Time Optimal Control Problem of Semilinear Retarded Systems in Hilbert Spaces

힐버트 공간상에서 준선형 지연계의 시간최적제어 문제

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A Dissertation

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힐버트 공간상에서 준선형 지연계의 시간초적제어 문제

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요약

본 논문에서는 Hilbert 공간 H 상에서 다음과 같은 준선형 지연계의 포물형 형태의 미분방정식에 의한 시간최적제어 문제를 다루는데 있다.

(RSE)
$$\begin{cases} \dfrac{dx(t)}{dt} = A_0 x(t) + A_1 x(t-h) + \\ \qquad\qquad \int_{-h}^{0} a(s) A_2 x(t) ds + f(t.x(t)) + k(t), \\ x(0) = \phi^0, \qquad x(s) = \phi^1(s) \qquad -h \le s \langle 0. \end{cases}$$

또 다른 Hilbert 공간 V는 H공간에서 조밀성을 가지며 그의 공액공간을 V^*라 하면 작용소 A_0는 $V \times V$에서 Gårding 부등식을 만족하고 sesquilinear 형태로부터 정의되어 질때 A_0는 H와 V^*에서 해석적 반군 $S(t)$를 생성 하므로 (RSE)는 H와 V^*에서 고려되어진다. $A_i(i=1,2)$ 작용소들도 비슷한 성질을 가질 때 먼저 (RSE)의 해의 존재성, 유일성 그리고 정규성을 다루었으며 이러한 성질을 바탕으로 하여 주 작용소 $A_i(i=0,1,2)$ 들이 unbounded 일 때 기본해를 구축하여 그 성질을 다루어 (RSE)에서의 시간 최적 문제를 조사하였다.

1. Introduction

Let H be a complex Hilbert space. We assume that another Hilbert space V is embedded in H as a dense subspace and that V has a stronger topology than H. Let V^* be the dual space of V. In this paper we deal with the time optimal control problem governed by the semilinear parabolic type equation in a Hilbert space H as follows.

$$\text{(RSE)}\qquad \begin{cases} \dfrac{d}{dt}x(t) = A_0x(t) + A_1x(t-h) \\ \qquad\qquad + \displaystyle\int_{-h}^{0} a(s)A_2x(t+s)ds + f(t,x(t)) + k(t), \\ x(0) = \phi^0, \quad x(s) = \phi^1(s) \quad -h \le s < 0. \end{cases}$$

Let A_0 be the operator associated with a bounded sesquilinear form on $V \times V$, satisfies Gårding inequality. Then A_0 generates an analytic semigroup $S(t)$ in both H and V^* and so the equation (RSE) may be considered as an equation in both H and V^*.

Let $(\phi^0, \phi^1) \in H \times L^2(0,T;V)$ and $x(T;\phi,f,u)$ be a solution of the system (RSE) associated with nonlinear term f and control u at time T.

We now define the fundamental solution $W(t)$ of (RSE) by

$$W(t)\phi^0 = \begin{cases} x(t;(\phi^0,0),0,0), & t \ge 0 \\ 0 & t < 0. \end{cases}$$

According to the above definition $W(t)$ is a unique solution of

$$W(t) = S(t) + \int_0^t S(t-s)\{A_1W(s-h) + \int_{-h}^{0} a(\tau)A_2W(s+\tau)d\tau\}ds$$

for $t \ge 0$ (cf. Nakagiri [5]). Under the conditions that $a(\cdot) \in L^2(-h,0;\mathcal{R})$ and $A_i (i = 1,2)$ are bounded linear operators on H into itself, Nakagiri in [5] proved the standard optimal control problems and the time optimal control problem for linear retarded system (RSE) in case $f \equiv 0$ in Banach spaces. If $A_i (i = 0,1,2) : D(A_0) \subset H \to H$ are unbounded operators, Blasio, Kunish and Sinestrari in [2] obtained the global existence and uniqueness of a strict solution for the linear retarded system in Hilbert spaces. Under some general condition of the Lipschitz continuity of nonlinear operator f from $\mathcal{R} \times V$ to H, in [4] they established the problem for existences and uniqueness of solution of the given system. But we can not immediately obtain the time optimal control problem as in [5; section 8] without the condition for boundedness of the fundamental solution $W(t)$. Since the integral of $A_0S(t-s)$ has a singularity at $t = s$, we can not solve directly the integral equation of $W(t)$. In [6], Tanabe was investigated the fundamental solution $W(t)$ by constructing the resolvent operators for integrodifferential equations of Volterra type(see (3.14), (3.21) of [6]) under the condition that $a(\cdot)$ is real valued and Hölder continuous on $[-h, 0]$.

This paper deals with the time optimal control problem by using the construction of fundamental solution, which is the same results of [5], in case the principal operators $A_i (i = 0, 1, 2)$ are unbounded operators.

2. Retarded semilinear equations

The inner product and norm in H are denoted by $(\cdot, \cdot)$ and $|\cdot|$. The notations $||\cdot||$ and $||\cdot||_*$ denote the norms of V and V^* as usual, respectively. Hence we may regard that

$$||u||_* \leq |u| \leq ||u||, \quad u \in V. \tag{2.1}$$

Let $a(\cdot, \cdot)$ be a bounded sesquilinear form on $V \times V$, satisfies Gårding's inequality

$$\text{Re } a(u, u) \geq c_0||u||^2 - c_1|u|^2, \quad c_0 > 0, \quad c_1 \geq 0. \tag{2.2}$$

Let A_0 be the operator associated with the sesquilinear form $-a(\cdot, \cdot)$:

$$(A_0u, v) = -a(u, v), \quad u, \ v \in V.$$

It follows from (2.2) that for every $u \in V$

$$\text{Re}\,((c_1 - A_0)u, u) \geq c_0||u||^2.$$

Then A_0 is a bounded linear operator from V to V^*, and its realization in H which is the restriction of A_0 to

$$D(A_0) = \{u \in V; A_0u \in H\}$$

is also denoted by A_0. Here, we note that $D(A_0)$ is dense in V. Therefore, it is also dense in H. Then A_0 generates an analytic semigroup in both H and V^*. Hence we may assume that there exists a constant C_0 such that

$$||u|| \leq C_0 ||u||_{D(A_0)}^{1/2} |u|^{1/2} \tag{2.3}$$

for every $u \in D(A_0)$, where

$$||u||_{D(A_0)} = (|A_0 u|^2 + |u|^2)^{1/2}$$

is the graph norm of $D(A_0)$.

First, we introduce the following linear retarded functional differential equation:

$$\text{(RE)} \qquad \begin{cases} \dfrac{d}{dt}x(t) = A_0 x(t) + A_1 x(t-h) \\ \qquad\qquad + \displaystyle\int_{-h}^{0} a(s) A_2 x(t+s) ds + k(t), \\ x(0) = \phi^0, \quad x(s) = \phi^1(s) \quad -h \leq s < 0. \end{cases}$$

Here, the operators A_1 and A_2 are bounded linear from V to V^* such that their restrictions to $D(A_0)$ are bounded linear operators from $D(A_0)$ to H. The function $a(\cdot)$ is assumed to be a real valued and Hölder continuous in the interval $[-h, 0]$.

Let $W(\cdot)$ be the fundamental solution of the linear equation associated with (RE) which is the operator valued function satisfying

$$\begin{aligned} W(t) = S(t) + \int_0^t & S(t-s)\{A_1 W(s-h) \\ & + \int_{-h}^{0} a(\tau) A_2 W(s+\tau) d\tau\} ds, \quad t > 0, \\ W(0) = I, \qquad & W(s) = 0, \quad -h \leq s < 0, \end{aligned} \tag{2.4}$$

where $S(\cdot)$ is the semigroup generated by A_0. Then

(2.5)

$$x(t) = W(t)\phi^0 + \int_{-h}^{0} U_t(s)\phi^1(s) ds + \int_0^t W(t-s) k(s) ds,$$

$$U_t(s) = W(t-s-h) A_1 + \int_{-h}^{s} W(t-s+\sigma) a(\sigma) A_2 d\sigma.$$

Recalling the formulation of mild solutions, we know that the mild solution of (RE) is also represented by

$$x(t) = \begin{cases} S(t)\phi^0 + \int_0^t S(t-s)\{A_1 x(s-h) \\ + \int_{-h}^0 a(\tau)A_2 x(s+\tau)d\tau + k(s)\}ds, \ (t>0), \\ \phi(s), \quad -h \le s < 0. \end{cases}$$

From Theorem 1 in [6] it follows the following results.

Proposition 2.1. *The fundamental solution $W(t)$ to (RE) exists uniquely. The functions $A_0W(t)$ and $dW(t)/dt$ are strongly continuous except at $t = nh$, $n = 0, 1, 2, \ldots$, and the following inequalities hold: for $i = 0, 1, 2$ and $n = 0, 1, 2, \ldots$*

$$|A_iW(t)| \le C_n/(t-nh), \tag{2.6}$$

$$|dW(t)/dt| \le C_n/(t-nh), \tag{2.7}$$

$$|A_iW(t)A_0^{-1}| \le C_n \tag{2.8}$$

in $(nh, (n+1)h)$,

$$|\int_t^{t'} A_iW(\tau)d\tau| \le C_n \tag{2.9}$$

for $nh \le t < t' \le (n+1)h$. Let ρ be the order of Hölder continuity of $a(\cdot)$. Then for $nh \le t < t' \le (n+1)h$ and $0 < \kappa < \rho$

$$|W(t') - W(t)| \le C_{n,\kappa}(t'-t)^\kappa (t-nh)^{-\kappa}, \tag{2.10}$$

$$|A_i(W(t') - W(t))| \le C_{n,\kappa}(t'-t)^\kappa (t-nh)^{-\kappa-1}, \tag{2.11}$$

$$|A_i(W(t') - W(t))A_0^{-1}| \le C_{n,\kappa}(t'-t)^\kappa (t-nh)^{-\kappa}, \tag{2.12}$$

where C_n and $C_{n,\kappa}$ are constants dependent on n and n, κ, respectively, but not on t and t'.

Considering as an equation in V^* we also obtain the same norm estimates of (2.6)-(2.12) in the space V^*. By virtue of Theorem 3.3 of [2] we have the following result for the linear equation (RE).

Proposition 2.2. *1) Let $F = (D(A_0), H)_{\frac{1}{2},2}$ where $(D(A_0), H)_{1/2,2}$ denote the real interpolation space between $D(A_0)$ and H. For (ϕ^0, ϕ^1) $\in F \times L^2(-h, 0; D(A_0))$ and $k \in L^2(0, T; H)$, $T > 0$, there exists a unique solution x of (RE) belonging to*

$$L^2(-h, T; D(A_0)) \cap W^{1,2}(0, T; H) \subset C([0, T]; F)$$

and satisfying

$$\begin{aligned}(2.13) \qquad ||x||_{L^2(-h,T;D(A_0))\cap W^{1,2}(0,T;H)} &\leq C_1'(||\phi^0||_F \\ &+ ||\phi^1||_{L^2(-h,0;D(A_0))} + ||k||_{L^2(0,T;H)}),\end{aligned}$$

where C_1' is a constant depending on T.
2) Let $(\phi^0, \phi^1) \in H \times L^2(-h, 0; V)$ and $k \in L^2(0, T; V^*)$, $T > 0$. Then there exists a unique solution x of (RE) belonging to

$$L^2(-h, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H)$$

and satisfying

$$\begin{aligned}(2.14) \qquad ||x||_{L^2(-h,T;V)\cap W^{1,2}(0,T;V^*)} &\leq C_1'(|\phi^0| \\ &+ ||\phi^1||_{L^2(-h,0;V)} + ||k||_{L^2(0,T;V^*)}).\end{aligned}$$

In what follows we assume that

$$||W(t)|| \leq M, \quad t > 0$$

for the sake of simplicity.

Proposition 2.3. *Let $k \in L^2(0, T; H)$ and $x(t) = \int_0^t W(t - s)k(s)ds$. Then there exists a constant C_1' such that for $T > 0$*

$$(2.15) \qquad ||x||_{L^2(0,T;D(A_0))} \leq C_1'||k||_{L^2(0,T;H)},$$

$$(2.16) \qquad ||x||_{L^2(0,T;H)} \leq MT||k||_{L^2(0,T;H)},$$

and

$$(2.17) \qquad ||x||_{L^2(0,T;V)} \leq (C_1' MT)^{\frac{1}{2}}||k||_{L^2(0,T;H)}.$$

Proof. The assertion (2.15) is immediately obtained from Proposition 2.2 for the equation (RE) with $(\phi^0, \phi^1) = (0, 0)$. Since

$$\begin{aligned} ||x||^2_{L^2(0,T;H)} &= \int_0^T |\int_0^t W(t-s)k(s)ds|^2 dt \\ &\le M^2 \int_0^T (\int_0^t |k(s)|ds)^2 dt \\ &\le M^2 \int_0^T t \int_0^t |k(s)|^2 ds dt \\ &\le M^2 \frac{T^2}{2} \int_0^T |k(s)|^2 ds \end{aligned}$$

we have that

$$||x||_{L^2(0,T;H)} \le MT ||k||_{L^2(0,T;H)}.$$

From (2.3), (2.15), and (2.16) it follows that

$$||x||_{L^2(0,T;V)} \le (C_1' MT)^{\frac{1}{2}} ||k||_{L^2(0,T;H)}. \quad \square$$

Let f be a nonlinear mapping from $\mathcal{R} \times V$ into H. We assume that for any x_1, $x_2 \in V$ there exists a constant $L > 0$ such that

(F1) $$|f(t, x_1) - f(t, x_2)| \le L ||x_1 - x_2||,$$

(F2) $$f(t, 0) = 0.$$

The following result on (RSE) is obtained from Theorem 2.1 in [4].

Proposition 2.4. *Suppose that the assumptions (F1), (F2) are satisfied. Then for any $(\phi^0, \phi^1) \in H \times L^2(-h, 0; V)$ and $k \in L^2(0, T; V^*)$, $T > 0$, the solution x of (RE) exists and is unique in $L^2(-h, T; V) \cap W^{1,2}(0, T; V^*)$, and there exists a constant C_2' depending on T such that*

(2.18) $$\begin{aligned} ||x||_{L^2(-h,T;V) \cap W^{1,2}(0,T;V^*)} \le C_2' (1 + |\phi^0| \\ + ||\phi^1||_{L^2(-h,0;V)} + ||k||_{L^2(0,T;V^*)}). \end{aligned}$$

3. Lemmas for fundamental solutions

For the sake of simplicity we assume that $S(t)$ is uniformly bounded and the following inequalities hold:

$$
\begin{aligned}
(3.1) \qquad & |S(t)| \leq M_0 (t \geq 0),\ |A_0 S(t)| \leq M_0/t (t > 0), \\
& |A_0^2 S(t)| \leq M_0/t^2 (t > 0)
\end{aligned}
$$

for some constant M_0(e.g., [6]). Let us assume that $a(\cdot)$ is Hölder continuous of order ρ:

$$
(3.2) \qquad |a(\cdot)| \leq H_0, \quad |a(s) - a(\tau)| \leq H_1 (s - \tau)^\rho
$$

for some constants H_0, H_1.

According to Tanabe [6] we set

$$
(3.3) \qquad V(t) = \begin{cases} A_0(W(t) - S(t)), & \text{if } t \in (0, h], \\ A_0(W(t) - \int_{nh}^{t} S(t-s) A_1 W(s-h) ds), \\ \text{if } t \in (nh, (n+1)h] \quad (n = 0, 1, 2, \ ...). \end{cases}
$$

For $0 < t \leq h$

$$
W(t) = S(t) + A_0^{-1} V(t)
$$

and from (2.4) and (3.3) the exchange of the order of integration yields

$$
W(t) = S(t) + \int_0^t \int_\tau^t S(t-s) a(\tau - s) ds A_2 W(\tau) d\tau.
$$

Hence,

$$
V(t) = V_0(t) + \int_0^t A_0 \int_\tau^t S(t-s) a(\tau - s) ds A_2 A_0^{-1} V(\tau) d\tau,
$$

where

$$
V_0(t) = \int_0^t A_0 \int_\tau^t S(t-s) a(\tau - s) ds A_2 S(\tau) d\tau.
$$

For $nh \leq t \leq (n+1)h \quad (n = 0, 1, 2, \ \dots)$ the fundamental solution $W(t)$ is represented by

$$\begin{aligned} W(t) =& S(t) + \int_h^t S(t-s)A_1 W(s-h)ds \\ &+ \int_0^{t-h} \int_\tau^{\tau+h} S(t-s)a(\tau-s)dsA_2 W(\tau)d\tau \\ &+ \int_{t-h}^{nh} \int_\tau^t S(t-s)a(\tau-s)dsA_2 W(\tau)d\tau \\ &+ \int_{nh}^t \int_\tau^t S(t-s)a(\tau-s)dsA_2 W(\tau)d\tau. \end{aligned}$$

The integral equation to be satisfied by (3.3) is

$$V(t) = V_0(t) + \int_{nh}^t A_0 \int_\tau^t S(t-s)a(\tau-s)dsA_2 A_0^{-1} V(\tau)d\tau$$

where

$$\begin{aligned} V_0(t) =& A_0 S(t) + A_0 \int_h^{nh} S(t-s)A_1 W(s-h)ds \\ &+ \int_0^{t-h} A_0 \int_\tau^{\tau+h} S(t-s)a(\tau-s)dsA_2 W(\tau)d\tau \\ &+ \int_{t-h}^{nh} A_0 \int_0^t S(t-s)a(\tau-s)dsA_2 W(\tau)d\tau \\ &+ \int_{nh}^t A_0 \int_\tau^t S(t-s)a(\tau-s)dsA_2 \int_{nh}^\tau S(\tau-\sigma)A_1 W(\sigma-h)d\sigma d\tau. \end{aligned}$$

Thus, the integral equation (3.3) can be solved by successive approximation and $V(t)$ is uniformly bounded in $[nh, (n+1)h]$ (e.g. (3.16) and the preceding part of (3.40) in [6]). It is not difficult to show that for $n > 1$,

$$V(nh+0) \neq V(nh-0) \quad \text{and} \quad W(nh+0) = W(nh-0).$$

Lemma 3.1. *For $0 < s < t$ and $0 < \alpha < 1$*

$$|S(t) - S(s)| \le \frac{M_0}{\alpha}\left(\frac{t-s}{s}\right)^\alpha, \tag{3.4}$$

$$|A_0 S(t) - A_0 S(s)| \le M_0 (t-s)^\alpha s^{-\alpha-1}. \tag{3.5}$$

Proof. From (3.1) for $0 < s < t$

$$|S(t) - S(s)| = |\int_s^t A_0 S(\tau) d\tau| \le M_0 \log \frac{t}{s}. \tag{3.6}$$

It is easily seen that for any $t > 0$ and $0 < \alpha < 1$

$$\log(1+t) \le t^\alpha / \alpha. \tag{3.7}$$

Combining (3.7) with (3.6) we get (3.4). For $0 < s < t$

$$|A_0 S(t) - A_0 S(s)| = |\int_s^t A_0^2 S(\tau) d\tau| \le M_0 (t-s)/ts. \tag{3.8}$$

Noting that $(t-s)/t \le ((t-s)/t)^\alpha$ for $0 < \alpha < 1$,
we obtain (3.5) from (3.8). □

We define the operator $K_1(t', t) : H \to H$ by

$$K_1(t', t) = \int_t^{t'} S(t'-s) A_1 W(s-h) ds, \tag{3.9}$$

for $nh \le t < t' < (n+1)h$.

Lemma 3.2. *$K_1(t', t)$ is uniformly bounded for $0 < t < t'$.*

Proof. Let $nh < t < (n+1)h$, $n = 0, 1, 2, \ldots$. Then the proof is a consequence of the following estimate

(3.10)

$$\begin{aligned} &|\int_{nh}^t S(t-\xi) A_1 W(\xi - h) d\xi| \\ &= |\int_{nh}^t (S(t-\xi) - S(t-nh)) A_1 W(\xi - h) d\xi \\ &+ S(t-nh) \int_{nh}^t A_1 W(\xi - h) d\xi| \\ &\le \int_{nh}^t M_0 \log \frac{t-nh}{t-\xi} \frac{C_{n-1}}{\xi - nh} d\xi + M_0 C_{n-1} \\ &\le M_0 C_{n-1} c_0 + M_0 C_{n-1}. \end{aligned}$$

If $t < nh < t'$ and $0 < t' - t < h$, then

$$K_1(t', t) = \int_t^{nh} S(t'-s)A_1W(s-h)ds + \int_{nh}^{t'} S(t'-s)A_1W(s-h)ds. \tag{3.11}$$

The first term of right hand side of (3.11) is

$$\begin{aligned}
&\int_t^{nh} S(t'-s)A_1W(s-h)ds \\
&\quad = \int_t^{nh} (S(t'-s) - S(t'-(n-1)h))A_1W(s-h)ds \\
&\quad + S(t'-(n-1)h)\int_t^{nh} A_1W(s-h)ds.
\end{aligned}$$

Thus,

$$\begin{aligned}
&|\int_t^{nh} (S(t'-s) - S(t'-(n-1)h))A_1W(s-h)ds| \\
&\quad \leq M_0C_{n-1}\int_t^{nh} \log\frac{t'-(n-1)h}{t'-s}\frac{ds}{s-(n-1)h} \\
&\quad \leq M_0C_{n-1}\int_{(n-1)h}^{t'} \log\frac{t'-(n-1)h}{t'-s}\frac{ds}{s-(n-1)h} \\
&\quad = M_0C_{n-1}\int_0^1 \log\frac{1}{1-\tau}\frac{d\tau}{\tau}, \\
&|S(t'-(n-1)h)\int_t^{nh} A_1W(s-h)ds| \leq M_0C_{n-1},
\end{aligned}$$

and hence, it is bounded. The boundedness of the second term of right hand side of (3.11) is obtained from (3.10). □

Remark 1. Let $K_1^*(t', t)$ be the adjoint of $K_1(t', t)$. Let $x^* \in D(A_0^*)$. Then from the fact that

$$\begin{aligned}
< K_1(t', t)x, x^* > &= \int_t^{t'} < S(t'-s)A_1W(s-h)x, x^* > ds \\
&= \int_t^{t'} < x, W^*(s-h)A_1^*S^*(t'-s)x^* > ds
\end{aligned}$$

where A_1^* is the formal adjoint operator of A_1, we have

$$K_1^*(t',t)x^* = \int_t^{t'} W^*(s-h)A_1^*S^*(t'-s)x^*ds.$$

and if $x^* \in D(A_0^*)$ then

$$\lim_{t'\to t} K_1^*(t',t) = 0 \tag{3.12}$$

in the sense of strong convergence. Since $K_1(t',t)$ is uniformly bounded, so is $K_1^*(t',t)$. From that $D(A_0^*)$ is dense in H, we have (3.12) in H.

We introduce another operator $K_2(t',t) : H \to H$ by

$$K_2(t',t) = \int_t^{t'} S(t'-s)\int_{-h}^0 a(\tau)A_2W(s+\tau)d\tau ds$$

for $0 \le t < t'$.

To obtain the estimate of $K_2(t',t)$ we need the following result.

Lemma 3.3. *For $0 \le t < t'$ and $t'-t < h$, there exists a constant C such that*

$$|K_2(t',t)| \le C(t'-t). \tag{3.13}$$

Proof. In $[0,h]$, we transform $K_2(t',t)$ by suitable change of variables and Fubini's theorem as

$$K_2(t',t) = \int_t^{t'} S(t'-s)\int_0^s a(\tau-s)A_2W(\tau)d\tau ds.$$

From (3.3) it follows

$$\int_0^t a(\tau-s)A_iW(\tau)d\tau = \int_0^t a(\tau-s)A_iA_0^{-1}(A_0S(\tau)+V(\tau))d\tau$$
$$= \int_0^t (a(\tau-s)-a(-s))A_iA_0^{-1}A_0S(\tau)d\tau + \int_0^t a(-s)A_iA_0^{-1}A_0S(\tau)d\tau$$
$$+ \int_0^t a(\tau-s)A_iA_0^{-1}V(\tau)d\tau.$$

Noting that

$$|\int_0^t (a(\tau-s)-a(-s))A_iA_0^{-1}A_0S(\tau)d\tau| \le M_0H_1|A_iA_0^{-1}|\int_0^t \tau^{\rho-1}d\tau,$$

we have

$$\begin{aligned}|\int_0^t a(\tau-s)A_iW(\tau)d\tau| \le& |A_iA_0^{-1}|(\frac{h^\rho}{\rho}M_0H_1 + hH_0M_0 \\ &+ hH_0(\sup_{0\le t\le h}|V(t)|)).\end{aligned}$$

Thus the assertion (3.13) holds in $[0,h]$. In $[nh,(n+1)h)$, we get

$$\begin{aligned}K_2(t',t) &= \int_t^{t'} S(t'-s)\int_{-h}^0 a(\tau)A_2W(\tau+s)d\tau ds \\ &= \int_t^{t'} S(t'-s)\int_{s-h}^s a(\tau-s)A_2W(\tau)d\tau ds.\end{aligned}$$

If $nh \le t \le s \le t'$ then

(3.14)

$$\begin{aligned}&\int_{s-h}^s a(\tau-s)A_2W(\tau)d\tau \\ &\quad = \int_{s-h}^{nh} a(\tau-s)A_2W(\tau)d\tau + \int_{nh}^s a(\tau-s)A_2W(\tau)d\tau.\end{aligned}$$

The second term of right hand side (3.14) is bounded in terms of (2.9). The estimate of the first term of right hand side (3.14) is

$$\begin{aligned}|\int_{s-h}^{nh} a(\tau-s)A_2W(\tau)d\tau| = |&\int_{s-h}^{nh}(a(\tau-s)-a(-h))A_2W(\tau)d\tau \\ &+ a(-h)\int_{s-h}^{nh} A_2W(\tau)d\tau|.\end{aligned}$$

Since $s > nh$, noting that $0 \le \tau - s + h < \tau - (n-1)h$

$$
\begin{aligned}
&|\int_{s-h}^{nh}(a(\tau - s) - a(-h))A_2W(\tau)d\tau| \\
&\le H_1|A_2A_0|^{-1}\int_{s-h}^{nh}(\tau - s + h)^\rho(\tau - (n-1)h)^{-1}d\tau \\
&\le H_1|A_2A_0|^{-1}\int_{s-h}^{nh}(\tau - (n-1)h)^{\rho-1}d\tau \\
&\le H_1|A_2A_0|^{-1}\int_{(n-1)h}^{nh}(\tau - (n-1)h)^{\rho-1}d\tau \le H_1|A_2A_0|^{-1}h^\rho.
\end{aligned}
$$

The estimate of the second term of right hand side (3.14) is

$$
\begin{aligned}
&|\int_{nh}^{s} a(\tau - s)A_2W(\tau)d\tau| \\
&\quad \le |\int_{nh}^{s}(a(\tau - s) - a(nh - s))A_2W(\tau)d\tau| + |a(nh - s)\int_{nh}^{s} A_2W(\tau)d\tau| \\
&\quad \le H_1M_0|A_2A_0|^{-1}\int_{nh}^{s}\tau^{\rho-1}d\tau + H_0C_{n-1}.
\end{aligned}
$$

If $t < nh < t'$ then $(n-1)h < t < nh < t' < (n+1)h$. First, let $t < s < nh$, then

$$
\begin{aligned}
&\int_{s-h}^{s} a(\tau - s)A_2W(\tau)d\tau \\
&\quad = \int_{s-h}^{(n-1)h} a(\tau - s)A_2W(\tau)d\tau + \int_{(n-1)h}^{s} a(\tau - s)A_2W(\tau)d\tau \\
&\quad = \int_{s-h}^{(n-1)h}(a(\tau - s) - a(-h))A_2W(\tau)d\tau \\
&\qquad + a(-h)\int_{s-h}^{(n-1)h} A_2W(\tau)d\tau \\
&\qquad + \int_{(n-1)h}^{s}(a(\tau - s) - a((n-1)h - s))A_2W(\tau)d\tau \\
&\qquad + a((n-1)h - s)\int_{(n-1)h}^{s} A_2W(\tau)d\tau,
\end{aligned}
$$

in case $nh < s < t'$, we have

$$\int_{s-h}^{s} a(\tau - s)A_2W(\tau)d\tau$$

$$= \int_{s-h}^{nh} a(\tau - s)A_2W(\tau)d\tau + \int_{nh}^{s} a(\tau - s)A_2W(\tau)d\tau$$

$$= \int_{s-h}^{nh} (a(\tau - s) - a(-h))A_2W(\tau)d\tau$$

$$+ a(-h)\int_{s-h}^{nh} A_2W(\tau)d\tau$$

$$+ \int_{nh}^{s} (a(\tau - s) - a(nh - s))A_2W(\tau)d\tau$$

$$+ a(nh - s)\int_{nh}^{s} A_2W(\tau)d\tau.$$

Therefore, from (3.1), (3.2) and Lemma 3.1 it follows (3.13). □

4. Time optimal control for retarded systems

Let Y be a real Banach space. In what follows the admissible set U_{ad} be a weakly compact subset in $L^2(0,T;Y)$. Consider the following hereditary controlled system:

$$\text{(RSC)} \qquad \begin{cases} \dfrac{d}{dt}x(t) = A_0x(t) + A_1x(t-h) \\ \qquad\qquad + \displaystyle\int_{-h}^{0} a(s)A_2x(t+s)ds + f(t,x(t)) + Bu(t), \\ x(0) = \phi^0, \quad x(s) = \phi^1(s) \quad -h \le s < 0, \\ u \in U_{ad}. \end{cases}$$

Here the controller B is a bounded linear operator from Y to H. We denote the solution $x(t)$ in (RSC) by $x_u(t)$ to express the dependence on $u \in U_{ad}$. That is, x_u is a trajectory corresponding to the controll u. Suppose the target set W is weakly compact in H and define

$$U_0 = \{u \in U_{ad} : x_u(t) \in W \text{ for some } \ t \in [0,T]\}$$

for $T > 0$ and suppose that $U_0 \neq \emptyset$. The optimal time is defined by low limit t_0 of t such that $x_u(t) \in W$ for some admissible control u. For each $u \in U_0$ we can define the first time $\tilde{t}(u)$ such that $x_u(\tilde{t}) \in W$. Our problem is to find a control $\bar{u} \in U_0$ such that

$$\tilde{t}(\bar{u}) \leq \tilde{t}(u) \quad \text{for all } u \in U_0$$

subject to the constraint (RSC).
Since $x_u \in C([0,T]; H)$, the transition time $\tilde{t}(u)$ is well defined for each $u \in U_{ad}$.

Theorem 4.1. *1) Let $F = (D(A_0), H)_{1/2,2}$. If $(\phi^0, \phi^1) \in F \times L^2(-h, 0; D(A_0))$ and $k \in L^2(0,T;H)$, then the solution x of the equation (RSE) belonging to $L^2(-h,T;D(A_0)) \cap W^{1,2}(0,T;H)$, and the mapping $F \times L^2(-h,0;D(A_0\ )) \times L^2(0,T;H) \ni (\phi^0,\phi^1,k) \mapsto x \in L^2(-h,T;D(A_0)) \cap W^{1,2}(0,T;H)$ is continuous.*
2) If $(\phi^0,\phi^1) \in H \times L^2(-h,0;V)$ and $k \in L^2(0,T;V^*)$, then the solution x of the equation (RSE) belonging to $L^2(-h,T;V)) \cap W^{1,2}(0,T;V^*)$, and the mapping $H \times L^2(-h,0;V\ )) \times L^2(0,T;V^*) \ni (\phi^0,\phi^1,k) \mapsto x \in L^2(-h,T;V) \cap W^{1,2}(0,T;V^*)$ is continuous.

Proof. [1] We know that x belongs to $L^2(0,T;D(A_0)) \cap W^{1,2}(0,T;H)$ from Proposition 2.2. Let $(\phi_i^0,\phi_i^1,k_i) \in F \times L^2(-h,0;D(A_0)) \times L^2(0,T;H)$, and x_i be the solution of (RSE) with (ϕ_i^0,ϕ_i^1,k_i) in place of (ϕ^0,ϕ^1,k) for $i = 1,\ 2$. Then in view of Proposition 2.2 we have

(4.1)

$$\begin{aligned}
&||x_1 - x_2||_{L^2(-h,T;D(A_0)) \cap W^{1,2}(0,T;H)} \leq C_1'\{||\phi_1^0 - \phi_2^0||_F \\
&\quad + ||\phi_1^1 - \phi_2^1||_{L^2(-h,0:D(A_0))} + ||f(\cdot,x_1) - f(\cdot,x_2)||_{L^2(0,T;H)} \\
&\quad + ||k_1 - k_2||_{L^2(0,T;H)}\} \\
&\quad \leq C_1'\{||\phi_1^0 - \phi_2^0||_F + ||\phi_1^1 - \phi_2^1||_{L^2(-h,0:D(A_0))} + ||k_1 - k_2||_{L^2(0,T;H)} \\
&\quad + L||x_1 - x_2||_{L^2(0,T:V)}\}.
\end{aligned}$$

Since

$$x_1(t) - x_2(t) = \phi_1^0 - \phi_2^0 + \int_0^t (\dot{x}_1(s) - \dot{x}_2(s))ds,$$

we get

$$||x_1 - x_2||_{L^2(0,T;H)} \leq \sqrt{T}|\phi_0^1 - \phi_2^0| + \frac{T}{\sqrt{2}}||x_1 - x_2||_{W^{1,2}(0,T;H)}.$$

Hence arguing as in (2.3) we get

(4.2)

$$
\begin{aligned}
||x_1 - x_2||_{L^2(0,T;V)} &\le C_0 ||x_1 - x_2||^{1/2}_{L^2(0,T;D(A_0))} ||x_1 - x_2||^{1/2}_{L^2(0,T;H)} \\
&\le C_0 ||x_1 - x_2||^{1/2}_{L^2(0,T;D(A_0))} \\
&\times \{T^{1/4}|\phi_1^0 - \phi_2^0|^{1/2} + (\frac{T}{\sqrt{2}})^{1/2} ||x_1 - x_2||^{1/2}_{W^{1,2}(0,T;H)}\} \\
&\le C_0 T^{1/4} |\phi_1^0 - \phi_2^0|^{1/2} ||x_1 - x_2||^{1/2}_{L^2(0,T;D(A_0))} \\
&+ C_0 (\frac{T}{\sqrt{2}})^{1/2} ||x_1 - x_2||_{L^2(0,T;D(A_0)) \cap W^{1,2}(0,T;H)} \\
&\le 2^{-7/4} C_0 |\phi_1^0 - \phi_2^0| \\
&+ 2C_0 (\frac{T}{\sqrt{2}})^{1/2} ||x_1 - x_2||_{L^2(0,T;D(A_0)) \cap W^{1,2}(0,T;H)}.
\end{aligned}
$$

Combining (4.1) with (4.2) we obtain

(4.3)

$$
\begin{aligned}
||x_1 - x_2||_{L^2(-h,T;D(A_0)) \cap W^{1,2}(0,T;H)} &\le C_1' \{ ||\phi_1^0 - \phi_2^0||_F \\
&+ ||\phi_1^1 - \phi_2^1||_{L^2(-h,0:D(A_0))} + ||k_1 - k_2||_{L^2(0,T;H)} \\
&+ 2^{-7/4} C_0 L |\phi_1^0 - \phi_2^0| \\
&+ 2C_0 (\frac{T}{\sqrt{2}})^{1/2} L ||x_1 - x_2||_{L^2(0,T;D(A_0)) \cap W^{1,2}(0,T;H)} \}.
\end{aligned}
$$

Suppose that $(\phi_n^0, \phi_n^1, k_n) \to (\phi^0, \phi^1, k)$ in $F \times L^2(-h, 0; D(A_0)) \times L^2(0, T; H)$, and let x_n and x be the solutions (RSE) with $(\phi_n^0, \phi_n^1, k_n)$ and (ϕ^0, ϕ^1, k) respectively. Let $0 < T_1 \le T$ with

$$2C_0 C_1' (T_1/\sqrt{2})^{1/2} L < 1.$$

Then by virtue of (4.3) with T replaced by T_1, we see that $x_n \to x$ in $L^2(-h, T_1; D(A_0)) \cap W^{1,2}(0, T_1; H)$. This implies that $(x_n(T_1), (x_n)_{T_1}) \mapsto (x(T_1), x_{T_1})$ in $F \times L^2(-h, 0; D(A_0))$. Hence the same argument shows that $x_n \to x$ in

$$L^2(T_1, \min\{2T_1, T\}; D(A_0)) \cap W^{1,2}(T_1, \min\{2T_1, T\}; H).$$

Repeating this process we conclude that $x_n \to x$ in $L^2(-h,T;D(A_0))\cap W^{1,2}(0,T;H)$.
[2] From proposition 2.2 or 2.4 we have

$$\begin{aligned}||x_1 - x_2||_{L^2(-h,T;V)\cap W^{1,2}(0,T;V^*)} &\leq C_1'\{|\phi_1^0 - \phi_2^0| \\ &+ ||\phi_1^1 - \phi_2^1||_{L^2(-h,0:V)} + ||f(\cdot,x_1) - f(\cdot,x_2)||_{L^2(0,T;V^*)} \\ &+ ||k_1 - k_2||_{L^2(0,T;V^*)}\} \\ &\leq C_1'\{|\phi_1^0 - \phi_2^0| + ||\phi_1^1 - \phi_2^1||_{L^2(-h,0:V)} + ||k_1 - k_2||_{L^2(0,T;V^*)} \\ &+ L||x_1 - x_2||_{L^2(0,T:V)}\}.\end{aligned}$$

Hence, in virtue of (4.2) and since the embedding $L^2(-h,T;D(A_0))\cap W^{1,2}(0,T;H) \subset L^2(-h,T;V)\cap W^{1,2}(0,T;V^*)$ is continuous, by the similar way of 1) we can obtain the result of 2) □

Theorem 4.2. *Assume that $U_0 \neq \emptyset$. Then there exists a time optimal control for (RSC).*

Proof. Let $t_n \to t_0 + 0$, u_n be an admissible control and suppose that the trajectory x_n corresponding to u_n belongs to W. Let $\mathcal{F}$ and $\mathcal{B}$ be the Nemitsky operators corresponding to the maps f and B, which are defined by

$$(\mathcal{F}u)(\cdot) = f(\cdot, x_u) \quad \text{and} \quad (\mathcal{B}u)(\cdot) = Bu(\cdot),$$

respectively. Then

(4.4)
$$\begin{aligned}x_n(t_n) =&x(t_n;\phi,0) + \int_0^{t_0} W(t_n - s)((\mathcal{F}+\mathcal{B})u_n)(s)ds, \\ &+ \int_{t_0}^{t_n} W(t_n - s)((\mathcal{F}+\mathcal{B})u_n)(s)ds,\end{aligned}$$

where

$$x(t_n;\phi,0) = W(t_n)\phi^0 + \int_{-h}^0 U_{t_n}(s)\phi^1(s)ds.$$

From Proposition 2.4 it follows that

(4.5) $$x(t_n;\phi,0) \to x(t_0;\phi,0) \quad \text{strongly in } H.$$

The third term in (4.4) tends to zero as $t_n \to t_0 + 0$ from the fact that

(4.6)
$$|\int_{t_0}^{t_n} W(t_n - s)((\mathcal{F} + \mathcal{B})u_n)(s)ds|$$
$$\leq (\sup_{t\in[0,T]} ||W(t)||)\{LC_2'(|\phi^0| + ||\phi^1||_{L^2(0,T;V)} + ||u_n||_{L^2(0,T;Y)}) + |f(0)|$$
$$+ ||B|| ||u||_{L^2(0,T;Y)}\}(t_n - t_0)^{1/2}.$$

By the definition of fundamental solution $W(t)$ we have

$$\begin{aligned} W(t+\epsilon) - S(\epsilon)W(t) &= S(t+\epsilon) + \int_0^{t+\epsilon} S(t+\epsilon-s)\{A_1 W(s-h) \\ &+ \int_{-h}^0 a(\tau)A_2 W(s+\tau)d\tau\}ds \\ &- S(\epsilon)\{S(t) + \int_0^t S(t-s)\{A_1 W(s-h) \\ &+ \int_{-h}^0 a(\tau)A_2 W(s+\tau)d\tau\}ds \\ &= \int_t^{t+\epsilon} S(t+\epsilon-s)\{A_1 W(s-h) \\ &+ \int_{-h}^0 a(\tau)A_2 W(s+\tau)d\tau\}ds \\ &= K_1(t+\epsilon, t) + K_2(t+\epsilon, t). \end{aligned}$$

Hence, since

$$W(t_n - s) = S(t_n - t_0)W(t_0 - s) + K_1(t_n - s, t_0 - s) + K_2(t_n - s, t_0 - s)$$

the second term of (4.4) is represented as

(4.7)
$$\int_0^{t_0} S(t_n - t_0)W(t_0 - s)((\mathcal{F} + \mathcal{B})u_n)(s)ds$$
$$+ \int_0^{t_0} (K_1(t_n - s, t_0 - s) + K_2(t_n - s, t_0 - s))((\mathcal{F} + \mathcal{B})u_n)(s)ds.$$

The second term of the (4.7) tends to zero as $n \to 0$ in terms of Remark1 and Lemma 3.3.

We denote $x_n(t_n)$ by w_n. Since W and U_{ad} are weakly compact, there exist an $u_0 \in U_0$, $w_0 \in W$ such that we may assume that $w-\lim u_n = u$ in U_{ad} and $w - \lim w_n = w_0$ in $L^2 \cap W^{1,2}$.

Let $p \in H$. Then $S^*(t_n - t_0)p \to p$ strongly in H and by (F1) and Theorem 4.1,

$$(4.8) \qquad W(t_0 - \cdot)((\mathcal{F} + \mathcal{B})u_n)(\cdot) \to W(t_0 - \cdot)((\mathcal{F} + \mathcal{B})u_0)(\cdot)$$

weakly $L^2(0, T; V)$. Hence from (4.5)-(4.8) it follows that

$$(w_0, p) = (x(t_0; \phi, 0), p) + \int_0^{t_0} (W(t_0 - s)((\mathcal{F} + \mathcal{B})u_0)(s), p)ds$$

by tending $n \to \infty$. Since p is arbitrary, we have

$$w_0 = x(t_0; \phi, 0) + \int_0^{t_0} W(t_0 - s)((\mathcal{F} + \mathcal{B})u_0)(s)ds \in W$$

and hence w_0 is the trajectory corresponding to u_0, i.e., $u_0 \in U_0$. □

Now we consider the case where the target set W is singleton.

Consider that $W = w_0$ such that $\phi^0 \neq w_0$ and $\phi^1(s) \neq w_0$ for some $s \in [-h, 0)$. Then we can choose a decreasing sequence $\{W_n\}$ of weakly compact sets with nonempty interior such that

$$(4.9) \quad w_0 \in \bigcap_{n=1}^{\infty} W_n, \text{ and } \operatorname{dist}(w_0, W) = \sup_{x \in W_n} |x - w_0| \to 0 (n \to \infty).$$

Define

$$U_0^n = \{u \in U_{ad} : x_u(t) \in W_n \text{ for some } t \in [0, T]\}.$$

Then, we may assume that u_n is the time optimal control with the optimal time t_n to the target set W_n, $n = 1, 2, \ldots$

Theorem 4.3. *Let $\{W_n\}$ be a sequence of closed convex in X satisfying the condition (4.9) and $U_0^n \neq \emptyset$. Then there exists a time optimal control u_0 with the optimal time $t_0 = \sup_{n \geq 1}\{t_n\}$ to the point target* *set $\{w_0\}$ which is given by the weak limit of some subsequence of $\{u_n\}$ in $L^2(0, t_0; Y)$.*

Proof. Since (4.9) is satisfied and U_{ad} is weakly compact, there exists $w_n = x_n(t_n) \in W_n \to w_0$ strongly in H. Since U_{ad} is weakly compact, there exists $u_0 \in U_{ad}$ such that $u_n \to u_0$ weakly in $L^2(0, t_0; Y)$. Thus, from the similar argument used in the proof of Theorem 4.2 we can easily prove that u_0 is the time optimal control and t_0 is the optimal time to the target $\{w_0\}$. □

Remark 2. Let x_u be the solution of (RSC) corresponding to u. Then the mapping $u \mapsto x_u$ is compact from $L^2(0, T; Y)$ to $L^2(0, T; H)$. We define the solution mapping S from $L^2(0, T; Y)$ to $L^2(0, T; H)$ by

$$(Su)(t) = x_u(t), \quad u \in L^2(0, T; Y).$$

In virtue of Proposition 2.4

$$\begin{aligned} ||Su||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} &= ||x_u|| \le C_2'(1 + |\phi^0| \\ &+ ||\phi^1||_{L^2(-h,0;V)} + ||u||_{L^2(0,T;Y)}). \end{aligned}$$

Hence if u is bounded in $L^2(0, T; Y)$, then so is x_u in $L^2(0, T; V) \cap W^{1,2}(0, T; V^*)$. Since V is compactly embedded in H by assumption, the embedding $L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset L^2(0, T; H))$ is also compact in view of Theorem 2 of Aubin [1]. Hence, the mapping $u \mapsto Su = x_u$ is compact from $L^2(0, T; Y)$ to $L^2(0, T; H)$.

Since $\{x_n\}$ is bounded in $L^2 \cap W^{1,2}$ and $L^2 \cap W^{1,2} \subset L^2(0, T; H)$ compactively it holds $x_n \to x$ strongly in $L^2(0, T; H)$. Since $x_n \to x$ weakly in $L^2 \cap W^{1,2}$ we have $x_n \to x$ strongly in $L^2(0, T; H)$. From (F1) and Lemma 3.1 we see that $\mathcal{F}$ is a compact operator from $L^2(0, T; Y)$ to $L^2(0, T; H)$ and hence, it holds $\mathcal{F}u_n \to \mathcal{F}u$ strongly in $L^2(0, T; V^*)$. Therefore $(\mathcal{F}u_n, x^*) = (\mathcal{F}u_0, x^*)$.

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