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Error Estimates of Discontinuous Galerkin Methods for Elliptic Problems

by

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Error Estimates of Discontinuous Galerkin Methods for Elliptic Problems
(타원형문제에 대한 불연속 갈레르킨 방법의 오차 추정)

Advisor : Prof. Jun Yong Shin

by

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이민아의 이학박사 학위논문을 인준함.

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Error Estimates of Discontinuous Galerkin Methods for Elliptic Problems

A dissertation

by

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Contents

List of Tables

List of Figures

타원형문제에 대한 불연속 갈레르킨 방법의 오차 추정

이 민 아

부 경 대 학 교 대 학 원 응 용 수 학 과

요 약

본 학위논문에서는 과학과 공학에서 다양한 문제를 설명하는 동차 디리클레 경계조건을 가지는 1차원과 2차원 타원형문제들에 대해서 내부 페널티를 가지는 불연속 갈레르킨 방법에 의한 근사해를 구성하고, 근사해의 존재성을 보였다. 또한 근사해의 에너지 노름과 L^2 노름에 대한 오차가 최적이 됨을 입증하였다.

그리고 1차원과 2차원 타원형문제에 고안한 내부 페널티를 가지는 불연속 갈레르킨 방법을 적용하여 근사해의 L^2 노름에 대한 이론적 결과를 수치적으로 입증하였다.

Chapter 1 Introduction

Many problems arising in the various areas of sciences and engineerings are described by equations, specially the elliptic differential equations. To obtain the approximate solution of differential equations, finite element methods, finite difference methods, finite volume methods, and other numerical methods are widely used.

Recently, several discontinuous Galerkin methods are proposed for the approximate solution of differential equations. Discontinuous Galerkin (DG) methods can be classified into two groups. The first group is based on the hyperbolic approach. In 1973, Reed and Hill [38] introduced the first discontinuous Galerkin method for hyperbolic equations, and since that time there has been a remarkable development of DG methods for hyperbolic and nearly hyperbolic problems. In recent years, these methods have also been applied to purely elliptic problems; examples are the original method of Bassi and Rebay [10], its variations studied in [13, 14], and the local discontinuous Galerkin method introduced in [26] and further studied in [16, 22, 23].

The second group is based on the elliptic approach. In the 1970's, but independently on the hyperbolic approach, Galerkin methods for elliptic and parabolic equations using discontinuous finite elements were proposed and a lot of their variants were introduced and studied; see, for example, [1, 2, 8, 28, 44]. These DG methods were then usually called interior penalty (IP) methods and their development were remained independent of the development of the DG methods for hyperbolic equations. For the current works and the brief history of the DG methods, one is referred to [3, 25].

The main objectives of this dissertation are to introduce the discontinuous Galerkin methods for elliptic problems, which are closely related to the interior penalty (IP) methods, to establish the optimal error estimates in the energy and L^2 norms of the discontinuous Galerkin methods, which improves the previous results of [3, 5, 6], and finally to verify the optimality result in the L^2 norm of the discontinuous Galerkin methods numerically.

This dissertation is organized as follows. In Chapter 2, we introduce some definitions and some results which will be used later. In Chapter 3, we introduce the discontinuous Galerkin methods for one dimensional elliptic problems and establish the optimal error estimates in the energy and L^2 norms of the discontinuous Galerkin methods. And in Chapter 4, we introduce the discontinuous Galerkin methods for two dimensional elliptic problems and establish the optimal error estimates in the energy and L^2 norms of the discontinuous Galerkin methods. Finally, in Chapter 5, the discontinuous Galerkin methods introduced in Chapters 3 and 4 are numerically treated and the numerical results for one or two dimensional elliptic problems are given to verify the optimality result in the L^2 norm of the discontinuous Galerkin methods.

Chapter 2 Preliminaries

2.1 Function Spaces

This section is devoted to introducing the function spaces which will be used in the variational formulation of differential equations. Let Ω be a bounded domain in $\mathbb{R}^n$, $n = 1$ or 2.

Definition 2.1.1. *Let $1 \leqslant p \leqslant \infty$. $L^p(\Omega)$ will denote the Banach space of real-valued measurable functions defined on Ω with the following norm*

$$\|u\|_{L^p(\Omega)} = \Big(\int_\Omega |u|^p dx\Big)^{1/p}, \quad 1 \leqslant p < \infty,$$
$$\|u\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |u|, \quad p = \infty.$$

Definition 2.1.2. *Let m be a non-negative integer and $1 \leqslant p \leqslant \infty$. $W^{m,p}(\Omega)$ will denote the Sobolev space of measurable functions which together with their weak derivatives of order up to m are in $L^p(\Omega)$, i.e.,*

$$W^{m,p}(\Omega) = \{u; D^\alpha u \in L^p(\Omega), \quad 0 \leqslant |\alpha| \leqslant m\}$$

with the following norm

$$\|u\|_{W^{m,p}(\Omega)} = \Big(\sum_{0\leqslant|\alpha|\leqslant m} ||D^\alpha u||^p_{L^p(\Omega)}\Big)^{1/p}, \quad 1 \leqslant p < \infty,$$
$$\|u\|_{W^{m,\infty}(\Omega)} = \max_{|\alpha|\leqslant m} \|D^\alpha v\|_{L^\infty}, \quad p = \infty,$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ is a multi-index of nonnegative integers and $|\alpha| = \sum_{i=1}^n \alpha_i$.

We shall use the symbol $H^m(\Omega)$ for $W^{m,2}(\Omega)$ and $H_0^m(\Omega)$ for the set of functions in $H^m(\Omega)$ which vanish on the boundary $\partial\Omega$.

Remark 2.1.1. *For our convenience, we use the following notations*

$$\|\cdot\|_{m,p} := \|\cdot\|_{W^{m,p}(\Omega)},$$

$$\|\cdot\|_{m} := \|\cdot\|_{W^{m,2}(\Omega)},$$

$$\|\cdot\| := \|\cdot\|_{W^{0,2}(\Omega)},$$

$$\|\cdot\|_{m,p,S} := \|\cdot\|_{W^{m,p}(S)},$$

$$\|\cdot\|_{m,S} := \|\cdot\|_{W^{m,2}(S)}.$$

There are some well-known inequalities that hold for the functions defined above.

Theorem 2.1.1 (Hölder inequality [12]). *For $1 \leqslant p, q \leqslant \infty$ such that $1 = 1/p + 1/q$, if $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then $uv \in L^1(\Omega)$ and*

$$\int_\Omega |u(x)v(x)|dx \leqslant \|u\|_p \|v\|_q.$$

The discrete version of Hölder inequality can be stated as follows.

Remark 2.1.2 (Hölder inequality [12]). *For a finite or infinite sum*

$$\sum |a_k b_k| \leqslant \big(\sum |a_k|^p\big)^{1/p} \big(\sum |b_k|^q\big)^{1/q}, \quad 1 \leqslant p, q < \infty, \ 1/p + 1/q = 1,$$

$$\sum |a_k b_k| \leqslant (\sup |a_k|) \sum |b_k|, \qquad p = 1, \ q = \infty.$$

Theorem 2.1.2 (Schwarz inequality [12]). *If $u, v \in L^2(\Omega)$ then $uv \in L^1(\Omega)$ and*

$$\int_\Omega |u(x)v(x)|dx \leqslant \|u\|_2 \|v\|_2.$$

The discrete version of Schwarz inequality can be stated as follows.

Remark 2.1.3 (Schwarz inequality [12]). *For a finite or infinite sum*

$$\sum |a_k b_k| \leqslant \big(\sum |a_k|^2\big)^{1/2}\big(\sum |b_k|^2\big)^{1/2}.$$

Theorem 2.1.3 (Minkowski inequality [12]). *For $1 \leqslant p \leqslant \infty$ and $u, v \in L^p(\Omega)$, we have*

$$\|u + v\|_p \leqslant \|u\|_p + \|v\|_p.$$

The discrete version of Minkowski inequality can be stated as follows.

Remark 2.1.4 (Minkowski inequality [12]). *For a finite or infinite sum*

$$\Big(\sum (a_k + b_k)^p\Big)^{1/p} \leqslant \Big(\sum (a_k)^p\Big)^{1/p} + \Big(\sum (b_k)^p\Big)^{1/p}, \quad 1 \leqslant p < \infty,$$
$$\sup |a_k + b_k| \leqslant \sup |a_k| + \sup |b_k|, \qquad p = \infty.$$

2.2 Trace Inequalities

The following Theorems 2.2.1 and 2.2.2 are trace inequalities in $\mathbb{R}$ and $\mathbb{R}^2$, respectively.

Theorem 2.2.1. *Let K be a bounded open interval in $\mathbb{R}$. Then, for all $v \in H^1(K)$,*

$$||v||_{0,\partial K}^2 \leqslant C\Big(\frac{1}{h_K}||v||_{0,K}^2 + h_K||v||_{1,K}^2\Big),$$

where C is a positive constant and h_K is the length of K.

Proof. Let $K = (x_{i-1}, x_i)$. From the definition of a definite integral, we have

$$v^2(x) - v^2(x_{i-1}) = \int_{x_{i-1}}^{x} \frac{d}{dx}(v^2(x))dx,$$

$$v^2(x_{i-1}) = v^2(x) - \int_{x_{i-1}}^{x} \frac{d}{dx}(v^2(x))dx.$$

Then, integrating both sides we have

$$\int_{x_{i-1}}^{x_i} v^2(x_{i-1})dx \leqslant \int_{x_{i-1}}^{x_i} v^2(x)dx + \int_{x_{i-1}}^{x_i}\int_{x_{i-1}}^{s} |2vv_x|dxds$$

and so

$$\begin{aligned} h_K v^2(x_{i-1}) &\leqslant \|v\|_{0,K}^2 + \int_{x_{i-1}}^{x_i}\int_{x_{i-1}}^{s} \Big(\frac{1}{h_K}v^2 + h_K v_x^2\Big)dxds \\ &\leqslant \|v\|_{0,K}^2 + \|v\|_{0,K}^2 + h_K^2\|v_x\|_{0,K}^2 \\ &\leqslant C(\|v\|_{0,K}^2 + h_K^2\|v_x\|_{0,K}^2). \end{aligned}$$

Therefore,

$$v^2(x_{i-1}) \leqslant C\big(\frac{1}{h_K}\|v\|_{0,K}^2 + h_K\|v_x\|_{0,K}^2\big).$$

Similarly, we have

$$v^2(x_i) \leqslant C\big(\frac{1}{h_K}\|v\|_{0,K}^2 + h_K\|v_x\|_{0,K}^2\big).$$

This completes the proof. □

Theorem 2.2.2 [37]. *Let K be a triangle or a quadrilateral in $\mathbb{R}^2$. Then, for all $v \in H^1(K)$,*

$$||v||_{0,\partial K}^2 \leqslant C\Big(\frac{1}{h_K}||v||_{0,K}^2 + h_K||v||_{1,K}^2\Big),$$

where C is a positive constant and h_K is the diameter of K.

Theorem 2.2.3 (Poincare-Friedrich inequality [37]). *Let Ω be an open, bounded, connected domain of $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$. Let $v \in H^1(\Omega)$ such that*

$$\int_{\Omega} v dx = 0.$$

Then

$$\|v\| \leqslant C\|\nabla v\|$$

where $C = C(\Omega)$ is a positive constant.

2.3 Lax-Milgram Theorem

Definition 2.3.1 [12]. *Let H be a normed linear space and V a closed subspace of H. A bilinear form $a(\cdot,\cdot)$ on H is said to be bounded (or continuous) if there is $c_1 < \infty$ such that*

$$|a(v,w)| \leqslant c_1\|v\|_H\|w\|_H, \quad \forall v,w \in H$$

and coercive on V if there is $c_2 < \infty$ such that

$$a(v,v) \geqslant c_2\|v\|_H^2, \quad \forall v \in V.$$

Theorem 2.3.1 (Lax-Milgram [12]). *Given a Hilbert space $(V,(\cdot,\cdot))$, a continuous, coercive bilinear form $a(\cdot,\cdot)$ and a continuous linear functional F on V, there exists a unique $u \in V$ such that*

$$a(u,v) = F(v), \quad \forall v \in V.$$

Definition 2.3.2. *Let $m \geqslant 1$, $s \geqslant 2m$ and $H_0^m(\Omega) \subset V \subset H^m(\Omega)$, suppose that $a(\cdot,\cdot)$ is a coercive bilinear form on V. Then the variational problem*

$$a(u,v) = (f,v), \quad \forall v \in V$$

is called H^{s-2m}-regular provided that there exists a constant $c = c(\Omega, a, s)$ such that for every $f \in H^{s-2m}(\Omega)$, there is a solution $u \in H^s(\Omega)$ with

$$\|u\|_s \leqslant c\|f\|_{s-2m}.$$

2.4 Approximation Properties

Let Ω be a bounded convex polygonal domain in $\mathbb{R}^n$, $n = 1, 2$ and P_h a partition of the domain Ω, i.e., P_h a finite collection of N_e open subdomains (elements) K_i, $i = 1, 2, \ldots, N_e$, such that

$$\bar{\Omega} = \bigcup_{K_i \in P_h} \bar{K}_i, \quad \text{and} \quad K_i \cap K_j = \phi, \quad i \neq j.$$

And for a given element $K \in P_h$, let $P_{p_K}(K)$ be the space of polynomial of degree at most p_K on K where $p_K \geq 1$.

Theorem 2.4.1 [6]. *Let K be an interval element of the partition P_h and $u \in H^s(K)$, $s \geqslant 0$. There exist a positive constant C depending on s but independent of u, h_K, and p_K, and a polynomial $u_p \in P_{p_K}(K)$, $p_K \geqslant 1$, such that*

$$\|u - u_p\|_{r,K} \leqslant C \frac{h_K^{\mu - r}}{p_K^{s-r}} \|u\|_{s,K},$$

where $\mu = \min(p_K + 1, s)$.

Theorem 2.4.2 [37]. *Let K be a triangular or quadrilateral element of the partition P_h and $u \in H^s(K)$, $s \geqslant 2$. There exist a positive constant C depending on s but independent of u, h_K, and p_K, and a polynomial $u_p \in P_{p_K}(K)$, $p_K \geqslant 2$, such that*

$$\int_\gamma \mathbf{n} \cdot \nabla(u - u_p) ds = 0, \quad \forall \gamma \subset \partial K,$$

and

$$\|u - u_p\|_{0,K} \leqslant C \frac{h_K^\mu}{p_K^{s-3/2}} \|u\|_{s,K},$$

$$\|\nabla(u - u_p)\|_{0,K} \leqslant C \frac{h_K^{\mu-1}}{p_K^{s-3/2}} \|u\|_{s,K},$$

$$\|\nabla^2(u - u_p)\|_{0,K} \leqslant C \frac{h_K^{\mu-2}}{p_K^{s-2}} \|u\|_{s,K},$$

where $\mu = \min(p_K + 1, s)$.

2.5 Generalized Minimum Residual Method

In this section, we want to introduce briefly the Generalized Minimum Residual method (GMRES). For more detail, one is referred to [44]. The GMRES is one of the important iterative techniques to solve a large linear system $Ax = b$.

The GMRES is an orthogonal projection method which seeks an approximate solution x_m from the affine subspace

$$x_0 + K_m(A, r_0) := x_0 + \text{span}\{r_0, Ar_0, A^2 r_0, \cdots, A^{m-1} r_0\}$$

of dimension m by imposing the Galerkin condition

$$b - Ax_m \perp K_m$$

where x_0 is an initial guess to the solution of $Ax = b$ and $r_0 = b - Ax_0$. The subspace $K_m(A, r_0)$ is called the Krylov subspace.

Now, we describe the basic GMRES algorithm. Any vector x in $x_0 + K_m$ can be written as

$$x = x_0 + V_m y \tag{2.5.1}$$

where y is an m-vector and V_m is an $n \times m$ matrix whose column vectors form an orthonormal basis of the Krylov subspace. Define

$$J(y) = \|b - Ax\| = \|b - A(x_0 + V_m y)\|. \tag{2.5.2}$$

Then

$$\begin{aligned} b - Ax &= b - A(x_0 + V_m y) \\ &= r_0 - AV_m y \\ &= \beta v_1 - V_{m+1}\tilde{H}_m y \\ &= V_{m+1}(\beta e_1 - \tilde{H}_m y), \end{aligned} \tag{2.5.3}$$

where $\beta = \|r_0\|$, e_1 is the first column of the $n \times n$ identity matrix and $\tilde{H}_m$ is an $n \times (m+1)$ Hessenberg matrix [44]. Since the column vectors of V_{m+1} are orthonormal,

$$J(y) = \|\beta e_1 - \tilde{H}_m y\|. \tag{2.5.4}$$

The GMRES approximation is the unique vector of $x_0 + K_m$ which minimizes (2.5.2). By (2.5.1) and (2.5.4), this approximation can be obtained quite simply as $x_m = x_0 + V_m y_m$ where y_m minimizes the function

$$J(y) = \|\beta e_1 - \tilde{H}_m y\|,$$

i.e.,

$$x_m = x_0 + V_m y_m,$$

where

$$y_m = \min_y \|\beta e_1 - \tilde{H}_m y\|.$$

The minimizer y_m is inexpensive to compute since it requires the solution of an $(m+1) \times m$ least-squares problem where m is typically small.

Theorem 2.5.1 [15]. *Suppose that $\tilde{x}$ is an approximation to the solution of $Ax = b$, A is a nonsingular matrix, and $r = b - A\tilde{x}$ is the residual vector for $\tilde{x}$. Then*

$$\|x - \tilde{x}\| \leqslant \|r\| \cdot \|A^{-1}\|$$

and

$$\frac{\|x - \tilde{x}\|}{\|x\|} \leqslant \|A\| \cdot \|A^{-1}\| \frac{\|r\|}{\|b\|} \tag{2.5.5}$$

provided $x \neq 0$ and $y \neq 0$. Here $\|x\|$ is a vector norm and $\|A\|$ is a matrix norm of A induced by the vector norm, i.e., $\|A\| = \max_{\|x\|=1} \|Ax\|$.

The inequalities in Theorem 2.5.1 imply that the quantities $\|A^{-1}\|$ and $\|A\| \cdot \|A^{-1}\|$ provide an indication of the connection between the residual vector and the accuracy of the approximation. In general, the relative error $\|x - \tilde{x}\|/\|x\|$ is of most interest and, by inequality (2.5.5), this error is bounded by the product of $\|A\| \cdot \|A^{-1}\|$ with the relative residual for this approximation, $\|r\|/\|b\|$.

Definition 2.5.1. *The condition number of a nonsingular matrix A relative to a norm $\|\cdot\|$ is*

$$cond(A) = \|A\| \cdot \|A^{-1}\|.$$

Remark 2.5.1. *The inequalities in Theorem 2.5.1 become*

$$\|x - \tilde{x}\| \leqslant cond(A)\frac{\|r\|}{\|A\|}$$

and

$$\frac{\|x - \tilde{x}\|}{\|x\|} \leqslant cond(A)\frac{\|r\|}{\|b\|}.$$

For any nonsingular matrix A and a norm $\|\cdot\|$,

$$1 = \|I\| = \|A \cdot A^{-1}\| \leqslant \|A\| \cdot \|A^{-1}\| = cond(A)$$

where I is an identity matrix.

Chapter 3

One Dimensional Elliptic Problems

3.1 Introduction

Discontinuous Galerkin methods with an interior penalty for elliptic problems were introduced by several authors [2, 28, 45]. These methods, referred to as interior penalty Galerkin schemes but not locally mass conservative, generalized Nitche method in [35] to treat the Dirichlet boundary condition with a penalty term on the boundary of the domain.

New types of elementwise conservative discontinuous Galerkin methods for diffusion problems were introduced and a priori error estimates were analyzed in [30, 36, 37, 40].

Recently, Babuška et al. [6] introduced a discontinuous Galerkin method for one dimensional elliptic problem and analyzed a priori error estimates in the energy and L^2 norms whose error estimate in the L^2 norm was not optimal. And Larson and Niklasson [32] analyzed the error in the L^2 norm of a family of discontinuous Galerkin methods, depending on two real parameters, for one dimensional elliptic problem. In the case of $\tilde{\alpha} = -1$ the error in the L^2 norm is optimal and in the case of $\tilde{\alpha} \neq -1$ for uniform mesh one in the L^2 norm is optimal for p odd and suboptimal for p even.

In this chapter, we consider the following one dimensional elliptic problem:

$$-\frac{d}{dx}\Big(a(\frac{du}{dx} + bu)\Big) + du = f \quad \text{in } I = (\alpha, \beta)$$

with the boundary conditions

$$u(\alpha) = u(\beta) = 0,$$

where a is a positive, bounded smooth function, b is a bounded smooth function, and d is a bounded nonnegative function.

And we will introduce discontinuous Galerkin approximations of the one dimensional elliptic problem based on the elliptic approach and obtain optimal error estimates in the energy and L^2 norms for the approximate solutions of the problem which improve the previous results of Babuška et al. [6] and Larson and Niklasson [32].

3.2 Notations

Let $I = (\alpha, \beta)$ be a bounded open interval in $\mathbb{R}$ and P_h denote a partition of I, i.e., P_h a finite collection of N open subintervals $K_i = (x_{i-1}, x_i),\ \ x_{i-1} < x_i,\ \ i = 1, 2, \ldots, N$, such that

$$[\alpha, \beta] = \bigcup_{K_i \in P_h} \bar{K}_i, \quad \text{and} \quad K_i \cap K_j = \phi, \quad i \neq j,$$

and if $h_i = x_i - x_{i-1},\ \ i = 1, 2, \cdots, N,\ \ h_{\max} = \max\{h_i\}$ and $h_{\min} = \min\{h_i\}$, then $h_{\max}/h_{\min}$ is bounded below and above by positive constants, independent of partitions P_h. Given a partition P_h, we introduce the sets Γ and Γ_{int} as follows:

$$\Gamma = \bigcup_{K_i \in P_h} \partial K_i = \{x_0, x_1, \ldots, x_N\},$$
$$\Gamma_{int} = \Gamma - \partial I = \{x_1, x_2, \ldots, x_{N-1}\},$$

where $\partial K_i = \{x_{i-1}, x_i\}$ denotes the boundary of the interval K_i and $\partial I = \{x_0, x_N\}$. We define $h := h_{\max}$ and $\hat{h}_i$ as follows:

$$\hat{h}_i = \begin{cases} \frac{h_i}{2}, & x_i \in \partial K_i \cap \partial I, \\ \frac{h_i + h_{i+1}}{2}, & x_i \in (\partial K_i \cap \partial K_{i+1}) \subset \Gamma_{int}. \end{cases}$$

The unit normal vector outward from K_i is denoted by $n|_i$. For each point $x_i \in \Gamma$ we will associate a unit normal vector n. The unit normal vector n is defined as $n = -1$ if $x_i \in \Gamma_{int}$ and $n = -1$ or 1 for x_0 or x_N, respectively. Therefore,

$$n|_{i+1}(x_i) = -n|_i(x_i) = n, \quad x_i \in \Gamma_{int}$$

and

$$n|_1(x_0) = -1, \;\; n|_N(x_N) = 1.$$

Let l be a nonnegative integer. For any given open interval S (S may be the whole interval I or an element K_i of P_h), the space $H^l(S)$ will denote the usual Sobolev space with norm $\|\cdot\|_{l,S}$. Moreover, $H_0^l(S)$ is the set of functions in $H^l(S)$ which vanish on the boundary ∂S, i.e.,

$$H_0^l(S) = \{v \in H^l(S); v = 0 \;\text{ on }\; \partial S\}.$$

The so-called (mesh-dependent) broken spaces $H^l(P_h)$ and $H_0^l(P_h)$ will be defined as

$$H^l(P_h) = \{v \in L^2(I); v|_{K_i} \in H^l(K_i), \;\; \forall K_i \in P_h\},$$

$$H_0^l(P_h) = \{v \in H^l(P_h); v = 0 \text{ on } \partial I\}.$$

The norm associated with the space $H^l(P_h)$ is given as

$$\|v\|_{l,h} = \Big(\sum_{K_i \in P_h} \|v\|_{l,K_i}^2 \Big)^{1/2}.$$

Finite element subspaces V_h of polynomial functions will be defined as

$$V_h = \{v \in L^2(I);\ v|_{K_i} \in P_p(K_i),\ \forall K_i \in P_h \text{ and } v = 0 \text{ on } \partial I\},$$

where $P_p(K_i)$ is the space of polynomial of degree less than or equal to p on K_i for a given integer $p \geq 1$. Then it is easy to show that $V_h \subset H_0^l(P_h)$.

For any function $v \in H^l(K_i) \times H^l(K_{i+1})$, $l > 1/2$, we denote the jump and average of v at $x_i \in \Gamma_{int}$, by $[v]$ and $\{v\}$, respectively, i.e.,

$$[v](x_i) = v(x_i)|_{K_{i+1}} - v(x_i)|_{K_i}, \quad x_i \in \Gamma_{int},$$
$$\{v\}(x_i) = \frac{1}{2}(v(x_i)|_{K_{i+1}} + v(x_i)|_{K_i}), \quad x_i \in \Gamma_{int}.$$

And at x_0, x_N, we define

$$[v](x_0) = [v](x_N) = 0.$$

We define the following seminorms and norm: for $\forall v \in H_0^2(P_h)$

$$|v|_{1,h}^2 = \sum_{K_i \in P_h} |v|_{1,K_i}^2,$$
$$|v|_-^2 = \sum_{i=0}^{N} \frac{1}{\hat{h}_i}\big([v](x_i)\big)^2,$$
$$|v|_+^2 = \sum_{i=0}^{N} \frac{1}{\hat{h}_i^3}\big([v](x_i)\big)^2,$$
$$|||v|||_{\pm}^2 = ||v||_{1,h}^2 + \sum_{K_i \in P_h} h_i^2 |v|_{2,K_i}^2 + |v|_{\pm}^2.$$

3.3 A Discontinuous Weak Formulation

We consider the following one dimensional elliptic problem:

$$-\frac{d}{dx}\Big(a(\frac{du}{dx}+bu)\Big)+du=f \quad \text{in } I=(\alpha,\beta) \tag{3.3.1}$$

with the boundary conditions

$$u(\alpha)=u(\beta)=0, \tag{3.3.2}$$

where a is a positive, bounded smooth function, b is a bounded smooth function, and d is a bounded nonnegative function. Assume that the following inequality holds as in [19]: for $v,w\in L^2(I)$

$$(aw,w)+(abv,w)+(dv,v)\geqslant C\Big((w,w)+(dv,v)\Big), \tag{3.3.3}$$

where C is a positive constant and $(\cdot,\cdot)$ is the L^2 inner product on I.

Multiplying both sides of (3.3.1) by v with $v=0$ on ∂I and integrating both sides, we have

$$\int_I\Big(-\frac{d}{dx}\Big(a(\frac{du}{dx}+bu)\Big)v+duv\Big)dx=\int_I fvdx. \tag{3.3.4}$$

And decomposing (3.3.4) over K, we obtain

$$\sum_{K_i\in P_h}\int_{K_i}-\frac{d}{dx}\big(a(\frac{du}{dx}+bu)\big)vdx+\sum_{K_i\in P_h}\int_{K_i}duvdx=\sum_{K_i\in P_h}\int_{K_i}fvdx.$$

Then integration by parts gives us

$$\begin{aligned}&\sum_{K_i\in P_h}\int_{K_i}\Big(a(\frac{du}{dx}+bu)\frac{dv}{dx}+duv\Big)dx\\&-\sum_{i=0}^{N-1}\big(na(\frac{du}{dx}+bu)v\big)|_{K_{i+1}}(x_i)-\sum_{i=1}^{N}\big(na(\frac{du}{dx}+bu)v\big)|_{K_i}(x_i)\\&=\int_I fvdx.\end{aligned} \tag{3.3.5}$$

By analogy with the formula below where a, b, c and d are real numbers:

$$ac - bd = \frac{1}{2}(a+b)(c-d) + \frac{1}{2}(a-b)(c+d), \qquad (3.3.6)$$

and using the average and jump operators, we have

$$\begin{aligned}\big(na(\frac{du}{dx}+bu)v\big)|_{K_{i+1}}(x_i) + &\big(na(\frac{du}{dx}+bu)\big)|_{K_i}(x_i) \\ &= \Big(\Big\{na(\frac{du}{dx}+bu)\Big\}[v] + \Big[na(\frac{du}{dx}+bu)\Big]\{v\}\Big)(x_i),\end{aligned}$$

for a given point $x_i \in \Gamma_{int}$. Therefore, we have

$$\begin{aligned}\sum_{i=0}^{N-1}\big(na(\frac{du}{dx}+bu)v\big)|_{K_{i+1}}(x_i) &+ \sum_{i=1}^{N}\big(na(\frac{du}{dx}+bu)v\big)|_{K_i}(x_i) \\ &= \sum_{i=1}^{N-1}\Big(\Big\{na(\frac{du}{dx}+bu)\Big\}[v] + \Big[na(\frac{du}{dx}+bu)\Big]\{v\}\Big)(x_i) \\ &\quad + \Big(na(\frac{du}{dx}+bu)v\Big)(x_0) + \Big(na(\frac{du}{dx}+bu)v\Big)(x_N) \\ &= \sum_{i=1}^{N-1}\Big(\Big\{na(\frac{du}{dx}+bu)\Big\}[v]\Big)(x_i),\end{aligned}$$

because the jump of $a(\frac{du}{dx}+bu)$ is zero on Γ_{int} and v is zero on ∂I. Consequently, (3.3.5) can now be reduced to

$$\begin{aligned}&\sum_{K_i \in P_h}\int_{K_i}\Big(a\frac{du}{dx}\frac{dv}{dx} + abu\frac{dv}{dx} + duv\Big)dx \\ &\quad - \sum_{i=1}^{N-1}\Big(\Big\{na\frac{du}{dx}\Big\}[v] + \{nabu\}[v]\Big)(x_i) = \sum_{K_i \in P_h}\int_{K_i} fv dx.\end{aligned} \qquad (3.3.7)$$

We introduce the following bilinear form $B(\cdot,\cdot)$ defined on $H^2(P_h) \times H^2(P_h)$ and the linear form $F(\cdot)$ defined on $H^2(P_h)$ as follows:

$$B(u,v) = \sum_{K_i \in P_h} \int_{K_i} \Big(a\frac{du}{dx}\frac{dv}{dx} + abu\frac{dv}{dx} + duv \Big) dx, \tag{3.3.8}$$

$$F(v) = \sum_{K_i \in P_h} \int_{K_i} fv dx = \int_I fv dx. \tag{3.3.9}$$

And we introduce the bilinear form $J(\cdot,\cdot)$ defined on $H^2(P_h) \times H^2(P_h)$ as follows:

$$\begin{aligned} J(u,v) &= \sum_{i=1}^{N-1} \Big(\Big\{ na\frac{du}{dx} \Big\} [v] + \{nabu\}[v] \Big)(x_i) \\ &\equiv J_1(u,v) + J_2(u,v), \quad \forall u, v \in H^2(P_h), \end{aligned} \tag{3.3.10}$$

where

$$J_1(u,v) = \sum_{i=1}^{N-1} \Big(\Big\{ na\frac{du}{dx} \Big\} [v] \Big)(x_i)$$

and

$$J_2(u,v) = \sum_{i=1}^{N-1} \Big(\{nabu\}[v] \Big)(x_i).$$

Thus, we define a discontinuous weak formulation of the problem (3.3.1) and (3.3.2) as follows: find $u \in H_0^2(P_h)$ such that

$$B(u,v) - J_1(u,v) - J_2(u,v) = F(v), \quad \forall v \in H_0^2(P_h). \tag{3.3.11}$$

3.4 DGFEMs with an Interior Penalty

To enforce the continuity of the solution at any $x_i \in \Gamma_{int}$, a penalty term will be added to the formulation. We introduce the following penalty terms

$$J_-^\sigma(u,v) = \sum_{i=0}^{N} \Big(\frac{\sigma}{\hat{h}_i} [u][v] \Big)(x_i), \tag{3.4.1}$$

$$J_{+}^{\sigma}(u,v)=\sum_{i=0}^{N}\Big(\frac{\sigma}{\hat{h}_i^3}[u][v]\Big)(x_i) \qquad (3.4.2)$$

where σ represents a penalty parameter with $\sigma_0 = inf_{x_i\in\Gamma_{int}}\sigma > 0$.

Define the bilinear forms $B_{\pm}^{\sigma}(\cdot,\cdot)$ on $H^2(P_h)\times H^2(P_h)$ as follows:

$$B_{\pm}^{\sigma}(u,v)=B(u,v)-J_1(u,v)-J_2(u,v)\pm J_1(v,u)+J_{\pm}^{\sigma}(u,v). \qquad (3.4.3)$$

Note that the bilinear $B_{-}^{\sigma}(\cdot,\cdot)$ is similar to one in [32] with $\tilde{\alpha}=-1$ and $B_{+}^{\sigma}(\cdot,\cdot)$ is similar to one in [32] with $\tilde{\alpha}=1$. Then, we obtain the discontinuous weak formulations of the problem (3.3.1) and (3.3.2) with an interior penalty: find $u\in H_0^2(P_h)$ such that

$$B_{\pm}^{\sigma}(u,v)=F(v), \quad \forall v\in H_0^2(P_h). \qquad (3.4.4)$$

And discontinuous Galerkin methods of the problem (3.3.1) and (3.3.2) with an interior penalty are: find $u_h\in V_h$ such that

$$B_{\pm}^{\sigma}(u_h,v)=F(v), \quad \forall v\in V_h. \qquad (3.4.5)$$

We now show that the bilinear forms $B_{\pm}^{\sigma}(\cdot,\cdot)$ are continuous with respect to the norms $|||\cdot|||_{\pm}$, respectively.

Theorem 3.4.1. *Let $B_{\pm}^{\sigma}(u,v)$ be the forms defined in (3.4.3). Then there exists a positive constant C such that*

$$|B_{\pm}^{\sigma}(u,v)|\leqslant C|||u|||_{\pm}|||v|||_{\pm}, \quad \forall u,v\in H_0^2(P_h). \qquad (3.4.6)$$

Proof. Let $u, v \in H_0^2(P_h)$. First we will prove that

$$|B_-^\sigma(u,v)| \leqslant C|||u|||_-|||v|||_-.$$

From the definition of B_-^σ, we have

$$\begin{aligned} |B_-^\sigma(u,v)| =& |B(u,v) - J_1(u,v) - J_2(u,v) - J_1(v,u) + J_-^\sigma(u,v)| \\ \leqslant& |B(u,v)| + |J_1(u,v)| + |J_2(u,v)| \\ &+ |J_1(v,u)| + |J_-^\sigma(u,v)|. \end{aligned} \tag{3.4.7}$$

It is clear that

$$\begin{aligned} |B(u,v)| \leqslant& \sum_{K_i \in P_h} \int_{K_i} \left| a(\frac{du}{dx} + bu)\frac{dv}{dx} + duv \right| dx \\ \leqslant& \sum_{K_i \in P_h} \Big(\int_{K_i} \left| a\frac{du}{dx}\frac{dv}{dx} \right| dx + \int_{K_i} \left| abu\frac{dv}{dx} \right| dx \\ &+ \int_{K_i} |duv| dx \Big) \\ \leqslant& \sum_{K_i \in P_h} \Big(C_1 \Big(\int_{K_i} \left| \frac{du}{dx} \right|^2 dx \Big)^{\frac{1}{2}} \Big(\int_{K_i} \left| \frac{dv}{dx} \right|^2 dx \Big)^{\frac{1}{2}} \\ &+ C_2 \Big(\int_{K_i} |u|^2 dx \Big)^{\frac{1}{2}} \Big(\int_{K_i} \left| \frac{dv}{dx} \right|^2 dx \Big)^{\frac{1}{2}} \\ &+ C_3 \Big(\int_{K_i} |u|^2 dx \Big)^{\frac{1}{2}} \Big(\int_{K_i} |v|^2 dx \Big)^{\frac{1}{2}} \Big) \\ \leqslant& C \big(|u|_{1,h}|v|_{1,h} + ||u||_{0,I}|v|_{1,h} + ||u||_{0,I}||v||_{0,I} \big) \\ \leqslant& C||u||_{1,h}||v||_{1,h} \end{aligned} \tag{3.4.8}$$

for some positive constant C. For the second and fourth terms in the right hand side of (3.4.7), we obtain

$$\begin{aligned}
|J_1(u,v)| &\leqslant \sum_{i=1}^{N-1} \left| \Big(\{na\frac{du}{dx}\}[v] \Big)(x_i) \right| \\
&\leqslant C\Big(\sum_{i=1}^{N-1} \hat{h}_i \left| \{a\frac{du}{dx}\}(x_i) \right|^2 \Big)^{\frac{1}{2}} \Big(\sum_{i=1}^{N-1} \frac{1}{\hat{h}_i} |[v](x_i)|^2 \Big)^{\frac{1}{2}} \\
&\leqslant C\Big(\sum_{K_i \in P_h} \big(|u|^2_{1,K_i} + h_i^2 |u|^2_{2,K_i} \big) \Big)^{\frac{1}{2}} |v|_- \\
&\leqslant C|||u|||_- |v|_-
\end{aligned} \tag{3.4.9}$$

and

$$\begin{aligned}
|J_1(v,u)| &\leqslant \sum_{i=1}^{N-1} \left| \Big(\{na\frac{dv}{dx}\}[u] \Big)(x_i) \right| \\
&\leqslant C\Big(\sum_{i=1}^{N-1} \hat{h}_i \left| \{a\frac{dv}{dx}\}(x_i) \right|^2 \Big)^{\frac{1}{2}} \Big(\sum_{i=1}^{N-1} \frac{1}{\hat{h}_i} |[u](x_i)|^2 \Big)^{\frac{1}{2}} \\
&\leqslant C\Big(\sum_{K_i \in P_h} \big(|v|^2_{1,K_i} + h_i^2 |v|^2_{2,K_i} \big) \Big)^{\frac{1}{2}} |u|_- \\
&\leqslant C|||v|||_- |u|_-.
\end{aligned} \tag{3.4.10}$$

And for the third term in the right hand side of (3.4.7), we get

$$\begin{aligned}
|J_2(u,v)| &\leqslant \sum_{i=1}^{N-1} \left| \Big(\{nabu\}[v] \Big)(x_i) \right| \\
&\leqslant C\Big(\sum_{i=1}^{N-1} \hat{h}_i |\{abu\}(x_i)|^2 \Big)^{\frac{1}{2}} \Big(\sum_{i=1}^{N-1} \frac{1}{\hat{h}_i} |[v](x_i)|^2 \Big)^{\frac{1}{2}} \\
&\leqslant C\Big(\sum_{K_i \in P_h} (|u|^2_{0,K_i} + h_i^2 |u|^2_{1,K_i}) \Big)^{\frac{1}{2}} |v|_- \\
&\leqslant C|||u|||_- |v|_-.
\end{aligned} \tag{3.4.11}$$

The last term is bounded by

$$
\begin{aligned}
|J_-^\sigma(u,v)| &= \left|\sum_{i=0}^{N} \frac{\sigma}{h}\big([u][v]\big)(x_i)\right| \\
&\leqslant C\Big(\sum_{i=0}^{N} \frac{1}{\hat{h}_i}([u](x_i))^2\Big)^{\frac{1}{2}}\Big(\sum_{i=0}^{N} \frac{1}{\hat{h}_i}([v](x_i))^2\Big)^{\frac{1}{2}} \\
&\leqslant C|u|_-|v|_-.
\end{aligned}
\tag{3.4.12}
$$

Substituting the above results (3.4.8)-(3.4.12) into (3.4.7), we have

$$
\begin{aligned}
|B_-^\sigma(u,v)| &\leqslant C\Big(||u||_{1,h}||v||_{1,h} + |||u|||_-|v|_- + |u|_-|||v|||_- + |u|_-|v|_-\Big) \\
&\leqslant C|||u|||_-|||v|||_-,
\end{aligned}
$$

which completes the proof of the boundedness for B_-^σ. And we can also prove the following result in a similar way:

$$
|B_+^\sigma(u,v)| \leqslant C|||u|||_+|||v|||_+.
$$

This completes the proof. □

Now we will show that the bilinear forms $B_\pm^\sigma(\cdot,\cdot)$ satisfy the stability condition with respect to the norms $|||\cdot|||_\pm$, respectively.

Theorem 3.4.2. *For a sufficiently large penalty parameter σ, there exists a positive constant C such that*

$$
B_\pm^\sigma(v,v) \geqslant C|||v|||_\pm^2, \quad \forall v \in V_h. \tag{3.4.13}
$$

Proof. Let $v \in V_h$. First, we will prove that $B_-^\sigma(v,v) \geqslant C|||v|||_-^2$. From the definition of $B_-^\sigma(v,v)$, we get

$$\begin{aligned} B_-^\sigma(v,v) &= B(v,v) - 2J_1(v,v) - J_2(v,v) + J_-^\sigma(v,v) \\ &= \sum_{K_i \in P_h} \int_{K_i} \Big(a\big(\frac{dv}{dx} + bv\big)\frac{dv}{dx} + dv^2 \Big) \\ &\quad - 2\sum_{i=1}^{N-1} \Big(\{na\frac{dv}{dx}\}[v] \Big)(x_i) - \sum_{i=1}^{N-1} \big(\{nabv\}[v] \big)(x_i) \\ &\quad + \sum_{i=0}^{N} \frac{\sigma}{\hat{h}_i}([v](x_i))^2. \end{aligned} \tag{3.4.14}$$

The first term in the right hand side of (3.4.14) can be written as

$$\begin{aligned} &\sum_{K_i \in P_h} \int_{K_i} \Big(a\big(\frac{dv}{dx} + bv\big)\frac{dv}{dx} + dv^2 \Big) dx \\ &\qquad = \big(a\frac{dv}{dx}, \frac{dv}{dx}\big) + \big(abv, \frac{dv}{dx}\big) + (dv, v). \end{aligned} \tag{3.4.15}$$

For the second and the third terms in the right hand side of (3.4.14), we obtain

$$\begin{aligned} \Big| \sum_{i=1}^{N-1} \Big(\{na\frac{dv}{dx}\}[v] \Big)(x_i) \Big| &\leqslant C\Big(\sum_{i=1}^{N-1} \hat{h}_i \big|\{a\frac{dv}{dx}\}(x_i)\big|^2 \Big)^{\frac{1}{2}} \\ &\qquad \Big(\sum_{i=1}^{N-1} \frac{1}{\hat{h}_i} |[v]|^2 ds \Big)^{\frac{1}{2}} \\ &\leqslant C\Big(\sum_{K_i \in P_h} \big(|v|_{1,K_i}^2 + h_i^2 |v|_{2,K_i}^2 \big) \Big)^{\frac{1}{2}} |v|_- \\ &\leqslant C|||v|||_- |v|_- \end{aligned} \tag{3.4.16}$$

and

$$\Big|\sum_{i=1}^{N-1}\big(\{nabv\}[v]\big)(x_i)\Big| \leqslant \Big(\int_{\Gamma_{int}}\hat{h}_i|\{abv\}|^2ds\Big)^{\frac{1}{2}}\Big(\int_{\Gamma}\frac{1}{\hat{h}_i}|[v]|^2ds\Big)^{\frac{1}{2}}$$

$$\leqslant C\Big(\sum_{K_i\in P_h}(|v|^2_{0,K_i}+h_i^2|v|^2_{1,K_i})\Big)^{\frac{1}{2}}|v|_- \qquad (3.4.17)$$

$$\leqslant C|||v|||_-|v|_-.$$

For the last term in the right hand side of (3.4.14), we also obtain (3.4.17). Note that

$$\sum_{i=0}^{N}\frac{\sigma}{h}\big([v](x_i)\big)^2 \geqslant \sigma_0|v|^2_-. \qquad (3.4.18)$$

By using (3.3.3) and (3.4.14)-(3.4.18), we have

$$\begin{aligned}
B^{\sigma}_-(v,v) &\geqslant C_1(|v|^2_{1,h}+C_2||v||^2_{0,I}) - C_3|||v|||_-|v|_- + \sigma_0|v|^2_- \\
&\geqslant C_1(|v|^2_{1,h}+C_2||v||^2_{0,I}) - \frac{C_3}{2}(\epsilon|||v|||^2_- + \frac{1}{\epsilon}|v|^2_-) + \sigma_0|v|^2_- \\
&\geqslant C_1(\frac{1}{2}|v|^2_{1,h}+C_2||v||^2_{0,I}+\frac{1}{2}|v|^2_{1,h}) \\
&\quad - \frac{C_3}{2}(\epsilon|||v|||^2_- + \frac{1}{\epsilon}|v|^2_-) + \sigma_0|v|^2_- \\
&\geqslant C_1(\frac{1}{2}|v|^2_{1,h}+C_2||v||^2_{0,I}+\frac{1}{2C_0}\sum_{K_i\in P_h}h_i^2|v|^2_{2,K_i}) \\
&\quad - \frac{C_3}{2}(\epsilon|||v|||^2_- + \frac{1}{\epsilon}|v|^2_-) + \sigma_0|v|^2_- \\
&\geqslant (C_1^* - \frac{\epsilon C_3}{2})|||v|||^2_- + (\sigma_0 - \frac{C_1}{2} - \frac{C_3}{2\epsilon})|v|^2_- \\
&\geqslant C|||v|||^2_-,
\end{aligned}$$

for a sufficiently small ϵ and a sufficiently large σ. Similarly, we can prove the following result

$$B^{\sigma}_+(v,v) \geqslant C|||v|||^2_+.$$

This completes the proof. $\square$

3.5 Error Estimates

In the error analysis, we need a bound on the approximation error $|||u - u_I|||_\pm$ when $u_I \in V_h$ is a suitable interpolant of the exact solution u. It is convenient to take an interpolant u_I which is discontinuous at $x_i \in \Gamma_{int}$. For the interpolant u_I, we just require the local approximation property:

$$|u - u_I|_{s,K_i} \leqslant C h_i^{p+1-s} |u|_{p+1,K_i}, \quad \forall K_i \in P_h, \quad s = 0, 1, 2, \tag{3.5.1}$$

where C depends only on p.

Theorem 3.5.1. *If $u \in H^{p+1}(I)$ and u_I, the interpolant of u, satisfies the local approximation property (3.5.1), then there exists a positive constant C such that*

$$|||u - u_I|||_- \leqslant C h^p |u|_{p+1,I}. \tag{3.5.2}$$

Proof. From the definition of $|||\cdot|||_-$, we obtain

$$\begin{aligned} |||u - u_I|||_-^2 &= ||u - u_I||_{1,h}^2 + \sum_{K_i \in P_h} h_i^2 |u - u_I|_{2,K_i}^2 + |u - u_I|_-^2 \\ &= ||u - u_I||_{1,h}^2 + \sum_{K_i \in P_h} h_i^2 |u - u_I|_{2,K_i}^2 \\ &\quad + \sum_{i=0}^{N} \frac{1}{\hat{h}_i} \big([u - u_I](x_i)\big)^2. \end{aligned} \tag{3.5.3}$$

Recall that

$$\|v\|_{0,\partial K_i}^2 \leqslant C(h_i^{-1} \|v\|_{0,K_i}^2 + h_i \|v\|_{1,K_i}^2), \qquad \forall v \in H^1(K_i). \tag{3.5.4}$$

Therefore, using (3.5.3) and (3.5.4), we obtain

$$|||u-u_I|||_-^2 \leqslant C\Big(||u-u_I||_{1,h}^2 + \sum_{K_i\in P_h} h_i^2|u-u_I|_{2,K_i}^2 + \sum_{K_i\in P_h}\frac{1}{h_i^2}\|u-u_I\|_{0,K_i}^2\Big) \tag{3.5.5}$$

and so, from (3.5.1) and (3.5.5), we have

$$|||u-u_I|||_- \leqslant Ch^p|u|_{p+1,I}. \tag{3.5.6}$$

This completes the proof. □

Theorem 3.5.2. *If $u \in H^{p+1}(I)$ and u_I, the continuous interpolant of u, satisfies the local approximation property (3.5.1), then there exists a positive constant C such that*

$$|||u-u_I|||_+ \leqslant Ch^p|u|_{p+1,I}. \tag{3.5.7}$$

Proof. Since u_I is the continuous interpolant of u, we obtain from the definition of $|||\cdot|||_+$

$$\begin{aligned}|||u-u_I|||_+^2 &= ||u-u_I||_{1,h}^2 + \sum_{K_i\in P_h} h_i^2|u-u_I|_{2,K_i}^2 + |u-u_I|_+^2\\ &= ||u-u_I||_{1,h}^2 + \sum_{K_i\in P_h} h_i^2|u-u_I|_{2,K_i}^2\\ &\quad + \sum_{i=0}^{N}\hat{h}_i^{-3}\big([u-u_I](x_i)\big)^2\\ &= ||u-u_I||_{1,h}^2 + \sum_{K_i\in P_h} h_i^2|u-u_I|_{2,K_i}^2.\end{aligned} \tag{3.5.8}$$

Therefore, from (3.5.1) and (3.5.8), we have

$$|||u - u_I|||_+ \leqslant Ch^p|u|_{p+1,I}. \tag{3.5.9}$$

This completes the proof. $\square$

Theorem 3.5.3. *If u is a solution to (3.4.4) and u_h is the finite element solution to (3.4.5), then there exists a positive constant C such that*

$$|||u - u_h|||_\pm \leqslant Ch^p|u|_{p+1,I}. \tag{3.5.10}$$

Proof. Let $u_I \in P_p$ be a piecewise interpolant of u which satisfies the approximation property (3.5.1). Then, from (3.4.4), (3.4.5), (3.5.1), Theorems 3.5.1 and 3.5.2, we have

$$\begin{aligned} c|||u_I - u_h|||_-^2 &\leqslant B_-^\sigma(u_I - u_h, u_I - u_h) \\ &= B_-^\sigma(u_I - u_h + u - u, u_I - u_h) \\ &= B_-^\sigma(u_I - u, u_I - u_h) + B_-^\sigma(u - u_h, u_I - u_h) \\ &= B_-^\sigma(u_I - u, u_I - u_h) \\ &\leqslant C|||u_I - u|||_-|||u_I - u_h|||_- \\ &\leqslant Ch^p|u|_{p+1,I}|||u_I - u_h|||_-. \end{aligned}$$

In the case of $B_+^\sigma(\cdot,\cdot)$, we use the continuous interpolant $u_I \in P_p$ of u satisfying approximation property (3.5.1). Then we can obtain the following result

$$c|||u_I - u_h|||_+^2 \leqslant Ch^p|u|_{p+1,I}|||u_I - u_h|||_+.$$

Hence, Theorem 3.5.2 and the triangle inequality give the optimal error estimate

$$|||u - u_h|||_{\pm} \leqslant Ch^p |u|_{p+1,I}.$$

This completes the proof. $\square$

Remark 3.5.1. *When $a(x) = 1$ and $b(x) = d(x) = 0$ in the problem (3.3.1) and (3.3.2), our result of Theorem 3.5.3 for B_{+}^{σ} is the same as one for a discontinuous Galerkin method without a penalty term [6].*

Next, we want to obtain an optimal error estimate in the L^2 norm. Let us consider the following dual or adjoint problem: find $\psi \in H_0^2(P_h)$ such that

$$B_{-}^{\sigma}(v, \psi) = (u - u_h, v), \quad \forall v \in H_0^2(P_h) \tag{3.5.11}$$

with the elliptic regularity

$$\|\psi\|_{2,I} \leqslant C\|u - u_h\|_{0,I}, \tag{3.5.12}$$

where C depends only on the domain I. Note that this corresponds to the concept of the adjoint consistency discussed in [3]. Assume that the dual problem (3.5.11) has a solution satisfying (3.5.12). The existence of a solution of (3.5.11) satisfying (3.5.12) can be guaranteed by the existence of a solution of the problem

$$\begin{gathered} -\frac{d}{dx}\left(a\frac{d\psi}{dx}\right) + ab\frac{d\psi}{dx} + d\psi = u - u_h \quad \text{in} \quad I = (\alpha, \beta), \\ \psi(\alpha) = \psi(\beta) = 0. \end{gathered} \tag{3.5.13}$$

Theorem 3.5.4. *If u is a solution to (3.4.4) and u_h is the finite element solution to (3.4.5) for B^σ_-, then there exists a positive constant C such that*

$$||u-u_h||_{0,I} \leqslant Ch^{p+1}|u|_{p+1,I}.$$

Proof. We take $\psi_I \in V_h$ as a piecewise linear interpolant of ψ satisfying the local approximation property (3.5.1). Then, taking $v = u - u_h$ in (3.5.11), and using (3.4.4), (3.4.5) and the elliptic regularity (3.5.12), we obtain

$$\begin{aligned} ||u-u_h||^2_{0,I} &= B^\sigma(u-u_h,\psi) \\ &= B^\sigma(u-u_h,\psi-\psi_I) + B^\sigma(u-u_h,\psi_I) \\ &\leqslant C|||u-u_h|||_-|||\psi-\psi_I|||_- \qquad (3.5.14) \\ &\leqslant Ch|\psi|_{2,I}|||u-u_h|||_- \\ &\leqslant Ch||u-u_h||_{0,I}|||u-u_h|||_-. \end{aligned}$$

Substituting (3.5.10) into the above result (3.5.14), we get the optimal error estimate

$$||u-u_h||_{0,I} \leqslant Ch^{p+1}|u|_{p+1,I}.$$

This completes the proof. □

In the case of $B^\sigma_+(\cdot,\cdot)$, we consider the following dual or adjoint problem: find $\psi \in H^2_0(P_h)$ such that

$$B^\sigma_+(v,\psi) - 2\sum_{i=0}^{N}\Big(\{n\frac{d\psi}{dx}\}[v]\Big)(x_i) = (u-u_h,v), \quad \forall v \in H^2(P_h) \qquad (3.5.15)$$

with the elliptic regularity (3.5.12). Assume that the dual problem (3.5.15) has a solution satisfying (3.5.12). The existence of a solution of (3.5.15) satisfying (3.5.12) can be guaranteed by the existence of a solution of the problem (3.5.13)

Theorem 3.5.5. *If u is a solution to (3.4.4) and u_h is the finite element solution to (3.4.5) for B_+^σ, then there exists a positive constant C such that*

$$||u - u_h||_{0,I} \leqslant Ch^{p+1}|u|_{p+1,I}.$$

Proof. Let ψ be the solution of the problem (3.5.15) satisfying (3.5.12) and take $\psi_I \in V_h$ as a continuous interpolant of ψ satisfying the local approximation property (3.5.1). Then taking $v = u - u_h$ in (3.5.15) and using the elliptic regularity (3.5.12), we obtain

$$\begin{aligned}
||u - u_h||^2 &= B_+^\sigma(u - u_h, \psi) - 2\sum_{i=1}^{N-1}\Big(\{\frac{d\psi}{dx}\}[u - u_h]\Big)(x_i) \\
&= B_+^\sigma(u - u_h, \psi - \psi_I + \psi_I) - 2\sum_{i=1}^{N-1}\Big(\{\frac{d\psi}{dx}\}[u - u_h]\Big)(x_i) \\
&\leqslant B_+^\sigma(u - u_h, \psi - \psi_I) - 2\sum_{i=1}^{N-1}\Big(\{\frac{d\psi}{dx}\}[u - u_h]\Big)(x_i) \\
&\leqslant C|||u - u_h|||_+|||\psi - \psi_I|||_+ \\
&\quad + C\Big(\sum_{i=1}^{N-1}\hat{h}_i^3\big(\{\frac{d\psi}{dx}\}(x_i)\big)^2\Big)^{\frac{1}{2}}\Big(\sum_{i=1}^{N-1}\frac{1}{\hat{h}_i^3}\big([u - u_h](x_i)\big)^2\Big)^{\frac{1}{2}} \\
&\leqslant C|||u - u_h|||_+ h||\psi||_{2,I} + Ch^{\frac{3}{2}}||\frac{d\psi}{dx}||_{0,\Gamma_{int}}|u - u_h|_+ \\
&\leqslant Ch|||u - u_h|||_+||u - u_h||_{0,I} \\
&\quad + Ch^{\frac{3}{2}}\Big(\sum_{i=1}^{N-1}\frac{1}{h_i}||\frac{d\psi}{dx}||_{0,K_i}^2 + h_i||\frac{d\psi}{dx}||_{1,K_i}^2\Big)^{\frac{1}{2}}|||u - u_h|||_+
\end{aligned}$$

$$
\begin{aligned}
&\leqslant Ch|||u-u_h|||_+||u-u_h||_{0,I} \\
&\quad + Ch\Big(\sum_{i=1}^{N-1}||\frac{d\psi}{dx}||^2_{0,K_i} + h_i^2||\frac{d\psi}{dx}||^2_{1,K_i}\Big)^{\frac{1}{2}}|||u-u_h|||_+ \\
&\leqslant Ch|||u-u_h|||_+||u-u_h||_{0,I} + Ch||\psi||_{2,I}|||u-u_h|||_+ \\
&\leqslant Ch|||u-u_h|||_+||u-u_h||_{0,I} + Ch||u-u_h||_{0,I}|||u-u_h|||_+.
\end{aligned}
$$

Therefore we obtain the optimal error estimate from (3.5.10)

$$
\begin{aligned}
||u-u_h|| &\leqslant Ch|||u-u_h|||_+ \\
&\leqslant Ch^{p+1}||u||_{p+1,I}.
\end{aligned}
$$

This completes the proof. □

Remark 3.5.2. *When $a(x) = 1$ and $b(x) = d(x) = 0$ in the problem (3.3.1) and (3.3.2), our problem for B^σ_+ is the same as one in [6]. And, it is stated in [32] that in the case of $\tilde{\alpha} = -1$ the error in the L^2 norm is order $O(h^{p+1})$ and in the case of $\tilde{\alpha} \neq -1$ for uniform mesh one in the L^2 norm is order $O(h^{p+1})$ for p odd and order $O(h^p)$ for p even. However, we obtain optimal error estimate in the L^2 norm for $B^\sigma_\pm$ when $p \geqslant 1$.*

Chapter 4
Two Dimensional Elliptic Problems

4.1 Introduction

In this chapter, we consider the following two dimensional elliptic problem in the "conservation" (or "strong divergence") form:

$$-\nabla \cdot \big(\boldsymbol{a}(\nabla u + \mathbf{b}u)\big) + du = f \quad \text{in } \Omega$$

with the homogeneous Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\Omega,$$

where $\bar{\Omega}$ is a closed bounded polygonal domain in $\mathbb{R}^2$, $\boldsymbol{a}$ is a symmetric, positive definite, bounded smooth matrix function, $\mathbf{b}$ is a bounded smooth vector function, and d is a bounded nonnegative function. It is assumed that the problem has a unique solution.

When $\mathbf{b} = \mathbf{0}$, based on the elliptic approach, Arnold et al. [3], Brezzi et al. [14], Houston et al. [31], Prudhomme et al. [37], Riviere et al. [40, 41] introduced discontinuous Galerkin approximations of the problem with various types of boundary conditions and analyzed the error of the discontinuous Galerkin approximations of the problem. And based on the hyperbolic approach, Chen and Chen [17] studied mixed discontinuous finite element methods for the problem. Note that there are also many mixed discontinuous finite element methods for the problem based on the hyperbolic approach.

It is referred to Cockburn et al. [24] for the theory and application of discontinuous finite element methods for elliptic and hyperbolic problems.

When $\mathbf{b} \neq \mathbf{0}$, based on the hyperbolic approach, Chen [19] first introduced discontinuous finite element methods of the problem and analyzed the error of discontinuous finite element approximations of the problem. And Harriman et al. [30] introduced discontinuous Galerkin methods of the problem whose bilinear forms came from the hyperbolic approach and obtained optimal error estimates of the discontinuous Galerkin approximations of the problem under a positivity assumption on $\boldsymbol{a}, \mathbf{b}$ and d.

And, under the assumption on $\boldsymbol{a}, \mathbf{b}$ and d as in [19] which is different from one in [30], we will introduce discontinuous Galerkin approximations of the problem based on the elliptic approach and we will obtain optimal error estimates for the approximations of the problem which are very similar to ones in [30]. When $\mathbf{b} = \mathbf{0}$, our discontinuous Galerkin finite element method leads to a symmetric interior penalty discontinuous Galerkin finite element method (DGFEM) of the problem and our optimal error estimates are similar to those obtained by a symmetric interior penalty DGFEM or other DGFEMs for the problem and also similar to those obtained by DGFEMs for elliptic problems in the divergence form.

4.2 Notations

Let Ω be a bounded convex polygonal domain in $\mathbb{R}^2$ with its boundary $\partial\Omega$ and P_h denote a partition of the domain Ω, i.e., P_h a finite collection of N_e

open subdomains (elements) $K_i,\ \ i=1,2,\ldots,N_e$, such that

$$\bar{\Omega}=\bigcup_{K_i\in P_h}\bar{K}_i,\quad \text{and}\quad K_i\cap K_j=\phi,\quad i\neq j.$$

And it is assumed that all partitions P_h are measured in terms of two quantities h_K and ρ_K, defined as:

$$h_K=\operatorname{diam}(K),$$

$$\rho_K=\sup\{\operatorname{diam}(B);B\text{ is a ball contained in }K\}.$$

We also introduce the parameter h associated with the partition P_h:

$$h=\max_{K\in P_h}h_K.$$

Definition 4.2.1. *A family $\{P_h\}$ of partitions P_h is said to be shape-regular as h tends to zero if there exists a number $\rho>0$, independent of h and K such that*

$$\frac{h_K}{\rho_K}\leqslant\rho,\quad \forall K\in P_h.$$

We assume that all partitions P_h are shape-regular. Given a partition P_h, we introduce the sets Γ and Γ_{int} as:

$$\Gamma=\bigcup_{K\in P_h}\partial K\quad\text{and}\quad \Gamma_{int}=\Gamma-\partial\Omega,$$

where ∂K denotes the boundary of the element K. We shall denote the collection of edges of P_h by the set $E_h=\{e_l\},\ \ l=1,\ldots,N_\gamma$ and denote the collection of edges (interface) e_{ij} between two adjacent elements K_i and K_j, $i>j$ by $E_{h,int}$ where an edge represents here an open subset of either Ω or $\partial\Omega$.

The unit normal vector outward from K(respectively K_i) is denoted by $\mathbf{n}$ (respectively $\mathbf{n}|_i$). For each edge $e \in E_h$ we will associate a unit normal vector $\mathbf{n}$. If $e_{ij} \in E_{h,int}$ then the unit normal vector is simply defined as $\mathbf{n} = \mathbf{n}|_i = -\mathbf{n}|_j$ and if e is an edge associated with an element K_i adjacent to $\partial\Omega$ then $\mathbf{n}$ is the unit normal vector outward from $\partial\Omega$.

Let l be a nonnegative integer. For any given open set S (S may be the whole domain Ω, an element K of P_h, or an edge e of E_h), the space $H^l(S)$ will denote the usual Sobolev space with norm $\|\cdot\|_{l,S}$. Moreover, $H_0^1(S)$ is the set of functions in $H^1(S)$ which vanish on the boundary ∂S, i.e.,

$$H_0^1(S) = \{v \in H^1(S); v = 0 \text{ on } \partial S\}.$$

The so-called (mesh-dependent) broken spaces $H^l(P_h)$ and $H_0^l(P_h)$ will be defined as

$$H^l(P_h) = \{v \in L^2(\Omega); v|_K \in H^l(K),\ \ \forall K \in P_h\},$$

$$H_0^l(P_h) = \{v \in H^l(P_h); v = 0 \text{ on } \partial\Omega\}.$$

The norm associated with the space $H^l(P_h)$ is given as

$$\|v\|_{l,h} = \Big(\sum_{K \in P_h} \|v\|_{l,K}^2\Big)^{1/2}$$

where $\|v\|_{l,K}$ is the Sobolev norm on K.

Finite element subspaces V_h of polynomial functions will be defined as

$$V_h = \{v \in L^2(\Omega);\ \ v|_K = \hat{v} \circ F_K^{-1}, \hat{v} \in P_p(\hat{K}),\ \ \forall K \in P_h\},$$

where F_K is the affine mapping from the reference element $\hat{K}$ to the element K in the partition P_h and $P_p(\hat{K})$ is the space of polynomial of degree at most p on K for the given $p \geq 1$. Then it is easy to show that $V_h \subset H_0^l(P_h)$.

For any function $v \in H^l(K_i) \times H^l(K_j)$, $l > 1/2$, we denote the jump and average of v on an interior edge e_{ij}, $i > j$, by $[v]$ and $\{v\}$, respectively, i.e.,

$$[v] = v_i - v_j,$$
$$\{v\} = \frac{1}{2}(v_i + v_j),$$

where v_i and v_j denote the restrictions of v on the elements K_i and K_j, repectively. We define the following seminorms and norm: for $\forall v \in H_0^2(P_h)$

$$|v|_{1,h}^2 = \sum_{K \in P_h} |v|_{1,K}^2,$$
$$|v|_{-}^2 = \sum_{e \in E_h} \frac{1}{h_e} \|[v]\|_{0,e}^2,$$
$$|v|_{+}^2 = \sum_{e \in E_h} \frac{1}{h_e^3} \|[v]\|_{0,e}^2,$$
$$|||v|||_{\pm}^2 = ||v||_{1,h}^2 + \sum_{K \in P_h} h_K^2 |v|_{2,K}^2 + |v|_{\pm}^2,$$

where h_e denotes the length of the given edge $e \in E_h$. Notice that these definitions are the same as ones in [3].

4.3 A Discontinuous Weak Formulation

We consider the following two dimensional elliptic problem:

$$-\nabla \cdot \big(\boldsymbol{a}(\nabla u + \mathbf{b}u)\big) + du = f \quad \text{in } \Omega \tag{4.3.1}$$

with the homogeneous Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\Omega, \tag{4.3.2}$$

where $\boldsymbol{a}$ is a symmetric, positive definite, bounded smooth matrix function, $\mathbf{b}$ is a bounded smooth vector function, and d is a bounded nonnegative function. Assume that the following inequality holds as in [19]: for each $\tau \in (L^2(\Omega))^2$ and each $v \in L^2(\Omega)$

$$(\boldsymbol{a}\tau, \tau) + (\boldsymbol{a}\mathbf{b}v, \tau) + (dv, v) \geqslant C\Big(||\tau||^2_{(L^2(\Omega))^2} + (dv, v)\Big), \tag{4.3.3}$$

where C is a positive constant. Note that the assumption (4.3.3) is different from the one in [30]: there exists a constant vector $\xi \in \mathbb{R}^2$ such that

$$d(x) - \frac{1}{2}\nabla \cdot (\boldsymbol{a}\mathbf{b}) - \mathbf{b} \cdot \xi > 0, \quad \text{a.e.} \quad x \in \Omega.$$

Multiplying both sides of (4.3.1) by v with $v = 0$ on $\partial\Omega$ and integrating both sides, we have

$$\int_\Omega \Big(-\nabla \cdot \big(\boldsymbol{a}(\nabla u + \mathbf{b}u)\big)v + duv\Big)dx = \int_\Omega fvdx. \tag{4.3.4}$$

And decomposing (4.3.4) over K, we obtain

$$\sum_{K \in P_h} \int_K -\nabla \cdot \big(\boldsymbol{a}(\nabla u + \mathbf{b}u)\big)vdx + \sum_{K \in P_h} \int_K duvdx = \sum_{K \in P_h} \int_K fvdx.$$

Then integration by parts gives us

$$\begin{aligned} &\sum_{K \in P_h} \int_K \Big(\boldsymbol{a}(\nabla u + \mathbf{b}u) \cdot \nabla v + duv\Big)dx \\ &- \sum_{K \in P_h} \int_{\partial K} \big(\mathbf{n} \cdot \boldsymbol{a}(\nabla u + \mathbf{b}u)\big)vds = \int_\Omega fvdx. \end{aligned} \tag{4.3.5}$$

Notice that using the average and jump operators, we obtain

$$\begin{aligned} \big(\mathbf{n} \cdot \boldsymbol{a}(\nabla u + \mathbf{b}u)\big)_i v_i + &\big(\mathbf{n} \cdot \boldsymbol{a}(\nabla u + \mathbf{b}u)\big)_j v_j \\ &= \mathbf{n} \cdot \big(\boldsymbol{a}(\nabla u + \mathbf{b}u)\big)_i v_i - \mathbf{n} \cdot \big(\boldsymbol{a}(\nabla u + \mathbf{b}u)\big)_j v_j \\ &= \Big\{\mathbf{n} \cdot \boldsymbol{a}(\nabla u + \mathbf{b}u)\Big\}[v] + \Big[\mathbf{n} \cdot \boldsymbol{a}(\nabla u + \mathbf{b}u)\Big]\{v\}, \end{aligned}$$

for a given edge $e_{ij} \in E_{h,int}$, where $(\mathbf{n} \cdot a(\nabla u + \mathbf{b}u))_i$ and $(\mathbf{n} \cdot a(\nabla u + \mathbf{b}u))_j$ represent the restrictions of $\mathbf{n} \cdot a(\nabla u + \mathbf{b}u)$ on K_i and K_j, respectively. Therefore we have

$$\begin{aligned}
&\sum_{K \in P_h} \int_{\partial K} \mathbf{n} \cdot \boldsymbol{a}(\nabla u + \mathbf{b}u) v ds \\
&\qquad = \int_{\Gamma_{int}} \Big\{ \mathbf{n} \cdot \boldsymbol{a}(\nabla u + \mathbf{b}u) \Big\} [v] ds + \int_{\Gamma_{int}} \Big[\mathbf{n} \cdot \boldsymbol{a}(\nabla u + \mathbf{b}u) \Big] \{v\} ds \\
&\qquad\quad + \int_{\partial\Omega} \mathbf{n} \cdot \boldsymbol{a}(\nabla u + \mathbf{b}u) v ds \\
&\qquad = \int_{\Gamma_{int}} \Big\{ \mathbf{n} \cdot \boldsymbol{a}(\nabla u + \mathbf{b}u) \Big\} [v] ds,
\end{aligned}$$

because the jump of $\boldsymbol{a}(\nabla u + \mathbf{b}u)$ is zero on Γ_{int} and v is zero on $\partial\Omega$. Consequently, (4.3.5) can now be reduced to

$$\begin{aligned}
&\sum_{K \in P_h} \int_K \Big(\boldsymbol{a}(\nabla u + \mathbf{b}u) \cdot \nabla v + duv \Big) dx \\
&\qquad - \int_{\Gamma_{int}} \Big\{ \mathbf{n} \cdot \boldsymbol{a}(\nabla u + \mathbf{b}u) \Big\} [v] ds = \sum_{K \in P_h} \int_K f v dx.
\end{aligned} \tag{4.3.6}$$

We introduce the following bilinear form $B(\cdot,\cdot)$ defined on $H^2(P_h) \times H^2(P_h)$ and the linear form $F(\cdot)$ defined on $H^2(P_h)$ as follows:

$$B(u,v) = \sum_{K \in P_h} \int_K \Big(\boldsymbol{a}(\nabla u + \mathbf{b}u) \cdot \nabla v + duv \Big) dx, \tag{4.3.7}$$

$$F(v) = \sum_{K \in P_h} \int_K f v dx = \int_\Omega f v dx. \tag{4.3.8}$$

And we introduce the bilinear form $J(\cdot,\cdot)$ defined on $H^2(P_h) \times H^2(P_h)$ as follows:

$$\begin{aligned}
J(u,v) &= \int_{\Gamma_{int}} \Big\{ \mathbf{n} \cdot \boldsymbol{a}(\nabla u + \mathbf{b}u) \Big\} [v] ds \\
&\equiv J_1(u,v) + J_2(u,v), \quad \forall u,v \in H^2(P_h),
\end{aligned} \tag{4.3.9}$$

where

$$J_1(u,v) = \int_{\Gamma_{int}} \{\mathbf{n} \cdot \boldsymbol{a}\nabla u\}[v]ds$$

and

$$J_2(u,v) = \int_{\Gamma_{int}} \{\mathbf{n} \cdot \boldsymbol{a}\mathbf{b}u\}[v]ds.$$

Thus, we define a discontinuous weak formulation of the problem (4.3.1) and (4.3.2) as follows: find $u \in H_0^2(P_h)$ such that

$$B(u,v) - J_1(u,v) - J_2(u,v) = F(v), \quad \forall v \in H_0^2(P_h). \tag{4.3.10}$$

4.4 DGFEMs with an Interior Penalty

To enforce the continuity of the solution at the interface of the elements, a penalty term will be added to the formulation such as Arnold [1] and Wheeler [45]. We introduce the following penalty terms

$$\begin{aligned} J_-^\sigma(u,v) &= \int_\Gamma \frac{\sigma}{\hat{h}}[u][v]ds, \\ J_+^\sigma(u,v) &= \int_\Gamma \frac{\sigma}{\hat{h}^3}[u][v]ds, \end{aligned} \tag{4.4.1}$$

where σ represents a penalty parameter with $\sigma_0 = \inf_{e \in E_h} \sigma$ and $\hat{h} = h_e$ on each $e \in E_h$.

Define the bilinear forms $B_\pm^\sigma(\cdot,\cdot)$ on $H^2(P_h) \times H^2(P_h)$ as follows:

$$B_\pm^\sigma(u,v) = B(u,v) - J_1(u,v) - J_2(u,v) \pm J_1(v,u) + J_\pm^\sigma(u,v). \tag{4.4.2}$$

Then, we obtain the discontinuous weak formulations of the problem (4.3.1) and (4.3.2) with an interior penalty: find $u \in H_0^2(P_h)$ such that

$$B_{\pm}^{\sigma}(u, v) = F(v), \quad \forall v \in H_0^2(P_h). \tag{4.4.3}$$

And discontinuous Galerkin methods of the problem (4.3.1) and (4.3.2) with an interior penalty are: find $u_h \in V_h$ such that

$$B_{\pm}^{\sigma}(u_h, v) = F(v), \quad \forall v \in V_h. \tag{4.4.4}$$

Notice that the bilinear form B_{-}^{σ} in (4.4.4) is different from one with $\theta = -1$ in [30] and that our interior penalty discontinuous Galerkin method leads to a symmetric interior penalty DGFEM of the elliptic problem when $\mathbf{b} = \mathbf{0}$.

We shall prove the equivalence of the strong and weak formulations with the interior penalty. Existence of solutions of the discontinuous formulations is then somewhat guaranteed.

Theorem 4.4.1. *Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ be the solution of the problem (4.3.1)-(4.3.2). Then u satisfies the weak formulations (4.4.3). Conversely, if $u \in H^1(\Omega) \cap H_0^2(P_h)$ is a smooth solution of (4.4.3) and f is a continuous function, then u satisfies the problem (4.3.1)-(4.3.2).*

Proof. The first part of the theorem has been proved along with the derivation with the interior penalty, since (4.3.5) is satisfied when $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$.

The proof of the converse is the same as one in Riviere [39]. Let $D(K) \subset H_0^2(K)$ be the space of infinitely differentiable functions with compact support on the element K and let $v \in D(K)$. Letting $v = 0$ on $\Omega - K$, we obtain from B_-^σ in (4.4.3) that

$$
\begin{aligned}
\int_K fv dx = & \int_\Omega fv dx \\
& = \int_\Omega \Big(\boldsymbol{a}(\nabla u + \mathbf{b}u) \cdot \nabla v + duv \Big) dx - \int_\Gamma \{ \mathbf{n} \cdot \boldsymbol{a}(\nabla u + \mathbf{b}u) \} [v] ds \\
& \quad - \int_\Gamma \{ \mathbf{n} \cdot \boldsymbol{a} \nabla v \} [u] ds + \int_\Gamma \frac{\sigma}{\hat{h}} [u][v] ds \\
& = \int_K \Big(\boldsymbol{a}(\nabla u + \mathbf{b}u) \cdot \nabla v + duv \Big) dx \\
& = \int_K \Big((-\nabla \cdot \boldsymbol{a} \nabla u) v - \nabla \cdot (\boldsymbol{a}\mathbf{b}u) v + duv \Big) dx \\
& \quad + \int_{\partial K} \mathbf{n} \cdot \boldsymbol{a}(\nabla u + \mathbf{b}u) v ds \\
& = \int_K \Big((-\nabla \cdot \boldsymbol{a} \nabla u) v - \nabla \cdot (\boldsymbol{a}\mathbf{b}u) v + duv \Big) dx \\
& = \int_K \Big(-\nabla \cdot \boldsymbol{a}(\nabla u + \mathbf{b}u) \ + du \Big) v dx,
\end{aligned}
$$

which implies that

$$
-\nabla \cdot \boldsymbol{a}(\nabla u + \mathbf{b}u) + du = f \quad \text{in } K. \tag{4.4.5}
$$

Next, we consider an interior edge e_{ij} shared by the elements K_i and K_j. Since u is smooth, u satisfies (4.3.1) on the interior edge e_{ij}. And in the case of B_+^σ, we can also prove in a similar way. This completes the proof. □

We now show that the bilinear forms $B_\pm^\sigma(\cdot,\cdot)$ are continuous with respect to the norms $|||\cdot|||_\pm$, respectively.

Theorem 4.4.2. *Let $B^{\sigma}_{\pm}(u,v)$ be the forms defined in (4.4.2). Then there exists a positive constant C such that*

$$|B^{\sigma}_{\pm}(u,v)| \leqslant C|||u|||_{\pm}|||v|||_{\pm}, \quad \forall u,v \in H^2_0(P_h). \tag{4.4.6}$$

Proof. Let $u,v \in H^2_0(P_h)$. First, we will prove that

$$|B^{\sigma}_{-}(u,v)| \leqslant C|||u|||_{-}|||v|||_{-}.$$

From definition of B^{σ}_{-}, we obtain

$$\begin{aligned}
|B^{\sigma}_{-}(u,v)| &= |B(u,v) - J_1(u,v) - J_2(u,v) - J_1(v,u) + J^{\sigma}_{-}(u,v)| \\
&\leqslant |B(u,v)| + |J_1(u,v)| + |J_2(u,v)| \\
&\quad + |J_1(v,u)| + |J^{\sigma}_{-}(u,v)|.
\end{aligned} \tag{4.4.7}$$

It is clear that

$$\begin{aligned}
|B(u,v)| &\leqslant \sum_{K\in P_h} \int_K |\boldsymbol{a}(\nabla u + \mathbf{b}u)\cdot \nabla v + duv|dx \\
&\leqslant \sum_{K\in P_h} \Big(\int_K |\boldsymbol{a}\nabla u \cdot \nabla v|dx + \int_K |\boldsymbol{a}\mathbf{b}u \cdot \nabla v|dx \\
&\quad + \int_K |duv|dx \Big) \\
&\leqslant \sum_{K\in P_h} \Big(C_1 \Big(\int_K |\nabla u|^2 dx \Big)^{\frac{1}{2}} \Big(\int_K |\nabla v|^2 dx \Big)^{\frac{1}{2}} \\
&\quad + C_2 \Big(\int_K |u|^2 dx \Big)^{\frac{1}{2}} \Big(\int_K |\nabla v|^2 dx \Big)^{\frac{1}{2}} \\
&\quad + C_3 \Big(\int_K u^2 dx \Big)^{\frac{1}{2}} \Big(\int_K v^2 dx \Big)^{\frac{1}{2}} \Big) \\
&\leqslant C\big(|u|_{1,h}|v|_{1,h} + ||u||_{0,\Omega}|v|_{1,h} + ||u||_{0,\Omega}||v||_{0,\Omega} \big) \\
&\leqslant C||u||_{1,h}||v||_{1,h}
\end{aligned} \tag{4.4.8}$$

for some positive constant C. For the second and fourth terms in the right hand side of (4.4.7), we obtain

$$
\begin{aligned}
|J_1(u,v)| &\leqslant \int_{\Gamma_{int}} |\{\mathbf{n}\cdot \boldsymbol{a}\nabla u\}[v]|ds \\
&\leqslant C\Big(\int_{\Gamma_{int}} \hat{h}|\{\boldsymbol{a}\nabla u\}|^2 ds\Big)^{\frac{1}{2}}\Big(\int_{\Gamma} \frac{1}{\hat{h}}|[v]|^2 ds\Big)^{\frac{1}{2}} \\
&\leqslant C\Big(\sum_{K\in P_h} (|u|_{1,K}^2 + h_K^2|u|_{2,K}^2)\Big)^{\frac{1}{2}}|v|_- \\
&\leqslant C|||u|||_-|v|_-
\end{aligned} \tag{4.4.9}
$$

and

$$
|J_1(v,u)| \leqslant C|||v|||_-|u|_-. \tag{4.4.10}
$$

And for the third term in the right hand side of (4.4.7), we get

$$
\begin{aligned}
|J_2(u,v)| &\leqslant \int_{\Gamma_{int}} |\{\mathbf{n}\cdot \boldsymbol{a}\mathbf{b}u\}[v]|ds \\
&\leqslant C\Big(\int_{\Gamma_{int}} \hat{h}|\{\boldsymbol{a}\mathbf{b}u\}|^2 ds\Big)^{\frac{1}{2}}\Big(\int_{\Gamma} \frac{1}{\hat{h}}|[v]|^2 ds\Big)^{\frac{1}{2}} \\
&\leqslant C\Big(\sum_{K\in P_h} (|u|_{0,K}^2 + h_K^2|u|_{1,K}^2)\Big)^{\frac{1}{2}}|v|_- \\
&\leqslant C|||u|||_-|v|_-.
\end{aligned} \tag{4.4.11}
$$

The last term is bounded by

$$
\begin{aligned}
|J_-^{\sigma}(u,v)| &= \Big|\int_{\Gamma} \frac{\sigma}{\hat{h}}[u][v]ds\Big| \\
&\leqslant C\Big(\int_{\Gamma} \frac{1}{\hat{h}}[u]^2 ds\Big)^{\frac{1}{2}}\Big(\int_{\Gamma} \frac{1}{\hat{h}}[v]^2 ds\Big)^{\frac{1}{2}} \\
&\leqslant C|u|_-|v|_-.
\end{aligned} \tag{4.4.12}
$$

Substituting the above results (4.4.8)-(4.4.12) into (4.4.7), we have

$$
\begin{aligned}
|B_-^{\sigma}(u,v)| &\leqslant C\Big(||u||_{1,h}||v||_{1,h} + |||u|||_-|v|_- + |u|_-|||v|||_- + |u|_-|v|_-\Big) \\
&\leqslant C|||u|||_-|||v|||_-,
\end{aligned}
$$

which completes the proof of the boundedness for B_-^σ. And we can also prove the following result in similar way:

$$|B_+^\sigma(u,v)| \leqslant C|||u|||_+|||v|||_+.$$

Thus the proof is complete. $\square$

Now we will show that the bilinear forms $B_\pm^\sigma(\cdot,\cdot)$ satisfy the stability condition with respect to the norms $|||\cdot|||_\pm$, respectively.

Theorem 4.4.3. *For a sufficiently large penalty parameter σ, there exists a positive constant C such that*

$$B_\pm^\sigma(v,v) \geqslant C|||v|||_\pm^2, \quad \forall v \in V_h. \tag{4.4.13}$$

Proof. Let $v \in V_h$. First, we will prove that $B_-^\sigma(v,v) \geqslant C|||v|||_-^2$. From the definition of $B_-^\sigma(v,v)$, we get

$$\begin{aligned} B_-^\sigma(v,v) &= B(v,v) - 2J_1(v,v) - J_2(v,v) + J_-^\sigma(v,v) \\ &= \sum_{K\in P_h}\int_K \Big(\boldsymbol{a}(\nabla v + \mathbf{b}v)\cdot\nabla v + dv^2\Big)dx \\ &\quad - 2\int_{\Gamma_{int}}\{\mathbf{n}\cdot\boldsymbol{a}\nabla v\}[v]ds - \int_{\Gamma_{int}}\{\mathbf{n}\cdot\boldsymbol{a}\mathbf{b}v\}[v]ds \\ &\quad + \int_\Gamma \frac{\sigma}{\hat{h}}[v]^2 ds. \end{aligned} \tag{4.4.14}$$

The first term in the right hand side of (4.4.14) can be written as

$$\begin{aligned} &\sum_{K\in P_h}\int_K \Big(\boldsymbol{a}(\nabla v + \mathbf{b}v)\cdot\nabla v + dv^2\Big)dx \\ &\qquad = (\boldsymbol{a}\nabla v,\nabla v) + (\boldsymbol{a}\mathbf{b}v,\nabla v) + (dv,v). \end{aligned} \tag{4.4.15}$$

For the second and the third terms in the right hand side of (4.4.14), we obtain

$$\begin{aligned}\left|\int_{\Gamma_{int}}\{\mathbf{n}\cdot\boldsymbol{a}\nabla v\}[v]ds\right| &\leqslant \int_{\Gamma_{int}}|\{\mathbf{n}\cdot\boldsymbol{a}\nabla v\}[v]|ds \\ &\leqslant C\Big(\int_{\Gamma_{int}}\hat{h}|\{\boldsymbol{a}\nabla v\}|^2ds\Big)^{\frac{1}{2}}\Big(\int_{\Gamma}\frac{1}{\hat{h}}|[v]|^2ds\Big)^{\frac{1}{2}} \\ &\leqslant C\Big(\sum_{K\in P_h}\big(|v|_{1,K}^2+h_K^2|v|_{2,K}^2\big)\Big)^{\frac{1}{2}}|v|_- \\ &\leqslant C|||v|||_-|v|_-\end{aligned} \tag{4.4.16}$$

and

$$\begin{aligned}\left|\int_{\Gamma_{int}}\{\mathbf{n}\cdot\boldsymbol{a}\mathbf{b}v\}[v]ds\right| &\leqslant \Big(\int_{\Gamma_{int}}\hat{h}|\{\boldsymbol{a}\mathbf{b}v\}|^2ds\Big)^{\frac{1}{2}}\Big(\int_{\Gamma}\frac{1}{\hat{h}}|[v]|^2ds\Big)^{\frac{1}{2}} \\ &\leqslant C\Big(\sum_{K\in P_h}(|v|_{0,K}^2+h_K^2|v|_{1,K}^2)\Big)^{\frac{1}{2}}|v|_- \\ &\leqslant C|||v|||_-|v|_-.\end{aligned} \tag{4.4.17}$$

Note that

$$\int_{\Gamma}\frac{\sigma}{\hat{h}}[v]^2ds \geqslant \sigma_0|v|_-^2. \tag{4.4.18}$$

By using (4.3.3) and (4.4.14)-(4.4.18), we have

$$\begin{aligned}B_-^{\sigma}(v,v) &\geqslant C_1(|v|_{1,h}^2+C_2||v||_{0,\Omega}^2)-C_3|||v|||_-|v|_-+\sigma_0|v|_-^2 \\ &\geqslant C_1(|v|_{1,h}^2+C_2||v||_{0,\Omega}^2)-\frac{C_3}{2}(\epsilon|||v|||_-^2+\frac{1}{\epsilon}|v|_-^2)+\sigma_0|v|_-^2 \\ &= C_1(\frac{1}{2}|v|_{1,h}^2+C_2||v||_{0,\Omega}^2+\frac{1}{2}|v|_{1,h}^2) \\ &\quad -\frac{C_3}{2}(\epsilon|||v|||_-^2+\frac{1}{\epsilon}|v|_-^2)+\sigma_0|v|_-^2\end{aligned}$$

$$
\begin{aligned}
&\geqslant C_1(\frac{1}{2}|v|_{1,h}^2 + C_2||v||_{0,\Omega}^2 + \frac{1}{2C_0}\sum_{K\in P_h} h_k^2|v|_{2,K}^2) \\
&\quad - \frac{C_3}{2}(\epsilon|||v|||_-^2 + \frac{1}{\epsilon}|v|_-^2) + \sigma_0|v|_-^2 \\
&\geqslant (C_1^* - \frac{\epsilon C_3}{2})|||v|||_-^2 + (\sigma_0 - \frac{C_1}{2} - \frac{C_3}{2\epsilon})|v|_-^2 \\
&\geqslant C|||v|||_-^2,
\end{aligned}
$$

for a sufficiently small ϵ and a sufficiently large σ. In the similarly way, we can prove the following result

$$B_+^\sigma(v,v) \geqslant C|||v|||_+^2.$$

This completes the proof. □

4.5 Error Estimates

In the error analysis, we need a bound on the approximation error $|||u - u_I|||_\pm$ when $u_I \in V_h$ is a suitable interpolant of the exact solution u. It is convenient to take an interpolant u_I which is discontinuous across the inter-element boundaries. For the interpolant u_I, we just require the local approximation property:

$$|u - u_I|_{s,K} \leqslant Ch_K^{p+1-s}|u|_{p+1,K}, \quad \forall K \in P_h, \quad s = 0, 1, 2, \tag{4.5.1}$$

where C depends only on p and the minimum angle of K.

Theorem 4.5.1. *If $u \in H^{p+1}(\Omega)$ and u_I, the interpolant of u, satisfies the local approximation property (4.5.1), then there exists a positive constant C such that*

$$|||u - u_I|||_- \leqslant Ch^p|u|_{p+1,\Omega}. \tag{4.5.2}$$

Proof. From the definition of $|||\cdot|||_-$, we obtain

$$\begin{aligned} |||u-u_I|||_-^2 &= ||u-u_I||_{1,h}^2 + \sum_{K\in P_h} h_K^2|u-u_I|_{2,K}^2 + |u-u_I|_-^2 \\ &= ||u-u_I||_{1,h}^2 + \sum_{K\in P_h} h_K^2|u-u_I|_{2,K}^2 \\ &\quad + \sum_{e\in E_h}\frac{1}{h_e}\|[u-u_I]\|_{0,e}^2. \end{aligned} \tag{4.5.3}$$

Recall that

$$\|v\|_{0,e}^2 \leqslant C(h_e^{-1}\|v\|_{0,K}^2 + h_e\|v\|_{1,K}^2), \qquad \forall v \in H^1(K). \tag{4.5.4}$$

Therefore, using (4.5.3) and (4.5.4), we obtain

$$\begin{aligned} |||u-u_I|||_-^2 \leqslant C\Big(&||u-u_I||_{1,h}^2 + \sum_{K\in P_h} h_K^2|u-u_I|_{2,K}^2 \\ &+ \sum_{K\in P_h}\frac{1}{h_K^2}\|u-u_I\|_{0,K}^2\Big) \end{aligned} \tag{4.5.5}$$

and so, from (4.5.1) and (4.5.5), we have

$$|||u - u_I|||_- \leqslant Ch^p|u|_{p+1,\Omega}. \tag{4.5.6}$$

This completes the proof. $\square$

Theorem 4.5.2. *If $u \in H^{p+1}(\Omega)$ and u_I, the continuous interpolant of u, satisfies the local approximation property (4.5.1), then there exists a positive constant C such that*

$$|||u - u_I|||_+ \leqslant Ch^p |u|_{p+1,\Omega}. \tag{4.5.7}$$

Proof. Since u_I is the continuous interpolant of u, we obtain from the definition of $|||\cdot|||_+$

$$\begin{aligned} |||u - u_I|||_+^2 &= ||u - u_I||_{1,h}^2 + \sum_{K \in P_h} h_K^2 |u - u_I|_{2,K}^2 + |u - u_I|_+^2 \\ &= ||u - u_I||_{1,h}^2 + \sum_{K \in P_h} h_K^2 |u - u_I|_{2,K}^2 \\ &\quad + \sum_{e \in E_h} \frac{1}{h_e^3} ||[u - u_I]||_{0,e}^2 \\ &= ||u - u_I||_{1,h}^2 + \sum_{K \in P_h} h_K^2 |u - u_I|_{2,K}^2. \end{aligned} \tag{4.5.8}$$

Therefore, using (4.5.4) and (4.5.8), we obtain

$$|||u - u_I|||_+^2 \leqslant C\Big(||u - u_I||_{1,h}^2 + \sum_{K \in P_h} h_K^2 |u - u_I|_{2,K}^2\Big) \tag{4.5.9}$$

and so, from (4.5.1) and (4.5.9), we have

$$|||u - u_I|||_+ \leqslant Ch^p |u|_{p+1,\Omega}. \tag{4.5.10}$$

This completes the proof. □

Remark 4.5.1. *When* $\boldsymbol{a}(x) = I$, $\mathbf{b}(x) = \mathbf{0}$ *and* $d(x) = 0$ *in the problem (4.3.1) and (4.3.2), our results of Theorems 4.5.1 and 4.5.2 for* $B^\sigma_\pm$ *are the same as ones in [3].*

Theorem 4.5.3. *If* u *is the finite element solution to (4.4.3) and* u_h *is a solution to (4.4.4), then there exists a positive constant* C *such that*

$$|||u - u_h|||_\pm \leqslant Ch^p|u|_{p+1,\Omega}. \tag{4.5.11}$$

Proof. Let $u_I \in P_p$ be a piecewise interpolant of u satisfying the approximation property (4.5.1). Then, from (4.4.3), (4.4.4), (4.5.1), Theorems 4.5.1 and 4.5.2, we have

$$\begin{aligned} c|||u_I - u_h|||_-^2 &\leqslant B^\sigma_-(u_I - u_h, u_I - u_h) \\ &= B^\sigma_-(u_I - u_h + u - u, u_I - u_h) \\ &= B^\sigma_-(u_I - u, u_I - u_h) + B^\sigma_-(u - u_h, u_I - u_h) \\ &= B^\sigma_-(u_I - u, u_I - u_h) \\ &\leqslant C|||u_I - u|||_-|||u_I - u_h|||_- \\ &\leqslant Ch^p|u|_{p+1,\Omega}|||u_I - u_h|||_-. \end{aligned}$$

In the case of B^σ_+, we use the continuous interpolant $u_I \in P_p$ of u satisfying approximation property (4.5.1). Then we can obtain the following result

$$c|||u_I - u_h|||_+^2 \leqslant Ch^p|u|_{p+1,\Omega}|||u_I - u_h|||_+.$$

Hence, Theorem 4.5.2 and the triangle inequality give the optimal error estimate

$$|||u - u_h|||_\pm \leqslant Ch^p|u|_{p+1,\Omega}.$$

This completes the proof. $\square$

Remark 4.5.2. *When* $\boldsymbol{a}(x) = I$, $\mathbf{b}(x) = \mathbf{0}$ *and* $d(x) = 0$ *in the problem (4.3.1) and (4.3.2), our result for* B_-^σ *is similar to one in [3, 33]. In [3, 33], the energy norm error estimate is optimal in* h *as (4.5.11). Also, the result of [3] is similar to our one for* B_+^σ.

Next, we want to obtain an optimal error estimate in the L^2 norm. Let us consider the following dual or adjoint problem: find $\psi \in H_0^2(P_h)$ such that

$$B_-^\sigma(v, \psi) = (u - u_h, v), \quad \forall v \in H_0^2(P_h) \tag{4.5.12}$$

with the elliptic regularity

$$\|\psi\|_{2,\Omega} \leqslant C\|u - u_h\|_{0,\Omega}, \tag{4.5.13}$$

with C depending only on the domain Ω. Note that this corresponds to the concept of the adjoint consistency discussed in [3]. Assume that the dual problem (4.5.12) has a solution satisfying (4.5.13). The existence of a solution of (4.5.12) satisfying (4.5.13) can be guaranteed by the existence of a solution of the problem

$$\begin{aligned} -\nabla \cdot \boldsymbol{a}(\nabla\psi) + \mathbf{b}^T \boldsymbol{a}\nabla\psi + d\psi = u - u_h \quad &\text{in} \quad \Omega, \\ \psi = 0 \qquad\qquad &\text{on} \quad \partial\Omega. \end{aligned} \tag{4.5.14}$$

Theorem 4.5.4. *If u is a solution to (4.4.3) and u_h is the finite element solution to (4.4.4) for B_-^σ, then there exists a positive constant C such that*

$$||u - u_h||_{0,\Omega} \leqslant Ch^{p+1}|u|_{p+1,\Omega}.$$

Proof. We take $\psi_I \in V_h$ as a piecewise linear interpolant of ψ satisfying the local approximation property (4.5.1). Then, taking $v = u - u_h$ in (4.5.12) and using (4.4.3), (4.4.4), and the elliptic regularity (4.5.13), we obtain

$$\begin{aligned} \|u - u_h\|_{0,\Omega}^2 &= B_-^\sigma(u - u_h, \psi) \\ &= B_-^\sigma(u - u_h, \psi - \psi_I) + B_-^\sigma(u - u_h, \psi_I) \\ &\leqslant C|||u - u_h|||_- |||\psi - \psi_I|||_- \qquad (4.5.15) \\ &\leqslant Ch|\psi|_{2,\Omega}|||u - u_h|||_- \\ &\leqslant Ch\|u - u_h\|_{0,\Omega}|||u - u_h|||_-. \end{aligned}$$

Substituting (4.5.11) into (4.5.15), we get the optimal error estimate

$$\|u - u_h\|_{0,\Omega} \leqslant Ch^{p+1}|u|_{p+1,\Omega}.$$

This completes the proof. □

Remark 4.5.3. *When $\boldsymbol{a}(x) = I$, $\mathbf{b}(x) = \mathbf{0}$ and $d(x) = 0$ in the problem (4.3.1) and (4.3.2), our result of Theorem 4.5.4 for B_-^σ is the same as one in [3].*

In the case of $B_+^\sigma(\cdot, \cdot)$, we consider the following dual or adjoint problem: find $\psi \in H_0^2(P_h)$ such that

$$B_+^\sigma(v, \psi) - 2\int_{\Gamma_{int}} \{\mathbf{n} \cdot \nabla\psi\}[v]ds = (u - u_h, v), \quad \forall v \in H^2(P_h). \qquad (4.5.16)$$

with the elliptic regularity (4.5.13). Assume that the dual problem (4.5.16) has a solution satisfying (4.5.13). The existence of a solution of (4.5.16) satisfying (4.5.13) can be guaranteed by the existence of a solution of the problem (4.5.14)

Theorem 4.5.5. *If u is a solution to (4.4.3) and u_h is the finite element solution to (4.4.4) for B_+^σ , then there exists a positive constant C such that*

$$||u - u_h||_{0,\Omega} \leqslant Ch^{p+1}|u|_{p+1,\Omega}.$$

Proof. Let ψ be the solution of the problem (4.5.16) and take $\psi_I \in V_h$ as a continuous interpolant of ψ satisfying the local approximation property (4.5.1). Then taking $v = u - u_h$ and using the elliptic regularity (4.5.13), we obtain

$$\begin{aligned}
||u - u_h||_{0,\Omega}^2 &= B_+^\sigma(u - u_h, \psi) - 2\int_{\Gamma_{int}} \{\mathbf{n}\cdot\nabla\psi\}[u - u_h]ds \\
&= B_+^\sigma(u - u_h, \psi - \psi_I + \psi_I) - 2\int_{\Gamma_{int}} \{\mathbf{n}\cdot\nabla\psi\}[u - u_h]ds \\
&\leqslant B_+^\sigma(u - u_h, \psi - \psi_I) - 2\int_{\Gamma_{int}} \{\mathbf{n}\cdot\nabla\psi\}[u - u_h]ds \\
&\leqslant C|||u - u_h|||_+|||\psi - \psi_I|||_+ \\
&\quad + C\Big(\int_{\Gamma_{int}} \hat{h}^3\{\nabla\psi\}^2\Big)^{\frac{1}{2}}\Big(\int_{\Gamma_{int}} \frac{1}{\hat{h}^3}[u - u_h]^2\Big)^{\frac{1}{2}} \\
&\leqslant C|||u - u_h|||_+ h||\psi||_{2,\Omega} + Ch^{\frac{3}{2}}||\nabla\psi||_{0,\Gamma_{int}}|u - u_h|_+ \\
&\leqslant Ch|||u - u_h|||_+||u - u_h||_{0,\Omega} \\
&\quad + Ch^{\frac{3}{2}}\Big(\frac{1}{h}||\nabla\psi||_{0,K}^2 + h||\nabla\psi||_{1,K}^2\Big)^{\frac{1}{2}}|||u - u_h|||_+ \\
&\leqslant Ch|||u - u_h|||_+||u - u_h||_{0,\Omega} \\
&\quad + Ch\Big(||\nabla\psi||_{0,K}^2 + h^2||\nabla\psi||_{1,K}^2\Big)^{\frac{1}{2}}|||u - u_h|||_+
\end{aligned}$$

$$\leqslant Ch|||u-u_h|||_+||u-u_h||_{0,\Omega} + Ch||\psi||_{2,\Omega}|||u-u_h|||_+$$

$$\leqslant Ch|||u-u_h|||_+||u-u_h||_{0,\Omega} + Ch||u-u_h||_{0,\Omega}|||u-u_h|||_+$$

and so

$$\begin{aligned} ||u-u_h||^2_{0,\Omega} \leqslant Ch|||u-u_h|||_+||u-u_h||_{0,\Omega} \\ + Ch||u-u_h||_{0,\Omega}|||u-u_h|||_+. \end{aligned} \tag{4.5.12}$$

Therefore we obtain the optimal error estimate from (4.5.11) and (4.5.12)

$$\begin{aligned} ||u-u_h||_{0,\Omega} &\leqslant Ch|||u-u_h|||_+ \\ &\leqslant Ch^{p+1}|u|_{p+1,\Omega} \end{aligned}$$

This completes the proof. □

Remark 4.5.4. *When $\boldsymbol{a}(x) = I$, $\mathbf{b}(x) = \mathbf{0}$ and $d(x) = 0$ in the problem (4.3.1) and (4.3.2), our problem for B^σ_+ is the same as one in [3]. It is stated in [3] that the error in the L^2 norm is order $O(h^p)$. However, we can obtain an optimal error estimate in the L^2 norm.*

Chapter 5 Numerical Experiments

In this chapter, we present numerical results for one and two dimensional elliptic problems obtained by the DG methods which are given in Chapters 3 and 4. And we show numerically that the error estimate of $u - u_h$ in the L^2 norm is optimal.

5.1 One Dimensional Elliptic Problems

In this section, we consider the following one dimensional elliptic problem:

$$\begin{aligned} &-\frac{d}{dx}\Big(a(\frac{du}{dx} + bu)\Big) + du = f \quad \text{for } I = (0,1), \\ &u(0) = u(1) = 0, \end{aligned} \tag{5.1.1}$$

whose exact solution is given by

$$u(x) = (x - x^2)^2 \tag{5.1.2}$$

or

$$u(x) = \sin(4\pi x). \tag{5.1.3}$$

And the function f is chosen so that the problem (5.1.1) is satisfied with the appropriate choices of $a(x)$, $b(x)$, $d(x)$ and $u(x)$.

To verify numerically the optimal error estimate of $u - u_h$ in the L^2 norm for discontinuous Galerkin methods with an interior penalty in Section 3.5, we test the following three cases:

Case 1. $a(x) = 1$ and $b(x) = d(x) = 0$, Case 2. $d(x) = 1$, $a(x)$ and $b(x)$ are given as following:

$$a(x) = \begin{cases} 9(x-0.1)^2 + 0.1, & 0 \leqslant x < 0.1, \\ 0.1, & 0.1 \leqslant x < 0.9, \\ 9(x-0.9)^2 + 0.1, & 0.9 \leqslant x \leqslant 1, \end{cases} \tag{5.1.4}$$

and

$$b(x) = \begin{cases} -1, & 0 \leqslant x < 0.45, \\ 20^3((x-0.45)^3 - 0.05^3), & 0.45 \leqslant x < 0.5, \\ 20^3((x-0.55)^3 + 0.05^3), & 0.5 \leqslant x < 0.55, \\ 1, & 0.55 \leqslant x \leqslant 1. \end{cases} \tag{5.1.5}$$

The graphs of $a(x)$ in (5.1.4) and $b(x)$ in (5.1.5) are given in Figures 5.1.1 and 5.1.2, respectively.

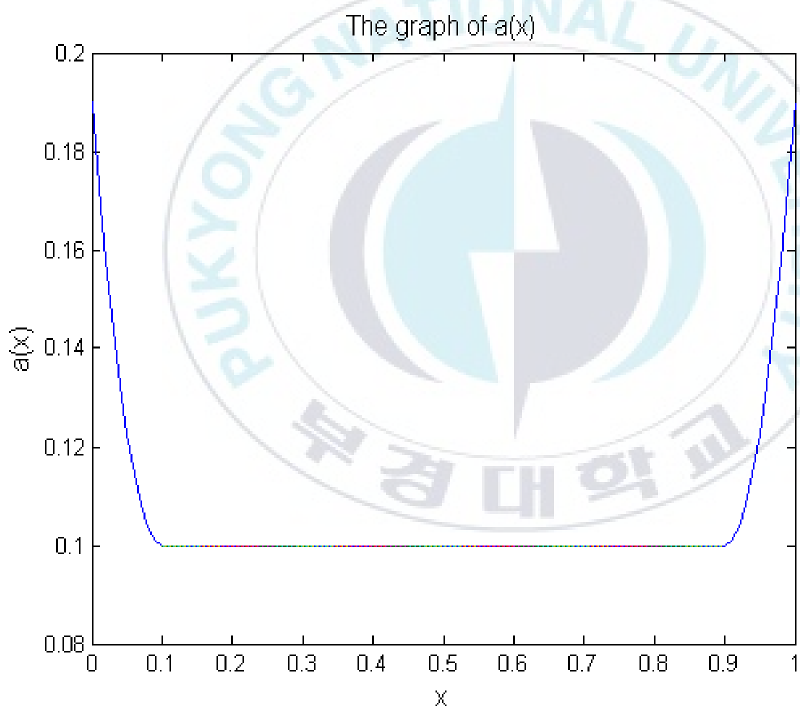

Figure 5.1.1. The graph of $a(x)$ in (5.1.4).

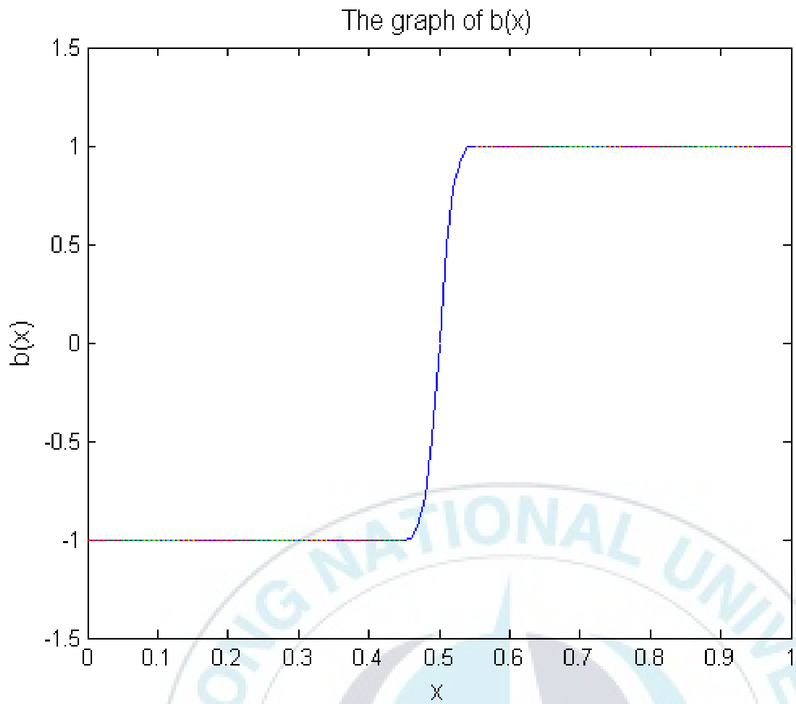

Figure 5.1.2. The graph of $b(x)$ in (5.1.5).

Case 3. $d(x) = 1$, $b(x)$ is the same as one in (5.1.5) and $a(x)$ is given as following:

$$a(x) = \begin{cases} 0.1, & 0 \leqslant x < 0.4, \\ 1000(x-0.4)^2 - 0.1, & 0.4 \leqslant x < 0.49, \\ 9000(x-0.49)(x-0.51) + 8.2, & 0.49 \leqslant x < 0.51, \\ 1000(x-0.6)^2 + 0.1, & 0.51 \leqslant x < 0.6, \\ 0.1, & 0.6 \leqslant x \leqslant 1. \end{cases} \tag{5.1.6}$$

The graph of $a(x)$ in (5.1.6) is given in Figures 5.1.3.

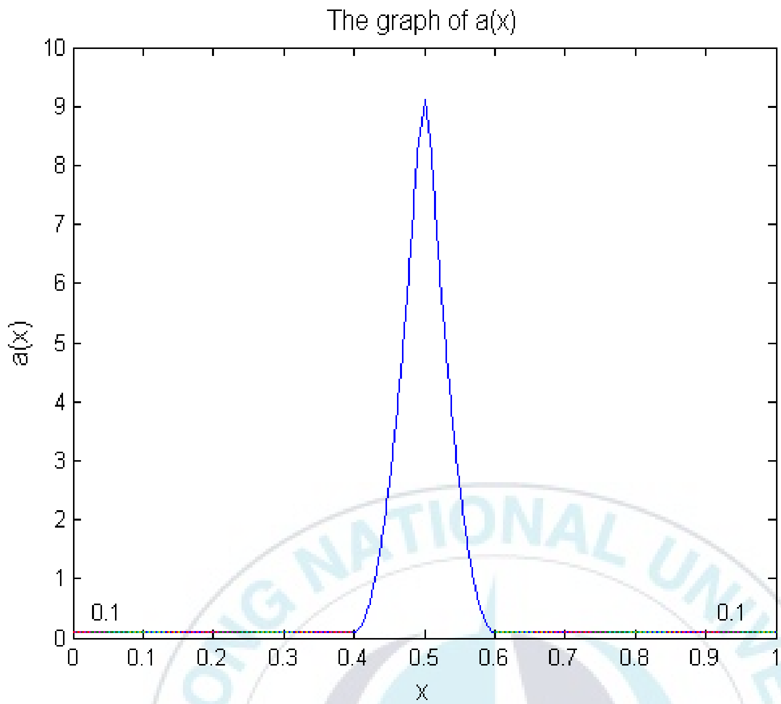

Figure 5.1.3. The graph of $a(x)$ in (5.1.6).

To implement the discontinuous Galerkin methods (3.4.5)

$$B_{\pm}^{\sigma}(u_h, v) = F(v), \quad \forall v \in V_h,$$

we take P_h as the collection of N uniform subintervals in I with its length $h = 1/N$. And finite element subspaces V_h of polynomial functions is defined as

$$V_h = \{v \in L^2(I);\ \ v|_{K_i} \in P_p(K_i),\ \ \forall K_i \in P_h \ \text{ and } \ v = 0 \ \text{ on } \ \partial I\},$$

where $P_p(K_i)$ is the space of polynomial of degree less than or equal to p on K_i for a given integer $p \geqslant 1$.

The linear systems in (3.4.5) are solved with the Generalized Minimum Residual method (GMRES) discussed in Section 2.5.

In Tables 5.1.1-5.1.4, we present the condition numbers of matrices of the linear systems for B_-^σ and B_+^σ in Cases 1 and 3 when $p=2$ and $\sigma=0.2,\ 5$.

Table 5.1.1. The condition numbers for B_-^σ in Case 1: $\sigma = 0.2, 5$.

N	$\sigma=0.2$	$\sigma=5$
5	22.7502	375.3400
10	38.7468	430.8489
20	144.9890	1.1961E+03
40	572.5427	4.6993E+03
80	2.2826E+03	1.8717E+04

Table 5.1.2. The condition numbers for B_+^σ in Case 1: $\sigma = 0.2, 5$.

N	$\sigma=0.2$	$\sigma=5$
5	203.4093	4.9973E+03
10	1.3922E+03	3.4553E+04
20	1.9973E+04	4.9865E+05
40	3.1331E+05	6.8968E+06

Table 5.1.3. The condition numbers for B_-^σ in Case 3: $\sigma = 0.2, 5$.

N	$\sigma=0.2$	$\sigma=5$
5	337.6965	1.9629E+03
10	2.0150E+04	2.2073E+03
20	6.0980E+04	6.8549E+03
40	2.5981E+05	2.6798E+04
80	1.0606E+06	1.0665E+05

Table 5.1.4. The condition numbers for B_+^σ in Case 3: $\sigma = 0.2, 5$.

N	$\sigma = 0.2$	$\sigma = 5$
5	1.8318E+03	4.7098E+04
10	8.8256E+03	2.2060E+05
20	1.1021E+05	2.7548E+06
40	1.7228E+06	4.3070E+07

We know form Tables 5.1.1-5.1.4 that the condition numbers for B_+^σ are always larger than ones for B_-^σ. And we know from Table 5.1.1 that when $\sigma = 0.2$ or $\sigma = 5$, the condition numbers for B_-^σ are not sufficiently large in Case 1 and that in Case 1 the condition numbers for B_-^σ with $\sigma = 5$ are larger than ones for B_-^σ with $\sigma = 0.2$. We know from Table 5.1.3 that in Case 3 the condition numbers for B_-^σ with $\sigma = 0.2$ are larger than ones for B_-^σ with $\sigma = 5$. And we know from Tables 5.1.2 and 5.1.4 that in Cases 1 and 3 the condition numbers for B_+^σ with $\sigma = 5$ are larger than ones for B_+^σ with $\sigma = 0.2$. Therefore, we choose $\sigma = 5$ for B_-^σ and $\sigma = 0.2$ for B_+^σ, respectively.

In Figures 5.1.4-5.1.7, we plot the exact solution u and the approximate solution u_h for B_-^σ in Case 3. Figures 5.1.4-5.1.5 correspond to the exact solution $u(x) = (x - x^2)^2$. Figures 5.1.6-5.1.7 correspond to the exact solution $u(x) = \sin(4\pi x)$. In Figures 5.1.8-5.1.11, we plot the exact solution u and the approximate solution u_h for B_+^σ in Case 3. Figures 5.1.8-5.1.9 correspond to the exact solution $u(x) = (x - x^2)^2$. Figures 5.1.10-5.1.11 correspond to the exact solution $u(x) = \sin(4\pi x)$.

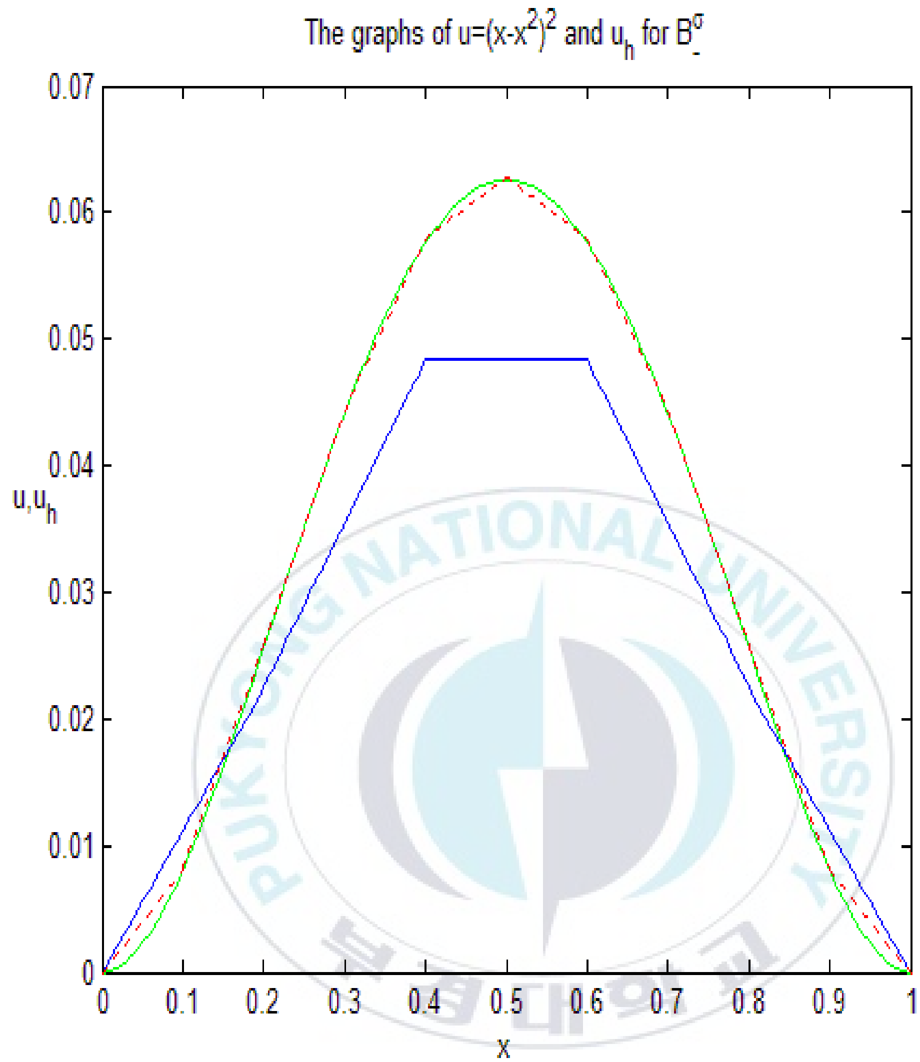

Figure 5.1.4. The graphs of the solution u and the approximation solution u_h for B_-^σ in Case 3: $p = 1$, $u(x) = (x - x^2)^2$, $h = 0.2, 0.1$. The solid red line (the solution u), the solid blue line (u_h, $h = 0.2$), the dotted green line (u_h, $h = 0.1$).

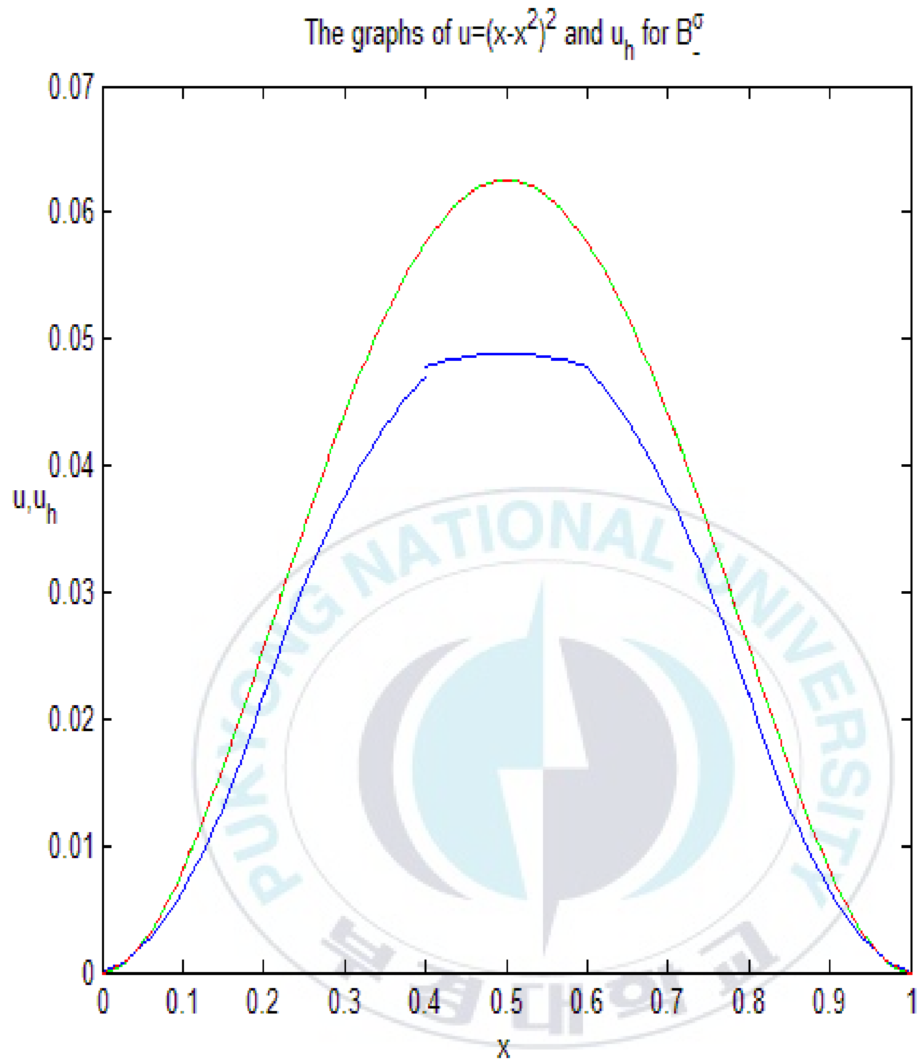

Figure 5.1.5. The graphs of the solution u and the approximation solution u_h for B^{σ}_{-} in Case 3: $p = 2$, $u(x) = (x - x^2)^2$, $h = 0.2, 0.1$. The solid red line (the solution u), the solid blue line (u_h, $h = 0.2$), the dotted green line (u_h, $h = 0.1$).

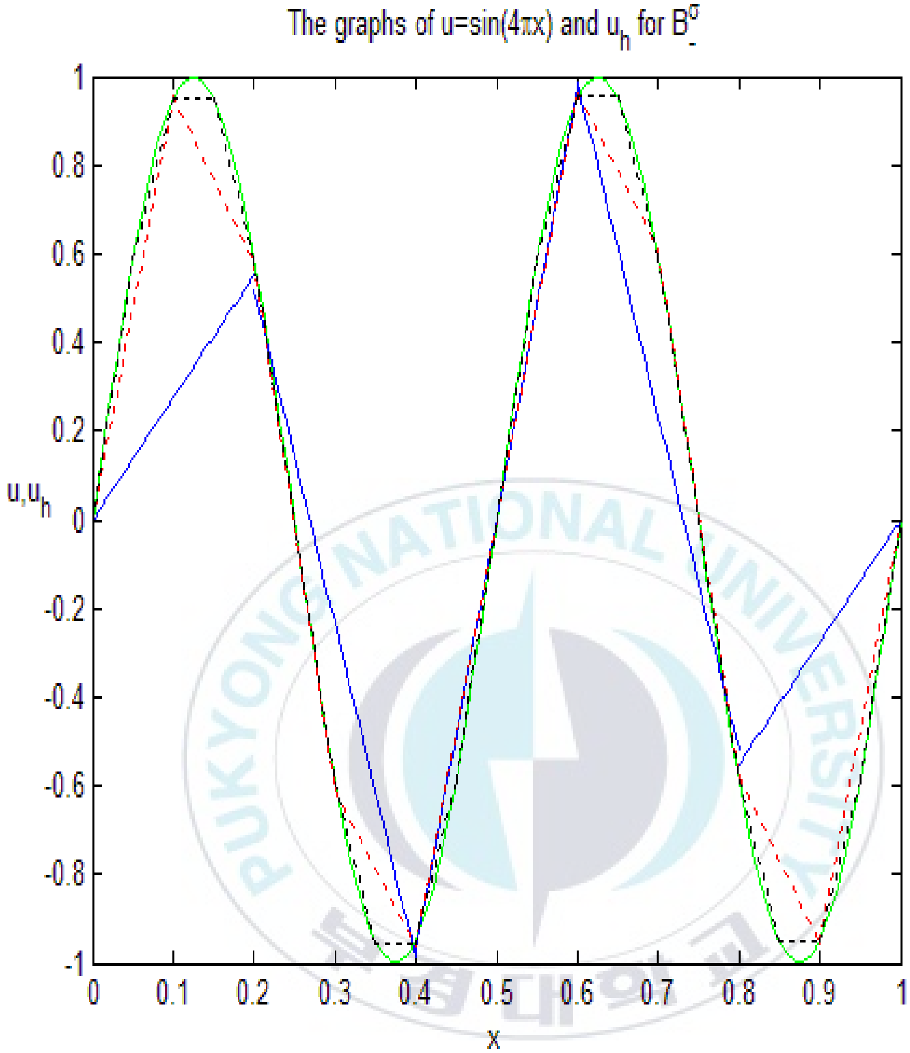

Figure 5.1.6. The graphs of the solution u and the approximation solution u_h for B^{σ}_{-} in Case 3: $p = 1$, $u(x) = \sin(4\pi x)$, $h = 0.2, 0.1, 0.05$. The solid red line (the solution u), the solid blue line (u_h, $h = 0.2$), the dotted green line (u_h, $h = 0.1$), the dotted black line (u_h, $h = 0.05$).

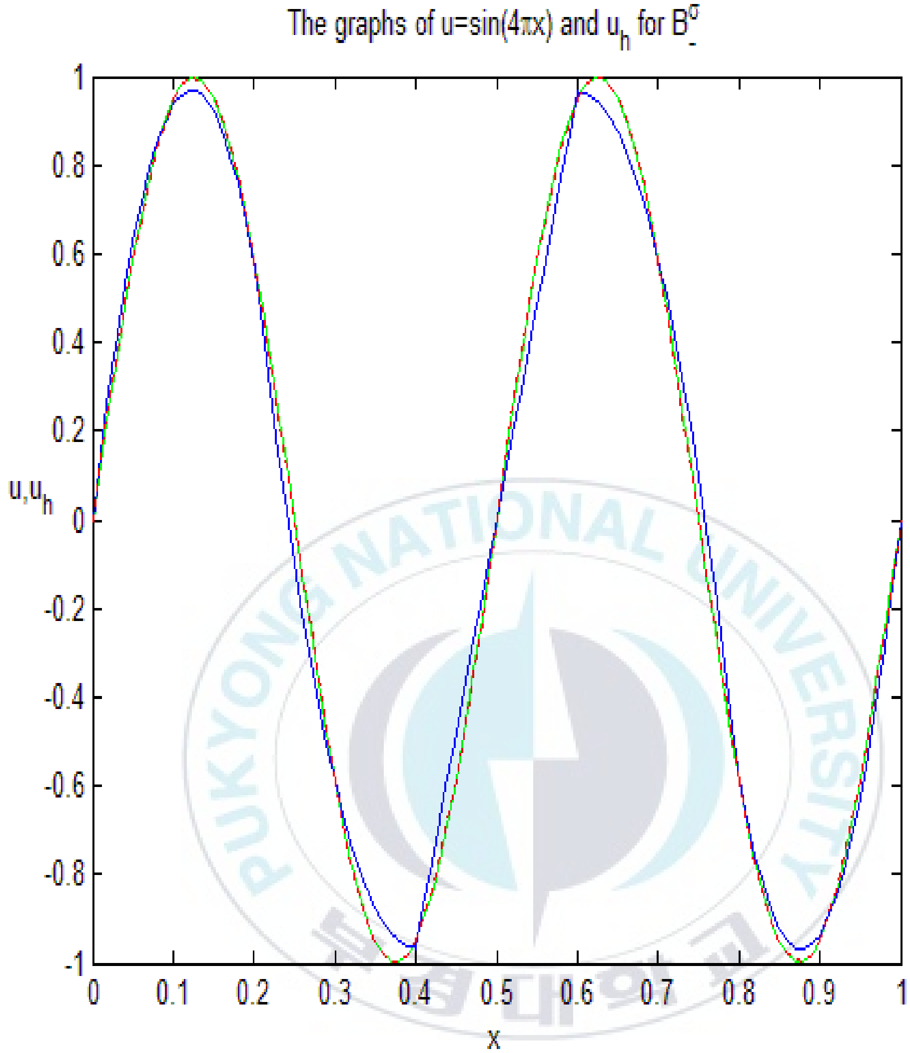

Figure 5.1.7. The graphs of the solution u and the approximation solution u_h for B^{σ}_{-} in Case 3: $p = 2$, $u(x) = \sin(4\pi x)$, $h = 0.2, 0.1$. The solid red line (the solution u), the solid blue line (u_h, $h = 0.2$), the dotted green line (u_h, $h = 0.1$).

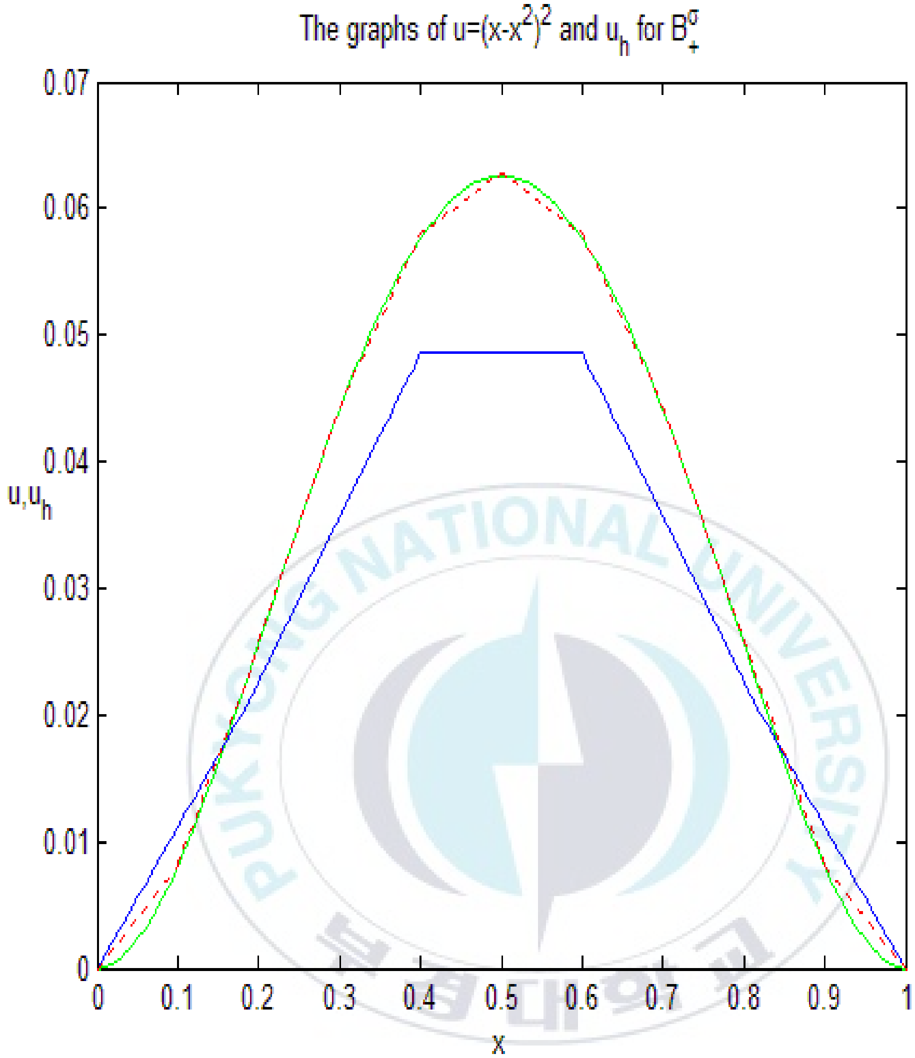

Figure 5.1.8. The graphs of the solution u and the approximation solution u_h for B_+^σ in Case 3: $p = 1$, $u(x) = (x - x^2)^2$, $h = 0.2, 0.1$. The solid red line (the solution u), the solid blue line (u_h, $h = 0.2$), the dotted green line (u_h, $h = 0.1$).

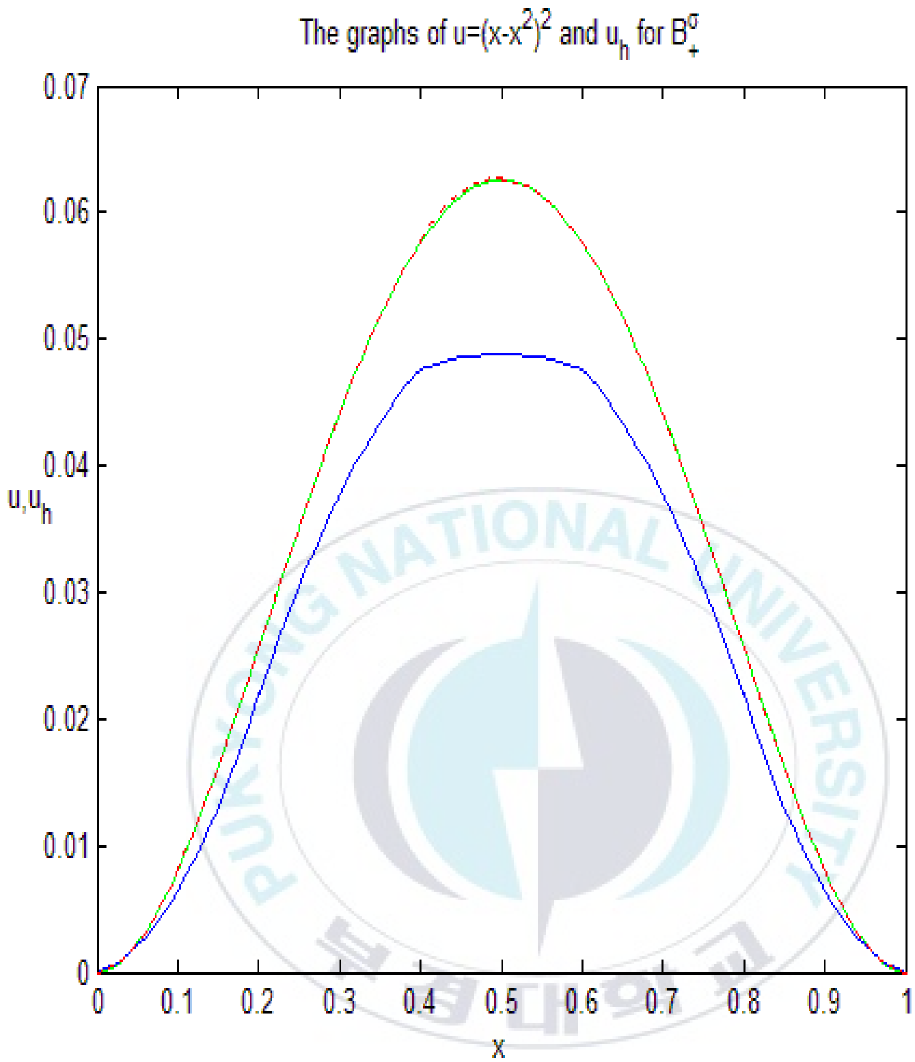

Figure 5.1.9. The graphs of the solution u and the approximation solution u_h for B_+^σ in Case 3: $p = 2$, $u(x) = (x - x^2)^2$, $h = 0.2, 0.1$. The solid red line (the solution u), the solid blue line (u_h, $h = 0.2$), the dotted green line (u_h, $h = 0.1$).

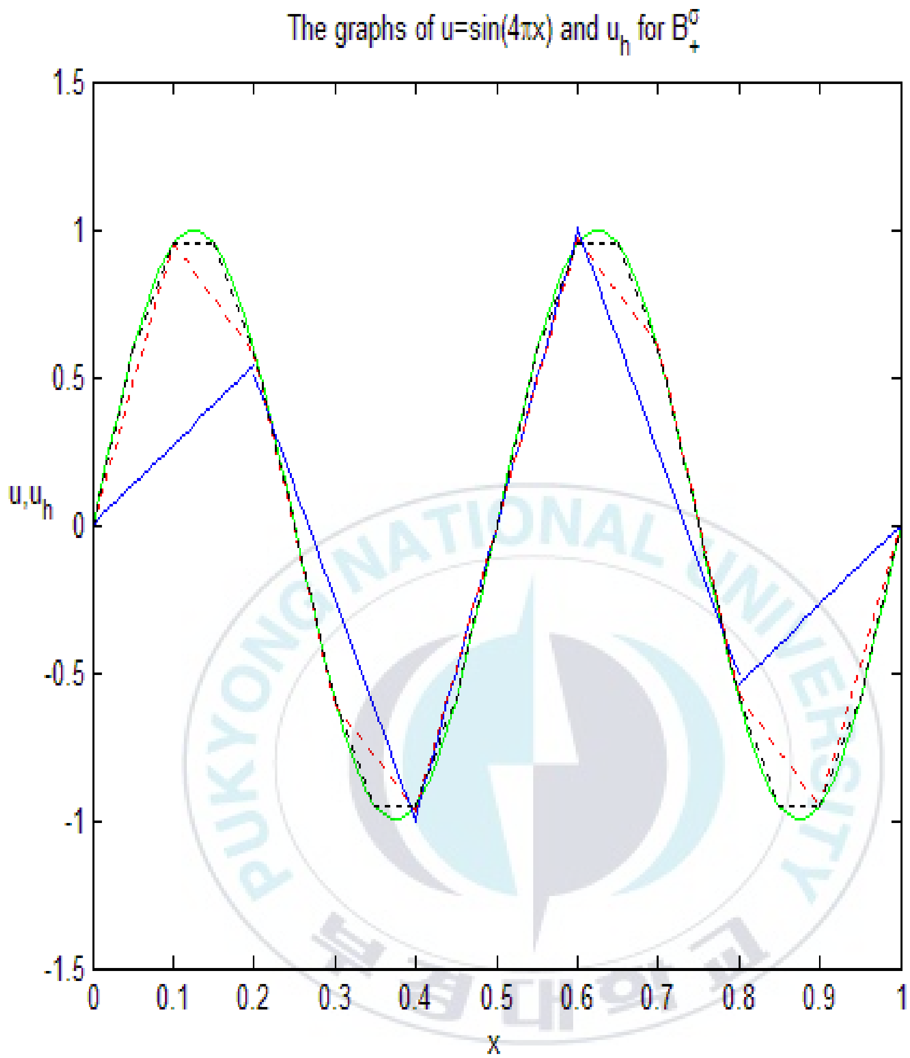

Figure 5.1.10. The graphs of the solution u and the approximation solution u_h for B_+^σ in Case 3: $p = 1$, $u(x) = \sin(4\pi x)$, $h = 0.2, 0.1, 0.05$. The solid red line (the solution u), the solid blue line (u_h, $h = 0.2$), the dotted green line (u_h, $h = 0.1$), the dotted black line (u_h, $h = 0.05$).

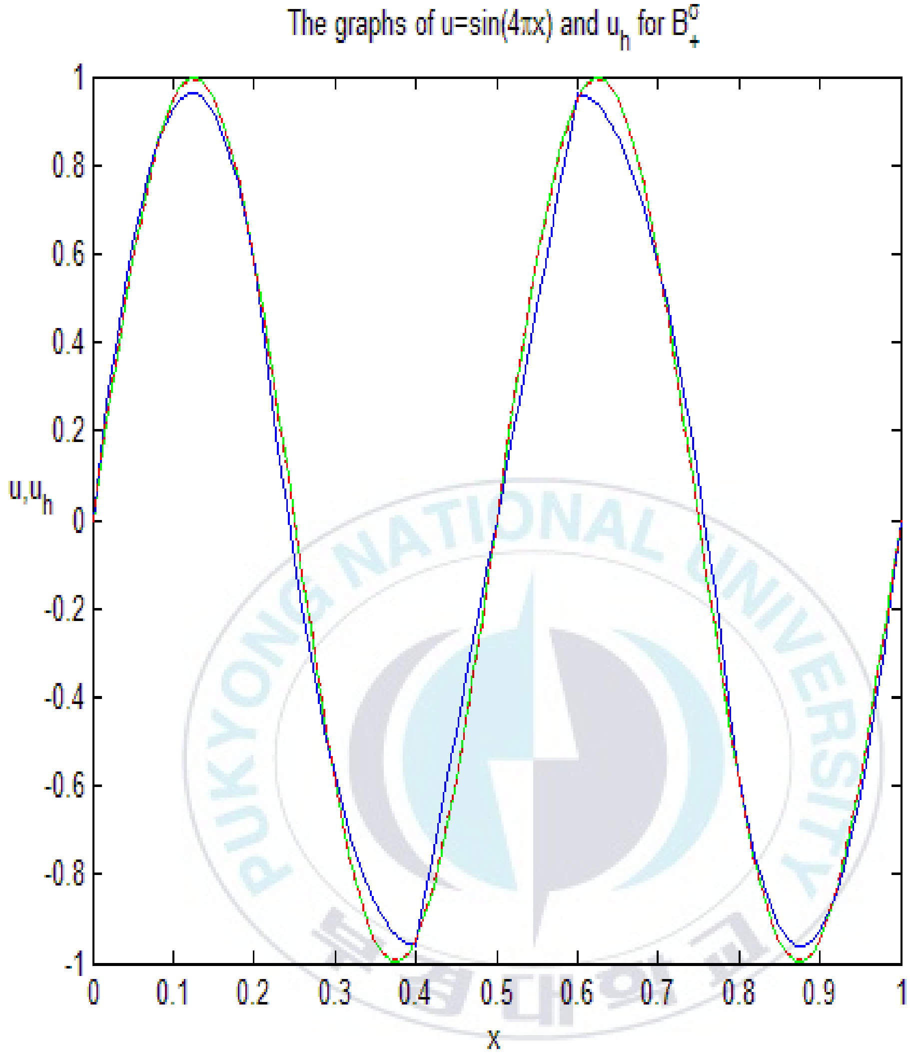

Figure 5.1.11. The graphs of the solution u and the approximation solution u_h for B_+^σ in Case 3: $p = 2$, $u(x) = \sin(4\pi x)$, $h = 0.2, 0.1$. The solid red line (the solution u), the solid blue line (u_h, $h = 0.2$), the dotted green line (u_h, $h = 0.1$).

In Tables 5.1.5-5.1.7, we present the error of $u - u_h$ in the L^2 norm for $p = 1, 2$ and $N = 5, 10, 20, 40, 80$ when the discontinuous Galerkin method (3.4.5) with B_-^σ is used to approximate the exact solution $u(x) = (x - x^2)^2$ and in Tables 5.1.8-5.1.10, the error of $u - u_h$ in the L^2 norm for $p = 1, 2$ and $N = 5, 10, 20, 40, 80$ when the discontinuous Galerkin method (3.4.5) with B_+^σ was used to approximate the exact solution $u(x) = (x - x^2)^2$.

Table 5.1.5. L^2 norm of the error $u - u_h$ for B_-^σ in Case 1: $p = 1, 2$, $u(x) = (x - x^2)^2$, $N = 5, 10, 20, 40, 80$.

N	p=1	p=2
5	2.612971416250891E-03	2.477122106346665E-04
10	7.664399237699947E-04	2.788869791480919E-05
20	2.006345475727433E-04	3.231004351910283E-06
40	5.080001242130203E-05	3.854283505225116E-07
80	1.274288396590847E-05	4.693432863630567E-08

Table 5.1.6. L^2 norm of the error $u - u_h$ for B_-^σ in Case 2: $p = 1, 2$, $u(x) = (x - x^2)^2$, $N = 5, 10, 20, 40, 80$.

N	p=1	p=2
5	8.746361442011283E-03	7.356967660094555E-03
10	8.066370779435818E-04	3.971915565871088E-05
20	1.987828259085093E-04	4.696224142664306E-06
40	4.970743984976005E-05	5.784883563418345E-07
80	1.242945225753404E-05	7.186628357347324E-08

Table 5.1.7. L^2 norm of the error $u - u_h$ for B_-^σ in Case 3: $p = 1, 2$, $u(x) = (x - x^2)^2$, $N = 5, 10, 20, 40, 80$.

N	p=1	p=2
5	7.768445588323121E-03	7.142835746836209E-03
10	7.343035186926095E-04	3.779838608619071E-05
20	1.867448478311510E-04	4.673012367963987E-06
40	4.690865829647428E-05	5.817674540127914E-07
80	1.174138247402690E-05	7.254416341448677E-08

Table 5.1.8. L^2 norm of the error $u - u_h$ for B_+^σ in Case 1: $p = 1, 2$, $u(x) = (x - x^2)^2$, $N = 5, 10, 20, 40, 80$.

N	p=1	p=2
5	2.770861950121497E-03	1.304750329139032E-03
10	7.159991273565740E-04	1.390595663743265E-04
20	1.954970636868840E-04	1.116120609575861E-05
40	5.044274252860372E-05	9.167095273947912E-07
80	1.272010835965560E-05	8.915612777659573E-08

Table 5.1.9. L^2 norm of the error $u - u_h$ for B_+^σ in Case 2: $p = 1, 2$, $u(x) = (x - x^2)^2$, $N = 5, 10, 20, 40, 80$.

N	p=1	p=2
5	8.599799004535568E-03	7.432627050223929E-03
10	7.975548043916715E-04	4.348219348513496E-05
20	1.981293181297888E-04	5.077997972147967E-06
40	4.966218763228582E-05	6.259793261124129E-07
80	1.242648846902750E-05	7.793212749529658E-08

Table 5.1.10. L^2 norm of the error $u - u_h$ for B_+^σ in Case 3: $p = 1, 2$, $u(x) = (x - x^2)^2$, $N = 5, 10, 20, 40, 80$.

N	p=1	p=2
5	7.635639254386270E-03	7.205578863629859E-03
10	7.329210857920233E-04	4.069000980131651E-05
20	1.867305983188443E-04	4.989959660296172E-06
40	4.691069091042560E-05	6.229347125314097E-07
80	1.174161835958162E-05	7.782970425733532E-08

In Tables 5.1.11-5.1.13, we present the error of $u - u_h$ in the L^2 norm for $p = 1, 2$ and $N = 5, 10, 20, 40, 80$ when the discontinuous Galerkin method (3.4.5) with B_-^σ is used to approximate the exact solution $u(x) = \sin(4\pi x)$ and in Tables 5.1.14-5.1.16, the error of $u - u_h$ in the L^2 norm for $p = 1, 2$ and $N = 5, 10, 20, 40, 80$ when the discontinuous Galerkin method (3.4.5) with B_+^σ was used to approximate the exact solution $u(x) = \sin(4\pi x)$.

Table 5.1.11. L^2 norm of the error $u - u_h$ for B_-^σ in Case 1: $p = 1, 2$, $u(x) = \sin(4\pi x)$, $N = 5, 10, 20, 40, 80$.

N	p=1	p=2
5	2.896202358919913E-01	5.753028391275423E-02
10	8.957976024560851E-02	6.088180043534613E-03
20	2.452262957038488E-02	6.611116170778919E-04
40	6.305499886596258E-03	7.761171607425504E-05
80	1.588479886793732E-03	9.447220307639147E-06

Table 5.1.12. L^2 norm of the error $u - u_h$ for B^{σ}_{-} in Case 2: $p = 1, 2$, $u(x) = \sin(4\pi x)$, $N = 5, 10, 20, 40, 80$.

N	p=1	p=2
5	3.426269413692835E-01	6.036143642566821E-02
10	8.752924151802116E-02	7.729999440587713E-03
20	2.269263897617622E-02	9.453883132872909E-04
40	5.716606089397737E-03	1.171870655709081E-04
80	1.431876842295169E-03	1.459218725323532E-05

Table 5.1.13. L^2 norm of the error $u - u_h$ for B^{σ}_{-} in Case 3: $p = 1, 2$, $u(x) = \sin(4\pi x)$, $N = 5, 10, 20, 40, 80$.

N	p=1	p=2
5	3.497854073475110E-01	5.539788040502378E-02
10	9.400752014160210E-02	7.499946851198857E-03
20	2.405988191746934E-02	9.422288125727790E-04
40	6.051238779631243E-03	1.175934157381394E-04
80	1.515115556054211E-03	1.467842370779159E-05

Table 5.1.14. L^2 norm of the error $u - u_h$ for B^{σ}_{+} in Case 1: $p = 1, 2$, $u(x) = \sin(4\pi x)$, $N = 5, 10, 20, 40, 80$.

N	p=1	p=2
5	2.770861950121497E-03	8.437970821164646E-02
10	7.159991273565740E-04	1.119872263125491E-02
20	1.954970636868840E-04	1.161911889891347E-03
40	5.044274252860372E-05	1.317655505147986E-04
80	1.272010835965560E-05	1.595028356895006E-05

Table 5.1.15. L^2 norm of the error $u - u_h$ for B_+^σ in Case 2: $p = 1, 2$, $u(x) = \sin(4\pi x)$, $N = 5, 10, 20, 40, 80$.

N	p=1	p=2
5	3.400587985307923E-01	6.036791396241847E-02
10	8.734578570319700E-02	8.135239061172634E-03
20	2.268266589429477E-02	1.011142073014070E-03
40	5.715891603576717E-03	1.261922046933027E-04
80	1.431829354085983E-03	1.576418558426913E-05

Table 5.1.16. L^2 norm of the error $u - u_h$ for B_+^σ in Case 3: $p = 1, 2$, $u(x) = \sin(4\pi x)$, $N = 5, 10, 20, 40, 80$.

N	p=1	p=2
5	3.491778241919659E-01	5.620792718643079E-02
10	9.382680422150219E-02	7.901415319455258E-03
20	2.404108764615146E-02	1.004682575405719E-03
40	6.049860944393975E-03	1.259684076119387E-04
80	1.515024139515292E-03	1.575664173188659E-05

To verify numerically the convergence rate of $u - u_h$ in the L^2 norm, we define CR_h by

$$CR_h = \frac{\log(||u - u_h||/||u - u_{h/2}||)}{\log 2}. \tag{5.1.4}$$

For Cases 1-3, we present the values of CR_h in Tables 5.1.17-5.1.21 for $p = 1, 2$ and $N = 5, 10, 20, 40, 80$ when the discontinuous Galerkin method (3.4.5) with B_-^σ is used to approximate the exact solution $u(x) = (x - x^2)^2$ or $\sin(4\pi x)$ and in Tables 5.1.22-5.1.28, for $p = 1, 2$ and $N = 5, 10, 20, 40, 80$ when the discontinuous Galerkin method (3.4.5) with B_+^σ is used to approximate the exact solution $u(x) = (x - x^2)^2$ or $\sin(4\pi x)$. From Tables

5.1.17-5.1.28, we know that the convergence rate of $u - u_h$ in the L^2 norm is order $O(h^{p+1})$ and these results agree with the theoretical results obtained in Theorems 3.5.4 and 3.5.5.

Table 5.1.17. Convergence rates of $u - u_h$ in the L^2 norm for B_-^σ in Case 1: $\sigma = 5$, $u(x) = (x - x^2)^2$.

N	p=1	p=2
10	1.77	3.15
20	1.93	3.11
40	1.98	3.07
80	2.00	3.04

Table 5.1.18. Convergence rates of $u - u_h$ in the L^2 norm for B_-^σ in Case 2: $\sigma = 5$, $u(x) = (x - x^2)^2$.

N	p=1	p=2
10	3.44	7.53
20	2.02	3.08
40	2.00	3.02
80	2.00	3.01

Table 5.1.19. Convergence rates of $u - u_h$ in the L^2 norm for B_-^σ in Case 3: $\sigma = 5$, $u(x) = (x - x^2)^2$.

N	p=1	p=2
10	3.40	7.56
20	1.98	3.02
40	1.99	3.01
80	2.00	3.00

Table 5.1.20. Convergence rates of $u - u_h$ in the L^2 norm for B_-^σ in Case 1: $\sigma = 5$, $u(x) = \sin(4\pi x)$.

N	p=1	p=2
10	1.69	3.24
20	1.87	3.20
40	1.96	3.09
80	1.99	3.04

Table 5.1.21. Convergence rates of $u - u_h$ in the L^2 norm for B_-^σ in Case 2: $\sigma = 5$, $u(x) = \sin(4\pi x)$.

N	p=1	p=2
10	1.97	2.97
20	1.95	3.03
40	1.99	3.01
80	2.00	3.01

Table 5.1.22. Convergence rates of $u - u_h$ in the L^2 norm for B_-^σ in Case 3: $\sigma = 5$, $u(x) = \sin(4\pi x)$.

N	p=1	p=2
10	1.90	2.88
20	1.90	2.99
40	1.99	3.00
80	2.00	3.00

Table 5.1.23. Convergence rates of $u - u_h$ in the L^2 norm for B_+^σ in Case 1: $\sigma = 0.2$, $u(x) = (x - x^2)^2$.

N	p=1	p=2
10	1.95	3.23
20	1.87	3.64
40	1.95	3.61
80	1.99	3.36

Table 5.1.24. Convergence rates of $u - u_h$ in the L^2 norm for B_+^σ in Case 2: $\sigma = 0.2$, $u(x) = (x - x^2)^2$.

N	p=1	p=2
10	3.43	7.42
20	2.01	3.11
40	2.00	3.02
80	2.00	3.01

Table 5.1.25. Convergence rates of $u - u_h$ in the L^2 norm for B_+^σ in Case 3: $\sigma = 0.2$, $u(x) = (x - x^2)^2$.

N	p=1	p=2
10	3.38	7.47
20	1.97	3.03
40	1.99	3.00
80	2.00	3.00

Table 5.1.26. Convergence rates of $u - u_h$ in the L^2 norm for B_+^σ in Case 1: $\sigma = 0.2$, $u(x) = \sin(4\pi x)$.

N	p=1	p=2
10	1.84	2.91
20	1.85	3.27
40	1.95	3.14
80	1.99	3.05

Table 5.1.27. Convergence rates of $u - u_h$ in the L^2 norm for B_+^σ in Case 2: $\sigma = 0.2$, $u(x) = \sin(4\pi x)$.

N	p=1	p=2
10	1.96	2.89
20	1.95	3.01
40	1.99	3.00
80	2.00	3.00

Table 5.1.28. Convergence rates of $u - u_h$ in the L^2 norm for B_+^σ in Case 3: $\sigma = 0.2$, $u(x) = \sin(4\pi x)$.

N	p=1	p=2
10	1.90	2.83
20	1.96	2.98
40	1.99	3.00
80	2.00	3.00

Remark 5.1.1. *When $a(x) = 1$, $b(x) = d(x) = 0$ in (5.1.1) and $u(x) = \sin(4\pi x)$, our test problem is the same as one in [6]. In [6], using a DG method without a penalty term, it is numerically verified that the convergence rate of $u - u_h$ in the L^2 norm is order $O(h^p)$ for $p \geqslant 2$. And there is no numerical result for $p = 1$. However, our DG methods with an interior penalty term are shown numerically that the convergence rate of $u - u_h$ in the L^2 norm is order $O(h^{p+1})$ for $p = 1, 2$.*

Remark 5.1.2. *When $b(x) = 0$ in (5.1.1) and $u(x) = \sin(4\pi x)$, our problem is the same as one in [18]. In [18], it is numerically verified that the convergence rate of $u - u_h$ in the L^2 norm for mixed discontinuous finite element (MDFE) method without a penalty term is order $O(h^p)$ for odd p and order $O(h^{p+1})$ for even p. Again, the convergence rate of $u - u_h$ in the L^2 norm is not optimal for $p = 1$.*

5.2 Two Dimensional Elliptic Problems

In this section, we consider the following two dimensional elliptic problem:

$$\begin{aligned} -\nabla \cdot (\boldsymbol{a}(\nabla u + \mathbf{b}u)) + du = f \quad \text{for} \quad \Omega = (0,1) \times (0,1), \\ u(0,y) = u(1,y) = u(x,0) = u(x,1) = 0, \quad \text{for} \quad 0 \leqslant x, y \leqslant 1, \end{aligned} \tag{5.2.1}$$

whose exact solution is given by

$$u(x,y) = 32x(1-x)y(1-y), \tag{5.2.2}$$

or

$$u(x,y) = \sin(\pi x)\sin(\pi y). \tag{5.2.3}$$

And the function f is chosen so that the problem (5.2.1) is satisfied with the appropriate choices of $\boldsymbol{a}(x)$, $\mathbf{b}(x)$, $d(x)$ and $u(x)$.

Like the one dimensional elliptic problem in Section 5.1, we test three cases to verify numerically the error estimate of $u - u_h$ in the L^2 norm for discontinuous Galerkin methods with an interior penalty discussed in Section 4.5:

Case 4. $\boldsymbol{a}(x,y)$ is an identity matrix, $\mathbf{b}(x,y) = (0,0)^T$ and $d(x,y) = 0$,

Case 5. $\boldsymbol{a}(x,y) = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$, $\mathbf{b}(x,y) = (0.5, -0.5)^T$ and $d(x,y) = 1$,

Case 6. $\boldsymbol{a}(x,y) = \begin{pmatrix} xy+1 & y \\ x & xy+1 \end{pmatrix}$, $\mathbf{b}(x,y) = (x+y, xy)^T$ and $d(x,y) = 1$.

To implement the discontinuous Galerkin method (4.4.4)

$$B^{\sigma}_{\pm}(u_h, v) = F(v), \quad \forall v \in V_h,$$

we take P_h as the collection of $N \times N$ uniform squares in Ω whose length of edge is $h = 1/N$. And finite element subspaces V_h of polynomial functions is defined as

$$V_h = \{v \in L^2(\Omega);\ v \in P_p(K),\ \forall K \in P_h \text{ and } v = 0 \text{ on } \partial\Omega\},$$

where $P_p(K)$ is the space of polynomial of degree less than or equal to p on K for a given integer $p \geqslant 1$.

As we do in Section 5.1, we first investigate the condition numbers for B^{σ}_{-} and B^{σ}_{+} in Cases 4 and 5.

Table 5.2.1. The condition numbers for B_-^σ and B_+^σ in Case 4: $\sigma = 10$.

N	B_-^σ	B_+^σ
5	1.2809E+03	2.5518E+04
10	1.5640E+03	1.3451E+05
20	3.1489E+03	1.2562E+06

Table 5.2.2. The condition numbers for B_-^σ and B_+^σ in Case 4: $\sigma = 20$.

N	B_-^σ	B_+^σ
5	2.3018E+03	5.0630E+04
10	2.9201E+03	2.6872E+05
20	6.3786E+03	2.5343E+06

Table 5.2.3. The condition numbers for B_-^σ and B_+^σ in Case 5: $\sigma = 10$.

N	B_-^σ	B_+^σ
5	1.6928E+03	8.5246E+03
10	3.5522E+03	4.4856E+04
20	4.5099E+03	4.4792E+05

Table 5.2.4. The condition numbers for B_-^σ and B_+^σ in Case 5: $\sigma = 20$.

N	B_-^σ	B_+^σ
5	903.6506	1.6981E+04
10	1.1672E+03	8.9695E+04
20	2.2323E+03	8.9418E+05

From the Tables 5.2.1-5.2.4, we know that the condition numbers for B_-^σ are not larger than ones for B_+^σ. But the condition numbers for B_+^σ get larger as h tends to zero. This fact agrees that the penalty term of B_+^σ increases significantly the condition numbers of the stiffness matrix in [3].

In Tables 5.2.5-5.2.7, we present the error of $u - u_h$ in the L^2 norm for $n = 5, 10, 20, 25, 30$, when the discontinuous Galerkin method (4.4.4) with only B_-^σ is used to approximate the exact solution $u(x) = 32x(1-x)y(1-y)$. In Tables 5.2.8-5.2.10, we present the error of $u - u_h$ in the L^2 norm for $n = 5, 10, 20, 25, 30$, when the discontinuous Galerkin method (4.4.4) with one is used to approximate the exact solution $u(x) = \sin(\pi x)\sin(\pi y)$.

Table 5.2.5. L^2 norm of the error $u - u_h$ for B_-^σ in Case 4: $p = 2$, $\sigma = 20$, $u(x) = 32x(1-x)y(1-y)$, $N = 5, 10, 20, 25, 30$.

N	Case 4
5	1.778188495369365E-02
10	1.368094620007076E-03
20	1.149573368876803E-04
25	5.366871115227838E-05
30	2.914733844414101E-05

Table 5.2.6. L^2 norm of the error $u - u_h$ for B_-^σ in Case 5: $p = 2$, $\sigma = 20$, $u(x) = 32x(1-x)y(1-y)$, $N = 5, 10, 20, 25, 30$.

N	Case 5
5	1.421938095167426E-02
10	1.172185188279127E-03
20	1.078211765012156E-04
25	5.132262089824426E-05
30	2.821737001089368E-05

Table 5.2.7. L^2 norm of the error $u - u_h$ for B_-^σ in Case 6: $p = 2$, $\sigma = 20$, $u(x) = 32x(1-x)y(1-y)$, $N = 5, 10, 20, 25, 30$.

N	Case 6
5	1.752570406324385E-02
10	1.333383038587379E-03
20	1.126431016510676E-04
25	5.284069954083651E-05
30	2.889612815560540E-05

Table 5.2.8. L^2 norm of the error $u - u_h$ for B_-^σ in Case 4: $p = 2$, $\sigma = 20$, $u(x) = \sin(4\pi x)\sin(4\pi y)$, $N = 5, 10, 20, 25, 30$.

N	Case 4
5	5.747545957409887E-03
10	4.426823192967139E-04
20	4.198104887890998E-05
25	2.036823179853948E-05
30	1.137639497455367E-05

Table 5.2.9. L^2 norm of the error $u - u_h$ for B_-^σ in Case 5: $p = 2$, $\sigma = 20$, $u(x) = \sin(4\pi x)\sin(4\pi y)$, $N = 5, 10, 20, 25, 30$.

N	Case 5
5	4.631075489716890E-03
10	3.881919382913525E-04
20	3.953538673836228E-05
25	1.939858764223667E-05
30	1.090378429386863E-05

Table 5.2.10. L^2 norm of the error $u - u_h$ for B_-^σ in Case 6: $p = 2$, $\sigma = 20$, $u(x) = \sin(4\pi x)\sin(4\pi y)$, $N = 5, 10, 20, 25, 30$.

N	Case 6
5	5.610024303727501E-03
10	4.241760657046240E-04
20	4.055800685437501E-05
25	2.988694089119666E-05
30	1.126456946113618E-05

Table 5.2.11. L^2 norm of the error $u - u_h$ for B_+^σ in Case 4: $p = 2$, $\sigma = 20$, $u(x) = 32x(1-x)y(1-y)$, $N = 5, 10, 20, 25, 30$.

N	Case 4
5	3.854252493410248E-02
10	8.557866088220699E-03
20	1.994029200550540E-03
25	1.256838627358809E-03
30	8.637405055789861E-04

Table 5.2.12. L^2 norm of the error $u - u_h$ for B_+^σ in Case 5: $p = 2$, $\sigma = 20$, $u(x) = 32x(1-x)y(1-y)$, $N = 5, 10, 20, 25, 30$.

N	Case 5
5	3.144173490747065E-02
10	6.268606973337158E-03
20	1.375117423451399E-03
25	8.559042529243409E-04
30	5.832419576820340E-04

Table 5.2.13. L^2 norm of the error $u - u_h$ for B_+^σ in Case 6: $p = 2$, $\sigma = 20$, $u(x) = 32x(1-x)y(1-y)$, $N = 5, 10, 20, 25, 30$.

N	Case 6
5	3.955492317086112E-02
10	8.595480341012460E-03
20	1.978514278774744E-03
25	1.244034829693771E-03
30	8.535784830961786E-04

Table 5.2.14. L^2 norm of the error $u - u_h$ for B_+^σ in Case 4: $p = 2$, $\sigma = 20$, $u(x) = \sin(4\pi x)\sin(4\pi y)$, $N = 5, 10, 20, 25, 30$.

N	Case 4
5	1.553189533436008E-02
10	3.362623889105508E-03
20	7.918010063225206E-04
25	5.015396979921996E-04
30	3.460143793714296E-04

Table 5.2.15. L^2 norm of the error $u - u_h$ for B_+^σ in Case 5: $p = 2$, $\sigma = 20$, $u(x) = \sin(4\pi x)\sin(4\pi y)$, $N = 5, 10, 20, 25, 30$.

N	Case 5
5	1.049589739278358E-02
10	2.194805255868131E-03
20	5.133676358502154E-04
25	3.246815805510620E-04
30	2.237387752254111E-04

Table 5.2.16. L^2 norm of the error $u - u_h$ for B_+^σ in Case 6: $p = 2$, $\sigma = 20$, $u(x) = \sin(4\pi x)\sin(4\pi y)$, $N = 5, 10, 20, 25, 30$.

N	Case 6
5	1.494298070131644E-02
10	3.249692250872653E-03
20	7.608045968234252E-04
25	4.809376236323910E-04
30	3.313139061910790E-04

To verify numerically the convergence rate of $u - u_h$ in the L^2 norm, we define $CR_{h,h'}$ by

$$CR_{h,h'} = \frac{\log(\|u - u_h\|/\|u - u_{h'}\|)}{\log(N/N')} \tag{5.2.4}$$

where $N = 1/h$ and $N' = 1/h'$. For Cases 4-5, we present the values of $CR_{h,h'}$ in Tables 5.2.17-5.2.20 for $N = 5, 10, 20, 25, 30$ when the discontinuous Galerkin method (4.4.4) with B_-^σ is used to approximate the exact solution $u(x) = 32x(1-x)y(1-y)$ or $\sin(\pi x)\sin(\pi y)$. From Tables 5.2.17-5.2.18, we know that the convergence rate of $u - u_h$ in the L^2 norm is order $O(h^{p+1})$ and these results agree with the theoretical results obtained in Theorem 4.5.4.

Table 5.2.17. Convergence rates of $u - u_h$ in the L^2 norm for B_-^σ in Cases 4-6: $\sigma = 20, u(x) = 32x(1-x)y(1-y)$.

N	Case 4	Case 5	Case 6
10	3.70	3.60	3.72
20	3.57	3.37	3.57
25	3.41	3.33	3.39
30	3.35	3.28	3.31

Table 5.2.18. Convergence rates of $u - u_h$ in the L^2 norm for B_-^σ in Cases 4-6: $\sigma = 20, u(x) = \sin(\pi x)\sin(\pi y)$.

N	Case 4	Case 5	Case 6
10	3.70	3.58	3.73
20	3.40	3.30	3.39
25	3.24	3.19	3.19
30	3.19	3.16	3.12

Remark 5.2.1. *When* $\boldsymbol{a}(x) = I$*,* $\mathbf{b}(x) = \mathbf{0}$ *and* $d(x) = 0$ *in (5.2.1) and* $u(x) = \sin(\pi x)\sin(\pi y)$*, our test problem is the same as one in [33]. In [33], it is verified numerically that the convergence rate of* $u - u_h$ *in the energy norm is optimal. And there is no numerical results on the convergence rate of* $u - u_h$ *in the* L^2 *norm. However, our DG method for* B_-^σ *is shown numerically that the convergence rate of* $u - u_h$ *in the* L^2 *norm is optimal.*

From Tables 5.2.19-5.2.20, we know that the convergence rate of $u - u_h$ in the L^2 norm is $O(h^p)$ and these results do not agree with the theoretical results obtained in Theorem 4.5.5. We think that this discrepancy comes from significantly large condition number for B_+^σ.

Table 5.2.19. Convergence rates of $u - u_h$ in the L^2 norm for B_+^σ in Cases 4-6: $\sigma = 10, u(x) = 32x(1-x)y(1-y)$.

N	Case 4	Case 5	Case 6
10	2.17	2.33	2.20
20	2.10	2.19	2.12
25	2.07	2.12	2.08
30	2.06	2.10	2.07

Table 5.2.20. Convergence rates of $u - u_h$ in the L^2 norm for B_+^σ in Cases 4-6: $\sigma = 10, u(x) = \sin(\pi x)\sin(\pi y)$.

N	Case 4	Case 5	Case 6
10	2.21	2.26	2.20
20	2.09	2.10	2.09
25	2.05	2.05	2.06
30	2.04	2.04	2.04

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