle$ ($\mathbb{P} \in \mathbf{X}$) generate $\pi_2(\wp)$ if and only if every spherical picture is equivalent (rel $\mathbf{X}$) to the empty picture.*

If the elements $\langle \mathbb{P} \rangle$ ($\mathbb{P} \in \mathbf{X}$) generate $\pi_2(\wp)$ then we say that $\mathbf{X}$ generates $\pi_2(\wp)$.

## 4.3 Short exact sequences

Let $\wp_1 = \langle\, \mathbf{x}_1 : \mathbf{r}_1 \,\rangle$ and $\wp_2 = \langle\, \mathbf{x}_2 : \mathbf{r}_2 \,\rangle$ be the presentations for $H$ and $K$ respectively and let $G = H *_U K$ be the amalgamated free product of groups $H$ and $K$ with subgroup $U$. Choose (disjoint) sets $\mathbf{y}_* = \{y_i : i \in I\}$. Let $\mathbf{w}_*$ be the set of all words (reduced or not) on $\mathbf{y}_*$. Let $F_*$ be the free group on $\mathbf{y}_*$ and let $N_*$ be the kernel of the epimorphism $\theta_* : F_* \longrightarrow U$ defined by $y_i \longmapsto a_i N_1$ $(i \in I)$, where $a_i$ is non-empty freely reduced words on $\mathbf{x}_1$ such that $U = gp_H\{a_i N_1 : i \in I\}$ and $U = gp_K\{b_i N_2 : i \in I\}$ with the property that the correspondence $a_i \longmapsto b_i$ induces the isomorphism $\gamma_*$. Then $G$ has a presentation

$$\wp = \langle\ \mathbf{x_1}, \mathbf{x_2} : \mathbf{r_1}, \mathbf{r_2}, \mathbf{s}\ \rangle$$

where $\mathbf{s} = \{a_i b_i^{-1} : i \in I\}$. Let $\mathbf{X_1}$ and $\mathbf{X_2}$ be the collections of all spherical pictures over $\wp_1$ and $\wp_2$ such that $\pi_2(\wp_1)$ and $\pi_2(\wp_2)$ are generated by $\mathbf{X_1}$ and $\mathbf{X_2}$ respectively and let $\mathbf{X} = \mathbf{X_1} \cup \mathbf{X_2}$.

If an element $W(y_i) \in \mathbf{w}_*$ defines an element of $N_*$ then $W(a_i)$ defines the identity in $H$. So there is a picture $\mathbb{A}_W$ over $\wp_1$ with the boundary label $W(a_i)$. We note that though $\mathbb{A}_W$ is not unique, it is unique up to equivalence (rel $\mathbf{X}$), because that the pictures $\mathbb{A}_W$ and $-\mathbb{A}_W$ can be combined to make a spherical picture over $\wp_1$. Thus we can make a collection $\mathbf{A}$ by choosing one picture $\mathbb{A}_W$ over $\wp_1$ with the boundary label $W(a_i)$ for each element $W(y_i) \in \mathbf{w}_*$ which defines an element of $N_*$. Since $\gamma_*$ is an isomorphism, if $W(a_i N_1)$ is 1 in $H$ then also $W(b_i N_2)$ is 1 in $K$. Thus for each element $W(y_i) \in \mathbf{w}_*$ which defines an element of $N_*$, we get another picture $\mathbb{B}_W$ over $\wp_2$ unique up to equivalence (rel $\mathbf{X}$) with the boundary label $W(b_i)$. Therefore we can get another collection $\mathbf{B}$ consisting of pictures $\mathbb{B}_W$ over $\wp_2$ with the boundary label $W(b_i)$ for each element $W(y_i) \in \mathbf{w}_*$ which defines an element of $N_*$. Let

$$W = W(y_i) = {y_{i_1}}^{\varepsilon_1} {y_{i_2}}^{\varepsilon_2} \cdots {y_{i_n}}^{\varepsilon_n} \quad (y_{i_j} \in \mathbf{y}_*,\ \ \varepsilon_j = \pm 1,\ \ j = 1, 2, \ldots, n).$$

Then we can construct a spherical picture $\mathbb{P}_W$ over $\wp$ of the form depicted in Figure 4.3.1. Let $\mathbf{Y}$ be the collection of all spherical pictures $\mathbb{P}_W$ ($W$ defines an element of $N_*$ ).

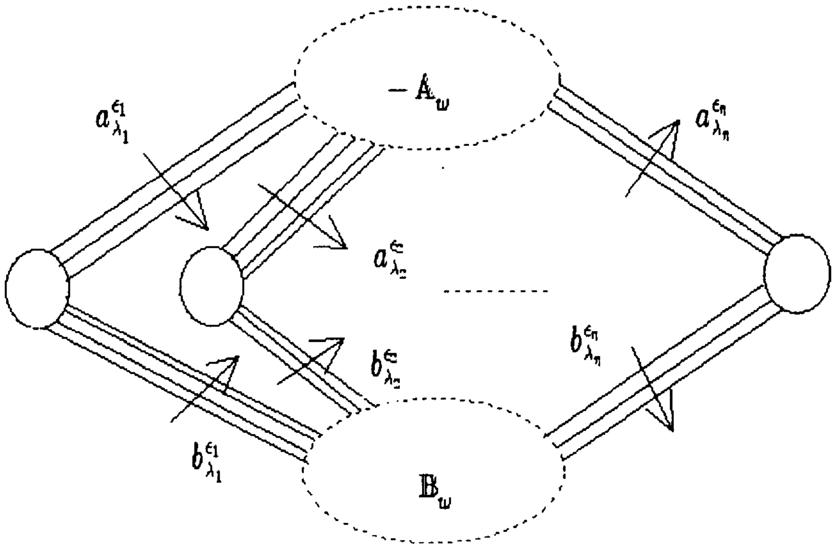


Figure 4.3.1. Spherical picture $\mathbb{P}_W$ .

**Theorem 4.3.1.** *[1] (Theorem 3.2.4)* $\mathbf{X} \cup \mathbf{Y}$ *generates* $\pi_2(\wp)$ .

**Example 4.3.1.** Let $\wp_1 =< a, b : [a, b], a^6, b^4 >$ and $\wp_2 =< c, d : [c, d], c^4, d^2 >$ be the presentations for $H$ and $K$ respectively. Let $U = gp_H\{a^3N_1, b^2N_1\}$, $V = gp_K\{c^2N_2, dN_2\}$ and let $\gamma : U \longrightarrow V$ be given by $a^3N_1 \longmapsto c^2N_2,\ b^2N_1 \longmapsto dN_2$ . Then $\wp =< a, b, c, d : [a, b], a^6, b^4, [c, d], c^4, d^2, a^3c^{-2}, b^2d^{-1} >$

Let $\theta : F_* \longrightarrow U$ be given by $x \longmapsto a^3 N_1$, $y \longmapsto b^2 N_1$, where $F_*$ is the free group on $\{x, y\}$. Then $N_*$ is the normal closure of $x^2, y^2$, and $[x, y]$. So $\pi_2(\wp)$ is generated by the following pictures like Figure 4.3.2, Figure 4.3.3, and Figure 4.3.4.

$$\mathbb{P}_1 \in \mathbf{X_1}, \quad \mathbb{P}_2 \in \mathbf{X_1}, \quad \mathbb{P}_3 \in \mathbf{X_1}, \quad \mathbb{P}_4 \in \mathbf{X_1}$$

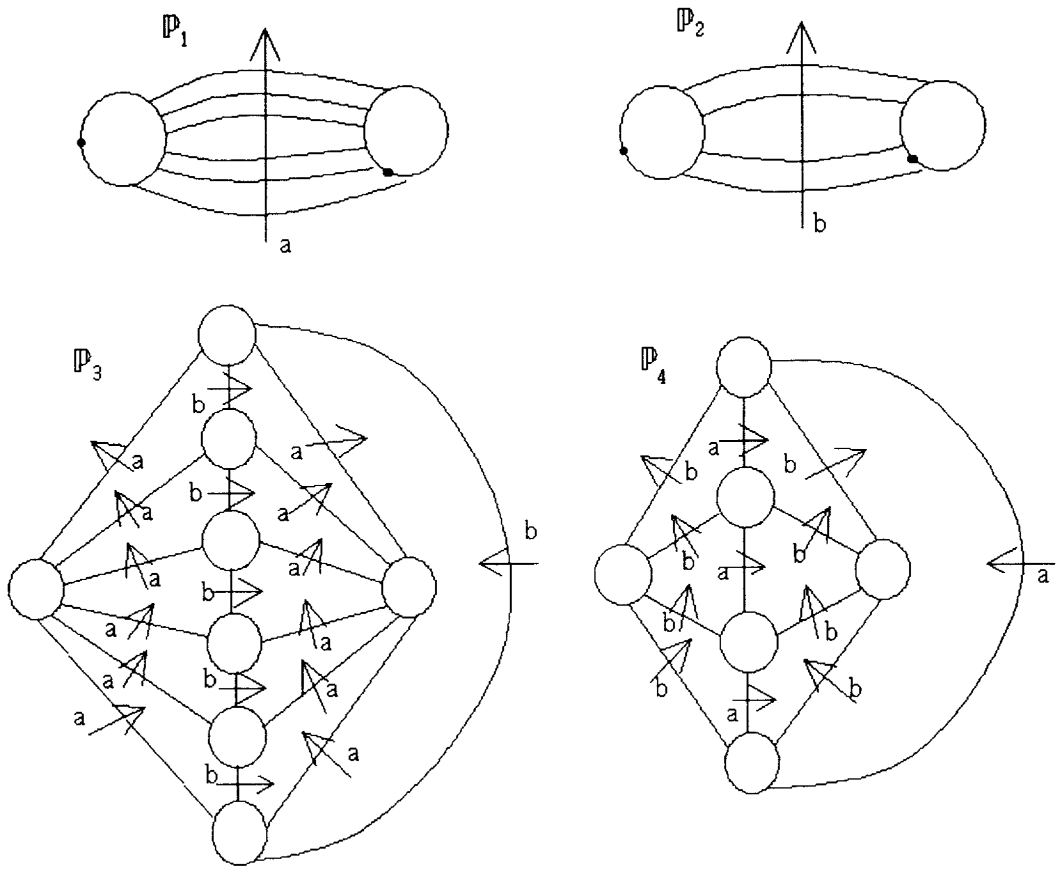


Figure 4.3.2. Generators which belong to $\mathbf{X_1}$.

$\mathbb{P}_5 \in \mathbf{X_2}, \quad \mathbb{P}_6 \in \mathbf{X_2}, \quad \mathbb{P}_7 \in \mathbf{X_2}, \quad \mathbb{P}_8 \in \mathbf{X_2}$

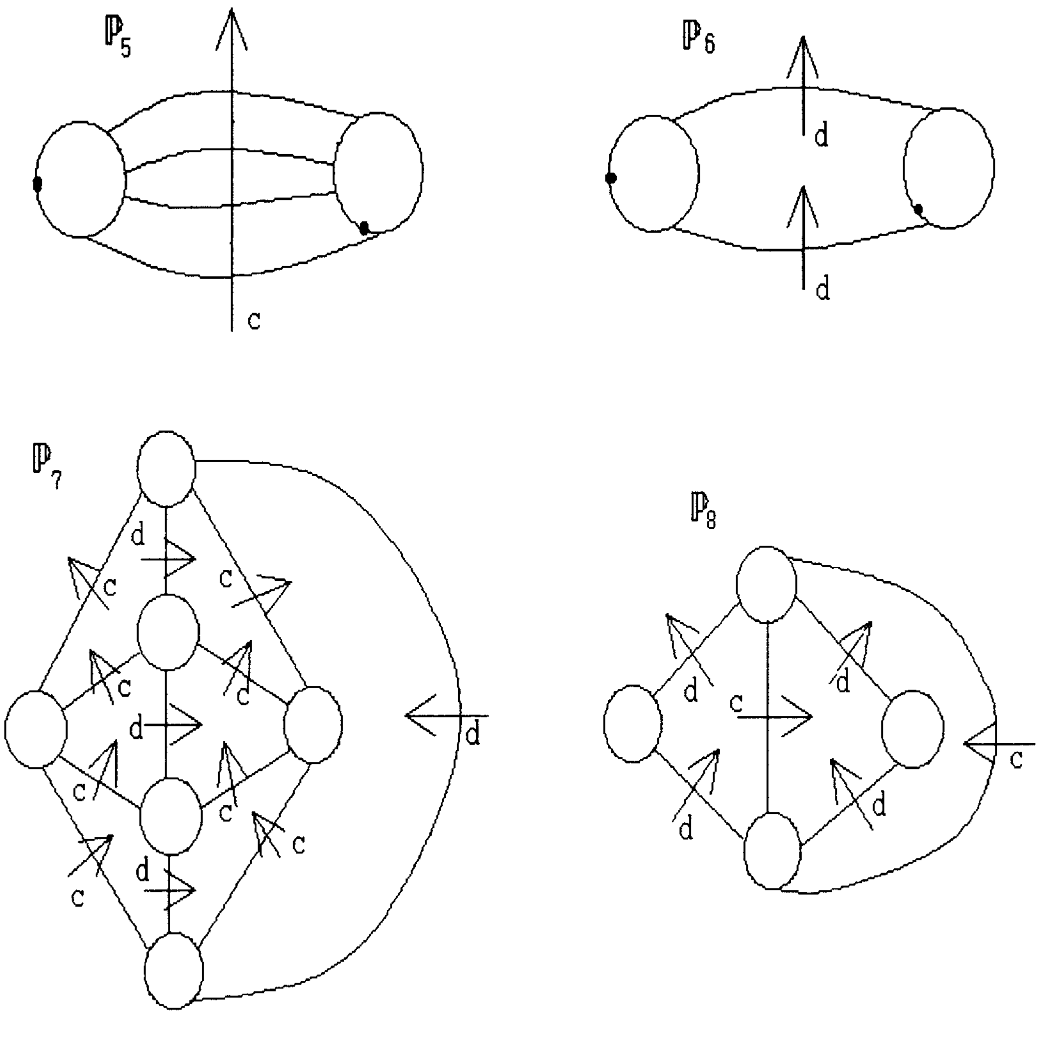


Figure 4.3.3. Generators which belong to $\mathbf{X_2}$.

$\mathbb{P}_9 \in \mathbf{Y}, \quad \mathbb{P}_{10} \in \mathbf{Y}, \quad \mathbb{P}_{11} \in \mathbf{Y}$

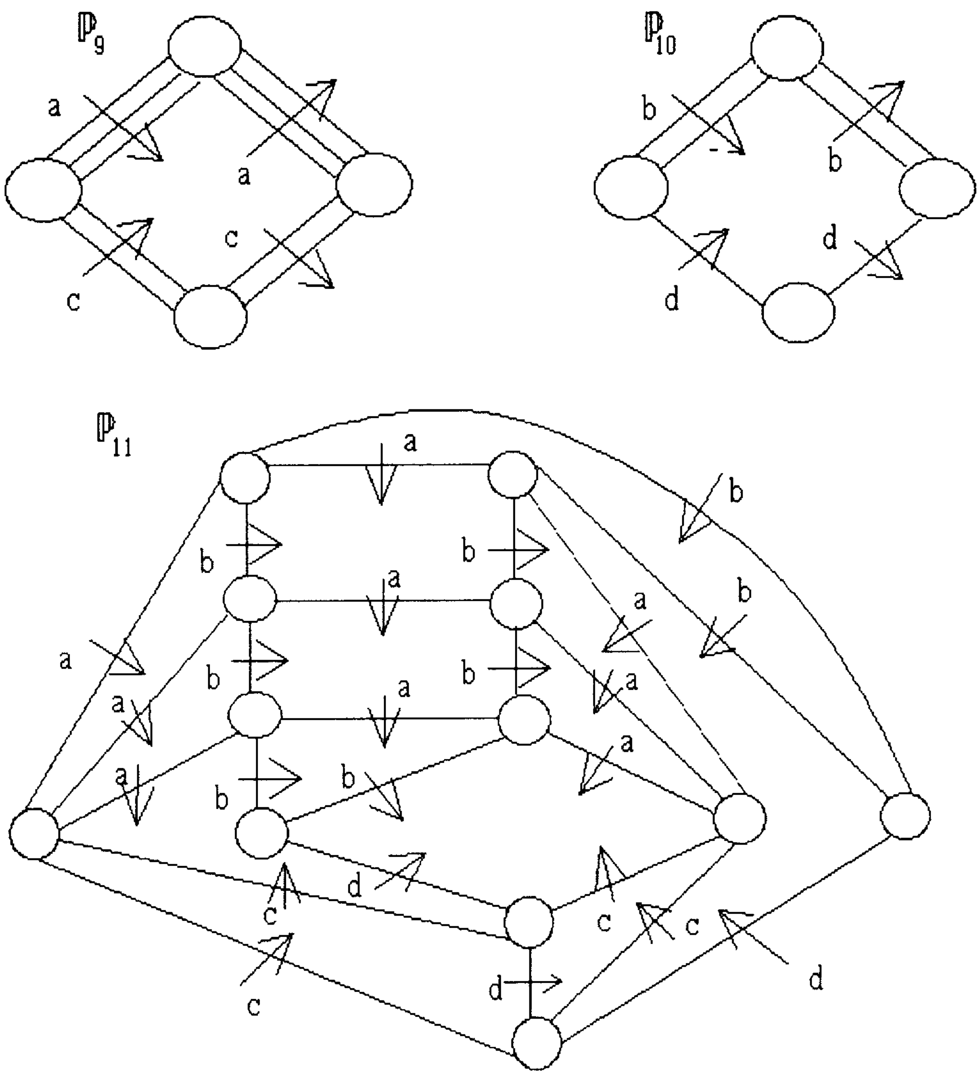


Figure 4.3.4. Generators which belong to $\mathbf{Y}$.

Let $\wp = \langle\, \mathbf{x} : \mathbf{r} \,\rangle$ and $\mathbf{X}$ a generating set for $\pi_2(\wp)$ and

$$P_3 = \oplus_{\mathbb{P}\in\mathbf{X}} \mathbb{Z}Gt_{\mathbb{P}},\ P_2 = \oplus_{R\in\mathbf{r}} \mathbb{Z}Gt_R,\ P_1 = \oplus_{x\in\mathbf{x}} \mathbb{Z}Gt_x,\ P_0 = \mathbb{Z}G.$$

Then we have the following short exact sequence (see [33]).

$$(4-1) \qquad 0 \longrightarrow \pi_2(\wp) \xrightarrow{\mu_2} P_2 \xrightarrow{\rho_2} N/N' \longrightarrow 0$$

$$\langle\mathbb{P}\rangle \longmapsto \sum_{i=1}^{n} \varepsilon_i W_i N t_{R_i} \quad (\mathbb{P}\in\mathbf{X})$$

$$t_R \longmapsto RN' \quad (R\in\mathbf{r})$$

where $\mathbb{P}$ represents

$$\sigma = (W_1 R_1^{\varepsilon_1} W_1^{-1}, \ldots, W_n R_n^{\varepsilon_n} W_n^{-1}).$$

We present short exact sequences concerned about relation modules and second homotopy modules associated with amalgamated free products.

Let $P_2^H = \oplus_{R\in\mathbf{r_1}}\mathbb{Z}H\bar{t}_R$ , $P_2^K = \oplus_{S\in\mathbf{r_2}}\mathbb{Z}K\bar{t}_S$ , $P_1^U = \oplus_{i\in I}\mathbb{Z}U\bar{t}_i$ ,

$P_2 = (\oplus_{R\in\mathbf{r_1}}\mathbb{Z}Gt_R) \oplus (\oplus_{S\in\mathbf{r_2}}\mathbb{Z}Gt_S) \oplus (\oplus_{i\in I}\mathbb{Z}Gt_i)$ ,

$M(\wp_1) = N_1/N_1{}'$ , $M(\wp_2) = N_2/N_2{}'$ , and $M(\wp) = N/N'$ and let $N_* = ker\theta_*$ where $\theta_* : F_* \longrightarrow U$ . Then we have:

**Theorem 4.3.2.** *There is a short exact sequence*

$$(4-2)\ \ 0 \to (\mathbb{Z}G\otimes_H \pi_2(\wp_1))\oplus(\mathbb{Z}G\otimes_K \pi_2(\wp_2)) \xrightarrow{\vartheta} \pi_2(\wp) \xrightarrow{\xi} \mathbb{Z}G\otimes_U (N_*/N_*{}') \to 0$$

*where* $\vartheta$ *and* $\xi$ *are defined by*

$$\vartheta : (1\otimes <\mathbb{P}_1>, 1\otimes <\mathbb{P}_2>) \longmapsto <\mathbb{P}_1+\mathbb{P}_2> \quad (\mathbb{P}_1\in\mathbf{X_1}, \mathbb{P}_2\in\mathbf{X_2})$$

$$\xi : \begin{cases} <\mathbb{P}> \longmapsto 0 \quad (\mathbb{P}\in\mathbf{X}) \\ <\mathbb{P}_W> \longmapsto 1\otimes WN_*{}' \quad (\mathbb{P}_W\in\mathbf{Y}) \end{cases}$$

**Theorem 4.3.3.** *There is a short exact sequence*

$$(4-3)\quad 0 \to (\mathbb{Z}G \otimes_H P_2^H) \oplus (\mathbb{Z}G \otimes_K P_2^K) \xrightarrow{\alpha} P_2 \xrightarrow{\beta} \mathbb{Z}G \otimes_U P_1^U \to 0$$

*where* $\alpha$ *and* $\beta$ *are defined by*

$$\alpha : \begin{cases} 1 \otimes \bar{t}_R \longmapsto t_R \\ 1 \otimes \bar{t}_S \longmapsto t_S \end{cases}$$

$$\beta : \begin{cases} t_R \longmapsto 0 \\ t_S \longmapsto 0 \\ t_i \longmapsto 1 \otimes \bar{t}_i \end{cases}$$

**Theorem 4.3.4.** *There is a short exact sequence*

$$(4-4)\quad 0 \to (\mathbb{Z}G \otimes_H \ M(\wp_1)) \oplus (\mathbb{Z}G \otimes_K \ M(\wp_2)) \xrightarrow{\lambda} M(\wp) \xrightarrow{\mu} \mathbb{Z}G \otimes_U IU \to 0$$

*where* $\lambda$ *and* $\mu$ *are defined by*

$$\lambda : (1 \otimes W_1 N_1{}', 1 \otimes W_2 N_2{}') \longmapsto W_1 N' + W_2 N' \qquad (W_1 \in N_1, W_2 \in N_2)$$

$$\mu : \begin{cases} RN' \mapsto 0 \ (R \in \mathbf{r_1}) \\ SN' \mapsto 0 \ (S \in \mathbf{r_2}) \\ TN' \mapsto 1 \otimes (a_i N_* - 1) \ (T \in \mathbf{s} = \{a_i b_i^{-1} : i \in I\}) \end{cases}$$

We now observe the relation among (4-2),(4-3), and (4-4) through the following theorem.

**Theorem 4.3.5.** *The following diagram is commutative with exact rows and* *columns.*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & (\mathbb{Z}G \otimes_H \pi_2(\wp_1)) \oplus (\mathbb{Z}G \otimes_K \pi_2(\wp_2)) & \stackrel{\vartheta}{\to} & \pi_2(\wp) & \stackrel{\xi}{\to} & \mathbb{Z}G \otimes_U N_*/N_*{}' & \to & 0 \\
 & & \downarrow^{\mu_2{}'} & & \downarrow^{\mu_2} & & \downarrow^{\mu_1{}'} & & \\
0 & \to & (\mathbb{Z}G \otimes_H P_2^H) \oplus (\mathbb{Z}G \otimes_K P_2^K) & \stackrel{\alpha}{\to} & P_2 & \stackrel{\beta}{\to} & \mathbb{Z}G \otimes_U P_1^U & \to & 0 \\
 & & \downarrow^{\rho_2{}'} & & \downarrow^{\rho_2} & & \downarrow^{\rho_1{}'} & & \\
0 & \to & (\mathbb{Z}G \otimes_H M(\wp_1)) \oplus (\mathbb{Z}G \otimes_K M(\wp_2)) & \stackrel{\lambda}{\to} & M(\wp) & \stackrel{\mu}{\to} & \mathbb{Z}G \otimes_U IU & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

*where*

$$\mu_2{}' : (1\otimes < \mathbb{P}_1 >, 1\otimes < \mathbb{P}_2 >) \mapsto (1 \otimes \sum_{R\in\mathbf{r_1}} \varepsilon_R W_R N \bar{t}_R \ , 1 \otimes \sum_{S\in\mathbf{r_2}} \varepsilon_S W_S N \bar{t}_S)$$

$$\mathbb{P}_1 \in \mathbf{X_1}, \quad \mathbb{P}_2 \in \mathbf{X_2}$$

$$\rho_2{}' : (1 \otimes \sum_{R\in\mathbf{r_1}} h_R \bar{t}_R, 1 \otimes \sum_{S\in\mathbf{r_2}} k_S \bar{t}_S) \mapsto (1 \otimes \sum_{R\in\mathbf{r_1}} h_R R N_1{}', 1 \otimes \sum_{S\in\mathbf{r_2}} k_S S N_2{}')$$

$$h_R \in H, \quad k_S \in K$$

$$\mu_1{}' : 1 \otimes W N_*{}' \longmapsto 1 \otimes \sum_{i\in I} \rho(\frac{\partial W}{\partial y_i})\bar{t}_i \quad (W \in N_*)$$

$$\rho_1{}' : 1 \otimes \sum_{i\in I} g_T \bar{t}_i \longmapsto 1 \otimes \sum_{i\in I} (a_i N_* - 1) \quad (g_T \in U).$$

*Proof.* We consider commutativity :

(1) $(1\otimes < \mathbb{P}_1 >, 1\otimes < \mathbb{P}_2 >) \mapsto < \mathbb{P}_1 + \mathbb{P}_2 > \mapsto \sum_{R\in\mathbf{r_1}} \varepsilon_R W_R N t_R + \sum_{S\in\mathbf{r_2}} \varepsilon_S W_S N t_S$

$$(1\otimes <\mathbb{P}_1>, 1\otimes <\mathbb{P}_2>) \mapsto (1\otimes \sum_{R\in \mathbf{r_1}} \varepsilon_R W_R N\bar{t}_R\ , 1\otimes \sum_{S\in \mathbf{r_2}} \varepsilon_S W_S N\bar{t}_S)$$

$$\mapsto \sum_{R\in \mathbf{r_1}} \varepsilon_R W_R N t_R + \sum_{S\in \mathbf{r_2}} \varepsilon_S W_S N t_S$$

Thus the left upper hand square is commutative.

(2) $\quad <\mathbb{P}> \longmapsto 0 \longmapsto 0$

$$<\mathbb{P}_W> \longmapsto 1\otimes W N_*{}' \longmapsto 1\otimes \sum_{i\in I} \rho(\frac{\partial W}{\partial y_i})\bar{t}_i$$

$$<\mathbb{P}> \longmapsto \sum_{R\in \mathbf{r_1}} g_R t_R + \sum_{S\in \mathbf{r_2}} g_S t_S \longmapsto 0$$

$$<\mathbb{P}_W> \mapsto \sum_{R\in \mathbf{r_1}} g_R t_R + \sum_{S\in \mathbf{r_2}} g_S t_S + \sum_{i\in I} \rho(\frac{\partial W}{\partial y_i}) t_i \mapsto 1\otimes \sum_{i\in I} \rho(\frac{\partial W}{\partial y_i})\bar{t}_i$$

Thus the right upper hand square is commutative.

(3) $\quad (1\otimes \sum_{R\in \mathbf{r_1}} h_R\bar{t}_R, 1\otimes \sum_{S\in \mathbf{r_2}} k_S\bar{t}_S) \mapsto \sum_{R\in \mathbf{r_1}} h_R t_R + \sum_{S\in \mathbf{r_2}} k_S t_S$

$$\mapsto \sum_{R\in \mathbf{r_1}} h_R R N' + \sum_{S\in \mathbf{r_2}} k_S S N'$$

$$(1\otimes \sum_{R\in \mathbf{r_1}} h_R\bar{t}_R, 1\otimes \sum_{S\in \mathbf{r_2}} k_S\bar{t}_S) \mapsto (1\otimes \sum_{R\in \mathbf{r_1}} h_R R N_1{}', 1\otimes \sum_{S\in \mathbf{r_2}} k_S S N_2{}')$$

$$\mapsto \sum_{R\in \mathbf{r_1}} h_R R N' + \sum_{S\in \mathbf{r_2}} k_S S N'$$

Thus the left lower hand square is commutative.

(4) $\quad \sum_{R\in \mathbf{r_1}} g_R t_R + \sum_{S\in \mathbf{r_2}} g_S t_S + \sum_{i\in I} g_T t_i \mapsto 1\otimes \sum_{i\in I} g_T\bar{t}_i \mapsto 1\otimes \sum_{i\in I}(a_i N_* - 1)$

$$\sum_{R\in \mathbf{r_1}} g_R t_R + \sum_{S\in \mathbf{r_2}} g_S t_S + \sum_{i\in I} g_T t_i \mapsto \sum_{R\in \mathbf{r_1}} g_R R N' + \sum_{S\in \mathbf{r_2}} g_S S N' + \sum_{i\in I} g_T T N'$$

$$\mapsto 1 \otimes \sum_{i \in I} (a_i N_* - 1)$$

Thus the right lower hand square is commutative.

Therefore we get the result by (1),(2),(3), and (4). □

As a consequence of the above theorem, we have the following corollary, which shows the evident relation that is to say, necessary and sufficient conditions between (4-2) and (4-4).

**Corollary 4.3.6.** *(4-2) is exact if and only if (4-4) is exact.*

*Proof.* The second column is given in (4-1). The first column is given from (4-1) and by tensoring $(\mathbb{Z}G \otimes_H -) \oplus (\mathbb{Z}G \otimes_K -)$. The third column is given from (3-8) and by tensoring $\mathbb{Z}G \otimes_U -$. Then by $3 \times 3$ Lemma and Theorem 4.3.3, we get the result. □

# Chapter 5

# Applications

In this chapter, it will be presented how the presentation of relation modules can be built up from a short exact sequence. We also compute the second integral (co)homology of $G$, and we investigate the efficiency of $G$ and Cockroft property.

## 5.1 Second integral (co)homology

In this section, we describe relation modules and higher (co)homology as a reason for computing generators of $\pi_2(\wp)$. We also compute the second integral (co)homology of $G$. Thus we define the $n$th *cohomology group* of $G$ with coefficients in the left $G$-module $A$ by

$$H^n(G, A) = Ext^n_G(\mathbb{Z}, A),$$

where $\mathbb{Z}$ is to be regarded as a trivial $G$-module. The $n$th *homology group* of $G$ with coefficients in the right $G$-module $B$ by

$$H_n(G, B) = Tor_n^G(B, \mathbb{Z}),$$

where again $\mathbb{Z}$ is to be regarded as a trivial $G$-module (see [23] p188).

We denote by $M(\wp)$ the relation module $N/N'$ of $\wp$. We often write $\mu_2(\mathbb{P})$ and $\sigma(\mathbb{P})$ instead of $\mu_2(< \mathbb{P} >)$ and $\sigma$ which is represented by $\mathbb{P}$ respectively. The short exact sequence (4-1) in section 4.3 of chapter 4 gives us a presentation

$$\langle\ t_R\ (R \in \mathbf{r}) : \mu_2(\mathbb{P}) = 0\ (\mathbb{P} \in \mathbf{X})\ \rangle$$

for $M(\wp)$ from $\mathbf{X}$. So we can sometimes know the structure of $M(\wp)$.

From now on, we observe through the following example that it is possible that their relation modules are different from each other between two isomorphic groups. Moreover, even though two groups are not isomorphic, their relation modules can be isomorphic from each other.

**Example 5.1.1.** (i) Let $G_1$ be the group defined by the presentation

$$\wp_1 =< x : R >, \text{ where } R = x^6$$

and $N$ is the normal closure of $R$ in $F$ free on $\{x\}$. Then $\pi_2(\wp_1)$ is generated by the following picture $\mathbb{P}$ like Figure 5.1.1. We also get through the following picture the identity sequence $\sigma(\mathbb{P})$ which is represented by $\mathbb{P}$ and calculate $\mu_2(\mathbb{P}) = 0$, so that we obtain a presentation for $M(\wp_1)$.

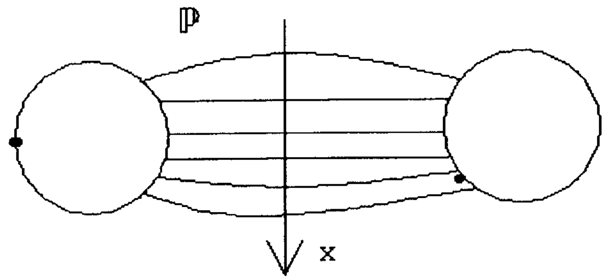


Figure 5.1.1. Generator of $\pi_2(\wp_1)$.

The above picture yields that

$$\sigma(\mathbb{P}) = (R, x^{-1}R^{-1}x)$$

$$\mu_2(\mathbb{P}) = Nt_R - x^{-1}Nt_R = 0.$$

Thus $x \in N$. Therefore $\wp_1{}' =< t_R : \ >$ is a presentation for $M(\wp_1)$.

(ii) Let $G_2$ be the group defined by the presentation

$$\wp_2 =< a, b : R_1, R_2, R_3 \ >$$

where $R_1 = a^3, R_2 = b^2$, and $R_3 = [a, b]$ and $N$ is the normal closure of $R_1, R_2$, and $R_3$ in $F$ free on $\{a, b\}$. Then $\pi_2(\wp_2)$ is generated by the following pictures like Figure 5.1.2. Through the following pictures $\mathbb{P}_i$ $(i = 1, 2, 3, 4)$, we get the identity sequence $\sigma(\mathbb{P}_i)$ which are represented by $\mathbb{P}_i$ and calculate the relation $\mu_2(\mathbb{P}_i) = 0$, so that we obtain a presentation for $M(\wp_2)$.

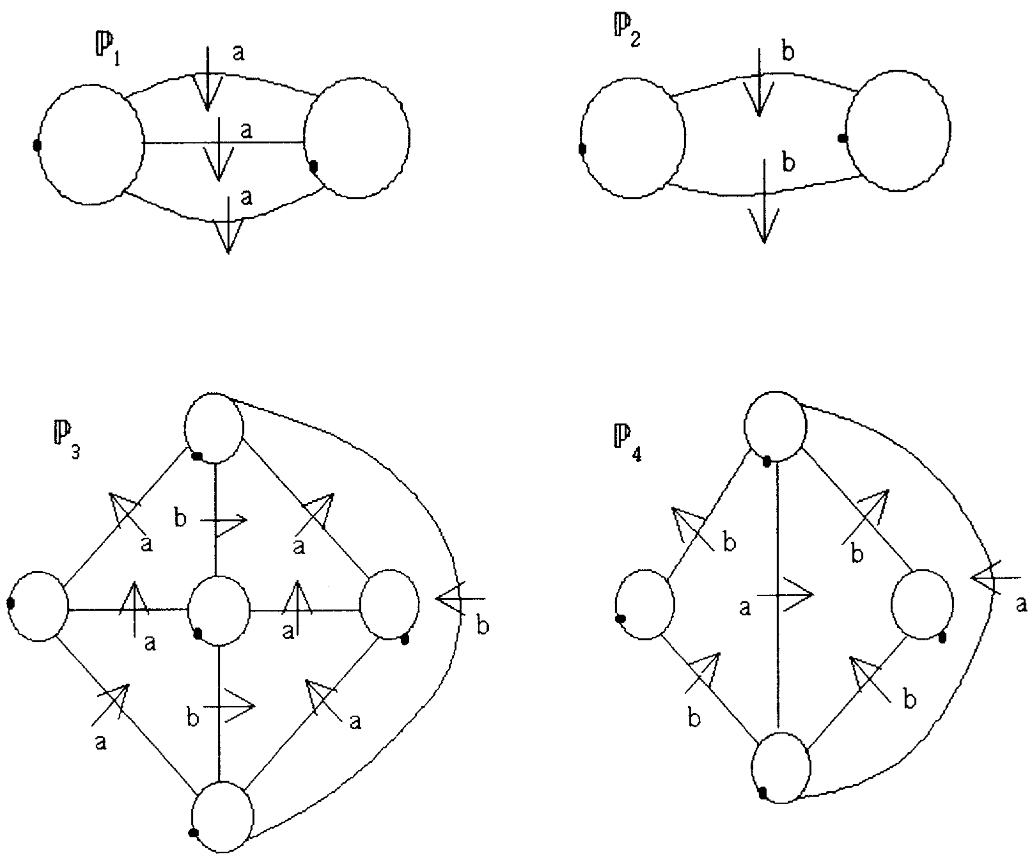


Figure 5.1.2. Generators of $\pi_2(\wp_2)$.

The spherical picture $\mathbb{P}_1$ yields that

$$\sigma(\mathbb{P}_1) = (R_1, a^{-1}R_1^{-1}a)$$

$$\mu_2(\mathbb{P}_1) = Nt_{R_1} - a^{-1}Nt_{R_1} = 0.$$

$$(1) \qquad a \in N$$

The spherical picture $\mathbb{P}_2$ yields that

$$\sigma(\mathbb{P}_2) = (R_2, b^{-1}R_2^{-1}b)$$
$$\mu_2(\mathbb{P}_2) = Nt_{R_2} - b^{-1}Nt_{R_2} = 0.$$

$$(2) \qquad b \in N$$

The spherical picture $\mathbb{P}_3$ yields that

$$\begin{aligned} \sigma(\mathbb{P}_3) &= (R_1^{-1}, a^2R_3a^{-2}, aR_3a^{-1}, R_3, bR_1b^{-1}) \\ \mu_2(\mathbb{P}_3) &= -Nt_{R_1} + a^2Nt_{R_3} + aNt_{R_3} + Nt_{R_3} + bNt_{R_1} \\ &= (-Nt_{R_1} + bNt_{R_1}) + (a^2Nt_{R_3} + aNt_{R_3} + Nt_{R_3}) = 0. \end{aligned}$$

From (2), we have $-Nt_{R_1}+bNt_{R_1} = -Nt_{R_1}+Nt_{R_1} = 0$. Thus $a^2Nt_{R_3}+aNt_{R_3}+Nt_{R_3} = 0$. From (1), we have $a^2Nt_{R_3} + aNt_{R_3} + Nt_{R_3} = Nt_{R_3} + Nt_{R_3} + Nt_{R_3} = 0$, i.e.,

$$(3) \qquad 3t_{R_3} = 0$$

The spherical picture $\mathbb{P}_4$ yields that

$$\begin{aligned} \sigma(\mathbb{P}_4) &= (R_2^{-1}, bR_3^{-1}b^{-1}, R_3^{-1}, aR_2a^{-1}) \\ \mu_2(\mathbb{P}_4) &= -Nt_{R_2} - bNt_{R_3} - Nt_{R_3} + aNt_{R_2} \\ &= (-Nt_{R_2} + aNt_{R_2}) + (-bNt_{R_3} - Nt_{R_3}) = 0. \end{aligned}$$

From (1), we have $-Nt_{R_2}+aNt_{R_2} = -Nt_{R_2}+Nt_{R_2} = 0$. Thus $bNt_{R_3}+Nt_{R_3} = 0$. From (2), we have $bNt_{R_3} + Nt_{R_3} = Nt_{R_3} + Nt_{R_3} = 0$, i.e.,

$$(4) \qquad 2t_{R_3} = 0$$

From (3) and (4), we have $t_{R_3} = 0$. Therefore $\wp_2{}' =< t_{R_1}, t_{R_2} : [t_{R_1}, t_{R_2}] >$ is a presentation for $M(\wp_2)$.

(iii) Let $G_3$ be the group defined by the presentation $\wp_3 =< c, d : S_1, S_2 >$, where $S_1 = c^3, S_2 = d^2$ and $N$ is the normal closure of $S_1, S_2$ in $F$ free on $\{c, d\}$. Then $\pi_2(\wp_3)$ is generated by the following pictures like Figure 5.1.3

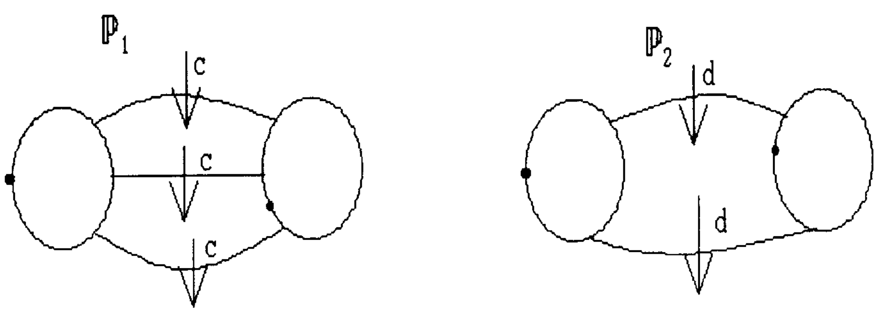


Figure 5.1.3. Generators of $\pi_2(\wp_3)$.

The spherical picture $\mathbb{P}_1$ yields that

$$\sigma(\mathbb{P}_1) = (S_1, c^{-1}S_1^{-1}c)$$

$$\mu_2(\mathbb{P}_1) = Nt_{S_1} - c^{-1}Nt_{S_1} = 0.$$

Thus $c \in N$. The spherical picture $\mathbb{P}_2$ yields that

$$\sigma(\mathbb{P}_2) = (S_2, d^{-1}S_2^{-1}d)$$

$$\mu_2(\mathbb{P}_2) = Nt_{S_2} - d^{-1}Nt_{S_2} = 0.$$

Thus $d \in N$. From the above procedures, it follows that

$$\wp_3{}' = < t_{S_1}, t_{S_2} : [t_{S_1}, t_{S_2}] >$$

is a presentation for $M(\wp_3)$. By (i), (ii), and (iii), we have

$$G_1 \cong \mathbb{Z}_6,\ G_2 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3,\ G_3 \cong \mathbb{Z}_2 * \mathbb{Z}_3,$$

$$M(\wp_1) = \mathbb{Z},\ M(\wp_2) = \mathbb{Z} \oplus \mathbb{Z},\ M(\wp_3) = \mathbb{Z} \oplus \mathbb{Z}$$

$$i.e.\ \ G_1 \cong G_2 \ \text{ but } \ M(\wp_1) \ncong M(\wp_2)$$

$$G_2 \ncong G_3 \ \text{ but } \ M(\wp_2) \cong M(\wp_3)$$

We may summarize the above procedures as follows. If two isomorphic groups have two different presentations, then it is possible that their relation modules are different from each other.

If we put the three sequences (3-1), (3-8), (4-1) together we get the exact sequence

$$(5-1) \qquad P_3 \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where

$$\partial_3 : t_{\mathbb{P}} \longmapsto \mu_2(\mathbb{P})$$

$$\partial_2 = \mu_1 \rho_2$$

$$\partial_1 = \rho_1$$

For any picture $\mathbb{P}$ over $\wp$ and for any $R \in \mathbf{r}$, the *exponent sum* of $R$ in $\mathbb{P}$, denoted by $exp_R(\mathbb{P})$ is the number of discs of $\mathbb{P}$ labelled $R$ minus the number of discs labelled $R^{-1}$.

For any word $W$ on $x$ and any $x \in \mathbf{x}$, the *exponent sum* of $x$ in $W$, denoted by $exp_x(W)$ is the number of occurences of $x$ in $W$ minus the number of occurences of $x^{-1}$.

We often write the following sequence instead of (3-8).

$$(5-2) \qquad 0 \longrightarrow M(\wp) \xrightarrow{\mu_1} P_1 \xrightarrow{\rho_1} P_0 \longrightarrow 0.$$

Then by dimension shifting we get

$$H^{n+2}(G,-) \cong Ext^n_G(M(\wp),-)$$
$$H_{n+2}(G,-) \cong Tor^G_n(-,M(\wp)), \qquad n \geq 1.$$

See [23], p189. So if we know the structure of $M(\wp)$ then we can compute the higher (co)homology groups of $G$. In particular, we can compute the $H_2(G)$ (Schur multiplier) and $H^2(G)$.

If $A$ and $B$ are any right and left $G$-modules respectively, then from (5-1) we have

$$H_2(G,A) = \frac{ker 1 \otimes \partial_2}{im 1 \otimes \partial_3}$$
$$H^2(G,B) = \frac{ker Hom_{\mathbb{Z}G}(\partial_3, 1)}{im Hom_{\mathbb{Z}G}(\partial_2, 1)}$$

In particular, taking $A = \mathbb{Z}$ and $B = \mathbb{Z}$ (with trivial $G$-action) we have

$$H_2(G) = ker\delta_2/im\delta_3$$
$$H^2(G) = ker{\delta_3}^*/im{\delta_2}^*$$

where

$$(5-3) \qquad \delta_2 : \oplus_{R\in \mathbf{r}} \mathbb{Z}t_R \longrightarrow \oplus_{x\in \mathbf{x}} \mathbb{Z}t_x \ , \qquad t_R \longmapsto \sum_{x\in \mathbf{x}} exp_x(R)t_x$$

$$(5-4) \qquad \delta_3 : \oplus_{\mathbb{P}\in \mathbf{X}} \mathbb{Z}t_{\mathbb{P}} \longrightarrow \oplus_{R\in \mathbf{r}} \mathbb{Z}t_R \ , \qquad t_{\mathbb{P}} \longmapsto \sum_{R\in \mathbf{r}} exp_R(\mathbb{P})t_R$$

$$(5-5) \qquad {\delta_2}^* : \oplus_{x\in \mathbf{x}} \mathbb{Z}{t_x}^* \longrightarrow \oplus_{R\in \mathbf{r}} \mathbb{Z}{t_R}^* \ , \qquad {t_x}^* \longmapsto \sum_{R\in \mathbf{r}} exp_x(R){t_R}^*$$

$$(5-6) \qquad {\delta_3}^* : \oplus_{R\in \mathbf{r}} \mathbb{Z}{t_R}^* \longrightarrow \oplus_{\mathbb{P}\in \mathbf{X}} \mathbb{Z}{t_{\mathbb{P}}}^* \ , \qquad {t_R}^* \longmapsto \sum_{\mathbb{P}\in \mathbf{X}} exp_R(\mathbb{P}){t_{\mathbb{P}}}^*$$

So we can compute them easily.

**Example 5.1.2.** We consider the same presentation as in Example 4.3.1.

$$\wp =< \ a, b, c, d : S_1, S_2, S_3, T_1, T_2, T_3, R_1, R_2 \ >$$

where

$S_1 = [a, b],\ S_2 = a^6,\ S_3 = b^4,\ T_1 = [c, d],\ T_2 = c^4,\ T_3 = d^2, R_1 = a^3c^{-2},\ R_2 = b^2d^{-1}.$

$\mathbb{P}_1, \ldots, \mathbb{P}_{11}$ are the same as in Example 4.3.1. Then we obtain the results as follows.

$$\delta_2 : \begin{cases} t_{S_1},\ t_{T_1} \longmapsto 0 \\ t_{S_2} \longmapsto 6t_a \\ t_{S_3} \longmapsto 4t_b \\ t_{T_2} \longmapsto 4t_c \\ t_{T_3} \longmapsto 2t_d \\ t_{R_1} \longmapsto 3t_a - 2t_c \\ t_{R_2} \longmapsto 2t_b - t_d \end{cases} \qquad {\delta_2}^* : \begin{cases} {t_a}^* \longmapsto 6{t_{S_2}}^* + 3{t_{R_1}}^* \\ {t_b}^* \longmapsto 4{t_{S_3}}^* + 2{t_{R_2}}^* \\ {t_c}^* \longmapsto 4{t_{T_2}}^* - 2{t_{R_1}}^* \\ {t_d}^* \longmapsto 2{t_{T_3}}^* - {t_{R_2}}^* \end{cases}$$

$$\delta_3 : \begin{cases} t_{\mathbb{P}_i} \longmapsto 0 \quad (i = 1, 2, 5, 6) \\ t_{\mathbb{P}_3} \longmapsto 6t_{S_1} \\ t_{\mathbb{P}_4} \longmapsto -4t_{S_1} \\ t_{\mathbb{P}_7} \longmapsto 4t_{T_1} \\ t_{\mathbb{P}_8} \longmapsto -2t_{T_1} \\ t_{\mathbb{P}_9} \longmapsto -t_{S_2} + t_{T_2} + 2t_{R_1} \\ t_{\mathbb{P}_{10}} \longmapsto -t_{S_3} + t_{T_3} + 2t_{R_2} \\ t_{\mathbb{P}_{11}} \longmapsto -6t_{S_1} + 2t_{T_1} \end{cases} \qquad \delta_3{}^* : \begin{cases} t_{S_1}{}^* \longmapsto 6t_{\mathbb{P}_3}{}^* - 4t_{\mathbb{P}_4}{}^* - 6t_{\mathbb{P}_{11}}{}^* \\ t_{S_2}{}^* \longmapsto -t_{\mathbb{P}_9}{}^* \\ t_{S_3}{}^* \longmapsto -t_{\mathbb{P}_{10}}{}^* \\ t_{T_1}{}^* \longmapsto 4t_{\mathbb{P}_7}{}^* - 2t_{\mathbb{P}_8}{}^* + 2t_{\mathbb{P}_{11}}{}^* \\ t_{T_2}{}^* \longmapsto t_{\mathbb{P}_9}{}^* \\ t_{T_3}{}^* \longmapsto t_{\mathbb{P}_{10}}{}^* \\ t_{R_1}{}^* \longmapsto 2t_{\mathbb{P}_9}{}^* \\ t_{R_2}{}^* \longmapsto 2t_{\mathbb{P}_{10}}{}^* \end{cases}$$

Suppose that

$$k_1(6t_a) + k_2(4t_b) + k_3(4t_c) + k_4(2t_d) + k_5(3t_a - 2t_c) + k_6(2t_b - t_d) = 0.$$

Then

$$\begin{cases} 2k_1 + k_5 = 0 \\ 2k_2 + k_6 = 0 \\ 2k_3 - k_5 = 0 \\ 2k_4 - k_6 = 0 \end{cases}$$

Thus we have

$$\begin{cases} k_1 = -k_3 \\ k_5 = 2k_3 \\ k_2 = -k_4 \\ k_6 = 2k_4 \end{cases}$$

Therefore $ker\delta_2$ is generated by

$$t_{S_1},\ t_{T_1},\ 2t_{R_1} + t_{T_2} - t_{S_2},\ \text{ and }\ 2t_{R_2} + t_{T_3} - t_{S_3}$$

$im\delta_3$ is generated by

$$2t_{R_1}+t_{T_2}-t_{S_2},\ 2t_{R_2}+t_{T_3}-t_{S_3},\ 6t_{S_1},\ 4t_{S_1},\ 4t_{T_1},\ 2t_{T_1}, \text{and } 2t_{T_1}-6t_{S_1}$$

So we get

$$H_2(G)\cong\ <\ x,\ y:[x,y],\ x^2,\ y^2,\ x^2y^{-6}\ >.$$

Now we calculate $H^2(G)$ . Suppose that

$k_1(6t_{\mathbb{P}_3}{}^* - 4t_{\mathbb{P}_4}{}^* - 6t_{\mathbb{P}_{11}}{}^*) + k_2(-t_{\mathbb{P}_9}{}^*) + k_3(-t_{\mathbb{P}_{10}}{}^*) + k_4(4t_{\mathbb{P}_7}{}^* - 2t_{\mathbb{P}_8}{}^* + 2t_{\mathbb{P}_{11}}{}^*)$

$+k_5t_{\mathbb{P}_9}{}^* + k_6t_{\mathbb{P}_{10}}{}^* + k_7(2t_{\mathbb{P}_9}{}^*) + k_8(2t_{\mathbb{P}_{10}}{}^*) = 0$

so we have

$$\begin{cases} k_1=k_4=0 \\ -k_2+k_5+2k_7=0 \\ -k_3+k_6+2k_8=0 \end{cases}$$

Then we have solutions ;

$$\begin{cases} (k_2,k_5,k_7)=(2,0,1)\ \text{ or }\ (1,1,0) \\ (k_3,k_6,k_8)=(2,0,1)\ \text{ or }\ (1,1,0) \end{cases}$$

Therefore $ker\delta_3{}^*$ is generated by

$$\omega_1=t_{S_2}{}^*+t_{T_2}{}^*,\ \omega_2=2t_{S_2}{}^*+t_{R_1}{}^*,\ \omega_3=t_{S_3}{}^*+t_{T_3}{}^*,\ \text{and } \omega_4=2t_{S_3}{}^*+t_{R_2}{}^*$$

$im\delta_2{}^*$ is generated by

$$3\omega_2,\ 2\omega_4,\ 2(2\omega_1-\omega_2),\ \text{and } 2\omega_3-\omega_4$$

So we get

$$H^2(G)\cong\ <\ \omega_1,\omega_2,\omega_3,\omega_4:[\omega_i,\omega_j](1\le i<j\le 4),{\omega_2}^3,{\omega_4}^2,({\omega_1}^2{\omega_2}^{-1})^2,{\omega_3}^2{\omega_4}^{-1}\ >.$$

## 5.2 Efficiency and Cockroft property

In this section, we investigate the efficiency and Cockroft property as a reason for computing generators of $\pi_2(\wp)$.

We can regard a finite presentation $\wp = < \mathbf{x} : \mathbf{r} >$ as a $2-CW$ complex with one vertex.

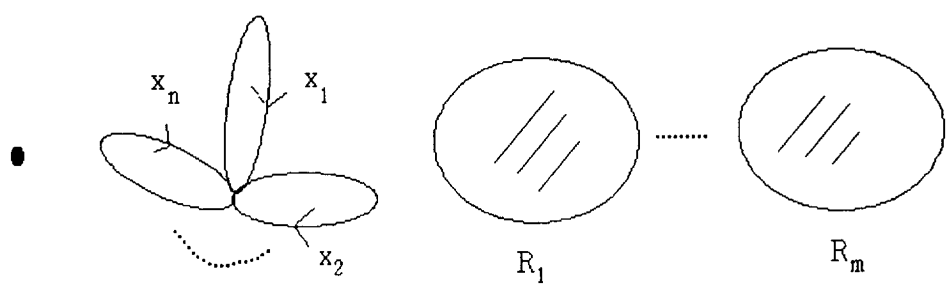


Figure 5.2.1. $2-CW$ complex with one vertex.

And the Euler characteristic $\chi(\wp)$, say

$$\chi(\wp) = 1 - |\mathbf{x}| + |\mathbf{r}|$$

is bounded below by

$$\nu(G) = 1 - rk(H_1(G)) + d(H_2(G))$$

where $G$ is the group defined by $\wp$, $rk(\ )$ means the rank of the torsion-free part and $d(\ )$ means the least number of generators. See [7].

**Definition 5.2.1.** *Consider the collection* $\mathbf{G}$ *of all finite presentations which define a group* $G$.

*(1)* $\wp_0 \in \mathbf{G}$ *is called minimal if* $\chi(\wp_0) \leq \chi(\wp)$ *for all* $\wp \in \mathbf{G}$.

*(2)* $\wp_0 \in \mathbf{G}$ *is called efficient if* $\chi(\wp_0) = \nu(G)$

*(3)* $G$ *is called efficient if there is an efficient presentation for* $G$.

*(4) A spherical picture* $\mathbb{P}$ *is called Cockroft if for all* $R \in \mathbf{r}$, $exp_R(\mathbb{P}) = 0$.

*(5)* $\wp$ *is called Cockroft if all* $\mathbb{P} \in \pi_2(\wp)$ *are Cockroft.*

*(6)* $\wp$ *is called Cockroft (mod p) where* $p > 1$ *is an integer if for each* $\mathbb{P} \in \pi_2(\wp)$ *and for all* $R \in \mathbf{r}$, $exp_R(\mathbb{P}) = 0$ $(modp)$.

**Remark 5.2.2.** *(i) Classes of efficient groups are given in [7], [13].*

*(ii) Examples of non-efficient groups were given by Swan [39], and their minimal presentations were given by Wamsley [41].*

**Example 5.2.1.** Let

$$\wp = < x, y, z, u : S_1, S_2, S_3, T_1, T_2, T_3, R_1, R_2 >$$

where $S_1 = [x, y]$, $S_2 = x^6$, $S_3 = y^9$, $T_1 = [z, u]$, $T_2 = z^9$, $T_3 = u^3$, $R_1 = x^2 z^{-3}$, and $R_2 = y^3 u^{-1}$. Then we get a set of generators for $\pi_2(\wp)$ consisting of the following eleven spherical pictures like Figure 5.2.2, Figure 5.2.3, and Figure 5.2.4. And also we compute the second integral (co)homology of $G$ and investigate the efficiency of $G$ as follows. The following pictures $\mathbb{P}_1$,

$\mathbb{P}_2$, $\mathbb{P}_3$, and $\mathbb{P}_4$ are spherical pictures consisting of $S$-discs with the property that there are no $T$-discs and $R$-discs.

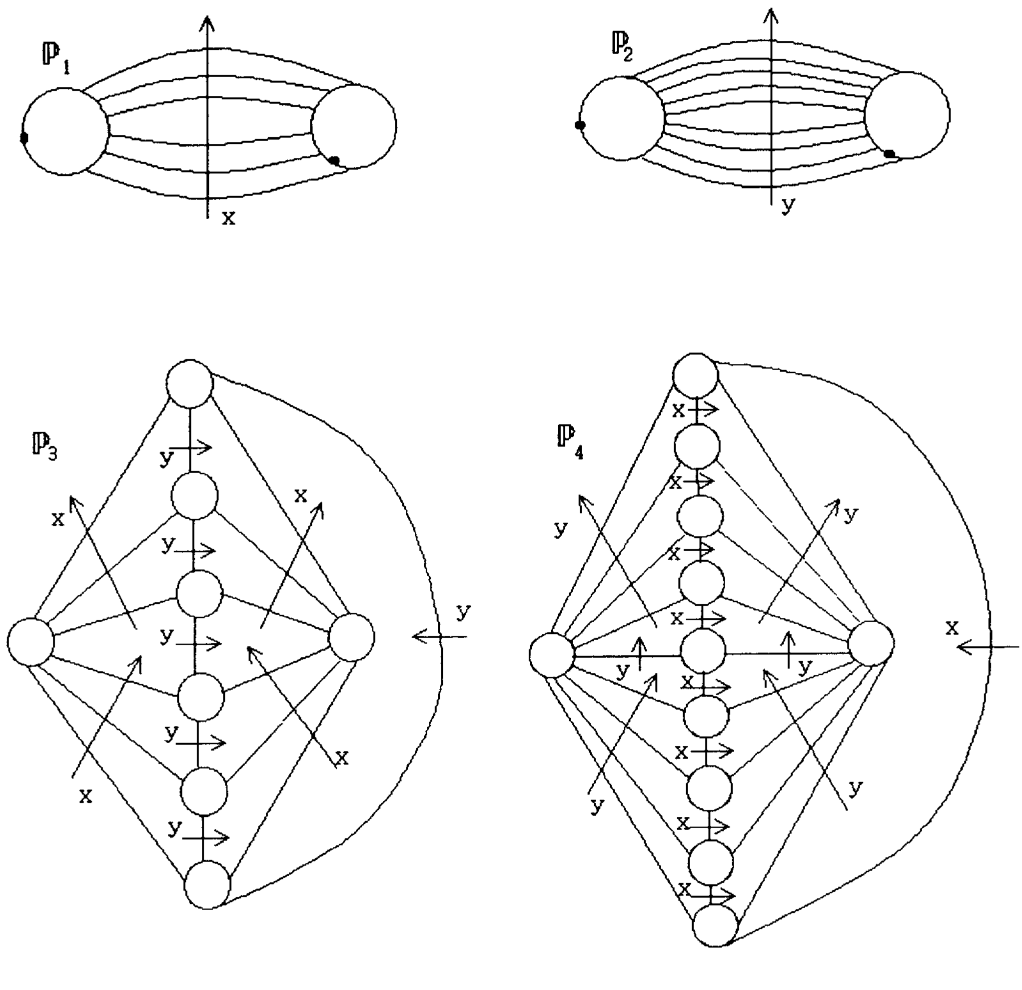


Figure 5.2.2. Generators of $\pi_2(\wp)$.

The following pictures $\mathbb{P}_5$, $\mathbb{P}_6$, $\mathbb{P}_7$, and $\mathbb{P}_8$ are spherical pictures consisting of $T$-discs with the property that there are no $S$-discs and $R$-discs.

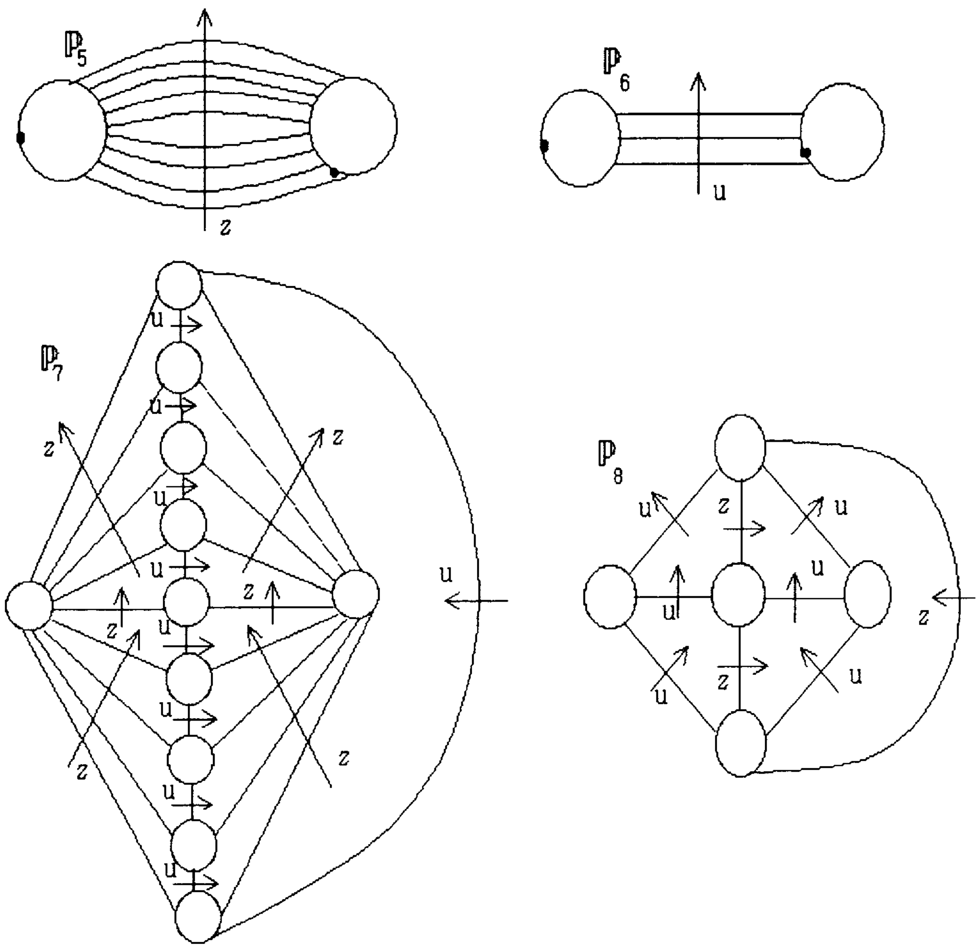


Figure 5.2.3. Generators of $\pi_2(\wp)$.

The following pictures $\mathbb{P}_9$, $\mathbb{P}_{10}$, and $\mathbb{P}_{11}$ are spherical pictures consisting of $S$-discs, $T$-discs, and $R$-discs.

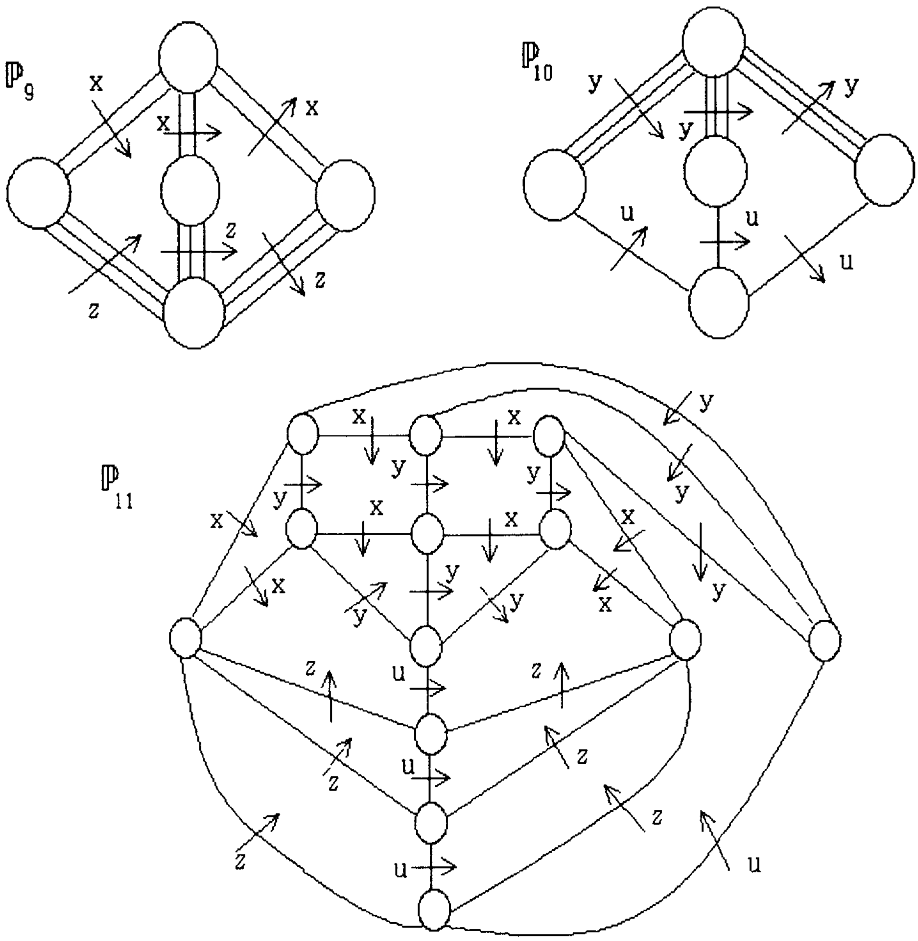


Figure 5.2.4. Generators of $\pi_2(\wp)$.

Now we compute the second integral (co)homology of $G$ and investigate the efficiency of $G$ and Cockroft property.

Since $exp_{S_2}(\mathbb{P}_1) = 0$, $exp_{S_3}(\mathbb{P}_2) = 0$, $exp_{T_2}(\mathbb{P}_5) = 0$, and $exp_{T_3}(\mathbb{P}_6) = 0$, it follows that $\mathbb{P}_1$, $\mathbb{P}_2$, $\mathbb{P}_5$, and $\mathbb{P}_6$ are Cockroft respectively.

$$\delta_2 : \begin{cases} t_{S_1},\ t_{T_1} \longmapsto 0 \\ t_{S_2} \longmapsto 6t_x \\ t_{S_3} \longmapsto 9t_y \\ t_{T_2} \longmapsto 9t_z \\ t_{T_3} \longmapsto 3t_u \\ t_{R_1} \longmapsto 2t_x - 3t_z \\ t_{R_2} \longmapsto 3t_y - t_u \end{cases} \qquad \delta_2{}^* : \begin{cases} t_x{}^* \longmapsto 6t_{S_2}{}^* + 2t_{R_1}{}^* \\ t_y{}^* \longmapsto 9t_{S_3}{}^* + 3t_{R_2}{}^* \\ t_z{}^* \longmapsto 9t_{T_2}{}^* - 3t_{R_1}{}^* \\ t_u{}^* \longmapsto 3t_{T_3}{}^* - t_{R_2}{}^* \end{cases}$$

$$\delta_3 : \begin{cases} t_{\mathbb{P}_i} \longmapsto 0 \quad (i = 1, 2, 5, 6) \\ t_{\mathbb{P}_3} \longmapsto 6t_{S_1} \\ t_{\mathbb{P}_4} \longmapsto -9t_{S_1} \\ t_{\mathbb{P}_7} \longmapsto 9t_{T_1} \\ t_{\mathbb{P}_8} \longmapsto -3t_{T_1} \\ t_{\mathbb{P}_9} \longmapsto -t_{S_2} + t_{T_2} + 3t_{R_1} \\ t_{\mathbb{P}_{10}} \longmapsto -t_{S_3} + t_{T_3} + 3t_{R_2} \\ t_{\mathbb{P}_{11}} \longmapsto -6t_{S_1} + 3t_{T_1} \end{cases} \qquad \delta_3{}^* : \begin{cases} t_{S_1}{}^* \longmapsto 6t_{\mathbb{P}_3}{}^* - 9t_{\mathbb{P}_4}{}^* - 6t_{\mathbb{P}_{11}}{}^* \\ t_{S_2}{}^* \longmapsto -t_{\mathbb{P}_9}{}^* \\ t_{S_3}{}^* \longmapsto -t_{\mathbb{P}_{10}}{}^* \\ t_{T_1}{}^* \longmapsto 9t_{\mathbb{P}_7}{}^* - 3t_{\mathbb{P}_8}{}^* + 3t_{\mathbb{P}_{11}}{}^* \\ t_{T_2}{}^* \longmapsto t_{\mathbb{P}_9}{}^* \\ t_{T_3}{}^* \longmapsto t_{\mathbb{P}_{10}}{}^* \\ t_{R_1}{}^* \longmapsto 3t_{\mathbb{P}_9}{}^* \\ t_{R_2}{}^* \longmapsto 3t_{\mathbb{P}_{10}}{}^* \end{cases}$$

Suppose that

$$k_1(6t_x) + k_2(9t_y) + k_3(9t_z) + k_4(3t_u) + k_5(2t_x - 3t_z) + k_6(3t_y - t_u) = 0.$$

Then

$$\begin{cases} 3k_1 + k_5 = 0 \\ 3k_2 + k_6 = 0 \\ 3k_3 - k_5 = 0 \\ 3k_4 - k_6 = 0 \end{cases}$$

Thus we have

$$\begin{cases} k_1 = -k_3 \\ k_5 = 3k_3 \\ k_2 = -k_4 \\ k_6 = 3k_4 \end{cases}$$

Therefore $ker\delta_2$ is generated by

$$t_{S_1},\ t_{T_1},\ 3t_{R_1} + t_{T_2} - t_{S_2},\ \text{ and }\ 3t_{R_2} + t_{T_3} - t_{S_3}\ .$$

$im\delta_3$ is generated by

$$3t_{R_1} + t_{T_2} - t_{S_2},\ 3t_{R_2} + t_{T_3} - t_{S_3},\ 6t_{S_1},\ 9t_{S_1},\ 9t_{T_1},\ 3t_{T_1},\ 3t_{T_1} - 6t_{S_1}\ .$$

So we get

$$H_2(G) \cong\ <\ a,\ b : [a,b],\ a^3,\ b^3,\ a^3b^{-6}\ >.$$

Now we calculate $H^2(G)$. Suppose that

$k_1(6t_{\mathbb{P}_3}{}^* - 9t_{\mathbb{P}_4}{}^* - 6t_{\mathbb{P}_{11}}{}^*) + k_2(-t_{\mathbb{P}_9}{}^*) + k_3(-t_{\mathbb{P}_{10}}{}^*) + k_4(9t_{\mathbb{P}_7}{}^* - 3t_{\mathbb{P}_8}{}^* + 3t_{\mathbb{P}_{11}}{}^*)$

$+k_5t_{\mathbb{P}_9}{}^* + k_6t_{\mathbb{P}_{10}}{}^* + k_7(3t_{\mathbb{P}_9}{}^*) + k_8(3t_{\mathbb{P}_{10}}{}^*) = 0$

so we have

$$\begin{cases} k_1 = k_4 = 0 \\ k_2 - k_5 - 3k_7 = 0 \\ k_3 - k_6 - 3k_8 = 0 \end{cases}$$

Then we have solutions ;

$$\begin{cases} (k_2, k_5, k_7) = (3, 0, 1) \quad \text{or} \quad (1, 1, 0) \\ (k_3, k_6, k_8) = (3, 0, 1) \quad \text{or} \quad (1, 1, 0) \end{cases}$$

Therefore $ker\delta_3{}^*$ is generated by

$$\begin{cases} a_1 = t_{S_2}{}^* + t_{T_2}{}^*, \\ a_2 = 3t_{S_2}{}^* + t_{R_1}{}^*, \\ a_3 = t_{S_3}{}^* + t_{T_3}{}^*, \\ a_4 = 3t_{S_3}{}^* + t_{R_2}{}^* \ . \end{cases}$$

$im\delta_2{}^*$ is generated by

$$2a_2,\ 3a_4,\ 3(3a_1 - a_2),\ \text{and}\ 3a_3 - a_4\ .$$

So we get

$$H^2(G) \cong \ < a_1, a_2, a_3, a_4 : [a_i, a_j](1 \le i < j \le 4), a_2{}^2, a_4{}^3, (a_1{}^3 a_2{}^{-1})^3, a_3{}^3 a_4{}^{-1} > .$$

Thus we have

$$\begin{cases} rk(H_1(G)) = 0 \\ d(H_2(G)) = 2 \end{cases}$$

So $\nu(G) = 1 - 0 + 2 = 3$ . Next we consider

$$\wp\,' = < x, y, z, u : [x,y], [z,u], x^6, y^9, x^2 z^{-3}, y^3 u^{-1} > .$$

Then $\chi(\wp\,') = 1 - 4 + 6 = 3$ . Since $\chi(\wp\,') = \nu(G)$ , it follows that $\wp\,'$ is an efficient presentation for $G$ .

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