Time optimal initial function problem for functional differential equations with time delay

시간지연을 가진 함수미분방정식에 대한 시간최적 초기함수문제

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A thesis submitted in partial fulfillment of the requirements
for the degree of

Master of Science

in the Department of Applied Mathematics, Graduate School,
Pukyyong National University

February 2004

노현희의 이학석사 학위논문을 인준함

2003년 12월 26일

주 심 이학박사 김 태 화

위 원 이학박사 조 낙 은

위 원 이학박사 정 진 문

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A Dissertation

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December 26, 2003

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요 약

이 논문은 주어진 Banach 공간 X 에서 시간지연항을 포함한 선형 함수미분방정식의 초기함수 문제에 대한 제어이론으로서 초기함수의 허용가능한 집합에서의 시간최적문제를 함수적 해석으로 다루고자 한다. 먼저 다음과 같이 주어진 지연항을 포함한 함수미분방정식:

$$\text{(REC)} \begin{cases} \dfrac{dx(t)}{dt} = A_0 x(t) + A_1 x(t-h) + \displaystyle\int_{-h}^{0} a(s) A_2 x(t+s) ds + f(t, x(t)) + f(t), & t>0 \\ x(0) = \phi^0, \quad x(s) = \phi^1(s), \quad s \in [-h, 0), & \phi = (\phi^0 \phi^1) \in U_{ad} \end{cases}$$

에서 $A_i (i=0,1,2)$는 주어진 공간에서 비유계인 선형작용소이고, U_{ad} 는 초기함수의 허용가능한 집합으로서 약 compact를 만족한다. $A_i (i=0,1,2)$의 작용소가 유계인 경우의 지연시스템의 경우에는 많은 연구가 되어져 있으나 비유계의 경우에는 해의 성질의 해석이 어려워 제어이론의 연구에 어려움이 있었다. 그러나 이논문에서는 $S(t) = \exp(tA_0)$라 할 때 다음의 적분방정식:

$$W(t) = S(t) + \int_0^t S(t-s) \{ A_1 W(s-h) + \int_{-h}^{0} a(\tau) W(t) A_2 W(s+\tau) d\tau \} ds$$

을 만족하는 기본해 $W(t)$ 를 해석하여 다음과 같은 최적문제의 결과를 얻을 수 있다.

(주결과) 초기함수의 허용가능한 집합 U_{ad} 는 $X \times L^p(0, T; D(A_0))$ 상에서 약 compact의 집합이고 $\phi = (\phi^0, \phi^1) \in U_{ad}$ 에 대한 (REC)의 해를 $x(t, \phi)$ 라 하자. 주어진 약 compact인 타켓집합 M 에 대하여 $U_0 = \{\phi \in U_{ad} : x(t, \phi) \in M \quad \text{for some} \quad t \in [0, T]\}$ 가 공집합이 아닐 때 $x(\hat{t}; \phi) \in M$ 을 만족하는 최초의 시간을 $\hat{t}(\phi)$ 로 정의할 수 있다. 이 때

$$\hat{t}(g) \le \hat{t}(\phi) \quad \text{for all} \quad \phi \in U_0$$

를 만족하는 시간최적 초기함수 $g = (g^0, g^1)$ 의 존재성을 증명하고 몇가지 응용가능한 문제를 조사하였다.

1. Introduction

Let X be a Banach space. In this paper we deal with the time optimal initial function problem governed by linear parabolic type equation in X as follows

$$\text{(REC)} \qquad \begin{cases} \frac{d}{dt}x(t) = A_0x(t) + A_1x(t-h) \\ \qquad\qquad + \int_{-h}^{0} a(s)A_2x(t+s)ds + f(t), \ t \geq 0 \\ x(0) = \phi^0, \quad x(s) = \phi^1(s) \quad -h \leq s < 0, \\ \phi = (\phi^0, \phi^1) \in U_{ad}. \end{cases}$$

Let the admissible set U_{ad} be assumed to be weakly compact in $X \times L^p(0,T;D(A_0))(1 < p < \infty)$.

Let $(\phi^0, \phi^1) \in X \times L^p(0,T;D(A_0))$ and $x(t;\phi)$ be a solution of the system (REC) associated with initial function $\phi \in U_{ad}$ at time t. Suppose the target set W is weakly compact in X and define

$$U_0 = \{\phi \in U_{ad} : x(t;\phi) \in W \text{ for some } \ t \in [0,T]\}$$

for $T > 0$ and suppose that $U_0 \neq \emptyset$. The our problem is to find a initial function $g \in U_0$ such that

$$\tilde{t}(g) \leq \tilde{t}(\phi) \quad \text{for all } \phi \in U_0$$

subject to the constraint (REC). We assume that A_0 is a densely closed linear operator which generates an analytic semigroup $S(t)$ in X, and A_1 and A_2 are closed linear operators with domains $D(A_1)$ and $D(A_2)$ containing the domain $D(A_0)$.

There exist many literatures which studies optimal control problems of control systems in Banach spaces. However, most studies have been devoted to the systems without delay and the papers treating the optimal initial functions for the retarded system with unbounded operators are not so many.

Under the conditions $a(\cdot) \in L^2(-h,0;\mathcal{R})$ and $A_i(i = 1,2)$ are bounded linear operators on Banach space X into itself, S. Nakariri in [5] proved the existence, uniqueness, and a variation of constant formular for mild solutions as given the initial data $(\phi^0, \phi^1) \in X \times L^2(0,T;X)$ and investigated the standard optimal control proplems and the time optimal control problem for linear retarded system (REC). If X is a Hilbert space and $A_i (i = 0,1,2) : D(A_0) \subset X \to H$ are unbounded operators, Di Blasio, Kunish and Sinestrari in [2] obtained L^2-regularity, global existence and uniqueness of the strict solution for linear retarded system in Hilbert spaces. Moreover, let X be ζ-convex, that is, the Hilbert transform is bounded from $L^2(0,\infty;X)(1 < p < \infty)$ to itself. Then, Dore and Venni as in [3] obtained the L^p-regurarity for the initial value problem (REC).

The main problem is the construction of the fundamental solution $W(t)$ in case $A_i(i = 1,2,3)$ are unbounded, which is defined by

$$W(t) = \begin{cases} x(t;(\phi^0, 0)), & t \geq 0 \\ 0 & t < 0. \end{cases}$$

The fundamental solution $W(t)$ is transformed to the integral equation

$$W(t) = S(t) + \int_0^t S(t-s)\{A_1 W(s-h) + \int_{-h}^0 a(\tau) A_2 W(s+\tau) d\tau\} ds$$

for $t \geq 0$. (cf. Nakagiri [6]).

In [7], H. Tanabe investigated the fundamental solution $W(t)$ by constructing the resolvent operators for integrodifferential equations of Volterra type (see (3.14), (3.21) of [7]) with the condition that $a(\cdot)$ is real valued and Hölder continuous on $[-h, 0]$.

This paper deals with the time optimal initial function problem by using the construction of fundamental solution in case where the principal operators $A_i(i = 0,1,2)$ are unbounded operators. Maximum principle and bang-bang principle for the time optimal initial function are also given.

2. Functional differential equations with time delay

Let X be a complex Banach space with norm $|\cdot|$. We assume that the principal operator $A_0 : D(A_0) \subset X \to X$ is a densely defined closed and unbounded linear operator which generates an analytic semigroup $S(t)$ in X. $D(A_0)$ will be regarded as a Banach space with the graph norm $||x||_{D(A_0)} = |x| + |A_0 x|$. $A_i (i = 1, 2)$ are closed linear operators with domains $D(A_i)$ containing the domain $D(A_0)$ of A_0.

The state space $M_p \equiv X \times L^p(-h, 0; D(A_0))$ of the equation (REC) is the Banach space with the norm

$$
\text{(2.1)} \qquad ||g||_{M_p} = \begin{cases} (|g^0|^p + \int_{-h}^0 ||g^1(s)||^p ds)^{1/p}, \text{ if } 1 \le p < \infty, \\ |g^0| + ||g^1||_\infty, \text{ if } p = \infty \end{cases}
$$

for every $g = (g^0, g^1) \in M_p$. Since X is reflexive and $1 < p < \infty$, the adjoint space $(M_p)^*$ of M_p is identified with the product space $X^* \times L_{p'}(-h, 0; D(A_0)^*)$ via the duality pairing

$$
< g, f >_{M_p} = < g^0, f^0 > + \int_{-h}^0 < g^1(s), f^1(s) >_{D(A_0)} ds
$$

for every $g = (g^0, g^1) \in M_p$ and $f = (f^0, f^1) \in (M_p)^*$ where $< \cdot, \cdot >$ denote the duality pairing between X and X^*. First, we introduce the following linear retarded functional differential equation:

$$
\text{(RE)} \qquad \begin{cases} \frac{d}{dt} x(t) = A_0 x(t) + A_1 x(t-h) \\ \qquad\qquad + \int_{-h}^0 a(s) A_2 x(t+s) ds + f(t), \\ x(0) = \phi^0, \quad x(s) = \phi^1(s) \quad -h \le s < 0 \end{cases}
$$

for every $\phi \in M_p$. The function $a(\cdot)$ is assumed to be a real valued and Hölder continous in the interval $[-h, 0]$.

Recalling the formulation of mild solutions, we know that the mild solution of (RE) is also represented by

$$
x(t) = \begin{cases} S(t)\phi^0 + \int_0^t S(t-s)\{A_1 x(s-h) \\ + \int_{-h}^0 a(\tau) A_2 x(s+\tau) ds + f(s)\} ds, \ (t > 0), \\ \phi(s), \quad -h \le s < 0. \end{cases}
$$

Let $W(\cdot)$ be the fundamental solution of the linear equaton associated with (RE) which is the operator valued function satisfying

(2.2)
$$W(t) = S(t) + \int_0^t S(t-s)\{A_1 W(s-h) + \int_{-h}^0 a(\tau)A_2 W(s+\tau)d\tau\}ds, \quad t > 0,$$
$$W(0) = I, \qquad W(s) = 0, \quad -h \le s < 0,$$

where $S(\cdot)$ is the semigroup generated by A_0. Then

(2.3)
$$x(t) = W(t)\phi^0 + \int_{-h}^0 U_t(s)\phi^1(s)ds + \int_0^t W(t-s)f(s)ds,$$
$$U_t(s) = W(t-s-h)A_1 + \int_{-h}^s W(t-s+\sigma)a(\sigma)A_2 d\sigma.$$

From Theorem 1 in [7] it follows the following results.

Proposition 2.1. *The fundamental solution $W(t)$ of (RE) exists uniquely. The functions $A_0W(t)$ and $dW(t)/dt$ are strongly continuous except at $t = nh,\ h = 0, 1, 2,\ \ldots$, and the following inequalities hold: for $i = 0, 1, 2$ and $n = 0, 1, 2,\ \ldots$*

(2.4) $$|A_i W(t)| \le C_n/(t-nh),$$

(2.5) $$|dW(t)/dt| \le C_n/(t-nh),$$

(2.6) $$|A_i W(t) A_0^{-1}| \le C_n$$

in $(nh, (n+1)h)$,

(2.7) $$\left|\int_t^{t'} A_i W(\tau)d\tau\right| \le C_n$$

for $nh \le t < t' \le (n+1)h$. Let ρ be the order of Hölder continuity of $a(\cdot)$. Then for $nh \le t < t' \le (n+1)h$ and $0 < \kappa < \rho$

(2.8) $$|W(t') - W(t)| \le C_{n,\kappa}(t'-t)^{\kappa}(t-nh)^{-\kappa},$$

(2.9) $$|A_i(W(t') - W(t))| \le C_{n,\kappa}(t'-t)^{\kappa}(t-nh)^{-\kappa-1},$$

(2.10)
$$|A_i(W(t') - W(t))A_0^{-1}| \le C_{n,\kappa}(t'-t)^{\kappa}(t-nh)^{-\kappa},$$

where C_n and $C_{n,\kappa}$ are constants depending on n and n, κ, respectively, but not on t and t'.

Let $\phi = (\phi^0, \phi^1) \in X \times L^2(-h, 0; D(A_0))$ and $x(T; \phi, f)$ be a solution of the system (RE) associated with forcing term f at time T. By virue of Proposition 2.1 we have immediately the following result in Banach space X.

Proposition 2.2. *Let $(\phi^0, \phi^1) \in M_p$ and $f \in L^p(0, T; X)$ for $1 < p < \infty$ and $T > 0$. Then there exists a solution x of (RE) satisfying that $x(\cdot, \phi, f)$ is strongly continuous on $[0, \infty)$, i.e.,*

$$x(\cdot, \phi, f) \in C([0, \infty), X),$$

and there exists a constant C_1 such that

$$|x(t; \phi, f)| \leq C_1(||\phi||_{M_p} + ||f||_{L^p(0,T;X)}).$$

If X is a Hilbert space then as in [2] we can derive L^2-regurarity for retarded equation (RE) in the highest-order derivative as follows.

Proposition 2.3. *1) Let $F = (D(A_0), X)_{\frac{1}{2},2}$ where $(D(A_0), X)_{1/2,2}$ denote the real interpolation space between $D(A_0)$ and X. For (ϕ^0, ϕ^1) $\in F \times L^2(-h, 0; D(A_0))$ and $f \in L^2(0, T; X)$, $T > 0$, there exists a unique solution x of (RE) belonging to*

$$L^2(-h, T; D(A_0)) \cap W^{1,2}(0, T; X) \subset C([0, T]; F)$$

and satisfying

$$\begin{aligned} ||x||_{L^2(-h,T;D(A_0)) \cap W^{1,2}(0,T;X)} &\leq C_1'(||\phi^0||_F \\ &+ ||\phi^1||_{L^2(0,T;D(A_0))} + ||f||_{L^2(0,T;X)}), \end{aligned} \tag{2.11}$$

where C_1' is a constant depending on T.

Moreover, let X be ζ-convex, that is, there exists a real valued function ζ on $X \times X$ having the properties

$\zeta(x, \cdot)$ and $\zeta(\cdot, y)$ are convex for all $x,\ y \in X$,
$\zeta(x, y) = \zeta(y, x)$,
$\zeta(x, y) \leq ||x + y||$ if $||x|| \leq 1 \leq ||y||$,
$\zeta(\cdot, \cdot) > 0$.

Then, with the aid of the maximal regularity result by G. Dore and A. Venni [3] we can obtain the regularity for the initial value problem (RE) in [4] as follows.

Proposition 2.4. *Let us assume that there exists a constants* $C > 0$ *such that*

$$||A^{is}||_{B(X)} \leq Ce^{\gamma|s|} \quad 0 \leq \gamma < \frac{\pi}{2}$$

for every $s \in \mathcal{R}$. *Let* $F = (D(A_0), X)_{1/p,p}(1 < p < \infty)$. *Then for* $\phi \in F \times L^p(-h, 0; D(A_0))$ *and* $f \in L^p(0, T; X)$, $T > 0$, *there exists a unique solution* x *of (RE) belonging to*

$$L^p(-h, T; D(A_0)) \cap W^{1,p}(0, T; X) \subset C([0, T]; F)$$

and satisfying

$$||x||_{L^p(-h,T;D(A_0)) \cap W^{1,p}(0,T;X)} \leq C_1'(||\phi^0||_F + ||\phi^1||_{L^p(0,T;D(A_0))} + ||f||_{L^p(0,T;X)}),$$

where C_1' *is a constant depending on* T.

3. Time optimal initial function

Let the admissible set $U_{ad} \subset M_p (1 < p < \infty)$ be assumed to be weakly compact, that is,

$$U_{ad} = U_{ad}^0 \times U_{ad}^1, \quad U_{ad}^0 \subset X \quad \text{and} \quad U_{ad}^1 \subset L^p(-h, 0; D(A_0))$$

and U_{ad}^0, U_{ad}^1 are weakly compact in X and $L^p(-h, 0, D(A_0))$, respectively. Consider the following hereditary controlled system:

$$\text{(REC)} \qquad \begin{cases} \frac{d}{dt} x(t) = A_0 x(t) + A_1 x(t-h) \\ \qquad\qquad + \int_{-h}^{0} a(s) A_2 x(t+s) ds + f(t), \ t \geq 0 \\ x(0) = \phi^0, \quad x(s) = \phi(s) \quad -h \leq s < 0, \\ \phi = (\phi^0, \phi^1) \in U_{ad}. \end{cases}$$

We denote the solution of (REC) by $x_\phi(t)$ to express the dependence on $\phi \in U_{ad}$. That is, x_ϕ is trajectory corresponding to the initial function ϕ. Suppose the target set W is weakly compact in X and define

$$U_0 = \{\phi \in U_{ad} : x_\phi(t) \in W \text{ for some } \ t \in [0, T]\}$$

for $T > 0$ and suppose that $U_0 \neq \emptyset$. The optimal time is defined by low limit t_0 of t such that $x_\phi(t) \in W$ for some admissible initial function ϕ. For each $\phi \in U_0$ we can define the first time $\tilde{t}(\phi)$ such that $x_\phi(\tilde{t}) \in W$. The our problem is to find a initial function $g \in U_0$ such that

$$\tilde{t}(g) \leq \tilde{t}(\phi) \quad \text{for all } \phi \in U_0$$

subject to the constraint (REC).
Since $x_\phi \in C([0, T]; X)$, the transition time $\tilde{t}(\phi)$ is well defined for each $\phi \in U_{ad}$.

Theorem 3.1. *Assume that $U_0 \neq \emptyset$. Then for $f \in L^p(0, T; X)$ there exists a time optimal initial function subject to constraint (REC).*

We will prove this theorem in the following sections.

Next we consider the maximal principle and bang-bang principle for time optimal initial functions as follows.

The structural operator F is defined by

(3.1)
$$Fg = ([Fg]^0, [Fg]^1),$$
$$[Fg]^0 = g^0,$$
$$[Fg]^1 = F_1 g^1(s) = A_1 g^1(-h-s) + \int_{-h}^{s} a(\tau) A_2 g^1(\tau - s) d\tau$$

for $g = (g^0, g^1) \in M_p$. It is easy to see that

(3.2)
$$F^*\phi = ([F^*\phi]^0, [F^*\phi]^1),$$
$$[F^*\phi]^0 = \phi^0,$$
$$[F^*\phi]^1(s) = F_1^*\phi^1(s) = A_1^*\phi^1(-h-s) + \int_{-h}^{s} a(\tau) A_2^*\phi^1(\tau - s) d\tau$$

With the aid of suitable changes of variables and Fubini's theorem we obtain
$$\int_{-h}^{0} U_t(s)\phi^1(s)ds = \int_{-h}^{0} W(t+s)[F\phi]^1(s)ds,$$

and hence, in virtue of (2.3) the mild solution $x_\phi(t)$ is represented by

$$(3.3)\quad x_\phi(t) = W(t)\phi^0 + \int_{-h}^{0} W(t+s)[F\phi]^1(s)ds + \int_0^t W(t-s)f(s)ds,$$

Theorem 3.2. *Let W be convex, closed and nonempty interior and g be a time optimal initial function subject to constraint (REC) with its optimal time t_0. Then there exists a nonzero $x^* \in X^*$ such that*

$$(3.4)\qquad \max_{\phi \in U_{ad}} < \phi, F^*p >_{M_p} = < g, F^*p >_{M_p}$$

where $p \in (M_p)^*$ satisfying

$$p^0 = W^*(t_0)x^*, \quad p^1(s) = W^*(t_0+s)x^* \quad s \in [-h, 0).$$

Proof. Let us define the reachable set $R(t_0)$ by

$$R(t_0) = \{y \in X : y = x_\phi(t_0) \text{ for some } \phi \in U_{ad}\}.$$

If there exists $y \in (\operatorname{Int} W) \cap R(t_0)$ then we have a initial function $\phi \in U_{ad}$ satisfying $x_\phi(t_0) \in \operatorname{Int} W$. Thus, there exists $t_1 < t_0$ such that $x_\phi(t_1) \in W$, which contradicts that t_0 is an optimal time. Thus we have $(\operatorname{Int} W) \cap R(t_0) = \emptyset$. Since W is closed and convex, $W = \operatorname{Cl}(\operatorname{Int} W)$. Therefore, by the separating hyperplane theorem and by continuity, there exists a nonzero $x^* \in X^*$ such that

$$(3.5) \qquad \sup_{y \in R(t_0)} < y, x^* > \leq \inf_{y \in W} < y, x^* > \leq < x_g(t_0), x^* > .$$

By the form of the trajectories, the inequality (3.5) is reduced to

$$\sup_{\phi \in U_{ad}} < W(t_0)\phi^0 + \int_{-h}^{0} W(t_0 + s)[F\phi]^1(s)ds, x^* >$$
$$\leq < W(t_0)g^0 + \int_{-h}^{0} W(t_0 + s)[Fg]^1(s)ds, x^* > .$$

It is equivalent to the fact that

$$\sup_{\phi \in U_{ad}} \{< \phi^0, W^*(t_0)x^* > + \int_{-h}^{0} < \phi^1(s), [F^*W(t_0 + \cdot)^*x^*]^1(s) >_{D(A_0)} ds$$
$$\leq < g^0, W^*(t_0)x^* > + \int_{-h}^{0} < g^1(s), [F^*W(t_0 + \cdot)^*x^*]^1(s) >_{D(A_0)} ds.$$

Hence, from the duality pairing between M_p and $(M_p)^*$, (3.4) follows. □

Proposition 3.1. *If $A_1 : D(A_0) \to X$ is an isomorphism, then $F : M_p \to X \times L_p(-h, 0; X)$ is an isomorphism.*

Proof. For $f = (f^0, f^1) \in X \times L_p(-h, 0; X)$ the element $g \in Z$ satisfying $g^0 = f^0$ and

$$g^1(-h - s) + \int_{-h}^{s} a(\tau)A_1^{-1}A_2 g^1(\tau - s)d\tau = A_1^{-1} f^1(s)$$

is the unique solution of $Fg = f$. The integral equation mentioned above is of Volterra type, and so it can be solved by successive approximation method. □

The following result is obtained from Lemma 5.1 in [6].

Lemma 3.1. *Let $f \in L^p(0,T;X)$, $1 \leq p \leq \infty$. If*

$$\int_0^t W(t-s)f(s)ds = 0, \quad 0 \leq t \leq T,$$

then $f(t) = 0$ a.e. $0 \leq t \leq T$.

Theorem 3.3. *Let A_1 be an isomorphism and $f \equiv 0$. Then the solution $x_\phi(t)$ is identically zero on a positive measure containing zero in $[-h,T]$ for $T \geq h$ if and only if $\phi^0 = 0$ and $\phi^1 \equiv 0$.*

Proof. With the change of variable and Fubini's theorem we obtain

$$\begin{aligned}
&\int_{-h}^0 U_t(s)\phi^1(s)ds \\
&\quad = \int_{-h}^0 W(t-s-h)A_1\phi^1(s)ds \\
&\quad + \int_{-h}^0 (\int_{-h}^s W(t-s+\tau)a(\tau)A_2 d\tau)\phi^1(s)ds \\
&\quad = \int_{-h}^0 W(t+s)\{A_1\chi_{(-h,0)}(s)\phi^1(-h-s) \\
&\quad + \int_{-h}^s a(\tau)A_2(\tau)\phi^1(\tau-s)d\tau\}ds \\
&\quad = \int_{-h}^0 W(t+s)F_1\phi^1(s)ds.
\end{aligned}$$

Thus the mild solution $x_\phi(t)$ is represented by

$$x(t) = W(t)\phi^0 + \int_{-h}^0 W(t+s)F_1\phi^1(s)ds.$$

Thus, we have that $x(0) = W(0)\phi^0 = \phi^0 = 0$ in X. Because that A_1 is an isomorphism and, we obtain that F_1 is isomorphism from Proposition 3.1. Therefore from Lemma 3.1 $x_\phi(t) = 0$ if and only if $\phi^0 = 0$ and $\phi^1 \equiv 0$. □

From Theorem 3.2 the following bang-bang principle follows immediately.

Corollary 3.1. *Let A_1 be one to one mapping and $W(t_0)g^0 \neq 0$. Then the time optimal initial function g is a bang-bang control, i.e, $g = (g^0, g^1)$ satisfies*

$$g^0 \in \partial U_{ad}^0 \quad \text{and} \quad g^1 \in \partial U_{ad}^1 \tag{3.6}$$

where ∂U_{ad}^0 and ∂U_{ad}^1 denote the boundary of U_{ad}^0 and ∂U_{ad}^1, respectively.

Proof. On account of Theorem 3.2, it is sufficient to show (3.6) that

$$p^0 = W^*(t_0)x^* \neq 0, \quad F_1^* p^1(s) = F_1^* W^*(t_0 + s)x^* \neq 0 \quad s \in [-h, 0).$$

Noting $x^* \neq 0$, by proposition 3.1 and Lemma 3.1, (3.6) follows. □

Now we consider the case where the target set W is singleton.

Consider that $W = w_0$. Then we can choose a decreasing sequence $\{W_n\}$ of weakly compact sets with nonempty interior such that

$$w_0 \in \bigcap_{n=1}^{\infty} W_n, \text{ and } \operatorname{dist}(w_0, W) = \sup_{x \in W_n} |x - w_0| \to 0 (n \to \infty). \tag{3.7}$$

Define

$$U_0^n = \{g \in U_{ad} : x_g(t) \in W_n \text{ for some } t \in [0, T]\}.$$

Then, we may assume that g_n is the time optimal initial function with the optimal time t_n to the target set W_n, $n = 1, 2, \ldots$.

Theorem 3.4. *Let $\{W_n\}$ be a sequence of closed convex subsets of X satisfying the condition (3.7) and $U_0^n = \emptyset$. Then there exists a time optimal initial function g_0 with the optimal time $t_0 = \sup_{n \geq 1}\{t_n\}$ to the point target set $\{w_0\}$ which is given by the weak limit of some subsequence of $\{g_n\}$ in M_p.*

Proof. Since (3.7) is satisfied and U_{ad} is weakly compact, there exists $w_n = x_n(t_n) \in W_n \to w_0$ strongly in H. Since U_{ad} is weakly compact, there exists $g_0 \in U_{ad}$ such that $g_n \to g_0$ weakly in M_p. Thus, from the similar argument used in the proof of Theorem 3.1 we can easily prove that g_0 is the time optimal initial function and t_0 is the optimal time to the target $\{w_0\}$. □

4. Preparations for the proof of main results

In what follows we assume that

$$||W(t)|| \leq M, \quad t > 0$$

for the sake of simplicity. We also assume that $S(t)$ is uniformly bounded. Then

$$(4.1) \qquad \begin{aligned} &|S(t)| \leq M_0(t \geq 0), \ |A_0 S(t)| \leq M_0/t(t > 0), \\ &|A_0^2 S(t)| \leq K/t^2(t > 0) \end{aligned}$$

for some constant M_0 (e.g., [7]). Let us assume that $a(\cdot)$ is Hölder continuous of order ρ:

$$(4.2) \qquad |a(\cdot)| \leq H_0, \quad |a(s) - a(\tau)| \leq H_1(s-\tau)^\rho$$

for some constants H_0, H_1. Set

$$(4.3) \qquad ||g^1||_\infty = \max_{s \in [-h,0]} |A_0 g^1(s)|.$$

According to Tanabe [7] we set

$$(4.4) \qquad V(t) = \begin{cases} A_0(W(t) - S(t)), \quad t \in (0,h] \\ A_0(W(t) - \displaystyle\int_{nh}^t S(t-s)A_1 W(s-h)ds), \end{cases}$$

where $t \in (nh, (n+1)h](n = 1, 2, \ ...\ )$ in the second line of the right term of (4.4). For $0 < t \leq h$

$$W(t) = S(t) + A_0^{-1} V(t)$$

and from (4.4) we have

$$W(t) = S(t) + \int_0^t \int_\tau^t S(t-s)a(\tau - s)ds A_2 W(\tau) d\tau.$$

Hence,

$$V(t) = V_0(t) + \int_0^t A_0 \int_\tau^t S(t-s)a(\tau-s)ds A_2 A_0^{-1} V(\tau) d\tau$$

where

$$V_0(t) = \int_0^t A_0 \int_\tau^t S(t-s)a(\tau-s)dsA_2S(\tau)d\tau.$$

For $nh \leq t \leq (n+1)h(n = 0, 1, 2, \ ... \ )$ the fundamental solution $W(t)$ is represented by

$$\begin{aligned} W(t) =& S(t) + \int_{nh}^t S(t-s)A_1W(s-h)ds \\ &+ \int_0^{t-h} \int_\tau^{\tau+h} S(t-s)a(\tau-s)dsA_2W(\tau)d\tau \\ &+ \int_{t-h}^{nh} \int_\tau^t S(t-s)a(\tau-s)dsA_2W(\tau)d\tau \\ &+ \int_{nh}^t \int_\tau^t S(t-s)a(\tau-s)dsA_2W(\tau)d\tau. \end{aligned}$$

The integral equation to be satisfied by (4.4) is

$$V(t) = V_0(t) + \int_{nh}^t A_0 \int_\tau^t S(t-s)a(\tau-s)dsA_2A_0^{-1}V(\tau)d\tau$$

where

$$\begin{aligned} V_0(t) =& A_0S(t) + A_0 \int_h^{nh} S(t-s)A_1W(s-h)ds \\ &+ \int_0^{t-h} A_0 \int_\tau^{\tau+h} S(t-s)a(\tau-s)dsA_2W(\tau)d\tau \\ &+ \int_{t-h}^{nh} A_0 \int_0^t S(t-s)a(\tau-s)dsA_2W(\tau)d\tau \\ &+ \int_{nh}^t A_0 \int_\tau^t S(t-s)a(\tau-s)dsA_2 \int_{nh}^\tau S(\tau-\sigma)A_1W(\sigma-h)d\sigma d\tau. \end{aligned}$$

Thus, the integral equation (4.4) can be solved by successive approximation and $V(t)$ is uniformly bounded in $[nh, (n+1)h]$ (e.g. (3.16) and the preceding part of (3.40) in [7]). It is not difficult to show that for $n > 1$,

$$V(nh+0) \neq V(nh-0), \quad \text{and} \quad W(nh+0) = W(nh-0).$$

Lemma 4.1. *For $0 < s < t$ and $0 < \alpha < 1$,*

$$|S(t) - S(s)| \le \frac{M_0}{\alpha}\left(\frac{t-s}{s}\right)^{\alpha}, \tag{4.5}$$

$$|A_0S(t) - A_0S(s)| \le M_0(t-s)^{\alpha}s^{-\alpha-1}. \tag{4.6}$$

Proof. It follows from (3.1) that for $0 < s < t$,

$$|S(t) - S(s)| = \left|\int_s^t A_0S(\tau)d\tau\right| \le M_0 \log\frac{t}{s}. \tag{4.7}$$

It is easily seen that for any $t > 0$ and $0 < \alpha < 1$,

$$\log(1+t) \le t^{\alpha}/\alpha. \tag{4.8}$$

Combining (4.8) with (4.7) we get (4.5). For $0 < s < t$,

$$|A_0S(t) - A_0S(s)| = \left|\int_s^t A_0^2S(\tau)d\tau\right| \le M_0(t-s)/ts. \tag{4.9}$$

Noting that $(t-s)/s \le ((t-s)/s)^{\alpha}$ for $0 < \alpha < 1$, we obtain (4.6) from (4.9). □

We define the operator $K_1(t', t) : X \to X$ by

$$K_1(t', t) = \int_t^{t'} S(t' - s)A_1W(s-h)ds, \tag{4.10}$$

for $nh \le t < t' < (n+1)h$.

Lemma 4.2. *Let $nh < t < (n+1)h$, $n = 0, \dots$. Then*

$$|K_1(t, nh)| \le M_0C_{n-1}c_0 + M_0C_{n-1}$$

where

$$c_0 = \int_0^1 \log\frac{1}{1-\sigma}\frac{d\sigma}{\sigma}.$$

Proof. The proof is a consequence of the following estimate

$$
\begin{aligned}
&\left|\int_{nh}^{t} S(t-\xi)A_1W(\xi-h)d\xi\right| \\
&\quad= \left|\int_{nh}^{t} (S(t-\xi)-S(t-nh))A_1W(\xi-h)d\xi\right. \\
&\quad\quad \left.+ S(t-nh)\int_{nh}^{t} A_1W(\xi-h)d\xi\right| \\
&\quad\le \int_{nh}^{t} M_0 \log\frac{t-nh}{t-\xi}\frac{C_{n-1}}{\xi-nh}d\xi + M_0C_{n-1} \\
&\quad\le M_0C_{n-1}c_0 + M_0C_{n-1}. \quad \square
\end{aligned}
$$

In terms of Lemma 4.2 $K_1(t',t)$ is uniformly bounded in $(nh,(n+1)h]$. And we remark that $K_1(t',t)$ converges to 0 as $t'\to t$ at any element of $D(A_0)$ in virtue of (2.6).

We introduce another operator $K_2(t',t) : X \to X$ by

$$
K_2(t',t) = \int_{t}^{t'} S(t'-s)\int_{-h}^{0} a(\tau)A_2W(s+\tau)d\tau ds, \tag{4.11}
$$

for $nh \le t < t' < (n+1)h$.

To obtain the estimate of $K_2(t',t)$ we obtain the following result.

Lemma 4.3. *There exists a constant $C_n' > 0$ such that*

$$
\left|\int_{nh}^{t} a(\tau-s)A_iW(\tau)d\tau\right| \le C_n', \quad i=1,2, \tag{4.12}
$$

for $n=0,1,2,\ ...,\ t\in[nh,(n+1)h]$ and $t\le s\le t+h$.

Proof. It follows from (3.4) that for $t\in[0,h]$(i.e.,$n=0$)

$$
\begin{aligned}
\int_0^t a(\tau-s)A_iW(\tau)d\tau &= \int_0^t a(\tau-s)A_iA_0^{-1}(A_0S(\tau)+V(\tau))d\tau \\
&= \int_0^t (a(\tau-s)-a(-s))A_iA_0^{-1}A_0S(\tau)d\tau + a(-s)A_iA_0^{-1}(S(t)-I) \\
&\quad+ \int_0^t a(\tau-s)A_iA_0^{-1}V(\tau)d\tau.
\end{aligned}
$$

Noting that

$$\left|\int_0^t (a(\tau-s)-a(-s))A_iA_0^{-1}A_0S(\tau)d\tau\right| \le M_0H_1|A_iA_0^{-1}|\int_0^t \tau^{\rho-1}d\tau,$$

we have

$$\begin{aligned}\left|\int_0^t a(\tau-s)A_iW(\tau)d\tau\right| \le & |A_iA_0^{-1}|\{h^\rho M_0H_1 + H_0(M+1) \\ & + hH_0(\sup_{0\le t\le h}|V(t)|)\}.\end{aligned}$$

Thus the assertion (4.12) holds in $[0,h]$. For $t\in[nh,(n+1)h],\ n\ge 1$,

(4.13)

$$\begin{aligned}\int_{nh}^t a(\tau-s)A_iW(\tau)d\tau = & \int_{nh}^t a(\tau-s)A_iA_0^{-1}V(\tau)d\tau \\ & + \int_{nh}^t a(\tau-s)A_i\int_{nh}^\tau S(\tau-\xi)A_1W(\xi-h)d\xi d\tau.\end{aligned}$$

The first term of the right of (4.13) is estimated as

$$\left|\int_{nh}^t a(\tau-s)A_iA_0^{-1}V(\tau)d\tau\right| \le hH_0|A_iA_0^{-1}|(\sup_{nh\le t\le(n+1)h}|V(t)|)\}.$$

Let $\sigma=(\tau+nh)/2$ for $nh<\tau<(n+1)h$. Then

(4.14)

$$\begin{aligned}&\left|A_0\int_{nh}^\tau S(\tau-\xi)A_1W(\xi-h)d\xi\right| \\ &\le \Big|\int_\sigma^\tau A_0S(\tau-\xi)(A_1W(\xi-h)-A_1W(\tau-h)d\xi \\ &\quad + (S((\tau-nh)/2)-I)A_1W(\tau-h) \\ &\quad + \int_{nh}^\sigma (A_0S(\tau-\xi)-A_0S(\tau-nh))A_1W(\xi-h)d\xi \\ &\quad + A_0S(\tau-nh)\int_{nh}^\sigma A_1W(\xi-h)d\xi\Big| \\ &\le \int_\sigma^\tau \frac{M_0}{\tau-\sigma}C_{n-1,\kappa}(\tau-\xi)^\kappa(\xi-nh)^{-\kappa-1}d\xi + (M_0+1)\frac{C_{n-1}}{\tau-nh} \\ &\quad + \int_{nh}^\sigma \frac{M_0(\xi-nh)}{(\tau-\xi)(\tau-nh)}\frac{C_{n-1}}{\xi-nh}d\xi + \frac{M_0C_{n-1}}{\tau-nh}\end{aligned}$$

$$\leq M_0 C_{n-1,\kappa} \int_{nh}^{\tau} (\tau-\xi)^{\kappa-1}(\xi-nh)^{-\kappa} d\xi \frac{2}{\tau-nh}$$
$$+ \frac{(2M_0+1)C_{n-1}}{\tau-nh} + \frac{M_0 C_{n-1}}{\tau-nh}\log 2$$
$$= \{2M_0 C_{n-1,\kappa} B(\kappa, 1-\kappa) + (2M_0+1+M_0\log 2)C_{n-1}\}/(\tau-nh)$$
$$\equiv C'_{n,\kappa}/(\tau-nh)$$

where $B(\cdot,\cdot)$ is the Beta function. Note that

$$\frac{d}{d\tau}\int_{nh}^{\tau} S(\tau-\xi)A_1 W(\xi-h)d\xi = A_1 W(\tau-h) + A_0 \int_{nh}^{\tau} S(\tau-\xi)A_1 W(\xi-h)d\xi.$$

Integrating this equality on $[nh,t]$ and by Lemma 3.1 and the induction hypothesis

(4.15)
$$\int_{nh}^{t} A_0 \int_{nh}^{\tau} S(\tau-\xi)A_1 W(\xi-h)d\xi d\tau$$
$$= \int_{nh}^{t} S(t-\xi)A_1 W(\xi-h)d\xi - \int_{nh}^{t} A_1 W(\tau-h)d\tau.$$

Thus, by Lemma 4.2, the induction hypothesis and combining the above inequality with (2.9) we get

$$(4.16)\quad \left|\int_{nh}^{t} A_0 \int_{nh}^{\tau} S(\tau-s)A_i W(s-h)ds d\tau\right| \leq (M_0 c_0 + M_0 + 1)C_{n-1}.$$

Therefore, from (4.14), (4.16) the second term of the right of (4.13) is estimated as

$$\left|\int_{nh}^{t} a(\tau-s)A_i \int_{nh}^{\tau} S(\tau-\xi)A_1 W(\xi-h)d\xi d\tau\right|$$
$$= \left|\int_{nh}^{t} (a(\tau-s)-a(s-nh))A_i \int_{nh}^{\tau} S(\tau-\xi)A_1 W(\xi-h)d\xi d\tau\right.$$
$$\left. + a(s-nh)\int_{nh}^{t} A_i \int_{nh}^{\tau} S(\tau-\xi)A_1 W(\xi-h)d\xi d\tau\right|$$
$$\leq \int_{nh}^{t} H_1(\tau-nh)^{\rho}|A_i A_0^{-1}|C'_{n.\kappa}(\tau-nh)^{-1}d\tau$$
$$+ |a(s-nh)||A_i A_0^{-1}|(M_0 c_0 + M_0 + 1)C_{n-1}$$
$$\leq H_1 C'_{n.\kappa}|A_i A_0^{-1}|(t-nh)^{\rho} + H_0|A_i A_0^{-1}|(Mc_0 + M + 1)C_{n-1}.$$

Hence, we get the assertion (4.12). □

Lemma 4.4. *Let $nh \le t < t' < (n+1)h$. Then there exists a constant C_n' such that*

$$|K_2(t',t)| \le 3M_0C_n'(t'-t). \tag{4.17}$$

Proof. In $[0,h]$, we transform $K_2(t',t)$ by suitable change of variables and Fubini's theorem as

$$\begin{aligned}
K_2(t',t) &= \int_t^{t'} S(t'-s)\int_0^s a(\tau-s)A_2W(\tau)d\tau ds \\
&= \int_0^t \int_t^{t'} S(t'-s)a(\tau-s)A_2W(\tau)dsd\tau \\
&\quad + \int_t^{t'} \int_\tau^{t'} S(t'-s)a(\tau-s)A_2W(\tau)dsd\tau \\
&= \int_t^{t'} S(t'-s)\int_0^t a(\tau-s)A_2W(\tau)d\tau ds \\
&\quad + \int_t^{t'} S(t'-s)\int_t^s a(\tau-s)A_2W(\tau)d\tau ds.
\end{aligned}$$

Thus from Lemma 4.3 we have

$$|K_2(t',t)| \le 2M_0C_n'(t'-t).$$

In $[nh,(n+1)h)$, by the similar way mentioned above we get

$$\begin{aligned}
K_2(t',t) &= \int_t^{t'} S(t'-s)\int_{-h}^0 a(\tau)A_2W(\tau+s)d\tau ds \\
&= \int_t^{t'} S(t'-s)\int_{s-h}^s a(\tau-s)A_2W(\tau)d\tau ds \\
&= \int_{t-h}^{t'-h} \int_t^{\tau+h} S(t'-s)a(\tau-s)A_2W(\tau)dsd\tau \\
&\quad + \int_{t'-h}^{t} \int_t^{t'} S(t'-s)a(\tau-s)A_2W(\tau)dsd\tau \\
&\quad + \int_t^{t'} \int_\tau^{t'} S(t'-s)a(\tau-s)A_2W(\tau)dsd\tau
\end{aligned}$$

$$= \int_{t}^{t'} S(t'-s) \int_{s-h}^{t'-h} a(\tau - s) A_2 W(\tau) d\tau ds$$

$$+ \int_{t}^{t'} S(t'-s) \int_{t'-h}^{t} a(\tau - s) A_2 W(\tau) d\tau ds$$

$$+ \int_{t}^{t'} S(t'-s) \int_{t}^{s} a(\tau - s) A_2 W(\tau) d\tau ds.$$

Therefore, by Lemma 4.1 it holds (4.17) □

5. The proof of main theorem

Throughout this section we will prove Theorem 3.1. Let $t_n \to t_0 + 0$ and g_n be an admissible initial function. Suppose that the trajectory x_n corresponding to g_n belongs to W. Then

$$
\begin{aligned}
x_n(t_n) =& W(t_n)g_n^0 + \int_{-h}^{0} W(t_n+s)F_1 g_n^1(s)ds \\
&+ \int_{t_0}^{t_n} W(t_n-s)f(s)ds \\
&+ \int_{0}^{t_0} W(t_n-s)f(s)ds.
\end{aligned} \tag{5.1}
$$

By the definition of fundamental solution $W(t)$ it holds

$$
\begin{aligned}
W(t+\epsilon) - S(\epsilon)W(t) = & S(t+\epsilon) + \int_0^{t+\epsilon} S(t+\epsilon-s)\{A_1W(s-h) \\
& + \int_{-h}^{0} a(\tau)A_2W(s+\tau)d\tau\}ds \\
& - S(\epsilon)\{S(t) + \int_0^t S(t-s)\{A_1W(s-h) \\
& + \int_{-h}^{0} a(\tau)A_2W(s+\tau)d\tau\}ds \\
& = \int_t^{t+\epsilon} S(t+\epsilon-s)\{A_1W(s-h) \\
& + \int_{-h}^{0} a(\tau)A_2W(s+\tau)d\tau\}ds \\
& = K_1(t+\epsilon,t) + K_2(t+\epsilon,t).
\end{aligned}
$$

Hence, since

$$
W(t_n) = S(t_n-t_0)W(t_0) + K_1(t_n,t_0) + K_2(t_n,t_0),
$$

for $x^* \in X^*$ we have

$$
\begin{aligned}
< W(t_n)g_n^0, x^* > =& < W(t_0)g_n^0, S^*(t_n-t_0)x^* > \\
&+ < K_1(t_n,t_0)g_n^0, x^* > \\
&+ < K_2(t_n,t_0)g_n^0, x^* > .
\end{aligned} \tag{5.2}
$$

The first term of right hand side of (5.2) tends to $< W(t_0)g^0, x^* >$ because of the strong continuity of $S^*(t_n - t_0)$ in X^*. The second and third terms in (5.2) tends to zero as $t_n \to t_0 + 0$ from in terms of Lemma 4.2, 4.4. Noting that

$$W(t_n+s) = S(t_n - t_0)W(t_0+s) + K_1(t_n+s, t_0+s) + K_2(t_n+s, t_0+s),$$

the second term of (5.1) is represented as

(5.3)

$$\begin{aligned}&\int_{-h}^{0} S(t_n - t_0)W(t_0 + s)F_1 g_n^1(s)ds \\ &+ \int_{-h}^{0} \{K_1(t_n + s, t_0 + s) + K_2(t_n + s, t_0 + s)\}F_1 g_n^1(s)ds.\end{aligned}$$

Since X^* is reflexive, we know that

$$S^*(t_n - t_0)x^* \to x^* \text{strongly in } X^*.$$

Noting that

$$|F_1 g_n^1(s)| \leq ||A_1 A_0^{-1}|| \, ||g||_\infty + hH_0 ||A_2 A_0^{-1}|| \, ||g||_\infty,$$

the second term of (5.3) is estimated as

$$\begin{aligned}&\Big|\int_{-h}^{0} \{K_1(t_n + s, t_0 - s) + K_2(t_n - s, t_0 - s)\}F_1 g_n^1(s)ds\Big| \\ &\leq h\{|K_1(t_n + s, t_0 - s)| + |K_2(t_n - s, t_0 - s)|\} \\ &(||A_1 A_0^{-1}|| \, ||g_n||_\infty + hH_0 ||A_2 A_0^{-1}|| \, ||g_n||_\infty) \\ &\to 0 (n \to \infty).\end{aligned}$$

Thus, since

$$\begin{aligned}&\int_{-h}^{0} < S(t_n - t_0)W(t_0 + s)F_1 g_n^1(s)ds, x^* > \\ &= \int_{-h}^{0} < W(t_0 + s)F_1 g_n^1(s)ds, S^*(t_n - t_0)x^* > \\ &\to \int_{-h}^{0} < W(t_0 + s)F_1 g^1(s)ds, x^* >,\end{aligned}$$

we have

$$S(t_n - t_0)W(t_0 + \cdot)F_1 g_n^1(\cdot) \to W(t_0 + \cdot)F_1 g^1(\cdot)$$

weakly in $L^p(-h, 0; D(A_0))$.

The third term of right hand side (5.1) is estimated as

$$\left| \int_{t_0}^{t_n} W(t_n - s)f(s)ds \right| \leq (t_n - t_0)^{1-1/p}(\sup_{t\in[0,T]} ||W(t)||)||f||_{L^p(0,T;X)}$$
$$\to 0 (n \to \infty).$$

By using the similar method to the fourth term of right hand side (5.1), we have

$$W(t_n - \cdot)f(\cdot) \to W(t_0 - \cdot)f(\cdot)$$

weakly in $L^p(0, t_0, X)$.

We denote $x_n(t_n)$ by w_n. Since W and U_{ad} are weakly compact, there exist $g \in U_0$, $w_0 \in W$ such that we may assume that $w - \lim g_n = g$ in U_{ad} and $w - \lim w_n = w_0$ in M_p and X, respectively.

Therefore, we obtain that

$$< w_0, x^* >= < W(t_0)g^0, x^* > + \int_{-h}^{0} < W(t_0 + s)F_1 g^1(s), x^* > ds$$
$$+ \int_0^{t_0} < W(t_0 - s)f(s), x^* > ds.$$

by tending $n \to \infty$. Since x^* is arbitrary, we have

$$w_0 = W(t_0)g^0 + \int_{-h}^{0} W(t_0 + s)F_1 g^1(s)ds$$
$$+ \int_0^{t_0} W(t_0 - s)f(s)ds.$$

and hence w_0 is the trajectory corresponding to g_0, i.e., $g_0 \in U_0$.

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