Space-like complex submanifolds in a semi-Kaehler manifold

반Kaehler다양체에서의 공간적 복소부분다양체

Advisor : Yong-Soo Pyo

by

Kyoung-Hwa Shin

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주 심 이학박사 이 일 용

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A Dissertation

by

Kyoung-Hwa Shin

Approved as to style and content by :

Chairman Il Yong Lee, Ph. D.

Member Joon-Sik Park, Ph. D.

Member Hyun-Jong Song, Ph. D.

Member Jin Han Park, Ph. D.

Member Yong-Soo Pyo, Ph. D.

June 28, 2002

Contents

Kaehler 다양체에서의 공간적 복소부분다양체

신 경 화

부경대학교 대학원 응용수학과

요 약

본 논문에서는 semi-Kaehler 다양체의 공간적 복소부분다양체를 중심으로 연구하여, 다음 정리들을 증명하였다.

정리 1. M이 상수정칙단면곡률 c와 지표 $2p(>0)$를 갖는 $(n+p)$차원 부정 복소쌍곡공간 $CH_p^{n+p}(c)$의 $n(\geq 3)$차원 완비공간적 복소부분다양체일 때, M이 전측지적 부분다양체가 아니면서 $p\leq n(n+1)/2$을 만족하면 M의 제 2기본형 α의 노름제곱 h_2는 $h_2\geq cnp(n+2)/2(n+2p)$를 만족한다. 단, 등호는 M이 복소사영공간 $CP^n(c/2)$, α가 평행이면서 $p=n(n+1)/2$일 때 성립한다.

정리 2. M이 지표가 4인 $(n+2)$차원 부정국소대칭 Kaehler 다양체 M'의 n차원 완비공간적 복소부분다양체이고, M의 정규연결은 고유하다고 가정하자. 만약, M'이 음이 아닌 공간적 정칙양측단면곡률과 양이 아닌 시간적 정칙양측단면곡률을 가지면, M은 전측지적 부분다양체이다.

정리 3. Kaehler 다양체 M의 모든 전실양측단면곡률이 음(양)의 상수에 의해 위(아래)로 유계되어 있으면서 Ryan조건을 만족하면, M은 Einstein이다.

정리 4. M이 상수정칙단면곡률 c와 지표 $2p(>0)$를 갖는 $(n+p)$차원 부정복소쌍곡공간 $CH_p^{n+p}(c)$의 $n(\geq 2)$차원 완비공간적 복소부분다양체라 하자. 만약, M이 Ryan조건을 만족하고, 여차원 p가 $n-1$ 보다 작으면, M은 Einstein이다.

정리 5. M은 상수정칙단면곡률 $c(>0)$를 갖는 $(n+p)$차원 Kaehler 다양체 M'의 $n(\geq 3)$차원 완비 Kaehler 부분다양체라 하자. $a(M)>d$이면, 항상 M이 전측지적 부분다양체가 되는 n과 c에 의존하는 상수 d가 존재한다. 단, $a(M)$은 M의 모든 전실양측단면곡률로 이루어진 집합의 하한이다.

Chapter 1

Introduction

The theory of indefinite complex submanifolds of an indefinite complex space form is one of interesting topics in differential geometry and it is investigated by many geometers from the various different points of view, see [1]-[4], [7], [11], [12], [18], [27] and [28] for examples. Romero [26] gave a nice survey in this direction. Their method in [1] and [3] seems to be interesting because they apply the Liouville-type inequality

$$\Delta f \geqq kf$$

for a non-negative function f, where k is a positive constant.

Let M be an n-dimensional submanifold of an $(n+p)$-dimensional complex space form $M^{n+p}(c)$ of constant holomorphic sectional curvature c. Let α be the second fundamental form on M of $TM \times TM$ into NM defined by

$$\alpha(X,Y) = \nabla'_X Y - \nabla_X Y$$

for any tangent vectors X and Y on M, where ∇' and ∇ are the Riemannian connections on M' and M, respectively. In particular, the submanifold M is said to be *totally geodesic* if $\alpha = 0$. Chern pointed out that it is interesting to study the distribution of the values of the squared norm h_2 of the second fundamental form α of M.

We introduce some definitions and basic formulas on semi-Kaehler manifolds in chapter 2. And in chapter 3, some basic formulas for the theory of semi-definite complex submanifolds of a semi-Kaehler manifold are prepared.

The purpose of chapter 4 is to investigate the Chern-type problem in the space-like Kaehler geometry. The Chern-type problem in the space-like Kaehler geometry can be written as follows :

Problem 1.1. *Let M be an n-dimensional complete space-like complex submanifold of an $(n+p)$-dimensional indefinite complex hyperbolic space $CH_p^{n+p}(c)$ of constant holomorphic sectional curvature c and of index $2p(>0)$. Then does there exists a constant d in such a way that if it satisfies $h_2 > d$, then M is totally geodesic ?*

In chapter 4, we prove the following

Theorem 4.2. *Let M be an $n(\geqq 3)$-dimensional complete space-like complex submanifold of an $(n+p)$-dimensional indefinite complex hyperbolic space $CH_p^{n+p}(c)$ of constant holomorphic sectional curvature c and of index $2p(>0)$. If M is not totally geodesic and $p \leqq \frac{n(n+1)}{2}$, then the squared norm h_2 of the second fundamental form α of M satisfies*

$$h_2 \geqq \frac{cnp(n+2)}{2(n+2p)},$$

where the equality holds if and only if M is a complex projective space $CP^n(\frac{c}{2})$, α is parallel and $p = \frac{n(n+1)}{2}$.

Let M be an n-dimensional space-like complex hypersurface of an $(n+1)$-dimensional indefinite Kaehler manifold M' of index 2. We denote by $H'(P', Q')$ the holomorphic bisectional curvature of M' for any holomorphic planes P' and Q'. In particular, the holomorphic bisectional curvature $H'(P', Q')$ for any two space-like holomorphic planes P' and Q' is said to be *space-like*, and that for any space-like holomorphic plane P' and any time-like holomorphic plane Q' is said to be *time-like*. We call it simply a *space-like* or *time-like holomorphic bisectional curvature*. Then the Kwon and Nakagawa proved the following theorem.

Theorem 1.2([16]). *Let M be an $n(\geqq 2)$-dimensional complete space-like complex hypersurface of an $(n+1)$-dimensional indefinite Kaehler manifold M' of index 2. If the ambient space is locally symmetric and if it has non-negative space-like holomorphic bisectional curvatures and non-positive time-like holomorphic bisectional curvatures, then M is totally geodesic.*

Let M' be an $(n+p)$-dimensional indefinite Kaehler manifold of constant holomorphic sectional curvature c and of index $2p$, and let M be an n-dimensional space-like complex submanifold of M'.

In Chapter 5, we introduce the concept of the normal curvature of M and the normal curvature operator of M. The time-like totally real bisectional curvature is closely related to the normal curvature of M. The purpose of chapter 5 is to prove the following theorem. In order to fulfill this theorem, we generalize Theorem A in the case where M is a space-like complex submanifold and then, by applying this result, research the Chen-type problem from the view point of the holomorphic bisectional curvatures.

Theorem 5.9. *Let M be an n-dimensional complete space-like complex submanifold of an $(n+2)$-dimensional indefinite locally symmetric Kaehler manifold M' of index 4. Assume that the normal connection of M is proper. If M' has non-negative space-like holomorphic bisectional curvatures and non-positive time-like holomorphic bisectional curvatures, then M is totally geodesic.*

For the curvatures of a Kaehler manifold M, we can consider two kinds of sectional curvature which are related to almost complex structure J and different from usual sectional curvatures, holomorphic sectional curvatures and totally real bisectional curvatures. The pinching problem for these three kinds, the sectional curvature, the holomorphic sectional curvature and the totally real bisectional curvature, is an interesting topic in Kaehler geometry.

As well known, Ryan [30] investigated complex hypersurfaces in a complex space form satisfying the condition

$$R(X,Y)S = 0$$

for any vector fields X and Y tangent to the hypersurface M, where R denote the Riemannian curvature tensor, S is the Ricci tensor on M and $R(X,Y)$ operates on the tensor algebra as a derivation. We call the equation $R(X,Y)S = 0$ the *Ryan condition.* Relative to the Ryan condition, Ryan [30] proved that these hypersurfaces are Einstein manifolds if the holomorphic sectional curvature of the ambient space does not vanish, which was generalized from two distinct directions. One of them is due to Takahashi [31], who verified that such hypersurfaces become cylindrical if the ambient space is complex Euclidean. Another extension is treated by Kon [15] in the case of complex submanifolds in a complex space form of constant negative holomorphic sectional curvature. On the other hand, independently of Kon's work, Aiyama, Kwon and Nakagawa [2] researched about properties on space-like complex submanifolds satisfying the Ryan condition in an indefinite complex space form. In the case of complex submanifolds in a complex space form of constant positive holomorphic sectional curvature, these submanifolds were determined by Nakagawa and Takagi [20].

On the other hand, Ki and Suh [13] observed the Ryan condition from the different point of view and obtained a nice theorem about Kaehler manifolds whose totally real bisectional curvature is bounded from below by a positive constant. Thus it seems for us to be interesting to investigate the space-like Kaehler submanifolds satisfying the Ryan condition of an indefinite complex space form.

Chapter 6 has two purposes, one of which is to give a more generalized property than the theorems by Kon [15], and Aiyama and et. al. (for examples [1], [2] and [3]) about the condition $RS = 0$ in terms of totally real bisectional curvatures and another is to prove the theorem related the Nakagawa and Takagi theorem([20]). Namely, the purpose is to prove the following two theorems.

Theorem 6.3. *Let M be a Kaehler manifold whose totally real bisectional curvature is bounded from above* (resp. *below*) by a negative (resp. *positive*) constant. If it satisfies the Ryan condition, then M is Einstein.

Theorem 6.8. *Let M be an $n(\geqq 2)$-dimensional complete space-like complex submanifold in an $(n+p)$-dimensional indefinite complex hyperbolic space $CH_p^{n+p}(c)$ of index $2p$ and of constant holomorphic sectional curvature c. If M satisfies the Ryan condition and if the codimension p is less than $n-1$, then M is Einstein.*

For a complex submanifold $M = M^n$ of a complex space form $M' = M^{n+p}(c)$, the set $B(M)$ of totally real bisectional curvatures satisfies $B(M) \leqq \frac{c}{2}$ by the Gauss equation. It is easily seen that a totally geodesic complex submanifold $M = M^n(c)$ of $M' = M^{n+p}(c)$ satisfies $B(M) = \frac{c}{2}$ again by the Gauss equation. On the other hand, a complex quadric $M = Q^n$ of $M' = M^{n+p}(c)$, $c > 0$, satisfies $0 \leqq B(M) \leqq \frac{c}{2}$ by Kobayashi and Nomizu [14]. By paying attention to this fact, the following theorem was proved by Ki and Suh for totally real bisectional curvatures.

Theorem 1.3([13]). *Let $M = M^n$ be an $n(\geqq 3)$-dimensional complete Kaehler submanifold of an $(n+p)$-dimensional Kaehler manifold $M' = M^{n+p}(c)$ of constant holomorphic sectional curvature $c(> 0)$. If $a(M) > a_1$, then M is totally geodesic, where*

$$a_1 = \frac{c(n^3 + 2n^2 + 2n - 2)}{2n(n^2 + 2n + 3)}$$

and $a(M)$ is the infimum of the set $B(M)$.

The purpose of chapter 7 is to prove the following theorem for an improvement on the above estimation.

Theorem 7.4. *Let $M = M^n$ be an $n(\geqq 3)$-dimensional complete Kaehler submanifold of an $(n+p)$-dimensional Kaehler manifold $M' = M^{n+p}(c)$ of constant holomorphic sectional curvature $c(> 0)$. Then there exists a constant a_2 depending only upon n and c so that if $a(M) > a_2$, then M is totally geodesic, where $a_2 < a_1$.*

Chapter 2

Semi-Kaehler manifolds

We begin with recalling some definitions and basic formulas on semi-Kaehler manifolds. Let M be a complex n-dimensional semi-Hermitian manifold with almost complex structure J and semi-Hermitian metric g. Let $\{z^1, \cdots, z^n\}$, $z^j = x^j + iy^j$, $x^j, y^j \in \mathbb{R}$, be a complex local coordinate system on a neighborhood U in M. We set

$$X_j = \frac{\partial}{\partial x^j}, \qquad Y_j = \frac{\partial}{\partial y^j},$$

$$JX_j = Y_j, \qquad JY_j = -X_j,$$

$$Z_j = \frac{\partial}{\partial z^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right),$$

$$\overline{Z}_j = \frac{\partial}{\partial \overline{z}^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right).$$

Given a semi-Hermitian metric g on M, we extend the Hermitian scalar product in each tangent space T_pM at any point p on M to a unique complex symmetric bilinear form in the complexification $T_p{}^cM$ of T_pM, which is denoted by the same symbol g and we set

$$g_{AB} = g(Z_A, Z_B),$$

where the small Latin indices $j, k, \cdots$ run from 1 to n, while the Capital Latin indices $A, B, \cdots$ run through the range $1, \cdots, n, \bar{1}, \cdots, \bar{n}$.

Lemma 2.1. *Let M be a semi-Hermitian manifold with semi-Hermitian metric g. Then we have*

$$g_{jk} = g_{\bar{j}\bar{k}} = 0.$$

Proof. Since g is semi-Hermitian, we have

$$\begin{aligned} g_{jk} &= g(Z_j, Z_k) = g(JZ_j, JZ_k) = g(iZ_j, iZ_k) \\ &= -g(Z_j, Z_k) = -g_{jk}. \end{aligned}$$

This gives us that the first formula follows.

It is similarly seen that the second one is derived.

Moreover we have

$$\bar{g}_{j\bar{k}} = \overline{g(Z_j, \overline{Z}_k)} = g(\overline{Z}_j, Z_k) = g_{k\bar{j}},$$

which means that $(g_{j\bar{k}})$ is the $n \times n$ Hermitian matrix. It is then customary to write the line element ds^2 as

$$ds^2 = 2\sum_{j,k} g_{j\bar{k}} dz^j \otimes d\bar{z}^k$$

for the semi-Hermitian metric g on M.

The *fundamental form* Φ of a semi-Hermitian manifold with semi-Hermitian metric g and almost complex structure J is defined by

$$\Phi(X,Y) = g(X, JY)$$

for any vector fields X and Y on M.

A *semi-Kaehler manifold* is a semi-Hermitian manifold with semi-Hermitian metric g and almost complex structure J, the fundamental 2-form Φ with which is closed.

Let M be an $n(\geqq 2)$-dimensional connected semi-Kaehler manifold equipped with a semi-Kaehler metric tensor g and almost complex structure J. For the semi-Kaehler structure $\{g, J\}$, it follows that J is integrable and the index of g is even, say $2s(0 \leqq s \leqq n)$. In the case where the index $2s$ is contained in the range $0 < s < n$, the structure $\{g, J\}$ is said to be *indefinite Kaehler structure* and, in particular, in the case where $s = 0$ or n, it is said to be *Kaehler structure.*

In this chapter, we shall consider M an $n(\geqq 2)$-dimensional connected semi-Kaehler manifold of index $2s$, $0 \leqq s \leqq n$. Then a local unitary frame field $\{U_j\} = \{U_1, \cdots, U_n\}$ on a neighborhood of M can be chosen. This is a complex linear frame which is orthonormal with respect to the semi-definite Kaehler metric g of M, that is, $g(U_j, U_k) = \epsilon_j \delta_{jk}$, where

$$\epsilon_j = -1 \text{ or } 1 \quad \text{according as} \quad 1 \leqq j \leqq s \text{ or } s+1 \leqq j \leqq n.$$

Its dual frame field $\{\omega_j\} = \{\omega_1, \cdots, \omega_n\}$ consists of complex valued 1-forms of $(1,0)$ on M such that $\omega_j(U_k) = \epsilon_j \delta_{jk}$ and $\omega_1, \cdots, \omega_n, \bar{\omega}_1, \cdots, \bar{\omega}_n$ are linearly independent. Thus the natural extension g^c of the semi-Kaehler metric g of M can be expressed as $g^c = 2\sum_j \epsilon_j \omega_j \otimes \bar{\omega}_j$. Associated with the frame field $\{U_j\}$, there exist complex valued forms $\omega_{ik,}$ where the indices i and k run over the range $1, \cdots, n$. They are usually called *connection forms* on M such that they satisfy the structure equations of M :

$$d\omega_i + \sum_j \epsilon_j \omega_{ij} \wedge \omega_j = 0, \quad \omega_{ij} + \bar{\omega}_{ji} = 0, \tag{2.1}$$

$$d\omega_{ij} + \sum_k \epsilon_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \tag{2.2}$$

$$\Omega_{ij} = \sum_{k,l} \epsilon_k \epsilon_l K_{\bar{i}jk\bar{l}} \omega_k \wedge \bar{\omega}_l, \tag{2.3}$$

where $\Omega = (\Omega_{ij})$ (resp. $K_{\bar{i}jk\bar{l}}$) denotes the curvature form (resp. the components of the semi-definite Riemannian curvature tensor R) of M. The second formula of (2.1) means the skew-Hermitian symmetricity of Ω_{ij}, which is equivalent to the symmetric condition

$$K_{\bar{i}jk\bar{l}} = \bar{K}_{\bar{j}il\bar{k}}. \tag{2.4}$$

Moreover the first Bianchi identity implies the further symmetric relations

$$K_{\bar{i}jk\bar{l}} = K_{\bar{i}kj\bar{l}} = K_{\bar{l}kj\bar{i}} = K_{\bar{l}jk\bar{i}}. \tag{2.5}$$

Next, relative to the frame field chosen above, the Ricci tensor S of M can be expressed as follows :

$$S = \sum_{i,j} \epsilon_i \epsilon_j (S_{i\bar{j}} \omega_i \otimes \bar{\omega}_j + S_{\bar{i}j} \bar{\omega}_i \otimes \omega_j), \tag{2.6}$$

where $S_{i\bar{j}} = \sum_k \epsilon_k K_{\bar{k}ki\bar{j}} = S_{\bar{j}i} = \bar{S}_{\bar{i}j}$. The scalar curvature r of M is also given by

$$r = 2 \sum_j \epsilon_j S_{j\bar{j}}. \tag{2.7}$$

The semi-Kaehler manifold M is said to be *Einstein* if the Ricci tensor S is given by

$$S_{i\bar{j}} = \frac{r}{2n}\epsilon_i \delta_{ij}. \tag{2.8}$$

Now, the components $K_{\bar{i}jk\bar{l}m}$ and $K_{\bar{i}jk\bar{l}\bar{m}}$ (resp. $S_{i\bar{j}k}$ and $S_{i\bar{j}\bar{k}}$) of the covariant derivative of the Riemannian curvature tensor R (resp. the Ricci tensor S) are obtained by

$$\begin{aligned} \sum_m \epsilon_m (K_{\bar{i}jk\bar{l}m}\omega_m + K_{\bar{i}jk\bar{l}\bar{m}}\bar{\omega}_m) &= dK_{\bar{i}jk\bar{l}} \\ &- \sum_m \epsilon_m (K_{\bar{m}jk\bar{l}}\bar{\omega}_{mi} + K_{\bar{i}mk\bar{l}}\omega_{mj} + K_{\bar{i}jm\bar{l}}\omega_{mk} + K_{\bar{i}jk\bar{m}}\bar{\omega}_{ml}), \end{aligned} \tag{2.9}$$

$$\sum_k \epsilon_k (S_{i\bar{j}k}\omega_k + S_{i\bar{j}\bar{k}}\bar{\omega}_k) = dS_{i\bar{j}} - \sum_k \epsilon_k (S_{k\bar{j}}\omega_{ki} + S_{i\bar{k}}\bar{\omega}_{kj}).$$

The second Bianchi formula is given by

$$K_{\bar{i}jk\bar{l}m} = K_{\bar{i}jm\bar{l}k}, \tag{2.10}$$

and hence we have

$$S_{i\bar{j}k} = S_{k\bar{j}i} = \sum_l \epsilon_l K_{\bar{j}ik\bar{l}l}, \quad r_i = 2\sum_j \epsilon_j S_{i\bar{j}j}, \tag{2.11}$$

where $dr = \sum_j \epsilon_j (r_j \omega_j + \bar{r}_j \bar{\omega}_j)$. The components $S_{i\bar{j}kl}$ and $S_{i\bar{j}k\bar{l}}$ of the covariant derivative of $S_{i\bar{j}k}$ are expressed by

$$\begin{aligned} &\sum_l \epsilon_l (S_{i\bar{j}kl}\omega_l + S_{i\bar{j}k\bar{l}}\bar{\omega}_l) \\ &\qquad = dS_{i\bar{j}k} - \sum_l \epsilon_l (S_{l\bar{j}k}\omega_{li} + S_{i\bar{l}k}\bar{\omega}_{lj} + S_{i\bar{j}l}\omega_{lk}). \end{aligned} \tag{2.12}$$

By the exterior differentiation of the definition of $S_{i\bar{j}k}$ and taking account of (2.12), the Ricci formula for the Ricci tensor S is given by

$$S_{i\bar{j}k\bar{l}} - S_{i\bar{j}\bar{l}k} = \sum_m \epsilon_m (K_{\bar{l}ki\bar{m}} S_{m\bar{j}} - K_{\bar{l}km\bar{j}} S_{i\bar{m}}).$$

Let M be an n-dimensional semi-Kaehler manifold of index $2s$ $(0 \leqq s \leqq n)$. A plane section P of the tangent space T_xM of M at any point x is said to be *non-degenerate*, provided that the restriction of $g_x|_{T_xM}$ to P is non-degenerate. It is easily seen that P is non-degenerate if and only if it has a basis $\{X, Y\}$ such that $g(X,X)g(Y,Y) - g(X,Y)^2 \neq 0$. If the non-degenerate plane P is invariant by the complex structure J, then it said to be *holomorphic*. It is also trivial that the plane P is holomorphic if and only if it contains a vector X such that $g(X,X) \neq 0$. For the non-degenerate plane P spanned by X and Y in P, the sectional curvature $K(P)$ of P is usually defined by

$$K(P) = K(X,Y) = \frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2}.$$

The holomorphic plane spanned by space-like or time-like vectors X and JX is said to be *space-like* or *time-like*, respectively. The sectional curvature $K(P)$ of the non-degenerate holomorphic plane P is called the *holomorphic sectional curvature,* which is denoted by $H(P)$. The semi-Kaehler manifold M is said to be *of constant holomorphic sectional curvature* if its holomorphic sectional curvature $H(P)$ is constant for any non-degenerate holomorphic plane P and any point on M. Then M is called a *semi-definite complex space form,* which is denoted by $M_s^n(c)$ provided that it is of constant holomorphic sectional curvature c, of complex dimension n and of index $2s(\geqq 0)$. It is seen in Wolf [33] that the standard models of semi-definite complex space forms are the following three kinds : the semi-definite complex projective space $CP_s^n(c)$, the semi-definite complex Euclidean space C_s^n or the semi-definite complex hyperbolic space $CH_s^n(c)$, according as $c > 0$, $c = 0$ or $c < 0$. For any integer s $(0 \leqq s \leqq n)$, it is also seen by [33] that they are complete simply connected semi-definite complex space forms of dimension n and of index $2s$. The Riemannian curvature tensor $K_{\bar{A}BC\bar{D}}$ of

$M = M_s^n(c)$ is given by

$$K_{\bar{A}BC\bar{D}} = \frac{c}{2}\epsilon_B\epsilon_C(\delta_{AB}\delta_{CD} + \delta_{AC}\delta_{BD}). \tag{2.13}$$

Given two holomorphic planes P and Q in T_xM at any point x in M, the holomorphic bisectional curvature $H(P,Q)$ determined by the two planes P and Q of M is defined by

$$H(P,Q) = \frac{g(R(X,JX)JY,Y)}{g(X,X)g(Y,Y)}, \tag{2.14}$$

where X (resp. Y) is a non-zero vector in P (resp. Q). It is a simple matter to verify that the right hand side in (2.14) depends only on P and Q, so, it is well defined. It may be also denoted by $H(P,Q) = H(X,Y)$. It is easily seen that $H(P,P) = H(P) = H(X,X) =: H(X)$ is the holomorphic sectional curvature determined by the holomorphic plane P, where X is a non-zero vector in P. We denote by P_A the holomorphic plane $[E_A, JE_A]$ spanned by E_A and $JE_A = E_{A^*}$. We set

$$H(P_A,P_B) = H(E_A,E_B) = H_{AB}, \quad A \neq B,$$
$$H(P_A,P_A) = H(P_A) = H_{AA} = H_A.$$

The holomorphic bisectional curvature H_{AB} $(A \neq B)$ and the holomorphic sectional curvature H_A are given by

$$H_{AB} = \frac{g(R(E_A,JE_A)JE_B,E_B)}{g(E_A,E_A)g(E_B,E_B)} = -\epsilon_A\epsilon_B K_{AA^*BB^*}, \quad A \neq B,$$
$$H_A = \frac{g(R(E_A,JE_A)JE_A,E_A)}{g(E_A,E_A)g(E_A,E_A)} = -K_{AA^*AA^*}.$$

We have

$$H_{AB} = \epsilon_A\epsilon_B K_{\bar{A}AB\bar{B}}\ (A \neq B), \qquad H_A = K_{\bar{A}AA\bar{A}}. \tag{2.15}$$

Chapter 3

Semi-definite complex submanifolds

This chapter is concerned with semi-definite complex submanifolds of a semi-Kaehler manifold.

Let (M', g') be an $(n+p)$-dimensional connected semi-Kaehler manifold of index $2(s+t)$ $(0 \leqq s \leqq n, 0 \leqq t \leqq p)$ and let M be an n-dimensional connected semi-definite complex submanifold of index $2s$ of M'. Then M is the semi-Kaehler manifold endowed with the induced metric tensor g. We choose a local unitary frame field $\{U_A\} = \{U_1, \cdots, U_{n+p}\}$ on a neighborhood of M' in such a way that restricted to M, $U_1, \cdots, U_n$ are tangent to M and the others are normal to M. Here and in the sequel, the following convention on the range of indices is used throughout this paper, unless otherwise stated :

$$A,\ B,\ \cdots = 1,\ \cdots\ ,\ n,\ n+1,\ \cdots\ ,\ n+p,$$
$$i,\ j,\ \cdots = 1,\ \cdots\ ,\ n,$$
$$x,\ y,\ \cdots = n+1,\ \cdots\ ,\ n+p.$$

With respect to the unitary frame field $\{U_A\}$, let $\{\omega_A\} = \{\omega_i,\ \omega_x\}$ be its dual frame field. Then the Kaehler metric tensor g' of M' is given by $g' = 2\sum_A \epsilon_A \omega_A \otimes \bar{\omega}_A$. The canonical forms ω_A and the connection forms ω_{AB} of the ambient space satisfy the structure equations

$$\begin{aligned} &d\omega_A + \sum_B \epsilon_B \omega_{AB} \wedge \omega_B = 0, \quad \omega_{AB} + \bar{\omega}_{AB} = 0, \\ &d\omega_{AB} + \sum_C \epsilon_C \omega_{AC} \wedge \omega_{CB} = \Omega'_{AB}, \\ &\Omega'_{AB} = \sum_{C,D} \epsilon_C \epsilon_D K'_{\bar{A}BC\bar{D}} \omega_C \wedge \bar{\omega}_D, \end{aligned} \tag{3.1}$$

where $\Omega' = (\Omega'_{AB})$ (resp. $K'_{\bar{A}BC\bar{D}}$) denotes the curvature form with respect to the unitary frame field $\{U_A\}$ (resp. the components of the semi-definite Riemannian curvature tensor R') of M'. Restricting these forms to the submanifold M, we have

$$\omega_x = 0, \tag{3.2}$$

and the induced semi-Kaehler metric tensor g of index $2s$ of M is given by $g = 2\sum_j \epsilon_j \omega_j \otimes \bar{\omega}_j$. Then $\{U_j\}$ is a local unitary frame field with respect to this metric and $\{\omega_j\}$ is a local dual frame field due to $\{U_j\}$, which consists of complex valued 1-forms of type (1.0) on M. Moreover, $\omega_1,\ \cdots,\ \omega_n,\ \bar{\omega}_1,\ \cdots\ ,\ \bar{\omega}_n$ are linearly independent, and they are said to be *canonical 1-forms* on M. It follows from (3.2) and Cartan's lemma that the exterior derivatives of (3.2) give rise to

$$\omega_{xi} = \sum_j \epsilon_j h^{x}_{ij} \omega_j, \quad h^x_{ij} = h^x_{ji}. \tag{3.3}$$

The quadratic form $\sum_{i,j,x} \epsilon_i \epsilon_j \epsilon_x h^x_{ij} \omega_i \otimes \omega_j \otimes E_x$ with values in the normal bundle is called the *second fundamental form* of the submanifold M. From the structure equations of M', it follows that the structure equations for M are similarly given by

$$\begin{aligned} &d\omega_i + \sum_j \epsilon_j \omega_{ij} \wedge \omega_j = 0, \quad \omega_{ij} + \bar{\omega}_{ji} = 0, \\ &d\omega_{ij} + \sum_k \epsilon_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \\ &\Omega_{ij} = \sum_{k,l} \epsilon_k \epsilon_l K_{\bar{i}jk\bar{l}} \omega_k \wedge \bar{\omega}_l, \end{aligned} \tag{3.4}$$

where $\Omega = (\Omega_{ij})$ (resp. $K_{\bar{i}jk\bar{l}}$) denotes the curvature form with respect to the unitary frame field $\{U_A\}$ (resp. the component of the semi-definite Riemannian curvature tensor R) of M.

Moreover, the following relationships are obtained :

$$\begin{aligned} &d\omega_{xy} + \sum_z \epsilon_z \omega_{xz} \wedge \omega_{zy} = \Omega_{xy}, \\ &\Omega_{xy} = \sum_{k,l} \epsilon_k \epsilon_l K_{\bar{x}yk\bar{l}} \omega_k \wedge \bar{\omega}_l, \end{aligned} \tag{3.5}$$

where Ω_{xy} is called the *normal curvature form* of M. For the Riemannian curvature tensors K and K' of M and M', respectively, it follows from (3.1), (3.3) and (3.4) that we have the Gauss equation

$$K_{\bar{i}jk\bar{l}} = K'_{\bar{i}jk\bar{l}} - \sum_x \epsilon_x h^x_{jk} \bar{h}^x_{il}, \tag{3.6}$$

and by means of (3.1), (3.3) and (3.5), we have

$$K_{\bar{x}yk\bar{l}} = K'_{\bar{x}yk\bar{l}} + \sum_j \epsilon_j h^x_{kj} \bar{h}^y_{jl}. \tag{3.7}$$

Using (2.6), (2.7) and (3.6), the components of the Ricci tensor S and the scalar curvature r of M are given by

$$\begin{aligned} S_{i\bar{j}} &= \sum_k \epsilon_k K'_{\bar{k}ki\bar{j}} - h_{i\bar{j}}{}^2, \\ r &= 2(\sum_{j,k} \epsilon_j \epsilon_k K'_{\bar{k}kj\bar{j}} - h_2), \end{aligned} \tag{3.8}$$

where $h_{i\bar{j}}{}^2 = h_{\bar{j}i}{}^2 = \sum_{k,x} \epsilon_k \epsilon_x h^x_{ik} \bar{h}^x_{kj}$ and $h_2 = \sum_j \epsilon_j h_{j\bar{j}}{}^2$.

Now, the components h^x_{ijk} and $h^x_{ij\bar{k}}$ of the covariant derivative of the second fundamental form of M are given by

$$\begin{aligned} \sum_k \epsilon_k (h^x_{ijk}\omega_k + h^x_{ij\bar{k}}\bar{\omega}_k) &= dh^x_{ij} \\ - \sum_k \epsilon_k (h^x_{kj}\omega_{ki} &+ h^x_{ik}\omega_{kj}) + \sum_y \epsilon_y h^y_{ij}\omega_{xy}. \end{aligned} \tag{3.9}$$

Then, substituting dh^x_{ij} into the exterior derivative of (3.3), we have

$$h^x_{ijk} = h^x_{jik} = h^x_{ikj}, \quad h^x_{ij\bar{k}} = -K'_{\bar{x}ij\bar{k}}. \tag{3.10}$$

Similarly the components h^x_{ijkl} and $h^x_{ijk\bar{l}}$ of the covariant derivative of h^x_{ijk} can be defined by

$$\begin{aligned} &\sum_m \epsilon_m (h^x_{ijkm}\omega_m + h^x_{ijk\bar{m}}\bar{\omega}_m) \\ &= dh^x_{ijk} - \sum_m \epsilon_m (h^x_{mjk}\omega_{mi} + h^x_{imk}\omega_{mj} + h^x_{ijm}\omega_{mk}) \\ &\quad + \sum_y \epsilon_y h^y_{ijk}\omega_{xy}, \end{aligned} \tag{3.11}$$

$$\sum_m \epsilon_m (h^x_{ij\bar{k}m}\omega_m + h^x_{ij\bar{k}\bar{m}}\bar{\omega}_m)$$

$$= dh^x_{ij\bar{k}} - \sum_m \epsilon_m (h^x_{mj\bar{k}}\omega_{mi} + h^x_{im\bar{k}}\omega_{mj} + h^x_{ij\bar{m}}\bar{\omega}_{mk}) \tag{3.12}$$

$$+ \sum_y \epsilon_y h^y_{ij\bar{k}}\omega_{xy}.$$

and by the simple calculation the Ricci formula for the second fundamental form are given by

$$h^x_{ijkl} = h^x_{ijlk},$$

$$h^x_{ijk\bar{l}} - h^x_{ij\bar{l}k} = \sum_r \epsilon_r (K_{\bar{l}ki\bar{r}}h^x_{rj} + K_{\bar{l}kj\bar{r}}h^x_{ir}) - \sum_y \epsilon_y K_{\bar{x}yk\bar{l}}h^y_{ij}. \tag{3.13}$$

In particular, let the ambient space be an $(n+p)$-dimensional semi-definite complex space form $M^{n+p}_{s+t}(c)$ of constant holomorphic sectional curvature c and of index $2(s+t)$ $(0 \leqq s \leqq n,\ 0 \leqq t \leqq p)$. Then, from (2.13), (3.6) and (3.8), we get

$$K_{\bar{i}jk\bar{l}} = \frac{c}{2}\epsilon_j\epsilon_k(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) - \sum_x \epsilon_x h^x_{jk}\bar{h}^x_{il}, \tag{3.14}$$

$$S_{i\bar{j}} = \frac{(n+1)c}{2}\epsilon_i\delta_{ij} - h_{i\bar{j}}{}^2, \qquad h^x_{ij\bar{k}} = 0. \tag{3.15}$$

And hence from (3.14) we obtain

$$h^x_{ijk\bar{l}} = \frac{c}{2}(\epsilon_k h^x_{ij}\delta_{kl} + \epsilon_i h^x_{jk}\delta_{il} + \epsilon_j h^x_{ki}\delta_{jl})$$
$$- \sum_{r,y} \epsilon_r\epsilon_y (h^x_{ri}h^y_{jk} + h^x_{rj}h^y_{ki} + h^x_{rk}h^y_{ij})\bar{h}^y_{rl}. \tag{3.16}$$

Here, we calculate the Laplacian of the squared norm $h_2 = \|\alpha\|_2$ of the second fundamental form α on M. The Laplacian Δh_2 of the function h_2 is by definition given as

$$\Delta h_2 = 2 \sum_{i,j,k,x} \epsilon_i\epsilon_j\epsilon_k\epsilon_x (h^x_{ij}\bar{h}^x_{ij})_{k\bar{k}}$$
$$= 2 \sum_{i,j,k,x} \epsilon_i\epsilon_j\epsilon_k\epsilon_x (h^x_{ijk\bar{k}}\bar{h}^x_{ij} + h^x_{ijk}\bar{h}^x_{ijk} + h^x_{ij\bar{k}}\bar{h}^x_{ij\bar{k}} + h^x_{ij}\bar{h}^x_{ij\bar{k}k}).$$

Hence we have by the second equation of (3.15) and (3.16)

$$\begin{aligned}\Delta h_2 &= c(n+2)h_2 - 4h_4 - 2\mathrm{Tr}\ A^2 \\ &\quad + 2\sum_{i,j,k,x} \epsilon_i\epsilon_j\epsilon_k\epsilon_x h^x_{ijk}\bar{h}^x_{ijk},\end{aligned} \tag{3.17}$$

where $h_4 = \sum_{i,j} \epsilon_i\epsilon_j {h_{i\bar{j}}}^2 {h_{j\bar{i}}}^2$, Tr A^2 is the trace of the matrix A^2 and $A = (A^x_y) = \sum_{i,j} \epsilon_i\epsilon_j h^x_{ij}\bar{h}^y_{ij}$.

Chapter 4

Chern-type problems

Let $M' = CH_p^{n+p}(c)$ be an $(n+p)$-dimensional indefinite complex hyperbolic space of index $2p(>0)$ and let M be an $n(\geqq 2)$-dimensional space-like complex submanifold of M'. First of all, we will estimate the Laplacian of the squared norm h_2 of the second fundamental form. By (3.14), we have

$$K_{\bar{j}jk\bar{k}} = \frac{c}{2} - \sum_x \epsilon_x h_{jk}^x \bar{h}_{jk}^x \geqq \frac{c}{2}, \quad j \neq k, \tag{4.1}$$

where we have used the fact that $\epsilon_x = -1$.

Since M is space-like, the normal space of M is time-like. So, the matrix $H = ({h_{j\bar{k}}}^2)$ is a negative semi-definite Hermitian one and hence all eigenvalues μ_j of H are non-positive real valued functions on M. The matrix $A = (A_y^x)$ is a positive semi-definite Hermitian one and hence all eigenvalues μ_x of A are non-negative real valued functions on M. Thus it is easily seen that

$$\sum_j \mu_j = \text{Tr } H = h_2, \quad \sum_x \mu_x = \text{Tr } A = -h_2,$$
$$h_2{}^2 \geqq h_4 = \sum_j \mu_j^2 \geqq \frac{1}{n} h_2{}^2, \tag{4.2}$$
$$h_2{}^2 \geqq \text{Tr } A^2 = \sum_x \mu_x^2 \geqq \frac{1}{p} h_2{}^2.$$

Also from the estimating of the squared norm of

$$\sum_x \left\{ \epsilon_x h_{jk}^x \bar{h}_{il}^x - \frac{h_2}{n(n+1)}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) \right\},$$

it follows that

$$\text{Tr } A^2 \geqq \frac{2}{n(n+1)} h_2{}^2, \tag{4.3}$$

where the equality holds if and only if M is a complex space form. By (3.17), (4.2) and (4.3), we have

$$\begin{aligned} \Delta h_2 &\leqq c(n+2)h_2 - 4h_4 - 2\text{Tr } A^2 \\ &\leqq c(n+2)h_2 - \frac{4(n+2)}{n(n+1)} h_2{}^2, \end{aligned}$$

where the equality holds if and only if M is a complex space form and the second fundamental form of M is parallel. Let f be a non-negative function defined by $-h_2$. Then the above inequality is reduced to

$$\Delta f \geqq c(n+2)f + \frac{4(n+2)}{n(n+1)} f^2, \tag{4.4}$$

where the equality folds if and only if M is a complex space form and the second fundamental form of M is parallel.

On the other hand, the Laplacian Δh_2 of h_2 is also estimated in the different type by (3.17) and (4.2). That is, we have

$$\Delta h_2 \leqq c(n+2)h_2 - \frac{2(n+2p)}{np}{h_2}^2,$$

where the equality holds if and only if M is Einstein and the second fundamental form of M is parallel. So, the function f defined by $-h_2$ satisfies

$$\Delta f \geqq c(n+2)f + \frac{2(n+2p)}{np}f^2, \tag{4.5}$$

where the equality holds if and only if M is Einstein and the second fundamental form of M is parallel.

Now, applying the generalized maximum principle due to Omori [23] and Yau [35], Choi, Kwon and Suh [8] proved recently the following theorem.

Theorem 4.1. *Let M be a complete Riemannian manifold whose Ricci tensor is bounded from below and let F be any polynomial of one variable x with constant coefficients $c_0, \cdots, c_{k+1}$ such that*

$$F(x) = c_0x^n + c_1x^{n-1} + \cdots + c_kx^{n-k} + c_{k+1},$$

where $n \geqq 2$, $n-k>0$ and $c_0 > c_{k+1}$. If a C^2-function f satisfies $\Delta f \geqq F(f)$, then we have $F(\sup f) \leqq 0$.

Owing to the above theorem, we estimate the squared norm h_2 of the second fundamental form α of M.

Theorem 4.2. *Let M be an $n(\geqq 2)$-dimensional complete space-like complex submanifold of an $(n+p)$-dimensional indefinite complex hyperbolic space*

$CH_p^{n+p}(c)$ of constant holomorphic sectional curvature c and of index $2p(>0)$. Then the squared norm h_2 of the second fundamental form α of M satisfies

$$h_2 \geqq \frac{cn(n+1)}{4} \quad \text{if} \quad p \geqq \frac{n(n+1)}{2},$$

or

$$h_2 \geqq \frac{cnp(n+2)}{2(n+2p)} \quad \text{if} \quad p \leqq \frac{n(n+1)}{2},$$

where the both equalities hold if and only if M is a complex space form $M^n(\frac{c}{2})$, α is parallel and $p = \frac{n(n+1)}{2}$.

Proof. We can choose a suitable unitary frame field $\{U_1, \cdots, U_n\}$ so that the negative semi-definite Hermitian matrix $H = ({h_{j\bar{k}}}^2)$ can be diagonalized. Then the Ricci curvature $S_{j\bar{j}}$ of M is given by

$$S_{j\bar{j}} = \frac{c(n+1)}{2} - \mu_j,$$

where we have used (3.15) and μ_j is an eigenvalue of the negative semi-definite Hermitian matrix H. Thus the Ricci tensor is bounded from below. Moreover, the non-negative function $f = -h_2$ satisfies the Liouville type inequalities (4.4) and (4.5). If we define a polynomial $F(x)$ by

$$F(x) = \frac{n+2}{n(n+1)}\{cn(n+1) + 4x\}x$$

$$(\text{resp.} \quad F(x) = \frac{1}{np}\{cnp(n+2) + 2(n+2p)x\}x),$$

then F satisfies the conditions of Theorem 4.1. So, we can apply Theorem 4.1 to the function f and hence we obtain

$$F(\sup f) \leqq 0, \quad \text{i.e.,} \quad (\sup f)\{cn(n+1) + 4 \sup f\} \leqq 0$$

$$(\text{resp.}\quad (\sup f)\{cnp(n+2) + 2(n+2p)\sup f\} \leqq 0).$$

This means that if M is not totally geodesic, then

$$cn(n+1) + 4\sup f \leqq 0, \quad \text{i.e.,} \quad 4h_2 \geqq cn(n+1)$$

$$(\text{resp. } cnp(n+2) + 2(n+2p)\sup f \leqq 0,$$
$$\text{i.e.,} \quad 2(n+2p)h_2 \geqq cnp(n+2)),$$

where the equality holds if and only if M is a complex space form $M^n(c')$ (resp. Einstein) and α is parallel, then, since the scalar curvature r of M is given by

$$r = cn(n+1) - 2h_2. \tag{4.6}$$

Comparing this with (3.15), we see that the first equality holds if and only if $c' = \frac{c}{2}.$ On the other hand, the second equality holds if and only if h_2 is a constant $\frac{cnp(n+2)}{2(n+2p)}$ and α is parallel. It implies that

$$h_2 = \frac{cnp(n+2)}{2(n+2p)} = \frac{cnp(n+2)}{4},$$

from which it follows that

$$p = \frac{n(n+1)}{2}.$$

The proof is completed.

Remark 4.3. Under the same assumption as stated in Theorem 4.2, we get

$$h_2 \geqq \frac{cn(n+1)}{4} \quad \text{and} \quad h_2 \geqq \frac{cnp(n+2)}{2(n+2p)}.$$

Here, in order to prove our main theorem, we will consider the totally real bisectional curvature of M. A plane section P in the tangent space T_xM of M

at any point x in M is said to be *totally real* if P is orthogonal to JP. For the non-degenerate totally real plane P spanned by orthonormal vectors u and v, the *totally real bisectional curvature* $B(u,v)$ is defined by

$$B(u,v) = \frac{g(R(u,Ju)Jv,v)}{g(u,u)g(v,v)}. \tag{4.7}$$

For a space-like complex submanifold, using the first Bianchi identity to (4.7) and the fundamental properties of the Riemannian curvature tensor of a space-like complex submanifold, we get

$$\begin{aligned} B(u,v) =& g(R(u,v)v,u) + g(R(u,Jv)Jv,u) \\ =& K(u,v) + K(u,Jv), \end{aligned} \tag{4.8}$$

where $K(u,v)$ means the sectional curvature of the plane spanned by u and v.

From now on, we suppose that u and v are space-like orthonormal vectors in the non-degenerate totally real plane P. If we put $u' = \frac{1}{\sqrt{2}}(u+v)$ and $v' = \frac{1}{\sqrt{2}}(u-v)$, then it is easily seen that

$$g(u',u') = 1, \quad g(v',v') = 1, \quad g(u',v') = 0.$$

Thus we get

$$B(u',v') = g(R(u',Ju')Jv',v') = \frac{1}{4}\{H(u) + H(v) + 2B(u,v) - 4K(u,Jv)\},$$

where $H(u) = K(u,Ju)$ means the holomorphic sectional curvature of the holomorphic plane spanned by u and Ju. Hence we have

$$4B(u',v') - 2B(u,v) = H(u) + H(v) - 4K(u,Jv). \tag{4.9}$$

If we put $u'' = \frac{1}{\sqrt{2}}(u+Jv)$ and $v'' = \frac{1}{\sqrt{2}}(Ju+v)$, then we get

$$g(u'',u'') = 1, \quad g(v'',v'') = 1, \quad g(u'',v'') = 0.$$

Using the similar method as in (4.9), we have

$$4B(u'', v'') - 2B(u, v) = H(u) + H(v) - 4K(u, v). \tag{4.10}$$

Summing up (4.9) and (4.10) and taking account of (4.8), we obtain

$$2B(u', v') + 2B(u'', v'') = H(u) + H(v). \tag{4.11}$$

In the sequel, let $b(M)$ (resp. $a(M)$) be the supremum (resp. the infimum) of the set B of the totally real bisectional curvatures on M. Suppose that the totally real bisectional curvature is bounded from above (resp. below) by a constant b (resp. a). From (4.11), it follows that

$$H(u) + H(v) \leqq 4b \quad (\text{resp. } \geqq 4a). \tag{4.12}$$

We can choose a unitary frame field $\{U_1, U_2, \cdots, U_n\}$ on a neighborhood of M. With respect to this unitary frame field, let $\{\omega_1, \omega_2, \cdots, \omega_n\}$ be a dual frame field. The holomorphic sectional curvature $H(U_j)$ of the holomorphic plane defined by U_j is given by

$$H(U_j) = g(R(U_j, \bar{U}_j)\bar{U}_j, U_j) = K_{\bar{j}j j\bar{j}}.$$

On the other hand, it is easily seen that the plane spanned by U_j and U_k ($j \neq k$) is totally real and the totally real bisectional curvature $B(U_j, U_k)$ is given by

$$B(U_j, U_k) = g(R(U_j, \bar{U}_j)\bar{U}_k, U_k) = K_{\bar{j}jk\bar{k}}, \quad j \neq k. \tag{4.13}$$

From the inequality (4.12) for $u = U_j$ and $v = U_k$, we have

$$K_{\bar{j}jj\bar{j}} + K_{\bar{k}kk\bar{k}} \leqq 4b \quad (\text{resp. } \geqq 4a), \quad j \neq k. \tag{4.14}$$

Thus we have

$$\sum_{j<k}(K_{\bar{j}jj\bar{j}} + K_{\bar{k}kk\bar{k}}) \leqq 2bn(n-1) \quad (\text{resp. } \geqq 2an(n-1)), \tag{4.15}$$

which implies that

$$\sum_{j} K_{\bar{j}jj\bar{j}} \leqq 2bn \quad (\text{resp. } \geqq 2an), \tag{4.16}$$

where the equality holds if and only if $K_{\bar{j}jj\bar{j}} = 2b$ (resp. $= 2a$) for any index j.

Since the scalar curvature r is given by

$$r = 2\sum_{j,k} K_{\bar{j}jk\bar{k}} = 2(\sum_{j} K_{\bar{j}jj\bar{j}} + \sum_{j \neq k} K_{\bar{j}jk\bar{k}}),$$

we have by (4.15)

$$r \leqq 2\sum_{j} K_{\bar{j}jj\bar{j}} + 2bn(n-1) \quad (\text{resp. } \geqq 2\sum_{j} K_{\bar{j}jj\bar{j}} + 2an(n-1)),$$

from which we have

$$\sum_{j} K_{\bar{j}jj\bar{j}} \geqq \frac{r}{2} - bn(n-1) \quad (\text{resp. } \leqq \frac{r}{2} - an(n-1)), \tag{4.17}$$

where the equality holds if and only if $K_{\bar{j}jk\bar{k}} = b$ (resp. $= a$) for any distinct indices j and k. In this case, M is locally congruent to $M^n(b)$ (resp. $M^n(a)$) due to Houh [10]. Also (4.14) gives us

$$\sum_{j \neq k}(K_{\bar{j}jj\bar{j}} + K_{\bar{k}kk\bar{k}}) \leqq 4b(n-1) \quad (\text{resp. } \geqq 4a(n-1)),$$

so that

$$(n-2)K_{\bar{j}jj\bar{j}} + \sum_{k} K_{\bar{k}kk\bar{k}} \leqq 4b(n-1) \quad (\text{resp. } \geqq 4a(n-1)).$$

From this together with (4.17), it follows that we have

$$
\begin{aligned}
(n-2)K_{\bar{j}j j\bar{j}} &\leqq b(n-1)(n+4) - \frac{r}{2} \\
(\text{resp. } &\geqq a(n-1)(n+4) - \frac{r}{2}),
\end{aligned}
\tag{4.18}
$$

for any index j, so that the holomorphic sectional curvature $K_{\bar{j}jj\bar{j}}$ is bounded from above (resp. below) for $n \geqq 3$. Moreover, the equality holds for some index j if and only if M is locally congruent to $M^n(2b)$ (resp. $M^n(2a)$).

Since the Ricci curvature $S_{j\bar{j}}$ is given by

$$
S_{j\bar{j}} = K_{\bar{j}jj\bar{j}} + \sum_{k \neq j} K_{\bar{j}jk\bar{k}},
$$

we have by (4.13)

$$
S_{j\bar{j}} \leqq K_{\bar{j}jj\bar{j}} + b(n-1) \quad (\text{resp. } \geqq K_{\bar{j}jj\bar{j}} + a(n-1))
$$

and hence, from (4.18), we get

$$
\begin{aligned}
S_{j\bar{j}} &\leqq \frac{1}{2(n-2)}\{4b(n-1)(n+1) - r\} \\
(\text{resp. } &\geqq \frac{1}{2(n-2)}\{4a(n-1)(n+1) - r\}).
\end{aligned}
\tag{4.19}
$$

On the other hand, using (4.19), we get

$$
\begin{aligned}
r &= 2S_{j\bar{j}} + 2\sum_{k \neq j} S_{k\bar{k}} \\
&\leqq 2S_{j\bar{j}} + \frac{1}{n-2}(n-1)\{4b(n-1)(n+1) - r\} \\
(\text{resp. } &\geqq 2S_{j\bar{j}} + \frac{1}{n-2}(n-1)\{4a(n-1)(n+1) - r\}),
\end{aligned}
$$

and hence we have

$$
\begin{aligned}
S_{j\bar{j}} &\geqq \frac{1}{2(n-2)}\{(2n-3)r - 4b(n-1)^2(n+1)\} \\
(\text{resp. } &\leqq \frac{1}{2(n-2)}\{(2n-3)r - 4a(n-1)^2(n+1)\}).
\end{aligned}
\tag{4.20}
$$

Combining this with (4.18) and (4.20), we get

$$
\begin{aligned}
K_{\bar{j}jk\bar{k}} &\geqq \frac{1}{n-2}\{(n-1)r-(2n^3-3n+2)b\} \\
(\text{resp. } &\leqq \frac{1}{n-2}\{(n-1)r-(2n^3-3n+2)a\})
\end{aligned}
\tag{4.21}
$$

for any distinct indices j and k.

First of all, before we estimate the supremum of B, we treat here the infimum $a(M)$.

Theorem 4.4. *Let M be an $n(\geqq 3)$-dimensional complete space-like complex submanifold of $CH_p^{n+p}(c)$, $p>0$. Then we have*

$$(1)\quad a(M) \leqq \frac{c}{4}.$$

$$(2)\quad a(M) \leqq \frac{cn(n+p+1)}{2(n+1)(n+2p)}.$$

Proof. Since the totally real bisectional curvatures are bounded from below by (4.1), there exists a constant a such that

$$K_{\bar{j}jk\bar{k}} \geqq a \quad \text{for any } j,\ k(j \neq k).$$

Hence, by (4.16), (4.17) and (4.6), we have

$$2an \leqq \sum_j K_{\bar{j}jj\bar{j}} \leqq \frac{cn(n+1)}{2} - h_2 - an(n-1).$$

Thus we get

$$2h_2 \leqq (c-2a)n(n+1). \tag{4.22}$$

From the estimate of h_2 in Theorem 4.2 together with (4.22), it follows that we get

$$(4a-c)n(n+1) \leqq 0.$$

The proof of the first assertion is completed.

Also, from Theorem 4.2 and (4.22), we can easily prove the second assertion.

Remark 4.5. (1) The above first assertion is essentially proved by Ki and Suh [13]. But it is unfortunately incomplete for applications of apply another Liouville-type theorem, so the gap is here recovered.

(2) Theorem 4.4 can be restated by the following

$$a(M) \leqq \frac{c}{4} \quad \text{if} \quad p \geqq \frac{n(n+1)}{2},$$

$$a(M) \leqq \frac{cn(n+p+1)}{2(n+1)(n+2p)} \quad \text{if} \quad p \leqq \frac{n(n+1)}{2}.$$

Next, we estimate the supremum $b(M)$ of the totally real bisectional curvatures of the space-like complex submanifold M.

Theorem 4.6. *Let M be an $n(\geqq 3)$-dimensional complete space-like submanifold of an $(n+p)$-dimensional indefinite complex hyperbolic space $CH_p^{n+p}(c)$ of constant holomorphic sectional curvature c and of index $2p(>0)$. Then the supremum $b(M)$ of the totally real bisectional curvatures of M satisfies*

$$b(M) < -\frac{c(n^3-2n+2)}{2(n-2)}.$$

Proof. By Remark 4.3, it is seen that the squared norm h_2 of the second fundamental form of M is restricted by

$$0 \geqq h_2 \geqq \frac{cn(n+1)}{4}, \tag{4.23}$$

where the second equality holds if and only if M is a complex space form $M^n(\frac{c}{2})$ and the second fundamental form of M is parallel. By (4.21), we see that any totally real bisectional curvature $K_{\bar{j}jk\bar{k}}(j \neq k)$ satisfies

$$K_{\bar{j}jk\bar{k}} \leqq \frac{1}{n-2}\{(n-1)r - (2n^3 - 3n + 2)a(M)\}, \tag{4.24}$$

where the equality holds if and only if $a(M) = \frac{c}{4}$. By the definition of $b(M)$, we have

$$b(M) \leqq \frac{1}{n-2}\{(n-1)r - (2n^3 - 3n + 2)a(M)\}.$$

Together with (4.6) and the result $a(M) \geqq \frac{c}{2}$ by (4.1), we obtain

$$b(M) \leqq \frac{c}{2} - \frac{2(n-1)}{n-2}h_2. \tag{4.25}$$

where the equality holds if and only if $a(M) = \frac{c}{2}$. From (4.23) and (4.25), it turns out to be

$$b(M) \leqq -\frac{c(n^3 - 2n + 2)}{2(n-2)}.$$

Hence the proof is completed by the conditions for the equalities of (4.24) and (4.25).

Chapter 5

Indefinite locally symmetric Kaehler manifolds

5.1 The Laplacian of the squared norm of the second fundamental form

In this section, the Laplacian of the squared norm of the second fundamental form on a space-like complex submanifold of an indefinite Kaehler manifold is calculated. Let M' be an $(n+p)$-dimensional indefinite Kaehler manifold of index $2p$ and let M be an n-dimensional space-like complex submanifold of M'. Let f be any smooth C^2-function on M. The components f_i and $f_{\bar{i}}$ of the exterior derivative df of f are given by

$$df = \sum_i \epsilon_i (f_i \omega_i + f_{\bar{i}} \bar{\omega}_i). \tag{5.1}$$

Moreover, the components f_{ij} and $f_{i\bar{j}}$ (resp. $f_{\bar{i}j}$ and $f_{\bar{i}\bar{j}}$) of the covariant derivative of f_i (resp. $f_{\bar{i}}$) can be defined by

$$\begin{aligned} \sum_j \epsilon_j (f_{ij}\omega_j + f_{i\bar{j}}\bar{\omega}_j) &= df_i - \sum_j f_j \omega_{ji}, \\ \sum_j \epsilon_j (f_{\bar{i}j}\omega_j + f_{\bar{i}\bar{j}}\bar{\omega}_j) &= df_{\bar{i}} - \sum_j f_{\bar{j}} \bar{\omega}_{ji}. \end{aligned} \tag{5.2}$$

The Laplacian of the function f is by definition given as

$$\Delta f = 2 \sum_j \epsilon_j f_{j\bar{j}}. \tag{5.3}$$

Now, we calculate the Laplacian of the squared norm $h_2 = \|\alpha\|_2$ of the second fundamental form α on M. By (3.12) and the second equation of (3.10), we have

$$\begin{aligned} &\sum_m \epsilon_m (h^x_{ij\bar{k}m}\omega_m + h^x_{ij\bar{k}\bar{m}}\bar{\omega}_m) \\ &= dh^x_{ij\bar{k}} - \sum_m \epsilon_m (h^x_{mj\bar{k}}\omega_{mi} + h^x_{im\bar{k}}\omega_{mj} + h^x_{ij\bar{m}}\bar{\omega}_{mk}) \\ &\quad + \sum_y \epsilon_y h^y_{ij\bar{k}}\omega_{xy} \\ &= - dK'_{\bar{x}ij\bar{k}} + \sum_m \epsilon_m (K'_{\bar{x}mj\bar{k}}\omega_{mi} + K'_{\bar{x}im\bar{k}}\omega_{mj} + K'_{\bar{x}ij\bar{m}}\bar{\omega}_{mk}) \\ &\quad - \sum_y \epsilon_y K'_{\bar{y}ij\bar{k}}\omega_{xy} \\ &= - dK'_{\bar{x}ij\bar{k}} + \sum_A \epsilon_A (K'_{\bar{x}Aj\bar{k}}\omega_{Ai} + K'_{\bar{x}iA\bar{k}}\omega_{Aj} + K'_{\bar{x}ij\bar{A}}\bar{\omega}_{Ak}) \\ &\quad - \sum_A \epsilon_A K'_{\bar{A}ij\bar{k}}\omega_{xA} - \sum_y \epsilon_y (K'_{\bar{x}yj\bar{k}}\omega_{yi} + K'_{\bar{x}iy\bar{k}}\omega_{yj} + K'_{\bar{x}ij\bar{y}}\bar{\omega}_{yk}) \\ &\quad + \sum_m \epsilon_m K'_{\bar{m}ij\bar{k}}\omega_{xm}, \end{aligned}$$

from which together with (2.9), (3.1) and (3.3), it follows that we have

$$\begin{aligned}\sum_m \epsilon_m (h^x_{ij\bar{k}m}\omega_m + h^x_{ij\bar{k}\bar{m}}\bar{\omega}_m) \\ = -\sum_A \epsilon_A (K'_{\bar{x}ij\bar{k}:A}\omega_A + K'_{\bar{x}ij\bar{k}:\bar{A}}\bar{\omega}_A) \\ -\sum_{y,m} \epsilon_y\epsilon_m (K'_{\bar{x}yj\bar{k}}h^y_{im}\omega_m + K'_{\bar{x}iy\bar{k}}h^y_{jm}\omega_m + K'_{\bar{x}ij\bar{y}}\bar{h}^y_{km}\bar{\omega}_m) \\ +\sum_{m,n} \epsilon_m\epsilon_n K'_{\bar{n}ij\bar{k}}h^x_{nm}\omega_m.\end{aligned}$$

Comparing the coefficients of ω_m in the above equation, we have

$$\begin{aligned}h^x_{ij\bar{k}m} = &- K'_{\bar{x}ij\bar{k}:m} - \sum_y \epsilon_y (K'_{\bar{x}yj\bar{k}}h^y_{im} + K'_{\bar{x}iy\bar{k}}h^y_{jm}) \\ &+ \sum_n \epsilon_n K'_{\bar{n}ij\bar{k}}h^x_{nm}.\end{aligned} \tag{5.4}$$

On the other hand, from (3.13), we get

$$\begin{aligned}h^x_{ijk\bar{m}} - h^x_{ij\bar{m}k} = &\sum_n \epsilon_n (K'_{\bar{n}ki\bar{m}}h^x_{nj} + K'_{\bar{n}kj\bar{m}}h^x_{ni}) \\ &- \sum_y \epsilon_y K'_{\bar{m}ky\bar{x}}h^y_{ij} \\ &- \sum_{y,n} \epsilon_y\epsilon_n (h^y_{ik}\bar{h}^y_{nm}h^x_{nj} + h^y_{jk}\bar{h}^y_{nm}h^x_{ni}) \\ &- \sum_{y,n} \epsilon_y\epsilon_n h^x_{kn}\bar{h}^y_{nm}h^y_{ij},\end{aligned} \tag{5.5}$$

where we have used (3.6) and (3.7). By (5.4) and (5.5), we obtain

$$\begin{aligned}h^x_{ijk\bar{m}} = &- K'_{\bar{x}ij\bar{m}:k} - \sum_y \epsilon_y (K'_{\bar{x}yj\bar{m}}h^y_{ik} + K'_{\bar{x}yi\bar{m}}h^y_{jk} + K'_{\bar{x}yk\bar{m}}h^y_{ij}) \\ &+ \sum_n \epsilon_n (K'_{\bar{n}jk\bar{m}}h^x_{ni} + K'_{\bar{n}ik\bar{m}}h^x_{nj} + K'_{\bar{n}ij\bar{m}}h^x_{nk}) \\ &- \sum_{y,n} \epsilon_y\epsilon_n (h^y_{ik}\bar{h}^y_{nm}h^x_{nj} + h^y_{jk}\bar{h}^y_{nm}h^x_{ni}) - \sum_{y,n} \epsilon_y\epsilon_n h^x_{kn}\bar{h}^y_{nm}h^y_{ij}.\end{aligned} \tag{5.6}$$

The matrix $A = (A_y{}^x)$ of order p defined by $A_y{}^x = \sum_{i,j} \epsilon_i\epsilon_j h_{ij}^x \bar{h}_{ij}^y$ is a Hermitian one. Since M is space-like and the normal space is time-like, it is a positive semi-definite Hermitian matrix of order p. Summing up $k = m$ in (5.6), we have

$$\begin{aligned}\sum_k \epsilon_k h_{ijk\bar{k}}^x = &- \sum_k \epsilon_k K'_{\bar{x}ij\bar{k}:k} \\ &- \sum_{y,k} \epsilon_y\epsilon_k (K'_{\bar{x}yj\bar{k}} h_{ik}^y + K'_{\bar{x}yi\bar{k}} h_{jk}^y + K'_{\bar{x}yk\bar{k}} h_{ij}^y) \\ &+ \sum_{k,m} \epsilon_k\epsilon_m (K'_{\bar{m}jk\bar{k}} h_{mi}^x + K'_{\bar{m}ik\bar{k}} h_{mj}^x + K'_{\bar{m}ij\bar{k}} h_{mk}^x) \\ &- \sum_k \epsilon_k (h_{i\bar{k}}^2 h_{kj}^x + h_{j\bar{k}}^2 h_{ki}^x) - \sum_y \epsilon_y A_y{}^x h_{ij}^y.\end{aligned} \tag{5.7}$$

By (5.3), we see

$$\Delta h_2 = \sum_{i,j,k} \epsilon_i\epsilon_j\epsilon_k \{(\sum_x \epsilon_x h_{ij}^x \bar{h}_{ij}^x)_{k\bar{k}} + (\sum_x \epsilon_x h_{ij}^x \bar{h}_{ij}^x)_{\bar{k}k}\}. \tag{5.8}$$

The first term in the right hand side of (5.8) is given by

$$\sum_{x,i,j,k} \epsilon_x\epsilon_i\epsilon_j\epsilon_k (h_{ijk\bar{k}}^x \bar{h}_{ij}^x + h_{ijk}^x \bar{h}_{ijk}^x + h_{ij\bar{k}}^x \bar{h}_{ij\bar{k}}^x + h_{ij}^x \bar{h}_{ij\bar{k}k}^x).$$

And the second term is expressed as

$$\sum_{x,i,j,k} \epsilon_x\epsilon_i\epsilon_j\epsilon_k (h_{ij\bar{k}k}^x \bar{h}_{ij}^x + h_{ij\bar{k}}^x \bar{h}_{ij\bar{k}}^x + h_{ijk}^x \bar{h}_{ijk}^x + h_{ij}^x \bar{h}_{ijk\bar{k}}^x).$$

On the other hand, by (5.4), we have

$$\begin{aligned}\sum_k \epsilon_k h_{ij\bar{k}k}^x = &- \sum_k \epsilon_k K'_{\bar{x}ij\bar{k}:k} - \sum_{y,k} \epsilon_y\epsilon_k (K'_{\bar{x}yj\bar{k}} h_{ik}^y + K'_{\bar{x}yi\bar{k}} h_{jk}^y) \\ &+ \sum_{k,n} \epsilon_k\epsilon_n K'_{\bar{n}ij\bar{k}} h_{nk}^x.\end{aligned}$$

Since A is the positive semi-definite Hermitian matrix of order p, its eigenvalues λ_x are all non-negative real valued functions on M and it is easily seen that we have

$$\sum_x \lambda_x = \text{Tr } A = -h_2, \quad {h_2}^2 \geqq \text{Tr } A^2 = \sum_x {\lambda_x}^2 \geqq \frac{1}{p}{h_2}^2. \tag{5.9}$$

Substituting these three relations into (4.8), we obtain the formula for the Laplacian of the squared norm h_2 of the second fundamental form on M. That is, we have

$$\begin{aligned} \Delta h_2 = {} & 2\|\nabla\alpha\|_2 - 4\sum_{i,j,k,x} \epsilon_i\epsilon_j\epsilon_k\epsilon_x K'_{\bar{x}ij\bar{k}:k}\bar{h}^x_{ij} \\ & - 8\sum_{x,y,i,j,k} \epsilon_x\epsilon_y\epsilon_i\epsilon_j\epsilon_k K'_{\bar{x}yj\bar{k}} h^y_{ki}\bar{h}^x_{ij} - 2\sum_{x,y,k} \epsilon_x\epsilon_y\epsilon_k {A_x}^y K'_{\bar{x}yk\bar{k}} \\ & + 4\sum_{x,i,j,k,m} \epsilon_x\epsilon_i\epsilon_j\epsilon_k\epsilon_m K'_{\bar{k}ij\bar{m}} h^x_{km}\bar{h}^x_{ij} \\ & + 4\sum_{i,j,k} \epsilon_i\epsilon_j\epsilon_k K'_{\bar{i}jk\bar{k}} {h_{i\bar{j}}}^2 - 4h_4 - 2\text{Tr } A^2, \end{aligned} \tag{5.10}$$

where we have used (2.4), (2.5), (5.7), (5.8) and $h_4 = \sum_{i,j} \epsilon_i\epsilon_j {h_{i\bar{j}}}^2 {h_{j\bar{i}}}^2$. The squared norm $\|\nabla\alpha\|_2$ of the covariant derivative $\nabla\alpha$ of the second fundamental form α on M is defined by

$$\|\nabla\alpha\|_2 = \sum_{x,i,j,k} \epsilon_x\epsilon_i\epsilon_j\epsilon_k (h^x_{ijk}\bar{h}^x_{ijk} + h^x_{ij\bar{k}}\bar{h}^x_{ij\bar{k}}). \tag{5.11}$$

5.2 Normal curvature tensors

In this section, we introduce the concept the normal curvature tensor on the space-like complex submanifold in an indefinite Kaehler manifold and research its properties.

Let M' be an $(n+p)$-dimensional indefinite Kaehler manifold of index $2p$ equipped with indefinite Kaehler structure $\{g', J'\}$ and let M be an n-dimensional space-like complex submanifold of M' endowed with induced Kaehler structure $\{g, J\}$ from the indefinite Kaehler structure $\{g', J'\}$. Let us denote by $\nabla^{\perp}$ the normal connection on M, namely, it is the mapping of $TM \times NM$ into NM defined by

$$\nabla^{\perp}(X, V) = \nabla^{\perp}{}_X V = \text{the normal part of } \nabla'_X V$$

for any tangent vector field X in TM and any normal vector field V in NM, where ∇' is the Kaehler connection on M', and TM and NM are the tangent bundle and the normal bundle of M, respectively (for details, see [26]). The normal curvature tensor $R^{\perp}$ on M is defined by

$$R^{\perp}(X, Y)V = (\nabla^{\perp}{}_X \nabla^{\perp}{}_Y - \nabla^{\perp}{}_Y \nabla^{\perp}{}_X - \nabla^{\perp}{}_{[X,Y]})V,$$

where $X, Y \in TM$ and $V \in NM$. If it satisfies

$$R^{\perp}(X, Y)V = f\, g(X, JY)J'V,$$

where f is any function on M, then the normal connection $\nabla^{\perp}$ is said to be *proper*. In particular, if f is a non-zero constant or equal to 0 on M, then it is said to be *semi-flat* or *flat*, respectively.

Remark 5.1. For the justification of the concept of flatness and semi-flatness, see Chen [6], and Yano and Kon [34], respectively.

On the other hand, the proper case is treated by Ki and Nakagawa [12].

Remark 5.2. In semi-Riemannian geometry, the shape operator A on the indefinite Einstein hypersurface M of index $2s$ in $M^{n+1}_{s+1}(c)$ can be not necessarily diagonalized. By the classification of the self-adjoint endomorphisms of a scalar product, we have the following properties ;

(1) A is diagonalizable,

(2) A is not diagonalizable, but either $\epsilon_{n+1}h_2 < 0$ or $h_2 = 0$ and not totally geodesic.

An indefinite Einstein hypersurface is said to be *proper* if the shape operator A is diagonalizable (see [12]).

Now, in order to consider the normal curvature transformation, we see the local version of the normal curvature tensor. By the property of (3.7) of the normal curvature tensor, we can define a linear transformation T_N on the np-dimensional complex vector space Ξ^{np} consisting of tensors (ξ_{xk}) at each point on M by

$$T_N(\xi_{xk}) = (\eta_{xk}), \quad \eta_{xk} = \sum_{y,m} \epsilon_y \epsilon_m K_{\bar{x}yk\bar{m}} \xi_{ym}.$$

We denote by $({R_{ym}}^{xk})$ the matrix of the linear transformation T_N. The linear operator defined by the $np \times np$ Hermitian matrix $({R_{ym}}^{xk})$ is called *the normal curvature operator* on M. Then T_N is the self-adjoint operator with respect to the definite metric canonically defined on Ξ^{np}. We assume that the matrix $({R_{ym}}^{xk})$ is diagonalizable. In this case, we can choose suitably an indefinite unitary frame field $\{U_A\} = \{U_j, U_y\}$ in such a way that, it satisfies

$$K_{\bar{x}yk\bar{m}} = \epsilon_x \epsilon_k f_{xk} \delta^{xk}_{ym} = \epsilon_x \epsilon_k f_{xk} \delta_{xy} \delta_{km}, \tag{5.12}$$

where every eigenvalue f_{xk} of T_N is a real valued function on M. By (3.7) and (5.12), we have

$$K'_{\bar{x}yk\bar{m}} = \epsilon_x \epsilon_k f_{xk} \delta_{xy} \delta_{km} - \sum_j \epsilon_j h^{x}_{kj} \bar{h}^{y}_{jm}. \tag{5.13}$$

Remark 5.3. In the space-like complex hypersurface M, the normal connection is always proper.

5.3 Locally symmetric spaces

In this section, let M' be an $(n+p)$-dimensional indefinite Kaehler manifold of index $2p$. For two holomorphic planes $P' = [X, J'X]$ and $Q' = [Y, J'Y]$, where X and Y are orthogonal vectors, the holomorphic bisectional curvature $H'(P', Q') = H'(X, Y)$ on M' is defined by

$$H'(P', Q') = H'(X, Y) = \frac{g'(R'(X, J'X)J'Y, Y)}{g'(X, X)g'(Y, Y)}.$$

Assume that M' is locally symmetric, the normal connection of M is proper and it satisfies the following two conditions :

$(*1)$ The space-like holomorphic bisectional curvature is bounded from below by a_1.

$(*2)$ The time-like holomorphic bisectional curvature is bounded from above by a_2.

Then M' is said to satisfy the condition $(*)$ if it satisfies the above conditions $(*1)$ and $(*2)$.

Let M be an n-dimensional space-like complex submanifold of M'. For the local field $\{E_A, E_{A^*}\}$ of orthonormal frames associated with the manifold chosen in chapter 2, we have by (2.15)

$$H'(P'_j, P'_k) = H'(E_j, E_k) = H'_{jk} = \epsilon_j \epsilon_k K'_{\bar{j}jk\bar{k}},$$
$$H'(P'_x, P'_k) = H'(E_x, E_k) = H'_{xk} = \epsilon_x \epsilon_k K'_{\bar{x}xk\bar{k}}$$

for the holomorphic plane $P'_A = [E_A, J'E_A]$. Then it satisfies

$$H'_{jk} \geqq a_1, \qquad H'_{xk} \leqq a_2.$$

Remark 5.4. Let M' be an $(n+p)$-dimensional indefinite complex space form $M_p^{n+p}(c)$ of index $2p$ and of constant holomorphic sectional curvature c. Then M' is locally symmetric and it satisfies the condition $(*)$ and we may consider $a_1 = a_2 = \frac{c}{2}$ if c is non-negative and $a_1 = c,\ a_2 = \frac{c}{2}$ if c is non-positive.

First of all, we estimate Δh_2 from the above on the space-like complex submanifold M. In order to estimate the fourth term and the fifth one in the right hand side of (5.10), we prepare for the basic formulas and a few of properties of the normal curvature operator T_N defined on the submanifold.

Let M' be an $(n+p)$-dimensional indefinite Kaehler manifold of index $2p$ and let M be an n-dimensional space-like complex submanifold of M'. and we check the relation between the normal curvature and the totally real bisectional curvature $H'(P', Q')$ for a space-like holomorphic plane P' and a time-like holomorphic plane Q' in M'. Accordingly, we have by (5.13)

$$H'_{xk} = -K'_{\bar{x}xk\bar{k}} = -\epsilon_x \epsilon_k f_{xk} + \sum_j \epsilon_j h^x_{kj} \bar{h}^x_{kj}. \tag{5.14}$$

Between the holomorphic bisectional curvature and the normal curvature, we get the following relation. By (5.14) and the condition $(*2)$, the normal curvature satisfies

$$f_{xk} = H'_{xk} - \sum_j \epsilon_j h^x_{kj} \bar{h}^x_{kj} \leqq a_2 - \sum_j \epsilon_j h^x_{kj} \bar{h}^x_{kj} \tag{5.15}$$

from which we can estimate the third term and the fourth one in the right hand side of (5.10) as follows :

$$\begin{aligned}
\frac{1}{8} \times \text{the third term} &= - \sum_{x,y,i,j,k} \epsilon_x \epsilon_y \epsilon_i \epsilon_j \epsilon_k K'_{\bar{x}yj\bar{k}} h^y_{ki} \bar{h}^x_{ij} \\
&= - \sum_{x,y,i,j,k} \epsilon_x \epsilon_y \epsilon_i \epsilon_j \epsilon_k (\epsilon_x \epsilon_j f_{xj} \delta_{xy} \delta_{jk} - \sum_m \epsilon_m h^x_{jm} \bar{h}^y_{mk}) h^y_{ki} \bar{h}^x_{ij} \\
&= - \sum_{x,i,j} \epsilon_x \epsilon_i \epsilon_j f_{xj} h^x_{ij} \bar{h}^x_{ij} + \sum_{j,k} \epsilon_j \epsilon_k {h_{j\bar{k}}}^2 {h_{k\bar{j}}}^2 \\
&\leqq \sum_{x,i,j} \epsilon_i (a_2 - \sum_k \epsilon_k h^x_{jk} \bar{h}^x_{jk}) h^x_{ij} \bar{h}^x_{ij} + h_4 \\
&= -a_2 h_2 + h_4 - \sum_{x,j} (\sum_k h^x_{jk} \bar{h}^x_{jk})^2,
\end{aligned}$$

where the second equality follows from (5.13) and the fourth inequality is derived by (5.15). For real numbers $x_1, \ldots, x_m$, since it is easily seen that $\sum_{\alpha=1}^m {x_\alpha}^2 \geqq \frac{1}{m} (\sum_{\alpha=1}^m x_\alpha)^2$, the last term of the above expression can be estimated from the above by $-\frac{1}{p} \sum_j (\sum_{x,k} h^x_{jk} \ \bar{h}^x_{jk})^2$, we have

$$\text{the third term} \leqq -8(a_2 h_2 - h_4 + \frac{1}{p} {h_2}^2). \tag{5.16}$$

On the other hand, let A be the positive semi-definite Hermitian matrix defined by $({A_y}^x) = (\sum_{j,k} h^x_{jk} \bar{h}^y_{jk})$ and let λ_x be its eigenvalue. Then the fourth term can be estimated as follows :

$$\text{the fourth term} = -2\sum_{x,y,k} \lambda_x \delta_{xy} K'_{\bar{x}yk\bar{k}} = -2\sum_{x,k} \lambda_x R'_{\bar{x}xk\bar{k}}$$
$$\leqq 2a_2 \sum_{x,k} \lambda_x = 2na_2 \sum_x \lambda_x,$$

from which it follows that we have by (5.9) and (*2)

$$\text{the fourth term} \leqq -2na_2h_2. \tag{5.17}$$

Next, we estimate the sixth term and the sixth one in the right hand side of (5.10). For the sake of the estimation, we consider the curvature operator T' on M'. From the symmetric relation (2.5), on the n^2-dimensional complex vector space $\Xi_x^{n^2} = T_xM^C \times T_xM^C$ at each point x on M which consists of symmetric tensor (ξ_{ij}), we can define a linear transformation T' by

$$T'(\xi_{ij}) = (\eta_{ij}), \quad \eta_{ij} = \sum_{m,k} \epsilon_m \epsilon_k K'_{\bar{k}ij\bar{m}} \xi_{km}.$$

We denote by $({K'}_{km}{}^{ij})$ the matrix of the linear transformation T'. The linear operator T' defined by the $n^2 \times n^2$ matrix $({K'}_{km}{}^{ij})$ is the called the *curvature operator* on the submanifold M. The curvature operator on the Kaehler manifold plays an important role in Nakagawa and Takagi [20]. Since T' is the self-adjoint operator with respect to the metric canonically induced on $\Xi_x^{n^2}$, every eigenvalue K'_{ik} of T' is a real valued function. So, we have

$$\begin{aligned} K'_{\bar{i}jk\bar{m}} &= {K'}_{km}{}^{ij} = \epsilon_i \epsilon_j K'_{ij} \delta_{ik} \delta_{jm}, \\ K'_{ij} &= K'_{\bar{i}ij\bar{j}} = H'(E_i, E_j). \end{aligned} \tag{5.18}$$

By (5.18) and the condition (*1), we have

$$K_{ij} = K_{\bar{i}ij\bar{j}} = H'(E_i, E_j) = H'_{ij} \geqq a_1.$$

Now, we estimate the fifth term of the right hand side of (5.10). By (5.18), we see

$$\text{the fifth term} = -4 \sum_{x,i,j,k,m} K'_{\bar{k}ij\bar{m}} h^x_{km} \bar{h}^x_{ij} = -4 \sum_{x,i,j,k,m} K'_{ki}\delta_{kj}\delta_{im} h^x_{km} \bar{h}^x_{ij}.$$

From which together with (5.18), it follows that we have

$$\text{the fifth term} = -4 \sum_{i,j} H'_{ij} \sum_{x} h^x_{ij} \bar{h}^x_{ij} \leqq 4a_1 h_2. \tag{5.19}$$

The matrix $(h_{i\bar{j}}{}^2)$ is negative semi-definite Hermitian one, whose eigenvalues λ_i 's are non-positive real functions, that is, $h_{i\bar{j}}{}^2 = \lambda_i \delta_{ij}$. Since $\sum_j \lambda_j = h_2$, the sixth term is estimated as follows :

$$\text{the sixth term} = 4 \sum_{i,j,k} K'_{\bar{i}jk\bar{k}} h_{i\bar{j}}{}^2 = 4 \sum_{j,k} \lambda_j K'_{\bar{j}jk\bar{k}}.$$

Thus, we have

$$\text{the sixth term} \leqq 4(n+1)a_1 h_2. \tag{5.20}$$

Now, we introduce here a fundamental property for the generalized maximum principal due to Omori [23] and Yau [35].

Lemma 5.5. *Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below on M. If a C^2-function f is bounded from above on M, then, for any positive constant ϵ, there exists a point p such that*

$$\|\nabla f(p)\| < \epsilon, \quad \Delta f(p) < \epsilon, \quad \sup f - \epsilon < f(p),$$

where ∇f is the gradient of the function f and Δ denotes the Laplacian operator on M, and $\sup f$ denotes the supremum of the given function f.

Under the above preparation, we can prove the following proposition.

Theorem 5.6. *Let M' be an $(n+p)$-dimensional indefinite Kaehler manifold of index $2p$ and let M be an n-dimensional complete space-like complex submanifold of M'. Assume that M' is locally symmetric and it satisfies the condition $(*)$. If the normal connection of M is proper, then the following statements hold :*

(1) *In the case where $p = 1$ or 2 and $2(n+3)a_1 - (n+4)a_2 \geqq 0$, M is totally geodesic.*

(2) *In the case where $p \geqq 3$ and $2(n+3)a_1 - (n+4)a_2 > 0$, if the scalar curvature on M is bounded from above, then there exists a negative constant h so that if $h_2 > h$, then M is totally geodesic.*

Proof. Since the ambient space is locally symmetric and the squared norm $\|\nabla\alpha\|_2$ of the covariant derivative $\nabla\alpha$ of the second fundamental form α is non-positive by (5.11), the equation (5.10) is estimated by (5.16), (5.17), (5.19) and (5.20) from the above as follows :

$$\begin{aligned}\Delta h_2 \leqq & -8(a_2h_2 - h_4 + \frac{1}{p}{h_2}^2) - 2na_2h_2 \\ & + 4a_1h_2 + 4na_1h_2 - 4h_4 - 2\mathrm{Tr}\ A^2\end{aligned}$$

Accordingly, by (5.9), we obtain

$$\Delta h_2 \leqq A_0{h_2}^2 + A_1h_2, \tag{5.21}$$

where the coefficients A_0 and A_1 are constants given by

$$A_0 = \frac{2}{p}(2p-5), \quad A_1 = 2\{2(n+3)a_1 - (n+4)a_2\}.$$

Now, since the space-like holomorphic bisectional curvature of M is bounded from below by a constant, the Ricci curvature of M is bounded from below. In fact, we have

$$S_{j\bar{j}} = \sum_k K_{\bar{j}jk\bar{k}} \geqq \sum_k K'_{\bar{j}jk\bar{k}} = \sum_k H'_{jk} \geqq na_1$$

with the help of (3.6). Let f be the non-negative function defined by $-h_2$. Then by (5.19), we have

$$\Delta f \geqq c_0 f^2 + c_1 f + c_2 =: F(f), \ c_0 = -A_0 \ , \ c_1 = A_1 \ , \ c_2 = 0,$$

where F is the polynomial of the variable f with the constant coefficients.

In the first assertion, the coefficients satisfy $c_0 > 0 = c_2$, which implies that we can apply Theorem 4.1 to the function f and hence we get $F(\sup f) \leqq 0$. Accordingly, we have $\sup f \leqq 0$. Since the function f is non-negative, it vanishes identically on M, which means that M is totally geodesic.

In the second assertion, we remark that the first coefficient c_0 is negative. Since the scalar curvature on M is bounded from above by the assumption, the function f is bounded from above. In fact, we see

$$r = 2\sum_{j,k} K_{\bar{j}jk\bar{k}} = 2\sum_{j,k} H'_{jk} + 2f \geqq 2n^2 a_1 + 2f$$

with the help of (3.8). Applying Lemma 5.5 to the function f, we obtain $F(\sup f) \leqq 0$, from which we get

$$\sup f = 0 \quad \text{or} \quad \sup f \geqq -\frac{c_1}{c_0} > 0.$$

For a negative constant h such that $h > \frac{c_1}{c_0}$, suppose that $h_2 > h$. Then we get $\inf h_2 \geqq h$ and hence $\sup f \leqq -h < -\frac{c_1}{c_0}$, which means that $\sup f = 0$. Hence f vanishes identically on M, which means that M is totally geodesic.

The proof is completed.

In the case where M is a hypersurface, it is natural that the normal connection is proper. So, the first assertion of Theorem 5.6 proves the following

Corollary 5.7. *Let M' be an $(n+1)$-dimensional indefinite Kaehler manifold of index 2 and let M be an n-dimensional complete space-like complex hypersurface of M'. Assume that M' is locally symmetric and it satisfies $(*)$ with $2(n+1)a_1 - (n+4)a_2 \geqq 0$. Then M is totally geodesic.*

Remark 5.8. Corollary 5.7 is given by Kown and Nakagawa [16].

Theorem 5.9. *Let M be an n-dimensional complete space-like complex submanifold of an $(n+2)$-dimensional indefinite locally symmetric Kaehler manifold M' of index 4. Assume that the normal connection of M is proper. If M' has non-negative space-like holomorphic bisectional curvatures and non-positive time-like holomorphic bisectional curvatures, then M is totally geodesic.*

Proof. Since the fact that M' has non-positive time-like holomorphic bisectional curvature is equivalent to the fact that it satisfies the condition $(*2)$ with $a_2 = 0$. Furthermore, it satisfies the condition $(*1)$ with $a_1 = 0$.

Accordingly, the proof is completed by Theorem 5.6(1).

Corollary 5.10. *Let M' be an $(n+2)$-dimensional indefinite Kaehler manifold of index 4 and let M be an n-dimensional complete space-like complex submanifold of M'. Assume that M' is locally symmetric and it satisfies $(*)$ with $a_1 = a_2 = 0$. If the normal connection of M is proper, then M is totally geodesic.*

Remark 5.11. For the complex coordinate system (z_A, z_{2n+1}) in C_s^{2n+1}, let $M = M(b_j)$ be the complex hypersurface in given by the equation

$$z_{2n+1} = \sum_j (z_j + b_j z_{j^*})^2, \qquad j^* = j + n$$

for any complex number b_j such that $|b_j| = 1$. Then it is seen in [4] and [28] that M is a family of complete indefinite complex hypersurfaces of index $2s$, which are Ricci flat and not flat. Thus we see $c_1 = 0$, but it is not totally geodesic.

Chapter 6

The Ryan condition

6.1 The Laplacian of the squared norm of the Ricci tensor

Taking the exterior derivative of (5.1) and using (5.2), we have

$$f_{ij} + f_{ji} = 0, \quad f_{\bar{i}\bar{j}} + f_{\bar{j}\bar{i}} = 0, \quad f_{i\bar{j}} = f_{\bar{j}i}. \tag{6.1}$$

Hence the Laplacian Δf of the function f is given as

$$\Delta f = 2\sum_j f_{j\bar{j}}. \tag{6.2}$$

We put $f = S_2 = \sum_{j,k} S_{j\bar{k}} S_{k\bar{j}}$. The components $S_{i\bar{j}k}$ and $S_{i\bar{j}\bar{k}}$ of the covariant derivative of the Ricci tensor S are obtained by

$$\sum_k (S_{i\bar{j}k}\omega_k + S_{i\bar{j}\bar{k}}\bar{\omega}_k) = dS_{i\bar{j}} - \sum_k (S_{k\bar{j}}\omega_{ki} + S_{i\bar{k}}\bar{\omega}_{kj}). \tag{6.3}$$

The components $S_{i\bar{j}kl}$ and $S_{i\bar{j}k\bar{l}}$ (resp. $S_{i\bar{j}\bar{k}l}$ and $S_{i\bar{j}\bar{k}\bar{l}}$) of the covariant derivative of $S_{i\bar{j}k}$ (resp. $S_{i\bar{j}\bar{k}}$) are expressed as

$$\begin{aligned}\sum_l (S_{i\bar{j}kl}\omega_l + S_{i\bar{j}k\bar{l}}\bar{\omega}_l) &= dS_{i\bar{j}k} - \sum_l (S_{l\bar{j}k}\omega_{li} + S_{i\bar{l}k}\bar{\omega}_{lj} + S_{i\bar{j}l}\omega_{lk}),\\ \sum_l (S_{i\bar{j}\bar{k}l}\omega_l + S_{i\bar{j}\bar{k}\bar{l}}\bar{\omega}_l) &= dS_{i\bar{j}\bar{k}} - \sum_l (S_{l\bar{j}\bar{k}}\omega_{li} + S_{i\bar{l}\bar{k}}\bar{\omega}_{lj} + S_{i\bar{j}\bar{l}}\bar{\omega}_{lk}).\end{aligned} \tag{6.4}$$

Taking the exterior derivative of (6.3), we obtain

$$\begin{aligned}&S_{i\bar{j}kl} = S_{i\bar{j}lk}, \quad S_{i\bar{j}\bar{k}\bar{l}} = S_{i\bar{j}\bar{l}\bar{k}},\\ &S_{i\bar{j}k\bar{l}} - S_{i\bar{j}\bar{l}k} = \sum_m (K_{\bar{m}ik\bar{l}}S_{m\bar{j}} - K_{\bar{j}mk\bar{l}}S_{i\bar{m}}),\end{aligned} \tag{6.5}$$

by help of (6.2) (6.3) and (6.4).

Now, we are in a position to calculate the Laplacian ΔS_2 of the squared norm of the Ricci tensor S on M. By (6.2) we have

$$\Delta S_2 = 2\sum_{i,j,k}\{2S_{i\bar{j}k}\overline{S}_{i\bar{j}k} + (S_{j\bar{i}}S_{i\bar{j}k\bar{k}} + S_{i\bar{j}}S_{j\bar{i}k\bar{k}})\},$$

and hence we have

$$\Delta S_2 = 2\{\|\nabla S\|_2 + \sum_{i,j,k} S_{i\bar{j}}(S_{j\bar{i}k\bar{k}} + S_{j\bar{i}\bar{k}k})\}, \tag{6.6}$$

where $\|\nabla S\|_2$ is the squared norm of the covariant derivative of the Ricci tensor S, that is, $\|\nabla S\|_2 = 2\sum_{i,j,k} S_{i\bar{j}k}\overline{S}_{i\bar{j}k}$. For the scalar curvature r on M the components r_j and $r_{\bar{j}}$ of the exterior differential dr are given by

$$dr = \sum_j (r_j\omega_j + r_{\bar{j}}\bar{\omega}_j),$$

and the components r_{jk} and $r_{j\bar{k}}$ (resp. $r_{\bar{j}k}$ and $r_{\bar{j}\bar{k}}$) of the covariant derivative of r_j (resp. $r_{\bar{j}}$) are given as

$$\sum_k (r_{jk}\omega_k + r_{j\bar{k}}\bar{\omega}_k) = dr_j - \sum_k r_k\omega_{kj},$$

$$\sum_k (r_{\bar{j}k}\omega_k + r_{\bar{j}\bar{k}}\bar{\omega}_k) = dr_{\bar{j}} - \sum_k r_k\bar{\omega}_{kj}.$$

On the other hand, we have

$$r = 2\sum_{i,j} K_{\bar{i}ij\bar{j}} = 2\sum_j S_{j\bar{j}}, \quad r_i = 2\sum_k S_{k\bar{k}i}, \quad r_{ij} = 2\sum_k S_{k\bar{k}ij}. \tag{6.7}$$

Summing up $j = k$ in (6.5) and using (6.7) and the components of the covariant derivative of the Riemannian curvature tensor R, we get

$$r_{i\bar{l}} - 2\sum_k S_{i\bar{l}\bar{k}k} = 2\sum_m (\sum_k K_{\bar{m}ik\bar{l}}S_{m\bar{k}} - S_{i\bar{m}}S_{m\bar{l}}).$$

Accordingly, we have by (2.5)

$$2\sum_k S_{i\bar{j}\bar{k}k} = r_{i\bar{j}} + 2\sum_m (S_{i\bar{m}}S_{m\bar{j}} - \sum_k K_{\bar{j}ik\bar{m}}S_{m\bar{k}}). \tag{6.8}$$

Next, summing up $i = l$ in (6.5), we obtain

$$\sum_l (S_{l\bar{j}k\bar{l}} - S_{l\bar{j}\bar{l}k}) = \sum_{l,m} (K_{\bar{m}lk\bar{l}}S_{m\bar{j}} - K_{\bar{j}mk\bar{l}}S_{l\bar{m}}),$$

from which it follows that we have similarly

$$2\sum_k S_{i\bar{j}k\bar{k}} = r_{\bar{j}i} + 2\sum_m (S_{i\bar{m}}S_{m\bar{j}} - \sum_k K_{\bar{j}ik\bar{m}}S_{m\bar{k}}), \tag{6.9}$$

Substituting (6.8) and (6.9) into (6.6), we obtain

$$\begin{aligned} \Delta S_2 = {} & 2\|\nabla S\|_2 + 2\sum_{i,j} S_{i\bar{j}}r_{j\bar{i}} \\ & + 4\sum_{m,i,j} S_{j\bar{i}}(S_{i\bar{m}}S_{m\bar{j}} - \sum_k K_{\bar{j}ik\bar{m}}S_{m\bar{k}}), \end{aligned} \tag{6.10}$$

where we have used (6.1). Since $(S_{i\bar{j}})$ is a Hermitian matrix, it can be diagonalized. Thus $S_{i\bar{j}} = \mu_i \delta_{ij}$, where μ_i is a real valued function. From this it follows that we have

$$r = 2\sum_j S_{j\bar{j}} = 2\sum_j \mu_j, \quad S_2 = \sum_{i,j} S_{i\bar{j}} S_{j\bar{i}} = \sum_j \mu_j^2, \tag{6.11}$$

$$S_2 - \frac{r^2}{4n} = \frac{1}{n}\sum_{i,j} (\mu_i - \mu_j)^2. \tag{6.12}$$

And, by (6.10) we get

$$\Delta S_2 \geqq 2\sum_{i,j} S_{i\bar{j}} r_{j\bar{i}} + 2\sum_{i,j} (\mu_i - \mu_j)^2 K_{\bar{i}ij\bar{j}}, \tag{6.13}$$

where the equality holds if and only if the Ricci tensor S is parallel.

And we know that the following lemma

Lemma 6.1. *Let M be an $n(\geqq 3)$-dimensional Kaehler manifold. If the totally real bisectional curvature is bounded from above* (resp. *below*) *by a constant, and if the scalar curvature on M is bounded from below* (resp. *above*), *then the following statements hold true ;*

(1) *the Ricci curvature on M is bounded.*

(2) *the totally real bisectional curvature is bounded.*

The following theorem is originally proved by Ki and Suh [13]. Here, we will give the simple proof of the theorem by using another technique.

Theorem 6.2. *Let M be an $n(\geqq 3)$-dimensional complete Kaehler manifold with constant scalar curvature. If the totally real bisectional curvature is bounded from below by a positive constant, then M is Einstein.*

Proof. Suppose that the totally real bisectional curvature is bounded from below by a positive constant a. So, we have

$$K_{\bar{i}ij\bar{j}} \geqq a > 0, \quad i \neq j.$$

Accordingly, (6.13) can be reduced to

$$\Delta S_2 \geqq 2a \sum_{i,j} (\mu_i - \mu_j)^2.$$

Let us consider a non-negative function $f = S_2 - \frac{r^2}{4n}$. Then, from (6.12) and the above inequality it follows that we have

$$\Delta f \geqq 2naf, \tag{6.14}$$

where the equality holds if and only if the Ricci tensor S is parallel on M. Since the totally real bisectional curvature is bounded from below and the scalar curvature is constant, Lemma 6.1 implies that the Ricci curvature is bounded, where the restriction of dimension is used. By the definition of the function f and (6.11), it implies that f is also bounded from above, because the Ricci curvatures on M is bounded from above, and hence we can apply Lemma 5.5 to the function f. For any positive sequence $\{\epsilon_m\}$ in such a way that it converges to zero as m tends to infinity, there exists a point sequence $\{p_m\}$ in M which satisfies the following properties.

$$\|\nabla f(p_m)\| < \epsilon_m, \quad \Delta f(p_m) < \epsilon_m, \quad \sup f - \epsilon_m < f(p_m).$$

By (6.14) and the above property, we have

$$\epsilon_m > \Delta f(p_m) \geqq 2naf(p_m) > 2na(\sup f - \epsilon_m),$$

which implies $0 \geqq 2na \sup f$. It turns out to be $\sup f = 0$. Since f is non-negative by (6.12), we see that the function f vanishes identically on M. It means that M is Einstein.

The proof is completed.

6.2 The Ryan condition

This section is concerned with Kaehler manifolds with the condition $RS = 0$. Namely, it satisfies

$$R(X, Y)S = 0 \tag{6.15}$$

for any vector fields X and Y.

Let M be a complex n-dimensional connected Kaehler manifold equipped with Kaehler metric tensor g and almost complex structure J, and let $\{U_j\}$ be a local unitary frame field on a neighborhood of M. For the canonical basis $\{U_j, U_{j^*}\}$ the Ricci tensor S and the Riemannian curvature tensor R are given by

$$S(U_j) = \sum_k S_{j\bar{k}} U_k, \quad R(U_i, \overline{U}_j)U_k = -\sum_l K_{\bar{i}jk\bar{l}} U_l,$$

where $U_{j^*} = JU_j$ and $j^* = n + j$. Accordingly, we have

$$\begin{aligned}(R(\overline{U}_i, U_j)S)U_k &= R(\overline{U}_i, U_j)(S(U_k)) - S(R(\overline{U}_i, U_j)U_k) \\ &= -\sum_{m,l}(K_{\bar{i}jm\bar{l}} S_{k\bar{m}} - K_{\bar{i}jk\bar{m}} S_{m\bar{l}})U_l.\end{aligned} \tag{6.16}$$

On the other hand, we see by (6.5)

$$S_{k\bar{l}j\bar{i}} - S_{k\bar{l}\bar{i}j} = \sum_m (K_{\bar{i}jk\bar{m}} S_{m\bar{l}} - K_{\bar{i}jm\bar{l}} S_{k\bar{m}}),$$

from which together with (6.16), it follows that we have

$$(R(\overline{U}_i, U_j)S)U_k = \sum_l (S_{k\bar{l}j\bar{i}} - S_{k\bar{l}\bar{i}j})U_l. \tag{6.17}$$

Thus it is easily seen that the following conditions is equivalent to (6.15).

$$S_{k\bar{l}j\bar{i}} - S_{k\bar{l}\bar{i}j} = 0, \tag{6.18}$$

$$\sum_m (K_{\bar{i}jk\bar{m}} S_{m\bar{l}} - K_{\bar{i}jm\bar{l}} S_{k\bar{m}}) = 0. \tag{6.19}$$

By the same argument as that the equation (6.13) is derived, we obtain

$$\sum_{j,k} (S_j - S_k)^2 K_{\bar{j}jk\bar{k}} = 0,$$

where S_j denotes the Ricci curvature on M. It implies that if the totally real bisectional curvature on M is bounded from above (resp. below) by a negative (resp. positive) constant, then we have

$$\sum_{j,k} (S_j - S_k)^2 = 0,$$

which means that M is Einstein. Thus we can prove

Theorem 6.3. *Let M be a Kaehler manifold. If the totally real bisectional curvature on M is bounded from above* (resp. *below*) *by a negative* (resp. *positive*) *constant, then the following are equivalent.*

(1) *M is Einstein.*

(2) $RS = 0$.

(3) *the Ricci tensor S is parallel.*

Proof. The condition (2) is equivalent to the equation (6.18). This means that (3) implies (2). It is trivial that (1) implies (2) and (3). Thus it is sufficient to prove that (2) implies (1). This already showed.

Hence the proof is completed.

Now, let (M', g') be an $(n+p)$-dimensional connected Kaehler manifold and let M be an n-dimensional connected complex submanifold of M' or let (M', g') be an $(n+p)$-dimensional connected indefinite Kaehler manifold of index $2p (p > 0)$ and let M be an n-dimensional connected space-like complex submanifold of M'. Then M is the Kaehler manifold endowed with the induced metric tensor g. Then, by (3.14) and (3.15) we have

$$K_{\bar{i}ij\bar{j}} = \frac{c}{2} - \sum_x \epsilon_x h_{ij}^x \bar{h}_{ij}^x, \quad i \neq j. \tag{6.20}$$

It implies that if $\epsilon_x = -1$ and if c is positive, then the totally real bisectional curvature is bounded from below by a positive constant. On the other hand, (6.20) implies that if $\epsilon_x = 1$ and if c is negative, then the totally real bisectional curvature is bounded from above by a negative constant. So, as a direct consequence of Theorem 6.3, we can get

Corollary 6.4. *Let M be a space-like complex submanifold of $M_p^{n+p}(c)$. If $c > 0$, then the following statements are equivalent.*

(1) *M is Einstein.*

(2) $RS = 0$.

(3) *the Ricci tensor S is parallel.*

Remark 6.5. This result is due to Aiyama, Kwon and Nakagawa [2].

Corollary 6.6. *Let M be a complex submanifold of $M^{n+p}(c)$. If $c < 0$, then the following statements are equivalent.*

(1) *M is Einstein.*

(2) *$RS = 0$.*

(3) *the Ricci tensor S is parallel.*

Remark 6.7. This result is due to Kon [15].

Finally, we shall prove the following

Theorem 6.8. *Let M be an $n(\geqq 2)$-dimensional space-like complex submanifold of an $(n+p)$-dimensional indefinite complex hyperbolic space $CH_p^{n+p}(c)$ of index $2p$. If M satisfies the condition (6.15) and if the codimension p is less than $n-1$, then M is Einstein.*

Proof. From the Gauss equation (3.14) (3.15) and (6.19), we have

$$c({h_{i\bar{l}}}^2\delta_{jk} - {h_{i\bar{j}}}^2\delta_{kl}) + 2\sum_{r,s,x}(h^x_{ik}\bar{h}^x_{lr}{h_{r\bar{j}}}^2 - h^x_{kr}{h_{r\bar{i}}}^2\bar{h}^x_{jl}) = 0. \tag{6.21}$$

Since $({h_{i\bar{j}}}^2)$ is a negative semi-definite Hermitian matrix, the eigenvalues $\lambda_1, \cdots, \lambda_n$ are non-positive real valued functions on M. Moreover we have

$${h_{i\bar{j}}}^2 = \lambda_i\delta_{ij}. \tag{6.22}$$

From (6.22) the equation (6.21) is reformed as

$$c(\lambda_i - \lambda_j)\delta_{il}\delta_{jk} + 2(\lambda_i - \lambda_j)\sum_x h^x_{ik}\bar{h}^x_{jl} = 0,$$

from which it follows that we have

$$(\lambda_i - \lambda_j)(\sum_x h^x_{ij}\bar{h}^x_{ij} + \frac{c}{2}) = 0,$$
$$(\lambda_i - \lambda_j)\sum_x h^x_{ik}\bar{h}^x_{jl} = 0 \text{ unless } i = l,\ j = k. \tag{6.23}$$

We may assume that $\lambda_1, \cdots, \lambda_q$ are all distinct eigenvalues of the matrix $(h_{i\bar{j}}{}^2)$. Let $n_1, \cdots, n_q$ be multiplicities of $\lambda_1, \cdots, \lambda_q$, respectively, where q is the function on M. If $q = 1$ everywhere on M, then M is Einstein. Suppose that there is a point x of M at which $q \geqq 2$. Then, at the point x there exist at least two distinct eigenvalues. For eigenvalues λ_i and λ_j such that $\lambda_i \neq \lambda_j$ it follows from (6.23) that we have

$$\sum_x h^x_{ij}\bar{h}^x_{ij} = -\frac{c}{2} \quad \text{if } \lambda_i \neq \lambda_j,$$
$$\sum_x h^x_{ik}\bar{h}^x_{jl} = 0 \quad \text{if } \lambda_i \neq \lambda_j \text{ and } (k, l) \neq (j, i). \tag{6.24}$$

Let h_{ij} be a vector in C^p defined by $h_{ij} = (h^{n+1}_{ij,} \cdots, h^{n+p}_{ij})$. Consider the subspace $\{h_{ij} | \lambda_i \neq \lambda_j\}$ consisting of $\sum^q_{r<s} n_r n_s$ vectors in C^p. The equation (6.24) means that they are linearly independent. Accordingly, because of $\sum^q_{r=1} n_r = n$, we have

$$p \geqq \sum^q_{r<s} n_r n_s \geqq n - 1,$$

where the second equality holds if and only if $q = 2$ and $n_1 = 1$ or $n_2 = 1$. In fact, the first inequality follows from the fact that the vectors h_{ij} are contained in C^p and the second inequality is derived from the following argument.

Let f be a function with variables $n_1, \cdots, n_{q-1}$ defined by $\sum^q_{r<s} n_r n_s$ with the condition $\sum^q_{r=1} n_r = n$. Namely, f is given by

$$f(n_1, \cdots, n_{q-1}) = \sum^{q-1}_{r<s} n_r n_s + (n_1 + \cdots + n_{q-1})(n - n_1 - \cdots - n_{q-1}).$$

Then it is easily seen that f is monotonically increasing with respect to the first variable n_1 and hence, because of $n_1 \geqq 1$, we have

$$f(n_1, \cdots, n_{q-1}) \geqq f(1, n_2, \cdots, n_{q-1}),$$

where the equality holds if and only if $n_1 = 1$. Similarly, we have

$$f(1, n_2, \cdots, n_{q-1}) \geqq f(1, 1, n_3, \cdots, n_{q-1})$$

where the equality holds if and only if $n_2 = 1$. Inductively, we have

$$f(n_1, \cdots, n_{q-1}) \geqq f(1, \cdots, 1)$$

where the equality holds if and only if $n_1 = \cdots = n_{q-1} = 1$.
On the other hand, we obtain

$$f(\underbrace{1, \cdots, 1}_{r-1}) - f(\underbrace{1, \cdots, 1}_{r-2}) = n - r + 1 > 0,$$

which implies that we have

$$f(1, \cdots, 1) \geqq f(1),$$

where the equality holds if and only if $q = 2$. Thus we obtain

$$f(n_1, \cdots, n_{q-1}) = \sum_{r<s}^{q} n_r n_s \geqq f(1) = n - 1,$$

where the equality holds if and only if $q = 2$ and $n_1 = 1$.

The proof is completed.

Remark 6.9. It is shown that the product manifold of a 1-dimensional complex hyperbolic space $CH^1(c)$ and an $(n-1)$-dimensional complex hyperbolic space $CH^{n-1}(c)$ is an n-dimensional Kaehler manifold and it is isometrically imbedded in a $(2n-1)$-dimensional indefinite complex hyperbolic space

$CH^{2n-1}_{n-1}(c)$ of index $2(n-1)$ (see [4] and [11]). Then it satisfies the condition (1.1), but it is not Einstein if $n \geqq 3$. This implies that the estimate of the codimension is best possible.

Chapter 7

Totally real bisectional curvature tensors

Let M^n be an n-dimensional Kaehler submanifold of an $(n+p)$-dimensional complex space form $M^{n+p}(c)$, $c > 0$. Let S and r be the Ricci tensor and the scalar curvature of M, respectively. The Ricci curvature of the complex quadric Q^n of $P^{n+1}(c)$ is equal to $\frac{cn}{2}$ and furthermore the set $B(M)$ is less than or equal to $\frac{c}{2}$ and if M is totally geodesic, then $B(M) = \frac{c}{2}$, where $B(M)$ is the set of all totally real bisectional curvatures. Paying attention to this fact, we consider whether or not in the value distribution of $B(M)$ the maximal value is discrete.

At the beginning of this chapter, we shall consider the following a generalized maximum principle.

Theorem 7.1. *Let M be a complete Riemannian manifold whose Ricci cur-*

vature is bounded from below. Let f be a non-negative function on M satisfies

$$\Delta f \geqq kf, \tag{7.1}$$

where k is a positive constant. If f is bounded, then f vanishes identically.

Proof. Under the assumption of the theorem, we can apply the generalized maximum principle due to Omori [23] and Yau [35] for the function f bounded from above. So, for any positive number ϵ, there exists a point $\{p\}$ in M which satisfies the following properties :

$$\|\nabla f(p)\| < \epsilon, \qquad \Delta f(p) < \epsilon, \qquad \sup f - \epsilon < f(p).$$

Thus, for any positive sequence $\{\epsilon_m\}$ in such a way that the sequence converges to zero as m tends to infinity, there exists a point sequence $\{p_m\}$ in M which satisfies the following properties :

$$\|\nabla f(p_m)\| < \epsilon_m, \qquad \Delta f(p_m) < \epsilon_m, \qquad \sup f - \epsilon_m < f(p_m). \tag{7.2}$$

By (7.1) and the above property (7.2), we have

$$\epsilon_m > \Delta f(p_m) \geqq kf(p_m) > k(\sup f - \epsilon_m),$$

which implies that we have $0 \geqq k \sup f$. It turns out to be $\sup f = 0$, because k is positive and f is non-negative. Accordingly, we see that the function f vanishes identically on M.

The proof is completed.

Remark 7.2. We do not know whether or not Theorem 7.1 holds without the condition that the function is bounded from above. Although it may be the difficult problem, it seems to be very interesting to wrestle with the problem.

Let M be an $n(\geqq 3)$-dimensional Kaehler submanifold of an $(n+p)$-dimensional complex space form $M' = M^{n+p}(c)$ of constant holomorphic sectional curvature c. Then by the equation (3.14) of Gauss, we have

$$K_{\bar{j}jk\bar{k}} = \frac{c}{2} - \sum_x h^x_{jk} \bar{h}^x_{jk} \leqq \frac{c}{2}, \qquad j \neq k.$$

Thus we see that for any totally real plane section $[X, Y]$, the totally real bisectional curvature $B(X, Y)$ satisfies

$$B(X, Y) \leqq \frac{c}{2}.$$

Now let $a(M)$ be the infimum of the set $B(M)$ of totally real bisectional curvatures of M. Though the set B is bounded from above, we have no information on $a(M)$. In their paper [13], Ki and Suh proved the following

Theorem 7.3. *Let $M = M^n$ be an $n(\geqq 3)$-dimensional complete Kaehler submanifold of an $(n+p)$-dimensional Kaehler manifold $M' = M^{n+p}(c)$ of constant holomorphic sectional curvature $c(> 0)$. If $a(M) > a_1$, then M is totally geodesic, where*

$$a_1 = \frac{c(n^3 + 2n^2 + 2n - 2)}{2n(n^2 + 2n + 3)}.$$

Since the matrix $H = (h_{j\bar{k}}{}^2)$ defined by $h_{j\bar{k}}{}^2 = \sum_{m,x} h^x_{jm} \bar{h}^x_{mk}$ and the matrix $A = (A^x_y)$ defined by $A^x_y = \sum_{j,k} h^x_{jk} \bar{h}^y_{jk}$ are both positive Hermitian ones, the eigenvalues λ_j of H and the eigenvalues λ_x of A are non-negative real valued functions on M. Thus it is easily seen that

$$\begin{aligned}
&\sum_j \lambda_j = \mathrm{Tr}\ H = h_2, \quad \sum_x \lambda_x = \mathrm{Tr}\ A = -h_2, \\
&h_2{}^2 \geqq h_4 = \sum_j \lambda_j{}^2 \geqq \frac{1}{n} h_2{}^2, \qquad (7.3)\\
&h_2{}^2 \geqq \mathrm{Tr}\ A^2 = \sum_x \lambda_x{}^2 \geqq \frac{1}{p} h_2{}^2,
\end{aligned}$$

where the second equality in the second relationship holds if and only if all eigenvalues of the matrix H are equal, and the second equality in the last relationship holds if and only if all eigenvalues of the matrix A are equal. It means that each equality holds if and only if the rank of matrices H and A are at most one.

Next, we will prove the following theorem.

Theorem 7.4. *Let $M = M^n$ be an $n(\geqq 3)$-dimensional complete Kaehler submanifold of an $(n+p)$-dimensional Kaehler manifold $M' = M^{n+p}(c)$ of constant holomorphic sectional curvature $c(> 0)$. Then there exists a constant a_2 depending only upon n and c so that if $a(M) > a_2$, then M is totally geodesic, where $a_2 < a_1$.*

Proof. By (3.17), we have

$$\Delta h_2 \geqq c(n+2)h_2 - 4h_4 - 2\mathrm{Tr}\, A^2,$$

where the equality holds if and only if the second fundamental form α of M is parallel. Together the above equality with the above properties about eigenvalues (7.3), it follows that

$$\Delta h_2 \geqq c(n+2)h_2 - 6{h_2}^2,$$

where the equality holds if and only if the second fundamental form of M is parallel and the rank of the matrices H and A are at most one. A non-negative function f is defined by h_2. Then the above inequality is reduced to

$$\Delta f \geqq -6f^2 + c(n+2)f, \tag{7.4}$$

where the equality holds if and only if the second fundamental form of M is parallel and the rank of the matrices H and A are at most one. By (4.16), (4.17) and (3.8), we have

$$2na(M) \leqq \frac{cn(n+1)}{2} - h_2 - n(n-1)a(M).$$

This yields that

$$f = \sum_j \lambda_j = h_2 \leqq \frac{n(n+1)}{2}(c - 2a(M)), \quad \lambda_j \geqq 0, \tag{7.5}$$

where the first equality holds if and only if $K_{\bar{j}jj\bar{j}} = 2a(M)$ and $K_{\bar{j}jk\bar{k}} = a(M)$ for any indices $j \neq k$. This means that each eigenvalue λ_j is bounded. On the other hand, since the Ricci curvature $S_{j\bar{j}}$ of M is given by (3.15) as

$$S_{j\bar{j}} = \frac{c(n+1)}{2} - \lambda_j,$$

it is also bounded. Applying the generalized maximum principle due to Omori [23] and Yau [35] to the bounded function f, we see that for any sequence $\{\epsilon_m\}$ of positive numbers which converges to 0 as m tends to infinity, there exists a point sequence $\{p_m\}$ such that

$$\|\nabla f(p_m)\| < \epsilon_m, \qquad \Delta f(p_m) < \epsilon_m, \qquad \sup f - \epsilon_m < f(p_m).$$

Thus, we have

$$\begin{aligned} &lim_{m\to\infty} \Delta f(p_m) \leqq lim_{m\to\infty} \epsilon_m = 0, \\ &lim_{m\to\infty} f(p_m) = \sup f. \end{aligned} \tag{7.6}$$

By (7.4) and (7.6), we see

$$(\sup f)\{\sup f - \frac{c(n+2)}{6}\} \geqq 0,$$

which means that

$$\sup f = 0 \quad \text{or} \quad \sup f \geqq \frac{c(n+2)}{6}.$$

If $\sup f = 0$, then f vanishes identically on M, because f is non-negative. Then M is totally geodesic.

Suppose that M is not totally geodesic. So, f satisfies $\sup f \geqq \frac{c(n+2)}{6}$. On the other hand, by (7.5), we have

$$\sup f \leqq \frac{n(n+1)}{2}(c - 2a(M)).$$

Thus, we see that

$$a(M) \leqq \frac{c(3n^2+2n-2)}{6n(n+1)}.$$

We denote the right hand side of the above inequality by a_2, which is the constant depending on n and c, which implies that if $a(M) > a_2$, then M is totally geodesic.

The proof is completed.

Remark 7.5. By straightforward calculations, we can easily show that $a_1 > a_2$. Since the estimation is rough and it is not the best possible, there may be room for further improvement. For the holomorphic pinching, Ros [29] determined the best possibility under the compact submanifolds. Under the condition $a(M) > a_2$, we have by (4.18) and (3.8)

$$\begin{aligned}(n-2)K_{\bar{j}j j\bar{j}} &\geqq (n-1)(n+4)a(M) - \frac{r}{2} \\ &= (n-1)(n+4)a(M) - \frac{c}{2}n(n+1) + h_2,\end{aligned}$$

and hence we have

$$K_{\bar{j}j j\bar{j}} > \frac{1}{2(n-2)}\{2(n-1)(n+4)a_2 - cn(n+1)\} > 0$$

for $n \geqq 3$. Therefore, M is compact because the holomorphic sectional curvatures of M are positive (see [5] and [32]).

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