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A Note on Vector Matrix Game

by

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A Note on Vector Matrix Game
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A dissertation

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벡터 행렬 게임에 대한 소고

홍 정 민

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요 약

본 논문에서는 행렬 게임의 확장인 두 개 이상의 반대칭 벡터 행렬 게임과 벡터 행렬 게임의 6종류의 해를 정의하고, 벡터 최적화 기법을 사용하여 6종류의 벡터 행렬 게임의 해를 특성화 한다. 특히, 두 개의 2×2 반대칭행렬에 의해 정의되는 벡터 행렬 게임의 6종류의 해를 완전하게 특성화하고, 두 개의 3×3 반대칭행렬에 의해 정의되는 벡터 행렬 게임의 6종류의 해가 서로 다르다는 것을 보여주는 예제를 제시한다. 나아가서 선형 벡터 최적 문제의 쌍대문제를 제시하고 쌍대문제와 그에 대응하는 벡터 행렬 게임 문제와의 동치관계가 성립함을 보여준다. 그리고 이러한 동치관계가 성립함을 예제를 통해 예증한다. 더욱이, 우리는 벡터 대칭 쌍대문제와 그에 대응하는 벡터 행렬게임 문제 사이에 동치관계가 성립한다는 사실을 얻는다.

Chapter 1
Introduction and Preliminaries

In 1994, Blum and Oettli [12] coined the terminology "Equilibrium Problem" for giving an unified formulation for optimization problem, saddle point problem, variational inequality, Nash equilibrium in noncooperative games and other problems related to equilibrium. Main theorems in nonlinear analysis [7, 8, 13, 14, 25, 26], for example, Brouwer fixed point theorem, Browder fixed point theorem, Kakutani fixed point theorem, Ky Fan's minimax inequality, Ky Fan's section theorem and Knaster-Kuratowski-Mazurkiewicz principle (KKM-Fan theorem), have supported mathematical tools for giving existence theorems for solutions of such equilibrium problems.

Most of decision making situations require a simultaneous consideration of more than two objectives which are in conflict or trade-off [30, 51]. Such requirements had led to multiobjective (vector) optimization problems. Many authors have formulated and studied vector equilibrium problems [1, 2, 3, 4, 5, 6, 11, 18, 21, 24, 27, 28, 29, 31, 35, 42, 43, 44, 45, 50, 56] which are vector versions of the equilibrium problem and which contain several kinds of vector variational inequalities and vector optimization problems as special cases.

Multiobjective optimization problems consist of conflicting objective functions and constraint sets and are to optimize the objective functions over the constraint sets under some concepts of solutions, for example, properly efficient solutions, efficient solutions and weakly efficient solutions.

Optimality criteria and duality theorems are very important topics in investigating optimization problems. In 1961, Wolfe [60] formulated a dual problem for a single objective optimization problem on the basis of the Kuhn-Tucker necessary optimality conditions, which is now called the Wolfe dual problem, and proved weak and strong duality theorems.

In 1981, Mond and Weir [47] gave another type dual problem for a single objective optimization problem on the basis of the Kuhn-Tucker necessary optimality condition, which is now called the Mond-Weir dual problem and proved weak, strong and converse duality theorems. Until now, many authors [9, 10, 23, 33, 34, 36, 37, 39, 40, 41, 46, 49, 57, 58, 59, 61] have formulated Wolfe type dual problems and Mond-Weir type dual problems for several kinds of optimization problems and have studied duality theorems.

A nonlinear programming problem and its dual are said to be symmetric if the dual of the dual is the original problem. Symmetric duality in nonlinear programming in which the dual of the dual is the primal was first introduced by Dorn [22]. Dantzig et al. [19] formulated a pair of symmetric dual nonlinear programs and established duality results for convex and concave functions with non-negative orthant as the cone. Mond and Weir [48] presented two pair of symmetric dual multiobjective programming problems for efficient solutions and obtained symmetric duality results concerning pseudoconvex and pseudoconcave functions. Chandra, Craven and Mond [16] formulated a pair of symmetric dual fractional programming problems under suitable convexity hypothesis.

In Dantzig [20], some equivalent relations between linear programming duality and symmetric matrix game are given. In the finite dimensional setting, Chandra, Craven and Mond [15] presented analogies of results from [20] for a certain class of nonlinear programming problems. And also Chandra, Mond and Prasad [17] studied some equivalent relations between continuous linear programs and continuous matrix games. Recently, many authors [15, 38, 52] have studied equivalent relations between optimization problems and its related matrix games.

In this dissertation, we consider vector matrix games with more than two skew symmetric matrices, which is an extension of the matrix game, define six kinds of solutions for vector matrix games and give an example which illustrate that such six kinds of solutions may be different. Using vector optimization techniques, we characterize solutions of vector matrix game. In particular, we calculate six kinds of solutions for vector matrix game with two 2×2 matrices. We formulate a dual problem for a linear vector optimization problem, give a duality result for the dual problem and establish equivalent relations between the dual problem and the corresponding vector matrix game. We give a numerical example for showing such equivalent relations. Furthermore, we obtain equivalent relations between the vector symmetric dual problems and the corresponding vector matrix game .

This dissertation is organized as follows;

In Chapter 2, we consider relations between solutions of a vector equilibrium problem and prove the existence theorem for the problem, and then we apply such results to vector saddle point problems and vector matrix games.

In Chapter 3, using vector optimization techniques, we characterize solutions of vector matrix game. In particular, we calculate six kinds of solutions for vector matrix game with two 2×2 matrices. Furthermore, we formulate a dual problem for a linear vector optimization problem, give a weak duality result for the dual problem and establish equivalent relations between the dual problem and the corresponding vector matrix game. Moreover, we give a numerical example for showing such equivalent relations.

In Chapter 4, we formulate vector symmetric dual problems and consider vector matrix games corresponding to the problems. We obtain equivalent relations between the vector matrix games and the vector symmetric dual problems.

In Chapter 5, we give examples which shows that six kinds of solutions of a vector matrix game may be different.

Now we give notations and preliminary results that will be used later. The following definitions are found in [53].

Definition 1.1. A subset $P \subset \mathbb{R}^n$ is said to be a polyhedral set if there exist $a_i \in \mathbb{R}^n,\ b_i \in \mathbb{R},\ i = 1, \cdots, k$ such that

$$P = \{x \mid a_i^T x \leqq b_i,\ i = 1, \cdots, k\},$$

where the symbol ^{T} denotes the transpose.

Definition 1.2. Let $C \subset \mathbb{R}^n$. The indicator function $\delta_C(\cdot)$ of C is defined by

$$\delta_C(x) = \begin{cases} 0 & \text{if} \quad x \in C \\ +\infty & \text{if} \quad x \notin C. \end{cases}$$

Definition 1.3. Let $H \subseteq \mathbb{R}^n$ be a closed and convex set. The normal cone to H at $\bar{x} \in H$ is defined as follows:

$$N_H(\bar{x}) := \{v \in \mathbb{R}^n \ : \ v^T(x - \bar{x}) \leqq 0 \ \ \forall x \in H\}.$$

Definition 1.4. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function.

(1) The epigraph of f, $epi f$, is defined by

$$epi f = \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leqq \alpha\}.$$

(2) The subdifferential of f at $a \in \mathbb{R}^n$ is defined as a nonempty convex set

$$\partial f(a) = \{v \in \mathbb{R}^n \mid f(x) - f(a) \geqq v^T(x - a) \ \ \forall x \in \mathbb{R}^n\}.$$

An important special case in the theory of subgradients is the case where f is the indicator of a non-empty convex set C. By definition, if $x \in C$, $x^* \in \partial\delta_C(x)$ if and only if $0 \geqq x^{*T}(z - x)$ for every $z \in C$, i.e., x^* is normal to C at x. Thus if $x \in C$, $\partial\delta_C(x)$ is the same as the normal cone to C at x, $N_C(x)$ defined in Definition 1.3.

Chapter 2
Vector Equilibrium Problem

2.1. Introduction

In this section, we obtain relations between solutions of a vector equilibrium problem and the existence theorem for the problem, and then we apply such results to vector saddle point problems and vector matrix games.

Now we consider the following convex vector optimization problem:

$$\text{(VP)} \qquad \text{Minimize} \quad f(x) := (f_1(x), \cdots, f_p(x))$$
$$\text{subject to} \quad x \in S,$$

where $f_i : \mathbb{R}^n \to \mathbb{R}, \quad i = 1, \cdots, p$, are convex functions and the constrained set S is a closed convex subset of $\mathbb{R}^n$.

Solving (VP) means to find (properly, weakly) efficient solutions defined as follows;

Definition 2.1.1. (1) A point $\bar{x} \in S$ is said to be an efficient solution of (VP) if for any $x \in S$,

$$(f_1(x) - f_1(\bar{x}), \cdots, f_p(x) - f_p(\bar{x})) \notin -\mathbb{R}^p_+ \setminus \{0\},$$

(2) A point $\bar{x} \in S$ is said to be a properly efficient solution of (VP) if $\bar{x} \in S$ is an efficient solution of (VP) and there exists a constant $M > 0$ such that for each $i = 1, \cdots, p$, we have

$$\frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} \leqq M$$

for some j such that $f_j(x) > f_j(\bar{x})$ whenever $x \in S$ and $f_i(x) < f_i(\bar{x})$.

(3) A point $\bar{x} \in S$ is said to be a weakly efficient solution of (VP) if for any $x \in S$,

$$(f_1(x) - f_1(\bar{x}), \cdots, f_p(x) - f_p(\bar{x})) \notin -int\mathbb{R}^p_+,$$

where $int\mathbb{R}^p_+$ is the interior of $\mathbb{R}^p_+$.

The quantity $\frac{f_i(\bar{x})-f_i(x)}{f_j(x)-f_j(\bar{x})}$ may be interpreted as the marginal trade-off for objective functions f_i and f_j between x and $\bar{x}$. Geoffrion [30] considered the concept of the proper efficiency to eliminate unbounded trade-off between objective functions of (VP).

Lemma 2.1.1 [7]. A point $\bar{x}$ is a weakly efficient solution of (VP) if and only if there exists $\lambda_i \geqq 0$, $i = 1, \cdots, p$, $(\lambda_1, \cdots, \lambda_p) \neq 0$ such that $\bar{x}$ is a solution of the following scalar optimization problem:

$$\begin{aligned} &\text{Minimize} \quad \sum_{i=1}^{p} \lambda_i f_i(x) \\ &\text{subject to} \quad x \in S. \end{aligned}$$

Now we recall definition of the KKM multifunction and KKM-Fan theorem [25] needed for the proofs of our existence theorems.

Definition 2.1.2. Let X be a vector space and K be a nonempty subset of X. Then a multifunction $G : K \to 2^X$ is called a KKM multifunction if for each finite subset $\{x_1, \cdots, x_n\}$ of K, $co\{x_1, \cdots, x_n\} \subset \bigcup_{i=1}^{n} G(x_i)$, where coA is the convex hull of a subset A of X.

Theorem 2.1.1 (KKM-Fan Theorem). Let X be a Hausdorff topological vector space, K be a nonempty subset of X and $G : K \to 2^X$ be a KKM multifunction. If all the sets $G(x)$ are closed in X and if one is compact, then $\bigcap_{x \in K} G(x) \neq \emptyset$.

Now we formulate vector equilibrium problems:

Let K be a closed convex subset of $\mathbb{R}^n$ and $f : (K \times K) \times (K \times K) \to \mathbb{R}^p$. Let $S_p = \{(x_1, x_2, \cdots, x_p) \in \mathbb{R}^p \mid x_i \geqq 0,\ i = 1, \cdots, p \text{ and } \sum_{i=1}^{p} x_i = 1\}$.

Vector Equilibrium Problems:

(1) **(scalarized equilibrium problem with respect to $\xi \in S_p$)**

$(\mathrm{SEP})_\xi$ Find $(\bar{x}, \bar{y}) \in K \times K$ such that

$$\sum_{i=1}^{p} \xi_i f_i((\bar{x}, \bar{y}), (z, \omega)) \geqq 0 \quad \forall (z, \omega) \in K \times K.$$

(2) **(vector equilibrium problem (VEP))**

(VEP) Find $(\bar{x}, \bar{y}) \in K \times K$ such that

$$f((\bar{x}, \bar{y}), (z, \omega)) \nleq 0 \quad \forall (z, \omega) \in K \times K.$$

(3) **(weak vector equilibrium problem (WVEP))**

(WVEP) Find $(\bar{x}, \bar{y}) \in K \times K$ such that

$$f((\bar{x}, \bar{y}), (z, \omega)) \nless 0 \quad \forall (z, \omega) \in K \times K.$$

We denote the solution sets of $(\text{SEP})_\xi$, (VEP) and (WVEP) by $sol(\text{SEP})_\xi$, sol(VEP) and sol(WVEP), respectively.

Now we formulate vector saddle point problems which are special cases of vector equilibrium problems:

Let K be a convex subset of $\mathbb{R}^n$ and $L := (L_1, \cdots, L_p) : K \times K \to \mathbb{R}^p$ be a vector-valued function.

Vector Saddle Point Problems:

(1) **(scalarized saddle point problem with respect to $\xi \in S_p$)**

$(\text{SSP})_\xi$ Find $(\bar{x}, \bar{y}) \in K \times K$ such that

$$\sum_{i=1}^{p} \xi_i L_i(x, \bar{y}) \leqq \sum_{i=1}^{p} \xi_i L_i(\bar{x}, \bar{y}) \leqq \sum_{i=1}^{p} \xi_i L_i(\bar{x}, y) \quad \forall (x, y) \in K \times K.$$

(2) **(vector saddle point problem (VSP))**

(VSP) Find $(\bar{x}, \bar{y}) \in K \times K$ such that

$$L(x, \bar{y}) \ngeq L(\bar{x}, \bar{y}) \ngeq L(\bar{x}, y) \quad \forall (x, y) \in K \times K.$$

(3) **(weak vector saddle point problem (WVSP))**

(WVSP) Find $(\bar{x}, \bar{y}) \in K \times K$ such that

$$L(x, \bar{y}) \not> L(\bar{x}, \bar{y}) \not> L(\bar{x}, y) \quad \forall (x, y) \in K \times K.$$

We denote the solution sets of the above inequality problems by $sol(\text{SSP})_\xi$, sol(VSP), sol(WVSP), respectively.

Denote the relative interior of the set S_n by $\overset{o}{S}_n$.

Throughout this paper, we will use the following conventions for vectors in the Euclidean space $\mathbb{R}^n$ for vectors $x := (x_1, \cdots, x_n)$ and $y := (y_1, \cdots, y_n)$:

$$x \leqq y \text{ if and only if } x_i \leqq y_i, \; i = 1, \cdots, n;$$

$$x < y \text{ if and only if } x_i < y_i, \; i = 1, \cdots, n;$$

$$x \leq y \text{ if and only if } x_i \leqq y_i, \text{ and } x \neq y; \text{ and}$$

$$x \nleq y \text{ is the negation of } x \leq y.$$

Now we define the following vector matrix game as special cases of vector saddle point problems:

Definition 2.1.3. Let $B_i, \; i = 1, \cdots, p,$ be real $n \times n$ skew-symmetric matrices.

(1) A point $\bar{x} \in S_n$ is said to be a vector solution of vector matrix game $(B_1, \cdots, B_p)$ if $(x^T B_1 \bar{x}, \cdots, x^T B_p \bar{x}) \ngeq (\bar{x}^T B_1 \bar{x}, \cdots, \bar{x}^T B_p \bar{x}) \ngeq (\bar{x}^T B_1 x, \cdots, \bar{x}^T B_p x)$ for any $x \in S_n$.

(2) A point $\bar{x} \in S_n$ is said to be a weakly vector solution of vector matrix game $(B_1, \cdots, B_p)$ if $(x^T B_1 \bar{x}, \cdots, x^T B_p \bar{x}) \ngtr (\bar{x}^T B_1 \bar{x}, \cdots, \bar{x}^T B_p \bar{x}) \ngtr (\bar{x}^T B_1 x, \cdots, \bar{x}^T B_p x)$ for any $x \in S_n$.

(3) A point $(\bar{x}, \bar{y}) \in S_n \times S_n$ is said to be an efficient solution of vector matrix game $(B_1, \cdots, B_p)$ if $(x^T B_1 \bar{y}, \cdots, x^T B_p \bar{y}) \ngeq (\bar{x}^T B_1 \bar{y}, \cdots, \bar{x}^T B_p \bar{y}) \ngeq (\bar{x}^T B_1 y, \cdots, \bar{x}^T B_p y)$ for any $x, \; y \in S_n$.

(4) A point $(\bar{x}, \bar{y}) \in S_n \times S_n$ is said to be a weakly efficient solution of vector matrix game $(B_1, \cdots, B_p)$ if $(x^T B_1 \bar{y}, \cdots, x^T B_p \bar{y}) \not> (\bar{x}^T B_1 \bar{y}, \cdots, \bar{x}^T B_p \bar{y}) \not> (\bar{x}^T B_1 y, \cdots, \bar{x}^T B_p y)$ for any $x,\ y \in S_n$.

(5) A point $(\bar{x}, \bar{y}) \in S_n \times S_n$ is said to be a scalarizing solution of vector matrix game $(B_1, \cdots, B_p)$ if there exists $\lambda \in \overset{o}{S}_p$ such that $x^T(\sum_{i=1}^p \lambda_i B_i)\bar{y} \leqq \bar{x}^T(\sum_{i=1}^p \lambda_i B_i)\bar{y} \leqq \bar{x}^T(\sum_{i=1}^p \lambda_i B_i)y$ for any $x,\ y \in S_n$.

(6) A point $(\bar{x}, \bar{y}) \in S_n \times S_n$ is said to be a weakly scalarizing solution of vector matrix game $(B_1, \cdots, B_p)$ if there exists $\lambda \in S_p$ such that $x^T(\sum_{i=1}^p \lambda_i B_i)\bar{y} \leqq \bar{x}^T(\sum_{i=1}^p \lambda_i B_i)\bar{y} \leqq \bar{x}^T(\sum_{i=1}^p \lambda_i B_i)y$ for any $x,\ y \in S_n$.

We denote the set of all the vector solutions, the set of all the weakly vector solutions, the set of all the efficient solutions, the set of all the weakly efficient solutions, the set of all the scalarizing solutions and the set of all the weakly scalarizing solutions for vector matrix game, by *sol*(VMG), *sol*(WVMG), *sol*(EVMG), *sol*(WEVMG), *sol*(SVMG) and *sol*(WSVMG), respectively.

2.2. Solution Sets of Vector Equilibrium Problem

Now we give existence theorems for $(\text{SEP})_\xi$ in compact settings.

Theorem 2.2.1. Let K be a convex and compact subset of $\mathbb{R}^n$. Let $f : (K \times K) \times (K \times K) \to \mathbb{R}^n$ be a continuous function. Assume that $f((x, y), (x, y)) = 0$ for any $(x, y) \in K \times K$. Then for any $\xi \in S_p$, $(\text{SEP})_\xi$ has a solution.

Proof. Let $\xi \in S_p$. Define a multifunction $F : K \times K \to 2^{K\times K}$ by for any $(z, \omega) \in K \times K$,

$$F(z,\omega) = \{(x,y) \in K \times K \mid \sum_{i=1}^{p} \xi_i f_i((x,y),(z,\omega)) \geqq 0\}.$$

(1) We will prove that F is a KKM multifunction.

Suppose that F is not a KKM multifunction. Then there exists $\{(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)\} \subset K \times K$ such that

$$co\{(x_1,y_1),(x_2,y_2),\cdots,(x_n,y_n)\} \not\subset \bigcup_{i=1}^{n} F(x_i,y_i).$$

Thus, there exist $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{R}$ such that $\alpha_i \geqq 0$, $\sum_{i=1}^{n} \alpha_i = 1$ and

$$\begin{aligned} \sum_{i=1}^{n} \alpha_i(x_i,y_i) &\in (\bigcup_{i=1}^{n} F(x_i,y_i))^C \\ &= \bigcap_{i=1}^{n} (F(x_i,y_i))^C. \end{aligned}$$

Since K is convex and $\{(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)\} \subset K \times K$,

$$\sum_{i=1}^{n} \alpha_i(x_i,y_i) \in K \times K.$$

Let $(z_0, \omega_0) = \sum_{i=1}^{n} \alpha_i (x_i, y_i)$. Since $(z_0, \omega_0) \notin F(x_i, y_i)$, $i = 1, 2, \cdots, n$ and $(z_0, \omega_0) \in K \times K$,

$$\sum_{j=1}^{p} \xi_j f_j((x_i, y_i), (z_0, \omega_0)) < 0, \quad i = 1, 2, \cdots, n.$$

So, we have

$$\begin{aligned} 0 &= \sum_{j=1}^{p} \xi_j f_j((z_0, \omega_0), (z_0, \omega_0)) \\ &= \sum_{j=1}^{p} \xi_j f_j(\sum_{i=1}^{n} \alpha_i (x_i, y_i), (z_0, \omega_0)) \\ &< 0. \end{aligned}$$

This is impossible. Hence F is a KKM multifunction.

(2) Let $(z, \omega) \in K \times K$. We will prove that $F(z, \omega)$ is compact. Let $\{(x_n, y_n)\}$ be a sequence in $F(z, \omega)$ converging (x_0, y_0). Since $x_n \in K$, $y_n \in K$ and K is compact, we may assume that $(x_0, y_0) \in K \times K$. Since

$\sum_{i=1}^{p} \xi_i f_i((x_n, y_n), (z, \omega)) \geqq 0$ and f_i is convex, $\sum_{i=1}^{p} \xi_i f_i((x_0, y_0), (z, \omega)) \geqq 0$. Thus $(x_0, y_0) \in F(z, \omega)$ and hence $F(z, \omega)$ is closed. Since $F(z, \omega) \subset K \times K$ and K is compact, $F(z, \omega)$ is compact.

By KKM-Fan Theorem, $\bigcap_{(z,\omega) \in K \times K} F(z, \omega) \neq \emptyset$. So, there exists $(\bar{x}, \bar{y}) \in$

$K \times K$ such that for any $(z, \omega) \in K \times K$, $(\bar{x}, \bar{y}) \in F(z, \omega)$. Hence there exists $(\bar{x}, \bar{y}) \in K \times K$ such that $\sum_{i=1}^{p} \xi_i f_i((\bar{x}, \bar{y}), (z, \omega)) \geqq 0 \quad \forall (z, \omega) \in K \times K$. □

We can give relations among the solution sets for vector equilibrium problems as follows.

Theorem 2.2.2. The following relations hold:

$$\bigcup_{\xi \in \overset{o}{S}_p} sol(\text{SEP})_\xi \subset sol(\text{VEP}) \subset sol(\text{WVEP}) = \bigcup_{\xi \in S_p} sol(\text{SEP})_\xi.$$

Proof. We want to prove that $\bigcup_{\xi \in \overset{o}{S}_p} sol(\text{SEP})_\xi \subset sol(\text{VEP})$.

Let $(\bar{x}, \bar{y}) \in \bigcup_{\xi \in \overset{o}{S}_p} sol(\text{SEP})_\xi$. Then there exist $\xi_i > 0$, $i = 1, \cdots, p$ and

$$\sum_{i=1}^{p} \xi_i f_i((\bar{x}, \bar{y}), (z, \omega)) \geqq 0, \quad \forall (z, \omega) \in K \times K.$$

Assume to the contrary that $(\bar{x}, \bar{y}) \notin sol(\text{VEP})$. Then there exists $(z^*, \omega^*) \in K \times K$ such that

$$f((\bar{x}, \bar{y}), (z^*, \omega^*)) \leq 0.$$

Since $\xi_i > 0$, $i = 1, \cdots, p$, we have

$$\sum_{i=1}^{p} \xi_i f_i((\bar{x}, \bar{y}), (z^*, \omega^*)) < 0.$$

This is a contradiction. Thus $(\bar{x}, \bar{y}) \in sol(\text{VEP})$.

It is clear that $sol(\text{VEP}) \subset sol(\text{WVEP})$.

Now we will prove that $sol(\text{WVEP}) = \bigcup_{\xi \in S_p} sol(\text{SEP})_\xi$. Let $(\bar{x}, \bar{y}) \in sol(\text{WVEP})$. Then $(\bar{x}, \bar{y}) \in K \times K$ and

$$f((\bar{x}, \bar{y}), (z, \omega)) \not< 0 \quad \forall (z, \omega) \in K \times K.$$

Since $f((\bar{x}, \bar{y}), (\bar{x}, \bar{y})) = 0$, $(\bar{x}, \bar{y})$ is a weakly efficient solution of the following vector optimization problem (WVP):

$$\begin{array}{lll} \text{(WVP)} & \text{Minimize} & (f_1((\bar{x}, \bar{y}), (z, \omega)), \cdots, f_p((\bar{x}, \bar{y}), (z, \omega))) \\ & \text{subject to} & (z, \omega) \in K \times K. \end{array}$$

Since K is a convex and compact subset of $\mathbb{R}^n$, by Lemma 2.1.1, there exists $\xi \in S_p$ such that $(\bar{x}, \bar{y}) \in K \times K$ is an optimal solution of the following optimization problem $(\text{SEP})_\xi$:

$$\begin{array}{lll} (\text{SEP})_\xi & \text{Minimize} & \displaystyle\sum_{i=1}^{p} \xi_i f_i((\bar{x}, \bar{y}), (z, \omega)) \\ & \text{subject to} & (z, \omega) \in K \times K. \end{array}$$

Thus $(\bar{x}, \bar{y}) \in sol(\text{SEP})_\xi$.

Let $(\bar{x}, \bar{y}) \in \bigcup_{\xi \in S_p} sol(\text{SEP})_\xi$. Then there exists $\xi_i \geqq 0$, $\sum_{i=1}^{p} \xi_i = 1$ and

$$\sum_{i=1}^{p} \xi_i f_i((\bar{x}, \bar{y}), (z, \omega)) \geqq 0, \quad \forall (z, \omega) \in K \times K.$$

Assume to the contrary that $(\bar{x}, \bar{y}) \notin sol(\text{WVEP})$. Then there exists $(z^*, \omega^*) \in K \times K$ such that

$$f((\bar{x}, \bar{y}), (z^*, \omega^*)) < 0.$$

Since $\xi_i \geqq 0$, $\sum_{i=1}^{p} \xi_i = 1$, we have

$$\sum_{i=1}^{p} \xi_i f_i((\bar{x}, \bar{y}), (z^*, \omega^*)) < 0.$$

This is a contradiction. Thus $(\bar{x}, \bar{y}) \in sol(\text{WVEP})$. □

From Theorems 2.2.1 and 2.2.2, we can obtain the following existence theorems for (VEP) and (WVEP):

Theorem 2.2.3. Let K be a convex and compact subset of $\mathbb{R}^n$. Let $f : (K \times K) \times (K \times K) \to \mathbb{R}^n$ be a continuous function. Assume that for any $(x, y) \in K \times K$, $f((x, y), (x, y)) = 0$. Then (VEP) and (WVEP) have solutions.

Now we give relations among the solution sets of vector saddle point problems and vector equilibrium problems and establish existence theorems of the problems.

Theorem 2.2.4. The following are true:

(1) $sol(\text{SEP})_\xi = sol(\text{SSP})_\xi$ for $\xi \in S_p$.

(2) $sol(\text{VEP}) \subset sol(\text{VSP})$.

(3) $sol(\text{WVEP}) \subset sol(\text{WVSP})$.

(4) (VSP) and (WVSP) have solutions.

Proof. (1) Let $\xi \in S_p$. Let $(\bar{x}, \bar{y}) \in sol(\text{SEP})_\xi$. Then $\sum_{i=1}^p \xi_i \Big[L_i(\bar{x}, \omega) - L_i(z, \bar{y}) \Big] \geqq 0 \quad \forall (z, \omega) \in K \times K$. Letting $z = \bar{x}$, $\sum_{i=1}^p \xi_i L_i(\bar{x}, \omega) \geqq \sum_{i=1}^p \xi_i L_i(\bar{x}, \bar{y})$ $\forall \omega \in K$. Letting $\omega = \bar{y}$, $\sum_{i=1}^p \xi_i L_i(\bar{x}, \bar{y}) \geqq \sum_{i=1}^p \xi_i L_i(z, \bar{y})$ $\forall z \in K$. Thus $(\bar{x}, \bar{y}) \in sol(\text{SSP})_\xi$. Hence $sol(\text{SEP})_\xi \subset sol(\text{SSP})_\xi$.

Let $(\bar{x}, \bar{y}) \in sol(\text{SSP})_\xi$. Then $\sum_{i=1}^p \xi_i \Big[L_i(\bar{x}, y) - L_i(x, \bar{y}) \Big] \geqq 0 \quad \forall (x, y) \in K \times K$, i.e., $\sum_{i=1}^p \xi_i f_i((\bar{x}, \bar{y}), (x, y)) \geqq 0 \quad \forall (x, y) \in K \times K$. Thus $(\bar{x}, \bar{y}) \in sol(\text{SEP})_\xi$. Hence $sol(\text{SSP})_\xi \subset sol(\text{SEP})_\xi$.

(2) Let $(\bar{x}, \bar{y}) \in sol(\text{VEP})$. Then $f((\bar{x}, \bar{y}), (z, \omega)) = L(\bar{x}, \omega) - L(z, \bar{y}) \not\leq 0 \quad \forall (z, \omega) \in K \times K$. Letting $z = \bar{x}$, $L(\bar{x}, \omega) - L(\bar{x}, \bar{y}) \not\leq 0 \quad \forall \omega \in K$. Letting $\omega = \bar{y}$, $L(\bar{x}, \bar{y}) - L(x, \bar{y}) \not\leq 0 \quad \forall x \in K$. Thus $L(\bar{x}, y) \not\leq L(\bar{x}, \bar{y}) \not\leq L(x, \bar{y}) \quad \forall (x, y) \in K \times K$. Therefore $(\bar{x}, \bar{y}) \in sol(\text{VSP})$.

(3) We can prove the inclusion by method similar to one in (2).

(4) From Theorem 2.2.3, it is clear. $\square$

We give relations among the solution sets of vector equilibrium problems and vector saddle point problems and establish existence theorems of the problems.

Let $L : K \times K \to \mathbb{R}^p$ be a function and let K be a convex and compact subset of $\mathbb{R}^n$. Then the following holds:

(1) $\displaystyle\bigcup_{\xi \in S_p} sol(\text{SSP})_\xi \subset sol(\text{VSP}) \subset sol(\text{WVSP})$.

(2) (VSP) and (WVSP) have solutions.

From Theorem 2.2.4, we can easily get the following theorem.

Theorem 2.2.5. Let $B_i,\ i = 1, \cdots, p$, be $n \times n$ skew symmetric matrices. Then the following holds:

(1) sol(SVMG) $\subset$ sol(EVMG) $\subset$ sol(WEVMG).

(2) (EVMG) and (WEVMG) have solutions.

Chapter 3
Vector Matrix Game

3.1. Introduction

A vector matrix game, which is a vector version of the usual matrix game, consists of skew symmetric matrices and vector ordering.

A matrix game is defined by B of $m \times n$ real matrix together with the Cartesian product $S_n \times S_m$ of all n-dimensional probability vectors S_n and all m-dimensional probability vectors S_m, that is, $S_n := \{x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n : x_i \geqq 0, \sum_{i=1}^n x_i = 1\}$.

A point $(\bar{x}, \bar{y})$ in $S_n \times S_m$ is an equilibrium point of matrix game B if $x^T B\bar{y} \leqq \bar{x}^T B\bar{y} \leqq \bar{x}^T By$ for all $x \in S_n$ and all $y \in S_m$. When $(\bar{x}, \bar{y}) \in S_n \times S_m$ is the equilibrium point, $v := \bar{x}^T B\bar{y}$ is called the value of the game.

If $n = m$ and B is skew symmetric, then $(\bar{x}, \bar{y}) \in S_n \times S_n$ is an equilibrium point of matrix game B if and only if $B\bar{x} \leqq 0$ and $B\bar{y} \leqq 0$. In this case, $\bar{x} \in S_n$ is called a solution of matrix game B if $B\bar{x} \leqq 0$.

Consider the following linear programming problem (LP) together with its dual (LD) as follows:

(LP) Minimize $c^T x$, subject to $Ax \geqq b,\ x \geqq 0$,

(LD) Maximize $b^T y$, subject to $A^T y \leqq c,\ y \geqq 0$,

where $c \in \mathbb{R}^n, x \in \mathbb{R}^n, b \in \mathbb{R}^m, y \in \mathbb{R}^m$ and $A = [a_{ij}]$ is a $m \times n$ real matrix.

Now consider the matrix game associated with the following $(n+m+1)\times(n+m+1)$ skew symmetric matrix B:

$$B = \begin{bmatrix} 0 & A^T & -c \\ -A & 0 & b \\ c^T & -b^T & 0 \end{bmatrix}.$$

The following results due to Dantzig [20] are well known: Theorem 3.1.1 and 3.1.2 give complete equivalence between linear programming duality and the matrix game B.

Theorem 3.1.1. Let $\bar{x}$ and $\bar{y}$ be optimal solutions to (LP) and (LD) respectively. Let $z^* = 1/(1+\sum_j \bar{x}_j + \sum_i \bar{y}_i)$, $x^* = z^*\bar{x}$, $y^* = z^*\bar{y}$. Then (x^*, y^*, z^*) is a solution of the matrix game B.

Theorem 3.1.2. Let (x^*, y^*, z^*) be a solution of the matrix game B with $z^* > 0$. Let $\bar{x}_j = x^*_j/z^*$, $\bar{y}_i = y^*_i/z^*$. Then $\bar{x}$ and $\bar{y}$ are optimal solutions to (LP) and (LD) respectively.

Many authors [15, 17, 52] have extended Theorems 3.1.1 and 3.1.2 to several kinds of (scalar) optimization problems.

In this section, we characterize solutions of vector matrix game, which was defined Definition 2.1.3. In particular, we calculate six kinds of solutions for vector matrix game with two 2×2 matrices. Furthermore, we formulate a dual problem for a linear vector optimization problem, give a weak duality result for the dual problem and establish equivalent relations between the dual problem and the corresponding vector matrix game. Moreover, we give a numerical example for showing such equivalent relations.

3.2. Characterizing Solutions of Vector Matrix Game

Consider a linear vector optimization problem:

$$\text{(LVP)} \quad \text{Minimize} \quad (c_1^T x, \cdots, c_p^T x)$$
$$\text{subject to} \quad x \in X,$$

where $X = \{x \in \mathbb{R}^n \mid Ax \geqq b,\ x \geqq 0\}$, $c_i \in \mathbb{R}^n$, $i = 1, \cdots, p$, $b \in \mathbb{R}^m$ and $A = [a_{ij}]$ is a $m \times n$ real matrix.

Definition 3.2.1. A point $\bar{x} \in X$ is said to be an efficient solution for (LVP) if there exists no other feasible point $x \in X$ such that $(c_1^T x, \cdots, c_p^T x) \leq (c_1^T \bar{x}, \cdots, c_p^T \bar{x})$.

Now we give well-known propositions which are needed in proving the following lemmas.

Proposition 3.2.1 [32]. Every efficient solution of (LVP) is properly efficient.

By Proposition 3.2.1 and results in [30], we have the following proposition.

Proposition 3.2.2 [7]. A point $\bar{x} \in X$ is an efficient solution of (LVP) if and only if there exists $\lambda \in \overset{o}{S}_p$ such that $\bar{x}$ is a solution of the following scalar optimization problem:

$$\begin{array}{ll} \text{Minimize} & \displaystyle\sum_{i=1}^{p} \lambda_i c_i^T x \\ \text{subject to} & x \in X. \end{array}$$

Proposition 3.2.3 [7]. A point $\bar{x} \in X$ is a weakly efficient solution of (LVP) if and only if there exists $\lambda \in S_p$ such that $\bar{x}$ is a solution of the following scalar optimization problem:

$$\begin{array}{ll} \text{Minimize} & \displaystyle\sum_{i=1}^{p} \lambda_i c_i^T x \\ \text{subject to} & x \in X. \end{array}$$

Using Propositions 3.2.2 and 3.2.3, we can give the following lemmas involving characterizations of vector solution and weakly vector solution of vector matrix game.

Lemma 3.2.1. Let $B_i,\ i = 1, \cdots, p$ be $n \times n$ skew symmetric matrices. Then $\bar{y} \in S_n$ is a vector solution of vector matrix game $(B_1, \cdots, B_p)$ if and only if there exists $\xi \in \overset{o}{S}_p$ such that $(\sum_{i=1}^{p} \xi_i B_i)\bar{y} \leqq 0$.

Proof. A point $\bar{y} \in S_n$ is a vector solution of vector matrix game $(B_1, \cdots, B_p)$.

$\Longleftrightarrow (y^T B_1 \bar{y}, \cdots, y^T B_p \bar{y}) \ngeq 0, \forall y \in S_n.$

$\Longleftrightarrow$ A point $\bar{y} \in S_n$ is an efficient solution of the following linear vector optimization problem:

$$\begin{aligned} \text{Maximize} \quad & (y^T B_1 \bar{y}, \cdots, y^T B_p \bar{y}) \\ \text{subject to} \quad & y \in S_n. \end{aligned}$$

$\Longleftrightarrow$ (by Proposition 3.2.2) there exists $\xi \in \overset{o}{S}_p$ such that $\bar{y}$ is optimal for the following linear scalar optimization:

$$\begin{aligned} \text{Maximize} \quad & y^T (\sum_{i=1}^{p} \xi_i B_i) \bar{y} \\ \text{subject to} \quad & y \in S_n. \end{aligned}$$

$\Longleftrightarrow$ there exists $\xi \in \overset{o}{S}_p$ such that $\forall y \in S_n, y^T(\sum_{i=1}^{p} \xi_i B_i)\bar{y} \leqq 0$.

$\Longleftrightarrow$ there exists $\xi \in \overset{o}{S}_p$ such that $(\sum_{i=1}^{p} \xi_i B_i)\bar{y} \leqq 0$. □

Using Proposition 3.2.3 and following the proof of Lemma 3.2.1, we can obtain the following lemma.

Lemma 3.2.2. Let $B_i, \ i = 1, \cdots, p$ be a $n \times n$ skew symmetric matrices. Then $\bar{y} \in S_n$ is a weakly vector solution of vector matrix game $(B_1, \cdots, B_p)$ if and only if there exists $\xi \in S_p$ such that $(\sum_{i=1}^{p} \xi_i B_i)\bar{y} \leqq 0$.

Using the skew-symmetry of the matrices B_i's, we can easily obtain the following lemmas showing characterizations of a scalarizing solution and a weakly scalarizing solution of the vector matrix game.

Lemma 3.2.3. Let B_i, $i = 1, \cdots, p$ be $n \times n$ skew symmetric matrices. Then $(\bar{x}, \bar{y}) \in S_n \times S_n$ is a scalarizing solution of vector matrix game $(B_1, \cdots, B_p)$ if and only if there exists $\xi \in \overset{o}{S}_p$ such that $(\sum_{i=1}^{p} \xi_i B_i)\bar{y} \leqq 0$ and $(\sum_{i=1}^{p} \xi_i B_i)\bar{x} \leqq 0$.

Proof. $(\Rightarrow)$ Let $(\bar{x}, \bar{y}) \in S_n \times S_n$ be a scalarizing solution of vector matrix game $(B_1, \cdots, B_p)$. Then there exists $\xi \in \overset{o}{S}_p$ such that

$$x^T(\sum_{i=1}^{p} \xi_i B_i)\bar{y} \leqq \bar{x}^T(\sum_{i=1}^{p} \xi_i B_i)\bar{y} \leqq \bar{x}^T(\sum_{i=1}^{p} \xi_i B_i)y, \tag{3.1}$$

for any $x, y \in S_n$. Replaying x by $\bar{y}$ and y by $\bar{x}$ in (3.1), $\bar{y}^T(\sum_{i=1}^{p} \xi_i B_i)\bar{y} \leqq \bar{x}^T(\sum_{i=1}^{p} \xi_i B_i)\bar{y} \leqq \bar{x}^T(\sum_{i=1}^{p} \xi_i B_i)\bar{x}$. Since $\sum_{i=1}^{p} \xi_i B_i$ is skew symmetric, $\bar{x}^T(\sum_{i=1}^{p} \xi_i B_i)\bar{x} = \bar{y}^T(\sum_{i=1}^{p} \xi_i B_i)\bar{y} = 0$. Therefore $\bar{x}^T(\sum_{i=1}^{p} \xi_i B_i)\bar{y} = 0$. From (3.1), $x^T(\sum_{i=1}^{p} \xi_i B_i)\bar{y} \leqq 0$ for any $x \in S_n$ and $\bar{x}^T(\sum_{i=1}^{p} \xi_i B_i)y \geqq 0$ for any $y \in S_n$. Thus for any $z \in \mathbb{R}^n_+ \setminus \{0\}$, $z^T(\sum_{i=1}^{p} \xi_i B_i)\bar{y} \leqq 0$ and hence $-(\sum_{i=1}^{p} \xi_i B_i)\bar{y} \in \mathbb{R}^n_+$. From (3.1), $0 \leqq \bar{x}^T(\sum_{i=1}^{p} \xi_i B_i)y = y^T(\sum_{i=1}^{p} \xi_i B_i)\bar{x} = -y^T(\sum_{i=1}^{p} \xi_i B_i)\bar{x} = y^T(\sum_{i=1}^{p} \xi_i B_i)(-\bar{x})$ for any $y \in S_n$. Thus $(\sum_{i=1}^{p} \xi_i B_i)(-\bar{x}) \in \mathbb{R}^n_+$. Consequently, there exists $\xi \in \overset{o}{S}_p$ such that $(\sum_{i=1}^{p} \xi_i B_i)\bar{y} \leqq 0$ and $(\sum_{i=1}^{p} \xi_i B_i)\bar{x} \leqq 0$.

$(\Leftarrow)$ Suppose that there exists $\xi \in \overset{o}{S}_p$ such that $(\sum_{i=1}^{p} \xi_i B_i)\bar{x} \leqq 0$ and $(\sum_{i=1}^{p} \xi_i B_i)\bar{y} \leqq 0$. Then for any $y \in S_n$, $\bar{x}^T(\sum_{i=1}^{p} \xi_i B_i)y = y^T(\sum_{i=1}^{p} \xi_i B_i)\bar{x} =$

$-y^T(\sum_{i=1}^p \xi_i B_i)\bar{x} \geqq 0$. Hence for any $y \in S_n$,

$$\bar{x}^T(\sum_{i=1}^p \xi_i B_i)y \geqq 0. \tag{3.2}$$

Clearly, for any $x \in S_n$,

$$x^T(\sum_{i=1}^p \xi_i B_i)\bar{y} \leqq 0. \tag{3.3}$$

From (3.2) and (3.3), $\bar{x}^T(\sum_{i=1}^p \xi_i B_i)\bar{y} = 0$. Therefore it follows from (3.2) and (3.3), $x^T(\sum_{i=1}^p \xi_i B_i)\bar{y} \leqq \bar{x}^T(\sum_{i=1}^p \xi_i B_i)\bar{y} \leqq \bar{x}^T(\sum_{i=1}^p \xi_i B_i)y$ for any $x,\ y \in S_n$. Thus $(\bar{x}, \bar{y})$ is a scalarizing solution of the vector matrix game $B_i,\ i = 1, \cdots, p$. □

Lemma 3.2.4. Let $B_i,\ i = 1, \cdots, p$ be $n \times n$ skew symmetric matrices. Then $(\bar{x}, \bar{y}) \in S_n \times S_n$ is a weakly scalarizing solution of vector matrix game $(B_1, \cdots, B_p)$ if and only if there exists $\xi \in S_p$ such that $(\sum_{i=1}^p \xi_i B_i)\bar{y} \leqq 0$, and $(\sum_{i=1}^p \xi_i B_i)\bar{x} \leqq 0$.

To characterize efficient solution and weakly efficient solution of vector matrix game, we consider the normal cone to a convex set.

Lemma 3.2.5 [54]. Let $h : \mathbb{R}^n \to \mathbb{R}$ be a differentiable convex function, and let $H \subseteq \mathbb{R}^n$ be a convex set. Then $\bar{x} \in H$ is a solution to

$$\inf_{x \in H} h(\bar{x})$$

if and only if

$$0 \in \nabla h(\bar{x}) + N_H(\bar{x}),$$

where $\nabla h(\bar{x})$ is the gradient of h at $\bar{x}$.

Proposition 3.2.4 [53]. Let $H = H_1 \cap H_2$, where $H_i,\ i = 1, 2$ are polyhedral sets. If $H_1 \cap \overset{o}{H}_2 \neq \emptyset$, where $\overset{o}{H}_2$ is the relative interior of H_2, then for any $x \in H$,

$$N_H(x) = N_{H_1}(x) + N_{H_2}(x).$$

Lemma 3.2.6. Let $H = \{x \in \mathbb{R}^n : Ax = b\}$ where $A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear operator and b is a vector in $\mathbb{R}^m$. At any $x \in H$, we have $N_H(x) = \{A^T y : y \in \mathbb{R}^m\}$.

Using Propositions 3.2.2 and 3.2.3 and Lemma 3.2.5, we can give the following lemmas involving characterizations of an efficient solution and a weakly efficient solution of vector matrix game.

Lemma 3.2.7. Let $B_i,\ i = 1, \cdots, p$ be $n \times n$ skew symmetric matrices. Then $(\bar{x}, \bar{y}) \in S_n \times S_n$ is an efficient solution of vector matrix game $(B_1, \cdots, B_p)$ if and only if there exist $\lambda \in \overset{o}{S}_p$ and $\mu \in \overset{o}{S}_p$ such that $(\sum_{i=1}^p \lambda_i B_i)\bar{y} \in N_{S_n}(\bar{x})$ and $(\sum_{i=1}^p \mu_i B_i)\bar{x} \in N_{S_n}(\bar{y})$.

Proof. A point $(\bar{x}, \bar{y}) \in S_n \times S_n$ is an efficient solution of vector matrix game $B_i, i = 1, \cdots, p$.

$\Longleftrightarrow$ $((-B_1\bar{y})^T x, \cdots, (-B_p\bar{y})^T x) \nleq ((-B_1\bar{y})^T \bar{x}, \cdots, (-B_p\bar{y})^T \bar{x})$ for any $x \in S_n$ and $((B_1^T\bar{x})^T y, \cdots, (B_p^T\bar{x})^T y) \nleq ((B_1^T\bar{x})^T \bar{y}, \cdots, (B_p^T\bar{x})^T \bar{y})$ for any $y \in S_n$.

$\Longleftrightarrow$ (by Proposition 3.2.2) there exists $\lambda \in \overset{o}{S}_p$ such that $\sum_{i=1}^{p} \lambda_i(-B_i\bar{y})^T x \geqq \sum_{i=1}^{p} \lambda_i(-B_i\bar{y})^T \bar{x}$ for any $x \in S_n$ and there exists $\mu \in \overset{o}{S}_p$ such that $\sum_{i=1}^{p} \mu_i(B_i^T\bar{x})^T y \geqq \sum_{i=1}^{p} \mu_i(B_i^T\bar{x})^T \bar{y}$ for any $y \in S_n$

$\Longleftrightarrow$ (by Lemma 3.2.5) there exists $\lambda \in \overset{o}{S}_p$ such that $(\sum_{i=1}^{p} \lambda_i B_i)\bar{y} \in N_{S_n}(\bar{x})$ and there exists $\mu \in \overset{o}{S}_p$ such that $(-\sum_{i=1}^{p} \mu_i B_i^T)\bar{x} \in N_{S_n}(\bar{y})$.

$\Longleftrightarrow$ there exists $\lambda \in \overset{o}{S}_p$ such that $(\sum_{i=1}^{p} \lambda_i B_i)\bar{y} \in N_{S_n}(\bar{x})$ and there exists $\mu \in \overset{o}{S}_p$ such that $(\sum_{i=1}^{p} \mu_i B_i)\bar{x} \in N_{S_n}(\bar{y})$. □

Lemma 3.2.8. Let B_i, $i = 1, \cdots, p$ be an $n \times n$ skew symmetric matrix. Then $(\bar{x}, \bar{y}) \in S_n \times S_n$ is a weakly efficient solution of vector matrix game $(B_1, \cdots, B_p)$ if and only if there exist $\lambda \in S_p$ and $\mu \in S_p$ such that $(\sum_{i=1}^{p} \lambda_i B_i)\bar{y} \in N_{S_n}(\bar{x})$ and $(\sum_{i=1}^{p} \mu_i B_i)\bar{x} \in N_{S_n}(\bar{y})$.

3.3. Characterizations of Solutions of Vector Matrix Game with Two 2×2 Matrices

In this section we characterize vector matrix game with two 2×2 matrices. Let

$$B_1 = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}.$$

Then we calculate the set of all the vector solutions, the set of all the weakly vector solutions, the set of all the efficient solutions, the set of all the weakly efficient solutions, the set of all the scalarizing solutions and the set of all the weakly scalarizing solutions for vector matrix game with B_1 and B_2 matrices.

(1) A point $\bar{x} \in S_2$ is a vector solution of vector matrix game (B_1, B_2) if

$$\left((x_1, x_2)\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}, (x_1, x_2)\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}\right)$$

$$\ngeq \left((\bar{x}_1, \bar{x}_2)\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}, (\bar{x}_1, \bar{x}_2)\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}\right)$$

$$\ngeq \left((\bar{x}_1, \bar{x}_2)\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, (\bar{x}_1, \bar{x}_2)\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)$$

$$\begin{aligned} \Longleftrightarrow \quad & (-ax_2\bar{x}_1 + ax_1\bar{x}_2,\ -bx_2\bar{x}_1 + bx_1\bar{x}_2) \\ & \ngeq (0,0) \ngeq (-a\bar{x}_2x_1 + a\bar{x}_1x_2,\ -b\bar{x}_2x_1 + b\bar{x}_1x_2) \\ \Longleftrightarrow \quad & (a\bar{x}_1x_2 - ax_1\bar{x}_2,\ b\bar{x}_1x_2 - bx_1\bar{x}_2) \nleq (0,0) \\ \Longleftrightarrow \quad & (\bar{x}_1x_2 - x_1\bar{x}_2)(a,b) \nleq (0,0) \end{aligned}$$

Thus the vector solution set of vector matrix game (B_1, B_2) is as follows;

(i) if $a > 0,\ b > 0$; the solution set is $\{(1,0)\}$.

(ii) if $a = 0,\ b > 0$; the solution set is $\{(1,0)\}$.

(iii) if $a > 0,\ b = 0$; the solution set is $\{(1,0)\}$.

(iv) if $a < 0,\ b < 0$; the solution set is $\{(0,1)\}$.

(v) if $a = 0,\ b < 0$; the solution set is $\{(0,1)\}$.

(vi) if $a < 0,\ b = 0$; the solution set is $\{(0,1)\}$.

(vii) if $a > 0,\ b < 0$; the solution set is $\{(x,y) : (x,y) \in S_2\}$.

(viii) if $a < 0,\ b > 0$; the solution set is $\{(x,y) : (x,y) \in S_2\}$.

(ix) if $a = 0,\ b = 0$; the solution set is $\{(x,y) : (x,y) \in S_2\}$.

(2) A point $\bar{x} \in S_2$ is a weakly vector solution of vector matrix game (B_1, B_2) if

$$\left((x_1,x_2)\begin{pmatrix}0 & a\\ -a & 0\end{pmatrix}\begin{pmatrix}\bar{x}_1\\ \bar{x}_2\end{pmatrix}, (x_1,x_2)\begin{pmatrix}0 & b\\ -b & 0\end{pmatrix}\begin{pmatrix}\bar{x}_1\\ \bar{x}_2\end{pmatrix}\right)$$

$$\not> \left((\bar{x}_1,\bar{x}_2)\begin{pmatrix}0 & a\\ -a & 0\end{pmatrix}\begin{pmatrix}\bar{x}_1\\ \bar{x}_2\end{pmatrix}, (\bar{x}_1,\bar{x}_2)\begin{pmatrix}0 & b\\ -b & 0\end{pmatrix}\begin{pmatrix}\bar{x}_1\\ \bar{x}_2\end{pmatrix}\right)$$

$$\not> \left((\bar{x}_1,\bar{x}_2)\begin{pmatrix}0 & a\\ -a & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}, (\bar{x}_1,\bar{x}_2)\begin{pmatrix}0 & b\\ -b & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}\right)$$

$$\begin{aligned}
\iff\quad & (-ax_2\bar{x}_1 + ax_1\bar{x}_2,\ -bx_2\bar{x}_1 + bx_1\bar{x}_2)\\
& \not> (0,0) \not> (-a\bar{x}_2x_1 + a\bar{x}_1x_2,\ -b\bar{x}_2x_1 + b\bar{x}_1x_2)\\
\iff\quad & (a\bar{x}_1x_2 - ax_1\bar{x}_2,\ b\bar{x}_1x_2 - bx_1\bar{x}_2) \not< (0,0)\\
\iff\quad & (\bar{x}_1x_2 - x_1\bar{x}_2)(a,b) \not< (0,0)
\end{aligned}$$

Thus the weakly vector solution set of vector matrix game (B_1, B_2) is as follows;

(i) if $a > 0,\ b > 0$; the solution set is $\{(1,0)\}$.

(ii) if $a = 0,\ b > 0$; the solution set is $\{(x, y) : (x, y) \in S_2\}$.

(iii) if $a > 0,\ b = 0$; the solution set is $\{(x, y) : (x, y) \in S_2\}$.

(iv) if $a < 0,\ b < 0$; the solution set is $\{(0, 1)\}$.

(v) if $a = 0,\ b < 0$; the solution set is $\{(x, y) : (x, y) \in S_2\}$.

(vi) if $a < 0,\ b = 0$; the solution set is $\{(x, y) : (x, y) \in S_2\}$.

(vii) if $a > 0,\ b < 0$; the solution set is $\{(x, y) : (x, y) \in S_2\}$.

(viii) if $a < 0,\ b > 0$; the solution set is $\{(x, y) : (x, y) \in S_2\}$.

(ix) if $a = 0,\ b = 0$; the solution set is $\{(x, y) : (x, y) \in S_2\}$.

(3) A point $(\bar{x}, \bar{y}) \in S_2 \times S_2$ is a scalarizing solution of vector matrix game (B_1, B_2) if there exists $\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_1 + \lambda_2 = 1$ such that

$$(x_1, x_2)\left(\lambda_1\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} + \lambda_2\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}\right)\begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}$$

$$\leqq (\bar{x}_1, \bar{x}_2)\left(\lambda_1\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} + \lambda_2\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}\right)\begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}$$

$$\leqq (\bar{x}_1, \bar{x}_2)\left(\lambda_1\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} + \lambda_2\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}\right)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\iff \quad -(\lambda_1 a + \lambda_2 b)x_2\bar{y}_1 + (\lambda_1 a + \lambda_2 b)x_1\bar{y}_2 \leqq -(\lambda_1 a + \lambda_2 b)\bar{x}_2\bar{y}_1 + (\lambda_1 a + \lambda_2 b)\bar{x}_1\bar{y}_2 \leqq -(\lambda_1 a + \lambda_2 b)\bar{x}_2 y_1 + (\lambda_1 a + \lambda_2 b)\bar{x}_1 y_2$$

$$\iff \begin{cases} -(\lambda_1 a + \lambda_2 b)(x_2\bar{y}_1 - x_1\bar{y}_2 - \bar{x}_2\bar{y}_1 + \bar{x}_1\bar{y}_2) \leqq 0 \\ -(\lambda_1 a + \lambda_2 b)(\bar{x}_2\bar{y}_1 - \bar{x}_1\bar{y}_2 - \bar{x}_2 y_1 + \bar{x}_1 y_2) \leqq 0 \end{cases}$$

$$\iff \begin{cases} -(\lambda_1 a + \lambda_2 b)\left((x_2 - \bar{x}_2)\bar{y}_1 - (x_1 - \bar{x}_1)\bar{y}_2\right) \leqq 0 \\ -(\lambda_1 a + \lambda_2 b)\left((y_2 - \bar{y}_2)\bar{x}_1 - (y_1 - \bar{y}_1)\bar{x}_2\right) \leqq 0 \end{cases}$$

Thus the scalarizing solution set of vector matrix game (B_1, B_2) is as follows;

(i) if $a > 0,\ b > 0$; the solution set is $\{(1, 0, 1, 0)\}$.

(ii) if $a = 0,\ b > 0$; the solution set is $\{(1, 0, 1, 0)\}$.

(iii) if $a > 0,\ b = 0$; the solution set is $\{(1, 0, 1, 0)\}$.

(iv) if $a < 0,\ b < 0$; the solution set is $\{(0, 1, 0, 1)\}$.

(v) if $a = 0,\ b < 0$; the solution set is $\{(0, 1, 0, 1)\}$.

(vi) if $a < 0,\ b = 0$; the solution set is $\{(0, 1, 0, 1)\}$.

(vii) if $a > 0,\ b < 0$; the solution set is $\{(1, 0, 1, 0), (0, 1, 0, 1)\}$.

(viii) if $a < 0,\ b > 0$; the solution set is $\{(1, 0, 1, 0), (0, 1, 0, 1)\}$.

(ix) if $a = 0,\ b = 0$; the solution set is $S_2 \times S_2$.

(4) A point $(\bar{x}, \bar{y}) \in S_2 \times S_2$ is a weakly scalarizing solution of vector matrix game (B_1, B_2) if there exist $\lambda_1 \geqq 0,\ \lambda_2 \geqq 0,\ \lambda_1 + \lambda_2 = 1$ such that

$$(x_1, x_2)\left(\lambda_1\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} + \lambda_2\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}\right)\begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}$$

$$\leqq (\bar{x}_1, \bar{x}_2)\left(\lambda_1\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} + \lambda_2\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}\right)\begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}$$

$$\leqq (\bar{x}_1, \bar{x}_2)\left(\lambda_1\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} + \lambda_2\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}\right)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\begin{aligned} \Longleftrightarrow \quad & -(\lambda_1 a + \lambda_2 b)x_2\bar{y}_1 + (\lambda_1 a + \lambda_2 b)x_1\bar{y}_2 \\ & \leqq -(\lambda_1 a + \lambda_2 b)\bar{x}_2\bar{y}_1 + (\lambda_1 a + \lambda_2 b)\bar{x}_1\bar{y}_2 \\ & \leqq -(\lambda_1 a + \lambda_2 b)\bar{x}_2 y_1 + (\lambda_1 a + \lambda_2 b)\bar{x}_1 y_2 \end{aligned}$$

$$\Longleftrightarrow \begin{cases} -(\lambda_1 a + \lambda_2 b)(x_2\bar{y}_1 - x_1\bar{y}_2 - \bar{x}_2\bar{y}_1 + \bar{x}_1\bar{y}_2) \leqq 0 \\ -(\lambda_1 a + \lambda_2 b)(\bar{x}_2\bar{y}_1 - \bar{x}_1\bar{y}_2 - \bar{x}_2 y_1 + \bar{x}_1 y_2) \leqq 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} -(\lambda_1 a + \lambda_2 b)\left((x_2 - \bar{x}_2)\bar{y}_1 - (x_1 - \bar{x}_1)\bar{y}_2\right) \leqq 0 \\ -(\lambda_1 a + \lambda_2 b)\left((y_2 - \bar{y}_2)\bar{x}_1 - (y_1 - \bar{y}_1)\bar{x}_2\right) \leqq 0 \end{cases}$$

Thus the weakly scalarizing solution set of vector matrix game (B_1, B_2) is as follows;

(I) The case of $\lambda_1 = 1,\ \lambda_2 = 0$:

(i) if $a > 0$; the solution set is $\{(1, 0, 1, 0)\}$.

(ii) if $a < 0$; the solution set is $\{(0, 1, 0, 1)\}$.

(iii) if $a = 0$; the solution set is $S_2 \times S_2$.

(II) The case of $\lambda_1 = 0,\ \lambda_2 = 1$:

(i) if $b > 0$; the solution set is $\{(1, 0, 1, 0)\}$.

(ii) if $b < 0$; the solution set is $\{(0, 1, 0, 1)\}$.

(iii) if $b = 0$; the solution set is $S_2 \times S_2$.

(III) The case of $\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_1 + \lambda_2 = 1$: For the case, the solution sets are same as (3).

(5) A point $(\bar{x}, \bar{y}) \in S_2 \times S_2$ is an efficient solution of vector matrix game (B_1, B_2) if

$$\left((x_1, x_2)\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}\begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}, (x_1, x_2)\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}\begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}\right)$$

$$\ngeq \left((\bar{x}_1, \bar{x}_2)\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}\begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}, (\bar{x}_1, \bar{x}_2)\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}\begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}\right)$$

$$\ngeq \left((\bar{x}_1, \bar{x}_2)\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, (\bar{x}_1, \bar{x}_2)\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right)$$

$$\Longleftrightarrow \quad (-ax_2\bar{y}_1 + ax_1\bar{y}_2,\ -bx_2\bar{y}_1 + bx_1\bar{y}_2)$$

$$\ngeq (-a\bar{x}_2\bar{y}_1 + a\bar{x}_1\bar{y}_2,\ -b\bar{x}_2\bar{y}_1 + b\bar{x}_1\bar{y}_2)$$

$$\ngeq (-a\bar{x}_2 y_1 + a\bar{x}_1 y_2,\ -b\bar{x}_2 y_1 + b\bar{x}_1 y_2)$$

$$\iff \begin{cases} (-a\bar{y}_1(\bar{x}_2 - x_2) + a\bar{y}_2(\bar{x}_1 - x_1),\ -b\bar{y}_1(\bar{x}_2 - x_2) + b\bar{y}_2(\bar{x}_1 - x_1)) \nleq (0,0) \\ (-a\bar{x}_2(y_1 - \bar{y}_1) + a\bar{x}_1(y_2 - \bar{y}_2),\ -b\bar{x}_2(y_1 - \bar{y}_1) + b\bar{x}_1(y_2 - \bar{y}_2)) \nleq (0,0) \end{cases}$$

$$\iff \begin{cases} (a,b)((x_2 - \bar{x}_2)\bar{y}_1 - (x_1 - \bar{x}_1)\bar{y}_2) \nleq (0,0) \\ (a,b)((y_2 - \bar{y}_2)\bar{x}_1 - (y_2 - \bar{y}_2)\bar{x}_2) \nleq (0,0) \end{cases}$$

Thus the efficient solution set of vector matrix game (B_1, B_2) is as follows;

(i) if $a > 0,\ b > 0$; the solution set is $\{(1,0,1,0)\}$.

(ii) if $a = 0,\ b > 0$; the solution set is $\{(1,0,1,0)\}$.

(iii) if $a > 0,\ b = 0$; the solution set is $\{(1,0,1,0)\}$.

(iv) if $a < 0,\ b < 0$; the solution set is $\{(0,1,0,1)\}$.

(v) if $a = 0,\ b < 0$; the solution set is $\{(0,1,0,1)\}$.

(vi) if $a < 0,\ b = 0$; the solution set is $\{(0,1,0,1)\}$.

(vii) if $a > 0,\ b < 0$; the solution set is $S_2 \times S_2$.

(viii) if $a < 0,\ b > 0$; the solution set is $S_2 \times S_2$.

(ix) if $a = 0,\ b = 0$; the solution set is $S_2 \times S_2$.

(6) A point $(\bar{x}, \bar{y}) \in S_2 \times S_2$ is a weakly efficient solution of vector matrix game (B_1, B_2) if

$$\left((x_1,x_2)\begin{pmatrix}0 & a\\ -a & 0\end{pmatrix}\begin{pmatrix}\bar{y}_1\\ \bar{y}_2\end{pmatrix},(x_1,x_2)\begin{pmatrix}0 & b\\ -b & 0\end{pmatrix}\begin{pmatrix}\bar{y}_1\\ \bar{y}_2\end{pmatrix}\right)$$

$$\not> \left((\bar{x}_1,\bar{x}_2)\begin{pmatrix}0 & a\\ -a & 0\end{pmatrix}\begin{pmatrix}\bar{y}_1\\ \bar{y}_2\end{pmatrix},(\bar{x}_1,\bar{x}_2)\begin{pmatrix}0 & b\\ -b & 0\end{pmatrix}\begin{pmatrix}\bar{y}_1\\ \bar{y}_2\end{pmatrix}\right)$$

$$\not> \left((\bar{x}_1,\bar{x}_2)\begin{pmatrix}0 & a\\ -a & 0\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix},(\bar{x}_1,\bar{x}_2)\begin{pmatrix}0 & b\\ -b & 0\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}\right)$$

$$\begin{aligned}\Longleftrightarrow \quad & (-ax_2\bar{y}_1+ax_1\bar{y}_2,\ -bx_2\bar{y}_1+bx_1\bar{y}_2) \not> (-a\bar{x}_2\bar{y}_1+a\bar{x}_1\bar{y}_2,\ -b\bar{x}_2\bar{y}_1+b\bar{x}_1\bar{y}_2)\\ & \qquad \not> (-a\bar{x}_2y_1+a\bar{x}_1y_2,\ -b\bar{x}_2y_1+b\bar{x}_1y_2)\end{aligned}$$

$$\Longleftrightarrow \begin{cases}(-a\bar{y}_1(\bar{x}_2-x_2)+a\bar{y}_2(\bar{x}_1-x_1),\ -b\bar{y}_1(\bar{x}_2-x_2)+b\bar{y}_2(\bar{x}_1-x_1)) \not< (0,0)\\ (-a\bar{x}_2(y_1-\bar{y}_1)+a\bar{x}_1(y_2-\bar{y}_2),\ -b\bar{x}_2(y_1-\bar{y}_1)+b\bar{x}_1(y_2-\bar{y}_2)) \not< (0,0)\end{cases}$$

$$\Longleftrightarrow \begin{cases}(a,b)((x_2-\bar{x}_2)\bar{y}_1-(x_1-\bar{x}_1)\bar{y}_2) \not< (0,0)\\ (a,b)((y_2-\bar{y}_2)\bar{x}_1-(y_1-\bar{y}_1)\bar{x}_2) \not< (0,0)\end{cases}$$

Thus the weakly efficient solution set of vector matrix game (B_1, B_2) is as follows;

(i) if $a > 0,\ b > 0$; the solution set is $\{(1,0,1,0)\}$.

(ii) if $a = 0,\ b > 0$; the solution set is $S_2 \times S_2$.

(iii) if $a > 0,\ b = 0$; the solution set is $S_2 \times S_2$.

(iv) if $a < 0,\ b < 0$; the solution set is $\{(0,1,0,1)\}$.

(v) if $a = 0,\ b < 0$; the solution set is $S_2 \times S_2$.

(vi) if $a < 0,\ b = 0$; the solution set is $S_2 \times S_2$.

(vii) if $a > 0,\ b < 0$; the solution set is $S_2 \times S_2$.

(viii) if $a < 0,\ b > 0$; the solution set is $S_2 \times S_2$.

(ix) if $a = 0,\ b = 0$; the solution set is $S_2 \times S_2$. □

3.4. Vector Duality

We formulate a dual problem for the linear vector optimization problem (LVP) in Section 3.2, and prove equivalent relations between the dual problem and the corresponding vector matrix game.

The following is a dual problem for (LVP).

$$
\begin{aligned}
\text{(LVD)}\quad \text{Maximize}\quad & (c_1^T u, \cdots, c_p^T u) - \lambda^T (Au - b)e \\
\text{subject to}\quad & \sum_{i=1}^{p} \xi_i c_i - A^T \lambda \geqq 0, \\
& u^T [\sum_{i=1}^{p} \xi_i c_i - A^T \lambda] \leqq 0, \\
& \lambda \geq 0, \\
& \xi \in \overset{o}{S}_p, \\
& u \in \mathbb{R}^p,
\end{aligned}
$$

where $e = (1, 1, \cdots, 1) \in \mathbb{R}^p$.

Theorem 3.4.1 (Weak Duality). Let x and (u, λ, ξ) be feasible solutions to (LVP) and (LVD), respectively. Then

$$(c_1^T x, \cdots, c_p^T x) \nleq (c_1^T u, \cdots, c_p^T u) - \lambda^T (Au - b)e.$$

Proof. Suppose that there exist feasible solution x and (u, λ, ξ) such that $(c_1^T x, \cdots, c_p^T x) \leq (c_1^T u, \cdots, c_p^T u) - \lambda^T (Au - b)e$. Since $\xi_i > 0$, $\sum_{i=1}^p \xi_i (c_i^T x - c_i^T u) - \lambda^T b + \lambda^T Au < 0$. Since x is feasible to (LVP), so that

$$\sum_{i=1}^p \xi_i (c_i^T x - c_i^T u) - \lambda^T Ax + \lambda^T Au < 0. \tag{3.4}$$

But x and (u, λ, ξ) are feasible solutions to (LVP) and (LVD), respectively,

$$\begin{aligned}
&\sum_{i=1}^p \xi_i c_i^T x - \Big(\sum_{i=1}^p \xi_i c_i - A^T \lambda\Big)^T u - \lambda^T Ax \\
&= \Big(\sum_{i=1}^p \xi_i c_i - A^T \lambda\Big)^T x - \Big(\sum_{i=1}^p \xi_i c_i - A^T \lambda\Big)^T u \\
&\geqq 0,
\end{aligned}$$

which contradicts (3.4). Hence the result holds. □

Theorem 3.4.2 (Strong Duality). Let $\bar{x}$ be an efficient solution of (LVP). Then there exist $\bar{\xi} \in \overset{o}{S}_p$ and $\bar{\lambda} \in \mathbb{R}_m^+$ such that $(\bar{x}, \bar{\lambda}, \bar{\xi})$ is a feasible solution to (LVD) and the objective values of (LVP) and (LVD) are equal. Moreover, $(\bar{x}, \bar{\lambda}, \bar{\xi})$ is an efficient solution of (LVD).

Proof. Let $\bar{x}$ be an efficient solution of (LVP). By Kuhn-Tucker optimality condition in [55], there exist $\bar{\xi} \in \overset{o}{S}_p$ and $\bar{\lambda} \in \mathbb{R}^+_m$ such that

$$\sum_{i=1}^{p} \bar{\xi}_i c_i - A^T \bar{\lambda} \geqq 0,$$

$$\bar{x}^T \left[\sum_{i=1}^{p} \bar{\xi}_i c_i - A^T \bar{\lambda} \right] = 0,$$

$$\bar{\lambda}^T (A\bar{x} - b) = 0.$$

Thus $(\bar{x}, \bar{\lambda}, \bar{\xi})$ is a feasible solution to (LVD) with $c^T\bar{x} = c^T\bar{x} - \bar{\lambda}^T(A\bar{x} - b)e$. By weak duality,

$$(c_1^T\bar{x}, \cdots, c_p^T\bar{x}) - \bar{\lambda}^T(A\bar{x} - b)e \nleq (c_1^T u, \cdots, c_p^T u) - \lambda^T(Au - b)e,$$

for any feasible solution (u, λ, ξ) of (LVD). Hence $(\bar{x}, \bar{\lambda}, \bar{\xi})$ is an efficient solution of (LVD). □

Now consider the vector matrix game associated with the following $(m + n + 1) \times (m + n + 1)$ skew symmetric matrix $B_i, i = 1, \cdots, p$:

$$B_i = \begin{bmatrix} 0 & A^T & -c_i \\ -A & 0 & b \\ c_i^T & -b^T & 0 \end{bmatrix}.$$

Theorem 3.4.3. Let $\bar{x}$ and $(\bar{x}, \bar{\lambda}, \bar{\xi})$ be feasible solutions to (LVP) and (LVD), respectively, such that $\bar{\lambda}^T(A\bar{x} - b) = 0$. Let $z^* = 1/(1 + \sum_i \bar{x}_i + \sum_j \bar{\lambda}_j), x^* = z^*\bar{x}$ and $\lambda^* = z^*\bar{\lambda}$. Then (x^*, λ^*, z^*) is a vector solution of vector matrix game $(B_1, \cdots, B_p)$.

Proof. Let $(\bar{x}, \bar{\lambda}, \bar{\xi})$ be a feasible solution to (LVD). Then the following holds:

$$\sum_{i=1}^{p} \bar{\xi}_i c_i - A^T \bar{\lambda} \geqq 0, \tag{3.5}$$

$$\bar{x}^T \Big[\sum_{i=1}^{p} \bar{\xi}_i c_i - A^T \bar{\lambda} \Big] \leqq 0, \tag{3.6}$$

$$A\bar{x} \geqq b, \tag{3.7}$$

$$\bar{\lambda}^T (A\bar{x} - b) = 0, \tag{3.8}$$

$$\bar{\lambda} \geq 0, \ \ \bar{x} \geqq 0, \ \ \bar{\xi} \in \overset{o}{S}_p. \tag{3.9}$$

Since $z^* > 0$ by (3.9), from (3.5) and (3.7), we get:

$$z^* \Big[A^T \bar{\lambda} - \sum_{i=1}^{p} \bar{\xi}_i c_i \Big] \leqq 0, \tag{3.10}$$

$$-z^* (A\bar{x} - b) \leqq 0. \tag{3.11}$$

Now (3.6) and (3.8) give

$$\begin{aligned} z^* \Big[\textstyle\sum_{i=1}^{p} \bar{\xi}_i c_i^T \bar{x} - b^T \bar{\lambda} \Big] &= z^* \Big[\textstyle\sum_{i=1}^{p} \bar{\xi}_i c_i^T \bar{x} - \bar{x}^T A^T \bar{\lambda} \Big] \\ &= z^* \bar{x}^T \Big[\textstyle\sum_{i=1}^{p} \bar{\xi}_i c_i - A^T \bar{\lambda} \Big] \\ &\leqq 0. \end{aligned} \tag{3.12}$$

From (3.10), (3.11) and (3.12) we have the following form of inequality

$$\Big(\sum_{i=1}^{p} \bar{\xi}_i B_i \Big) \begin{pmatrix} z^* \bar{x} \\ z^* \bar{\lambda} \\ z^* \end{pmatrix} \leqq 0.$$

By Lemma 3.2.1, (x^*, λ^*, z^*) is a vector solution of vector matrix game $(B_1, \cdots, B_p)$. $\square$

Theorem 3.4.4. Let (x^*, λ^*, z^*) with $z^* > 0$ be a vector solution of vector matrix game $(B_1, \cdots, B_p)$. Let $\bar{x} = x^*/z^*$ and $\bar{\lambda} = \lambda^*/z^*$. Then there exists $\bar{\xi} \in \overset{o}{S}_p$ such that $(\bar{x}, \bar{\lambda}, \bar{\xi})$ is a feasible solution to (LVD) with $A\bar{x} \geqq b$ and $\bar{\lambda}^T(A\bar{x} - b) = 0$. Moreover $\bar{x}$ is an efficient solution of (LVP) and $(\bar{x}, \bar{\lambda}, \bar{\xi})$ is an efficient solution of (LVD).

Proof. Let (x^*, λ^*, z^*) with $z^* > 0$ be a vector solution of vector matrix game $B_i,\ i = 1, \cdots, p$. Then by Lemma 3.2.1, there exists $\bar{\xi} \in \overset{o}{S}_p$ such that

$$\left(\sum_{i=1}^{p} \bar{\xi}_i B_i\right) \begin{pmatrix} x^* \\ \lambda^* \\ z^* \end{pmatrix} \leqq 0.$$

Thus we get:

$$A^T\lambda^* - \sum_{i=1}^{p} \bar{\xi}_i c_i z^* \leqq 0, \tag{3.13}$$

$$-Ax^* + bz^* \leqq 0, \tag{3.14}$$

$$\sum_{i=1}^{p} \bar{\xi}_i c_i^T x^* - b^T\lambda^* \leqq 0, \tag{3.15}$$

$$x^* \geqq 0,\ \lambda^* \geq 0,\ z^* > 0. \tag{3.16}$$

Dividing (3.13), (3.14) and (3.15) by $z^* > 0$, we have

$$A^T\bar{\lambda} - \sum_{i=1}^{p} \bar{\xi}_i c_i \leqq 0, \tag{3.17}$$

$$- A\bar{x} + b \leqq 0, \tag{3.18}$$

$$\sum_{i=1}^{p} \bar{\xi}_i c_i^T \bar{x} - b^T\bar{\lambda} \leqq 0. \tag{3.19}$$

From (3.16),

$$\bar{x} \geqq 0, \ \bar{\lambda} \geqq 0. \tag{3.20}$$

From (3.17) and (3.20),

$$\bar{\lambda}^T A\bar{x} - \sum_{i=1}^{p} \bar{\xi}_i c_i^T \bar{x} \leqq 0. \tag{3.21}$$

From (3.19) and (3.21), $\bar{\lambda}^T(A\bar{x} - b) \leqq 0$. Using (3.18) and (3.20), we obtain $\bar{\lambda}^T(A\bar{x} - b) \geqq 0$. It implies that

$$\bar{\lambda}^T(A\bar{x} - b) = 0. \tag{3.22}$$

From (3.19) and (3.22), we receive $\bar{x}^T\left[\sum_{i=1}^{p} \bar{\xi}_i c_i - A^T\bar{\lambda}\right] \leqq 0$. Using (3.21), we obtain $\bar{x}^T\left[\sum_{i=1}^{p} \bar{\xi}_i c_i - A^T\bar{\lambda}\right] = 0$. Therefore $(\bar{x}, \bar{\lambda}, \bar{\xi})$ is a feasible solution to (LVD), with $c^T\bar{x} = c^T\bar{x} - \bar{\lambda}^T(A\bar{x} - b)e$. Since $(\bar{x}, \bar{\lambda}, \bar{\xi})$ is feasible for (LVD), by weak duality, $(c_1^T x, \cdots, c_p^T x) \nleq (c_1^T\bar{x}, \cdots, c_p^T\bar{x})$ and $(c_1^T\bar{x}, \cdots, c_p^T\bar{x}) - \bar{\lambda}^T(A\bar{x} - b)e \nleq (c_1^T u, \cdots, c_p^T u) - \lambda^T(Au - b)e$, for any feasible solution (u, λ, ξ) of (LVD).

Therefore $\bar{x}$ is an efficient solution of (LVP) and $(\bar{x}, \bar{\lambda}, \bar{\xi})$ is an efficient solution of (LVD). □

Now we give an example verifying Theorems 3.4.3 and 3.4.4.

Example 3.4.1. Let $c_1 = 1, c_2 = -1, A = [-1]$ and $b = -1$ for $x \in \mathbb{R}_+$. Consider the following linear vector optimization problem:

$$\begin{array}{lll} \text{(LVP)} & \text{Minimize} & (c_1^T x, c_2^T x) \\ & \text{subject to} & Ax \geqq b, \\ & & x \geqq 0. \end{array}$$

Then $\bar{x} = 0$ is an efficient solution of (LVP). Now we solve the Kuhn-Tucker system with respect to $\bar{x} = 0$ as follows:

$$\begin{cases} \sum_{i=1}^{2} \xi_i c_i - A^T \lambda \geqq 0 \\ \lambda(A \cdot 0 - b) = 0, \ \lambda \geq 0, \ (\xi_1, \xi_2) \in \overset{o}{S}_2 \end{cases}$$

$$\Longleftrightarrow \lambda = 0, \ \xi_1 - \xi_2 \geqq 0, \ (\xi_1, \xi_2) \in \overset{o}{S}_2$$

Therefore $(\bar{x}, \bar{\lambda}, \bar{\xi}) = (0, 0, \frac{1}{2}, \frac{1}{2})$ is a feasible solution for (LVD):

$$\begin{array}{lll} \text{(LVD)} & \text{Maximize} & (x, -x) - \lambda(-x + 1)e \\ & \text{subject to} & \xi_1 - \xi_2 + \lambda \geqq 0, \\ & & x[\xi_1 - \xi_2 + \lambda] \leqq 0, \\ & & \lambda \geqq 0, \\ & & \xi = (\xi_1, \xi_2) \in \overset{o}{S}_2. \end{array}$$

Consider the matrix game associated with the following skew symmetric matrix $B_i, i = 1, 2$.

$$B_1 = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

We can easily check that $\bar{x} = 0$ is a feasible solution for (LVP) and $(0, 0, \frac{1}{2}, \frac{1}{2})$ is feasible for (LVD) and $\bar{\lambda}^T(A\bar{x} - b) = 0$. Let $z^* = 1$. Then $x^* = 0, \lambda^* = 0$. By Theorem 3.4.3, $(0, 0, 1)$ is a vector solution of vector matrix game (B_1, B_2).

We know that $(0, 0, 1)$ solves the vector matrix game (B_1, B_2). Let $\bar{x} = \frac{x^*}{z^*} = 0, \bar{\lambda} = \frac{\lambda^*}{z^*} = 0$. By Theorem 3.4.4, there exists $\bar{\xi} \in \overset{o}{S}_2$ such that $(\bar{x}, \bar{\lambda}, \bar{\xi})$ is a feasible solution for (LVD). Since the weak duality holds between (LVP) and (LVD), by Theorem 3.4.4, $\bar{x} = 0$ is an efficient solution for (LVP) and $(0, 0, \frac{1}{2}, \frac{1}{2})$ is an efficient solution for (LVD). □

Chapter 4
Vector Matrix Game for Vector Symmetric Dual Problem

4.1. Introduction

A nonlinear programming problem and its dual are said to be symmetric if the dual of the dual is the original problem. Symmetric duality in nonlinear programming in which the dual of the dual is the primal was first introduced by Dorn [22]. Dantzig et al. [19] formulated a pair of symmetric dual nonlinear programs and established duality results for convex and concave functions with non-negative orthant as the cone. Mond and Weir [47] presented two pair of symmetric dual multiobjective programming problems for efficient solutions and obtained symmetric duality results concerning pseudoconvex and pseudoconcave functions. Chandra et al. [16] formulated a pair of symmetric dual fractional programming problems under suitable convexity hypothesis. Recently, Kim and Noh [38] established equivalent relations between certain matrix game and symmetric dual problems. In this section, we formulate vector symmetric dual problems and consider vector matrix game corresponding to the problems.

4.2. Equivalent Relations

Now we consider the nonlinear vector symmetric programming problem (VSP) together with its dual (VSD) as follows:

$$\text{(VSP)} \quad \text{Minimize} \quad \Big(f_1(x,y) - y^T\nabla_y(\lambda^T f)(x,y), \cdots, f_p(x,y) - y^T\nabla_y(\lambda^T f)(x,y)\Big)$$

$$\text{subject to} \quad -\nabla_y(\lambda^T f)(x,y) \geqq 0,$$

$$x \geqq 0, \ \lambda > 0,$$

$$\text{(VSD)} \quad \text{Maximize} \quad \Big(f_1(u,v) - u^T\nabla_u(\lambda^T f)(u,v), \cdots, f_p(u,v) - u^T\nabla_u(\lambda^T f)(u,v)\Big)$$

$$\text{subject to} \quad -\nabla_u(\lambda^T f)(u,v) \leqq 0,$$

$$v \geqq 0, \ \lambda > 0,$$

where $f := (f_1, \cdots, f_p) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$, $\lambda \in \mathbb{R}^p$.

The following Theorems 4.2.1 and 4.2.2 are well known, but for the completeness, we give proofs for the theorems.

Theorem 4.2.1 (Weak Duality) [47]. Let (x, y, λ) be feasible for (VSP) and (u, v, λ) be feasible for (VSD). If $f_i(\cdot, y)$ $(1 \leq i \leq p)$ are convex for fixed y and $f_i(x, \cdot)$ $(1 \leq i \leq p)$ are concave for fixed x, then the following cannot hold:

$$\Big(f_1(x,y) - y^T\nabla_y(\lambda^T f)(x,y), \cdots, f_p(x,y) - y^T\nabla_y(\lambda^T f)(x,y)\Big)$$

$$\leq \Big(f_1(u,v) - u^T\nabla_u(\lambda^T f)(u,v), \cdots, f_p(u,v) - u^T\nabla_u(\lambda^T f)(u,v)\Big). \quad (4.1)$$

Proof. Suppose contrary to the result that (4.1) holds,

$$\Big(f_1(x,y) - y^T\nabla_y(\lambda^T f)(x,y), \cdots, f_p(x,y) - y^T\nabla_y(\lambda^T f)(x,y)\Big)$$

$$\leq \Big(f_1(u,v) - u^T\nabla_u(\lambda^T f)(u,v), \cdots, f_p(u,v) - u^T\nabla_u(\lambda^T f)(u,v)\Big).$$

Then since $\lambda > 0$, we have

$$\sum_{i=1}^{p} \lambda_i f_i(x,y) - y^T \nabla_y (\lambda^T f)(x,y) < \sum_{i=1}^{p} \lambda_i f_i(u,v) - u^T \nabla_u (\lambda^T f)(u,v). \quad (4.2)$$

By the convexity of $f_i(\cdot, v)$ $(1 \leq i \leq p)$, we have

$$f_i(x,v) - f_i(u,v) \geqq (x-u)^T \nabla_x f_i(u,v).$$

It follows from $\lambda > 0$ that

$$\sum_{i=1}^{p} \lambda_i f_i(x,v) - \sum_{i=1}^{p} \lambda_i f_i(u,v) \geqq (x-u)^T \nabla_x (\lambda^T f)(u,v). \quad (4.3)$$

By the concavity of $f_i(x, \cdot)$ $(1 \leq i \leq p)$, we have

$$f_i(x,y) - f_i(x,v) \geqq (y-v)^T \nabla_y f_i(x,y).$$

It follows from $\lambda > 0$ that

$$\sum_{i=1}^{p} \lambda_i f_i(x,y) - \sum_{i=1}^{p} \lambda_i f_i(x,v) \geqq (y-v)^T \nabla_y (\lambda^T f)(x,y). \quad (4.4)$$

From (4.3) and (4.4), we have

$$\Big[\sum_{i=1}^{p} \lambda_i f_i(x,y) - y^T \nabla_y (\lambda^T f)(x,y)\Big] - \Big[\sum_{i=1}^{p} \lambda_i f_i(u,v) - u^T \nabla_x (\lambda^T f)(u,v)\Big]$$

$$\geqq x^T \nabla_x (\lambda^T f)(u,v) - v^T \nabla_y (\lambda^T f)(x,y)$$

$$\geqq 0,$$

which contradicts (4.2). □

Theorem 4.2.2 (Strong Duality) [47]. Let $(\bar{x}, \bar{y}, \bar{\lambda})$ be an efficient solution of (VSP). Suppose that $\nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})$ is positive definite and $\nabla_y f_1(\bar{x}, \bar{y}), \cdots,$ $\nabla_y f_p(\bar{x}, \bar{y})$ are linearly independent, then $(\bar{x}, \bar{y}, \bar{\lambda})$ is an efficient solution of (VSD).

Proof. Since $(\bar{x}, \bar{y}, \bar{\lambda})$ is an efficient solution of (VSP), by the Fritz John optimality condition in [45], there exist $\alpha \in \mathbb{R}^p,\ \beta \in \mathbb{R}^m,\ \gamma \in \mathbb{R}^n$ and $\omega \in \mathbb{R}^p$ such that

$$\alpha^T \nabla_x f(\bar{x}, \bar{y}) + (\beta - (\alpha^T e)\bar{y})^T \nabla_{yx}(\bar{\lambda}^T f)(\bar{x}, \bar{y}) - \gamma = 0, \tag{4.5}$$

$$(\alpha - (\alpha^T e)\bar{\lambda})^T \nabla_y f(\bar{x}, \bar{y}) + (\beta - (\alpha^T e)\bar{y})^T \nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y}) = 0, \tag{4.6}$$

$$(\beta - (\alpha^T e)\bar{y})^T \nabla_y f(\bar{x}, \bar{y}) - \omega^T = 0, \tag{4.7}$$

$$\beta^T \nabla_y(\bar{\lambda}^T f)(\bar{x}, \bar{y}) = 0, \tag{4.8}$$

$$\gamma^T \bar{x} = 0, \tag{4.9}$$

$$\omega^T \bar{\lambda} = 0, \tag{4.10}$$

$$(\alpha, \beta, \gamma, \omega) \geqq 0, \tag{4.11}$$

$$(\alpha, \beta, \gamma, \omega) \neq 0. \tag{4.12}$$

Multiplying (4.6) by $(\beta - (\alpha^T e)\bar{y})$,

$$(\alpha - (\alpha^T e)\bar{\lambda})^T \nabla_y f(\bar{x}, \bar{y})(\beta - (\alpha^T e)\bar{y}) + (\beta - (\alpha^T e)\bar{y})^T \nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})(\beta - (\alpha^T e)\bar{y}) = 0.$$

By (4.7) and (4.10),

$$\alpha^T\omega + (\beta - (\alpha^T e)\bar{y})^T \nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})(\beta - (\alpha^T e)\bar{y}) = 0.$$

Since $\alpha \geq 0,\ \omega \geq 0$ and hence

$$(\beta - (\alpha^T e)\bar{y})^T \nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})(\beta - (\alpha^T e)\bar{y}) \leqq 0.$$

Since $\nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})$ is positive definite, then

$$\beta = (\alpha^T e)\bar{y}. \tag{4.13}$$

By (4.7), $\omega = 0$. From (4.6), $(\alpha - (\alpha^T e)\bar{\lambda})^T \nabla_y f(\bar{x}, \bar{y}) = 0$. Since $\{\nabla_y f_1(\bar{x}, \bar{y}), \cdots, \nabla_y f_p(\bar{x}, \bar{y})\}$ is linearly independent,

$$\alpha = (\alpha^T e)\bar{\lambda}. \tag{4.14}$$

If $\alpha = 0$, then (4.13) implies $\beta = 0$ and by (4.5) $\gamma = 0$, $\omega = 0$, which contradicts (4.12). Hence $\alpha \neq 0$, since $\alpha > 0$. From (4.5), (4.13) and (4.14),

$$(\alpha^T e)\nabla_x(\bar{\lambda}^T f)(\bar{x}, \bar{y}) = \gamma \geqq 0. \tag{4.15}$$

Since $\nabla_x(\bar{\lambda}^T f)(\bar{x}, \bar{y}) \geqq 0$, from (4.9) and (4.15), $\bar{x}\nabla_x(\bar{\lambda}^T f)(\bar{x}, \bar{y}) = 0$, and from (4.8) and (4.13), $((\alpha^T e)\bar{y})^T \nabla_y(\bar{\lambda}^T f)(\bar{x}, \bar{y}) = 0$. Since $\alpha > 0$, $\bar{y}^T \nabla_y(\bar{\lambda}^T f)(\bar{x}, \bar{y}) = 0$. Thus $(\bar{x}, \bar{y}, \bar{\lambda})$ is feasible for (VSD). Clearly, $(\bar{x}, \bar{y}, \bar{\lambda})$ is efficient for (VSD), otherwise there exists $(\bar{u}, \bar{v}, \bar{\lambda})$ which is feasible for (VSD) such that

$$f(\bar{u}, \bar{v}) - \bar{u}^T \nabla_x(\bar{\lambda}^T f)(\bar{u}, \bar{v})e \geq f(\bar{x}, \bar{y}) - \bar{x}^T \nabla_x(\bar{\lambda}^T f)(\bar{x}, \bar{y})e.$$

Since $\bar{x}^T \nabla_x (\bar{\lambda}^T f)(\bar{x}, \bar{y}) = \bar{y}^T \nabla_y (\bar{\lambda}^T f)(\bar{x}, \bar{y}) = 0$, it follows that

$$f(\bar{u}, \bar{v}) - \bar{u}^T \nabla_x (\bar{\lambda}^T f)(\bar{u}, \bar{v}) e \geq f(\bar{x}, \bar{y}) - \bar{y}^T \nabla_y (\bar{\lambda}^T f)(\bar{x}, \bar{y}) e,$$

which also contradicts weak duality. □

Lemma 4.2.1. Let $(\bar{x}, \bar{y}, \bar{\lambda})$ be feasible for (VSP) and (VSD), and assume that $\bar{y}^T \nabla_y (\bar{\lambda}^T f)(\bar{x}, \bar{y}) = \bar{x} \nabla_x (\bar{\lambda}^T f)(\bar{x}, \bar{y}) = 0$. If there is weak duality between (VSP) and (VSD), then $(\bar{x}, \bar{y}, \bar{\lambda})$ is an efficient solution of (VSP) and (VSD).

Proof. By weak duality, $f(\bar{x}, \bar{y}) - \bar{y}^T \nabla_y (\bar{\lambda}^T f)(\bar{x}, \bar{y}) e \nleq f(u, v) - u^T \nabla_u (\bar{\lambda}^T f)(u, v) e$ for any feasible $(u, v, \bar{\lambda})$ of (VSD). Since $f(\bar{x}, \bar{y}) - \bar{y}^T \nabla_y (\bar{\lambda}^T f)(\bar{x}, \bar{y}) e = f(\bar{x}, \bar{y}) - \bar{x}^T \nabla_x (\bar{\lambda}^T f)(\bar{x}, \bar{y}) e$, $f(\bar{x}, \bar{y}) - \bar{x}^T \nabla_x (\bar{\lambda}^T f)(\bar{x}, \bar{y}) e \nleq f(u, v) - u^T \nabla_u (\bar{\lambda}^T f)(u, v) e$ for any feasible $(u, v, \bar{\lambda})$ of (VSD). Therefore, $(\bar{x}, \bar{y}, \bar{\lambda})$ is an efficient solution of (VSD).

By weak duality, $f(x, y) - y^T \nabla_y (\bar{\lambda}^T f)(x, y) e \nleq f(\bar{x}, \bar{y}) - \bar{x}^T \nabla_x (\bar{\lambda}^T f)(\bar{x}, \bar{y}) e$ for any feasible $(x, y, \bar{\lambda})$ of (VSP). Since $f(\bar{x}, \bar{y}) - \bar{x}^T \nabla_x (\bar{\lambda}^T f)(\bar{x}, \bar{y}) e = f(\bar{x}, \bar{y}) - \bar{y}^T \nabla_y (\bar{\lambda}^T f)(\bar{x}, \bar{y}) e$, $f(x, y) - y^T \nabla_y (\bar{\lambda}^T f)(x, y) e \nleq f(\bar{x}, \bar{y}) - \bar{y}^T \nabla_y (\bar{\lambda}^T f)(\bar{x}, \bar{y}) e$ for any feasible $(x, y, \bar{\lambda})$ of (VSP). Therefore, $(\bar{x}, \bar{y}, \bar{\lambda})$ is an efficient solution of (VSP). □

Consider the vector matrix game defined by the following $(n + m + 1) \times (n + m + 1)$ skew symmetric matrix $B_i(x, y), i = 1, \cdots, p$ related to (VSP) and (VSD):

$$B_i(x,y) = \begin{bmatrix} 0 & -x\nabla_y f_i(x,y)^T & -\nabla_x f_i(x,y) \\ \nabla_y f_i(x,y)x^T & 0 & \nabla_y f_i(x,y) \\ \nabla_x f_i(x,y)^T & -\nabla_y f_i(x,y)^T & 0 \end{bmatrix}.$$

Now we give equivalent relations between vector symmetric dual problem and vector matrix game $(B_1(\bar{x},\bar{y}),\cdots,B_p(\bar{x},\bar{y}))$.

Theorem 4.2.3. Let $(\bar{x},\bar{y},\bar{\xi})$ be feasible for (VSP) and (VSD), with $\bar{y}^T\nabla_y(\bar{\xi}^T f)(\bar{x},\bar{y}) = \bar{x}^T\nabla_x(\bar{\xi}^T f)(\bar{x},\bar{y}) = 0$. Let $z^* = 1/(1+\sum_i \bar{x}_i+\sum_j \bar{y}_j)$, $x^* = z^*\bar{x}$ and $y^* = z^*\bar{y}$. Then (x^*,y^*,z^*) is a vector solution of vector matrix game $(B_1(\bar{x},\bar{y}),\cdots,B_p(\bar{x},\bar{y}))$.

Proof. Let $(\bar{x},\bar{y},\bar{\xi})$ be feasible for (VSP) and (VSD). Then the following holds:

$$-\nabla_y(\bar{\xi}^T f)(\bar{x},\bar{y}) \geqq 0, \tag{4.16}$$

$$-\nabla_x(\bar{\xi}^T f)(\bar{x},\bar{y}) \leqq 0, \tag{4.17}$$

$$\bar{y}^T\nabla_y(\bar{\xi}^T f)(\bar{x},\bar{y}) = \bar{x}^T\nabla_x(\bar{\xi}^T f)(\bar{x},\bar{y}) = 0, \tag{4.18}$$

$$\bar{x} \geqq 0,\ \bar{y} \geqq 0,\ \bar{\xi} \in \overset{o}{S}_p. \tag{4.19}$$

Multiplying (4.18) by $\bar{x} \geqq 0$ gives $-\bar{x}\nabla_y(\bar{\xi}^T f)(\bar{x},\bar{y})^T\bar{y} = 0$ and from (4.17),

$$-\bar{x}\nabla_y(\bar{\xi}^T f)(\bar{x},\bar{y})^T\bar{y} - \nabla_x(\bar{\xi}^T f)(\bar{x},\bar{y}) \leqq 0. \tag{4.20}$$

Multiplying (4.16) by $\bar{x}^T\bar{x} \geqq 0$, $\nabla_y(\bar{\xi}^T f)(\bar{x},\bar{y})\bar{x}^T\bar{x} \leqq 0$. It implies that since $\nabla_y(\bar{\xi}^T f)(\bar{x},\bar{y}) \leqq 0$,

$$\nabla_y(\bar{\xi}^T f)(\bar{x},\bar{y})\bar{x}^T\bar{x} + \nabla_y(\bar{\xi}^T f)(\bar{x},\bar{y}) \leqq 0. \tag{4.21}$$

From (4.18) we have

$$\nabla_x(\bar{\xi}^T f)(\bar{x},\bar{y})^T\bar{x} - \nabla_y(\bar{\xi}^T f)(\bar{x},\bar{y})^T\bar{y} = 0. \tag{4.22}$$

But $z^* > 0$ by (4.19), from (4.20), (4.21) and (4.22), we get:

$$-\bar{x}\nabla_y(\bar{\xi}^T f)(\bar{x},\bar{y})^T y^* - \nabla_x(\bar{\xi}^T f)(\bar{x},\bar{y})z^* \leqq 0, \tag{4.23}$$

$$\nabla_y(\bar{\xi}^T f)(\bar{x},\bar{y})\bar{x}^T x^* + \nabla_y(\bar{\xi}^T f)(\bar{x},\bar{y})z^* \leqq 0, \tag{4.24}$$

$$\nabla_x(\bar{\xi}^T f)(\bar{x},\bar{y})^T x^* - \nabla_y(\bar{\xi}^T f)(\bar{x},\bar{y})^T y^* = 0, \tag{4.25}$$

$$x^* \geqq 0,\ \ y^* \geqq 0,\ \ z^* > 0. \tag{4.26}$$

From (4.23), (4.24) and (4.25) we have the following inequality

$$\Big(\sum_{i=1}^{p} \bar{\xi}_i B_i(\bar{x},\bar{y})\Big)\begin{pmatrix} x^* \\ y^* \\ z^* \end{pmatrix} \leqq 0.$$

By Lemma 3.2.1, (x^*, y^*, z^*) is a vector solution of the vector matrix game $(B_1(\bar{x},\bar{y}),\cdots,B_p(\bar{x},\bar{y}))$. □

Theorem 4.2.4. Let (x^*, y^*, z^*) with $z^* > 0$ be a vector solution of vector matrix game $(B_1(\bar{x},\bar{y}),\cdots,B_p(\bar{x},\bar{y}))$, where $\bar{x} = x^*/z^*$ and $\bar{y} = y^*/z^*$. Then there exists $\bar{\xi} \in \overset{o}{S}_p$ such that $(\bar{x}, \bar{y}, \bar{\xi})$ is feasible for (VSP) and (VSD), and $\bar{y}^T \nabla_y (\bar{\xi}^T f)(\bar{x}, \bar{y}) = \bar{x}^T \nabla_x (\bar{\xi}^T f)(\bar{x}, \bar{y}) = 0$.

Proof. Let (x^*, y^*, z^*) with $z^* > 0$ be a vector solution of vector matrix game $(B_1(\bar{x}, \bar{y}), \cdots, B_p(\bar{x}, \bar{y}))$. Then by Lemma 3.2.1, there exists $\bar{\xi} \in \overset{o}{S}_p$ such that

$$\Big(\sum_{i=1}^{p} \bar{\xi}_i B_i(\bar{x}, \bar{y})\Big) \begin{pmatrix} x^* \\ y^* \\ z^* \end{pmatrix} \leqq 0.$$

Thus we get:

$$-\bar{x}\nabla_y(\bar{\xi}^T f)(\bar{x}, \bar{y})^T y^* - \nabla_x(\bar{\xi}^T f)(\bar{x}, \bar{y}) z^* \leqq 0, \tag{4.27}$$

$$\nabla_y(\bar{\xi}^T f)(\bar{x}, \bar{y}) \bar{x}^T x^* + \nabla_y(\bar{\xi}^T f)(\bar{x}, \bar{y}) z^* \leqq 0, \tag{4.28}$$

$$\nabla_x(\bar{\xi}^T f)(\bar{x}, \bar{y})^T x^* - \nabla_y(\bar{\xi}^T f)(\bar{x}, \bar{y})^T y^* \leqq 0, \tag{4.29}$$

$$x^* \geqq 0, \;\; y^* \geqq 0, \;\; z^* > 0. \tag{4.30}$$

Dividing (4.27), (4.28) and (4.29) by $z^* > 0$, we have

$$-\bar{x}\nabla_y(\bar{\xi}^T f)(\bar{x}, \bar{y})^T \bar{y} - \nabla_x(\bar{\xi}^T f)(\bar{x}, \bar{y}) \leqq 0, \tag{4.31}$$

$$\nabla_y(\bar{\xi}^T f)(\bar{x}, \bar{y}) \bar{x}^T \bar{x} + \nabla_y(\bar{\xi}^T f)(\bar{x}, \bar{y}) \leqq 0, \tag{4.32}$$

$$\nabla_x(\bar{\xi}^T f)(\bar{x}, \bar{y})^T \bar{x} - \nabla_y(\bar{\xi}^T f)(\bar{x}, \bar{y})^T \bar{y} \leqq 0. \tag{4.33}$$

From (4.30),

$$\bar{x} \geqq 0, \;\; \bar{y} \geqq 0. \tag{4.34}$$

By (4.32), $\nabla_y(\bar{\xi}^T f)(\bar{x},\bar{y})(\bar{x}^T\bar{x}+1) \leqq 0$. It implies that since $\bar{x}^T\bar{x}+1>0$,

$$-\nabla_y(\bar{\xi}^T f)(\bar{x},\bar{y}) \geqq 0. \tag{4.35}$$

From (4.31), $-\bar{x}\nabla_y(\bar{\xi}^T f)(\bar{x},\bar{y})^T\bar{y} \leqq \nabla_x(\bar{\xi}^T f)(\bar{x},\bar{y})$. Using (4.34) and (4.35), we obtain $0 \leqq -\bar{x}\nabla_y(\bar{\xi}^T f)(\bar{x},\bar{y})^T\bar{y} \leqq \nabla_x(\bar{\xi}^T f)(\bar{x},\bar{y})$. It implies that

$-\nabla_x(\bar{\xi}^T f)(\bar{x},\bar{y}) \leqq 0$. From (4.33), $\bar{x}^T\nabla_x(\bar{\xi}^T f)(\bar{x},\bar{y}) \leqq \bar{y}^T\nabla_y(\bar{\xi}^T f)(\bar{x},\bar{y})$. But since $\bar{x} \geqq 0$ and $\nabla_x(\bar{\xi}^T f)(\bar{x},\bar{y}) \geqq 0$, $\bar{x}^T\nabla_x(\bar{\xi}^T f)(\bar{x},\bar{y}) \geqq 0$ and since $\bar{y} \geqq 0$ and $\nabla_y(\bar{\xi}^T f)(\bar{x},\bar{y}) \leqq 0$, $\bar{y}^T\nabla_y(\bar{\xi}^T f)(\bar{x},\bar{y}) \leqq 0$. Then we have

$$0 \leqq \bar{x}^T\nabla_x(\bar{\xi}^T f)(\bar{x},\bar{y}) \leqq \bar{y}^T\nabla_y(\bar{\xi}^T f)(\bar{x},\bar{y}) \leqq 0.$$

Hence $\bar{x}^T\nabla_x(\bar{\xi}^T f)(\bar{x},\bar{y}) = \bar{y}^T\nabla_y(\bar{\xi}^T f)(\bar{x},\bar{y})$. Thus $(\bar{x},\bar{y},\bar{\xi})$ is feasible for (VSP) and (VSD) with $f_i(\bar{x},\bar{y})-\bar{y}^T\nabla_y(\bar{\xi}^T f)(\bar{x},\bar{y}) = f_i(\bar{x},\bar{y})-\bar{x}^T\nabla_x(\bar{\xi}^T f)(\bar{x},\bar{y})$, $i = 1,\cdots,p$. Since $(\bar{x},\bar{y},\bar{\xi})$ is feasible for (VSD), by weak duality,

$$\begin{aligned}&\Big(f_1(x,y)-y^T\nabla_y(\xi^T f)(x,y),\cdots,f_p(x,y)-y^T\nabla_y(\xi^T f)(x,y)\Big)\\ &\quad\not\leq \Big(f_1(\bar{x},\bar{y})-\bar{y}^T\nabla_y(\bar{\xi}^T f)(\bar{x},\bar{y}),\cdots,f_p(\bar{x},\bar{y})-\bar{y}^T\nabla_y(\bar{\xi}^T f)(\bar{x},\bar{y})\Big)\end{aligned}$$

and

$$\begin{aligned}&\Big(f_1(\bar{x},\bar{y})-\bar{x}^T\nabla_x(\bar{\xi}^T f)(\bar{x},\bar{y}),\cdots,f_p(\bar{x},\bar{y})-\bar{x}^T\nabla_x(\bar{\xi}^T f)(\bar{x},\bar{y})\Big)\\ &\quad\not\leq \Big(f_1(u,v)-u^T\nabla_u(\xi^T f)(u,v),\cdots,f_p(u,v)-u^T\nabla_u(\xi^T f)(u,v)\Big)\end{aligned}$$

for any feasible (u,v,ξ) of (VSP) and (VSD). Therefore $(\bar{x},\bar{y},\bar{\xi})$ is an efficient solution of (VSP) and $(\bar{x},\bar{\lambda},\bar{\xi})$ is an efficient solution of (VSD). □

By Theorem 4.2.4 and Lemma 4.2.1, we give the following corollary.

Corollary 4.2.1. Let (x^*, y^*, z^*) with $z^* > 0$ be a vector solution of vector matrix game $(B_1(\bar{x}, \bar{y}), \cdots, B_p(\bar{x}, \bar{y}))$, where $\bar{x} = x^*/z^*$. Let $\bar{y} = y^*/z^*$. If weak duality holds, there exists $\bar{\xi} \in \overset{o}{S}_p$ so that $(\bar{x}, \bar{y}, \bar{\xi})$ is an efficient solution of (VSP) and (VSD).

Now we give an example illustrating Theorems 4.2.3 and 4.2.4. But the following example does not satisfy the assumption of strong duality the fact that $\nabla_y f_1(\bar{x}, \bar{y}), \cdots, \nabla_y f_p(\bar{x}, \bar{y})$ are linearly independent.

Example 4.2.1. Let $f_1(x, y) = x^2 - y^2$ and $f_2(x, y) = y - x$. Consider the following vector optimization problem (VSP) together with its dual (VSD) as follows:

$$
\begin{array}{lll}
\text{(VSP)} & \text{Minimize} & (x^2 - y^2 + 2\lambda_1 y^2 - \lambda_2 y,\ y - x + 2\lambda_1 y^2 - \lambda_2 y) \\
& \text{subject to} & 2\lambda_1 y - \lambda_2 \geqq 0, \\
& & x \geqq 0,\ \lambda = (\lambda_1, \lambda_2) \in \overset{o}{S}_2 \\
\text{(VSD)} & \text{Maximize} & (u^2 - v^2 - 2\lambda_1 u^2 + \lambda_2 u,\ v - u - 2\lambda_1 u^2 + \lambda_2 u) \\
& \text{subject to} & 2\lambda_1 u - \lambda_2 \geqq 0, \\
& & v \geqq 0,\ \lambda = (\lambda_1, \lambda_2) \in \overset{o}{S}_2.
\end{array}
$$

Now we determine the set of all vector solutions of vector matrix game $(B_1(\bar{x}, \bar{y}), \cdots, B_p(\bar{x}, \bar{y}))$. Let

$$
B_i(x, y) = \begin{pmatrix} 0 & -x\nabla_y f_i(x, y)^T & -\nabla_x f_i(x, y) \\ \nabla_y f_i(x, y) x^T & 0 & \nabla_y f_i(x, y) \\ \nabla_x f_i(x, y)^T & -\nabla_y f_i(x, y)^T & 0 \end{pmatrix}.
$$

Then

$$B_1(x,y) = \begin{pmatrix} 0 & 2xy & -2x \\ -2xy & 0 & -2y \\ 2x & 2y & 0 \end{pmatrix} \quad \text{and} \quad B_2(x,y) = \begin{pmatrix} 0 & -x & 1 \\ x & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}.$$

Let $(x, y) \in \mathbb{R}^2$ and $(x^*, y^*, z^*) \in S_3$ be a vector solution of vector matrix game $(B_1(\bar{x}, \bar{y}), B_2(\bar{x}, \bar{y}))2$, if and only if there exist $\xi_1 > 0,\ \xi_2 > 0,\ \xi_1+\xi_2 = 1$ and $(x, y) \in \mathbb{R}^2$ such that

$$\left(\xi_1 \begin{pmatrix} 0 & 2xy & -2x \\ -2xy & 0 & -2y \\ 2x & 2y & 0 \end{pmatrix} + \xi_2 \begin{pmatrix} 0 & -x & 1 \\ x & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix} \right) \begin{pmatrix} x^* \\ y^* \\ z^* \end{pmatrix} \leqq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$\iff$ there exist $\xi_1 > 0,\ \xi_2 > 0,\ \xi_1 + \xi_2 = 1$ such that

$$\begin{pmatrix} x(2y\xi_1 - \xi_2)y^* - (2x\xi_1 - \xi_2)z^* \\ -x(2y\xi_1 - \xi_2)x^* - (2y\xi_1 - \xi_2)z^* \\ (2x\xi_1 - \xi_2)x^* + (2y\xi_1 - \xi_2)y^* \end{pmatrix} \leqq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus we determine the set of all the vector solutions of the vector matrix game $(B_1(\bar{x}, \bar{y}), B_2(\bar{x}, \bar{y}))$.

(I) the cast that $x > 0$:

(a) $2x\xi_1 - \xi_2 > 0,\ 2y\xi_1 - \xi_2 > 0$: $(x^*, y^*, z^*) = (0, 0, 1)$.

(b) $2x\xi_1 - \xi_2 > 0,\ 2y\xi_1 - \xi_2 = 0$: $(x^*, y^*, z^*) = \{(0, \alpha, 1-\alpha) \mid 0 \leqq \alpha \leqq 1\}$.

(c) $2x\xi_1 - \xi_2 > 0,\ 2y\xi_1 - \xi_2 < 0$: $(x^*, y^*, z^*) = (0, 1, 0)$.

(d) $2x\xi_1 - \xi_2 = 0,\ 2y\xi_1 - \xi_2 > 0$: $(x^*, y^*, z^*) = \{(\alpha, 0, 1-\alpha) \mid 0 \leqq \alpha \leqq 1\}$.

(e) $2x\xi_1 - \xi_2 = 0,\ 2y\xi_1 - \xi_2 = 0$: $(x^*, y^*, z^*) = \{(x_1, x_2, x_3) \mid x_1 \geqq 0, x_2 \geqq 0, x_3 \geqq 0, x_1 + x_2 + x_3 = 1\}$.

(f) $2x\xi_1 - \xi_2 = 0,\ 2y\xi_1 - \xi_2 < 0$: $(x^*, y^*, z^*) = (0, 1, 0)$.

(g) $2x\xi_1 - \xi_2 < 0,\ 2y\xi_1 - \xi_2 > 0$: $(x^*, y^*, z^*) = (1, 0, 0)$.

(h) $2x\xi_1 - \xi_2 < 0,\ 2y\xi_1 - \xi_2 = 0$: $(x^*, y^*, z^*) = \{(\alpha, 1-\alpha, 0) \mid 0 \leqq \alpha \leqq 1\}$.

(i) $2x\xi_1 - \xi_2 < 0,\ 2y\xi_1 - \xi_2 < 0$: $(x^*, y^*, z^*) = (0, 1, 0)$.

(II) the cast that $x = 0$:

(a) $2y\xi_1 - \xi_2 > 0$: $(x^*, y^*, z^*) = \{(1-\alpha, \alpha, 0) \mid 0 \leqq \alpha \leqq \frac{\xi_2}{2y\xi_1}, y > 0, \xi_1 > 0, \xi_2 > 0, \xi_1 + \xi_2 = 1\}$.

(b) $2y\xi_1 - \xi_2 = 0$: $(x^*, y^*, z^*) = \{(\alpha, 1-\alpha, 0) \mid 0 \leqq \alpha \leqq 1\}$.

(c) $2y\xi_1 - \xi_2 < 0$: $(x^*, y^*, z^*) = \{(\alpha, 1-\alpha, 0) \mid 0 \leqq \alpha \leqq 1\}$.

(III) the cast that $x < 0$:

(a) $2y\xi_1 - \xi_2 > 0$: $(x^*, y^*, z^*) = \{(\frac{2y\xi_1 - \xi_2}{2y\xi_1 - 2x\xi_1 - 2xy\xi_1 + x\xi_2}, -\frac{2x\xi_1 - \xi_2}{2y\xi_1 - 2x\xi_1 - 2xy\xi_1 + x\xi_2}, -\frac{2xy\xi_1 - x\xi_2}{2y\xi_1 - 2x\xi_1 - 2xy\xi_1 + x\xi_2})\ :\ 2y\xi_1 - 2x\xi_1 - 2xy\xi_1 + x\xi_2 > 0, 2x\xi_1 - \xi_2 < 0\}$

(b) $2y\xi_1 - \xi_2 = 0$: $(x^*, y^*, z^*) = \{(\alpha, 1-\alpha, 0) \mid 0 \leqq \alpha \leqq 1\}$.

(c) $2y\xi_1 - \xi_2 < 0$: $(x^*, y^*, z^*) = (1, 0, 0)$.

Let $(x, y) \in \mathbb{R}^2$ and $S_{(x,y)}$ the set of vector solutions of vector matrix game

$(B_1(\bar{x},\bar{y}), B_2(\bar{x},\bar{y}))$. From (I), (II) and (III),

$$\begin{aligned} S_{(x,y)} &= \{(\alpha, 1-\alpha, 0) \mid 0 \leqq \alpha \leqq 1\} \cup \{(0, \alpha, 1-\alpha) \mid 0 \leqq \alpha \leqq 1\} \\ &\cup \{(\alpha, 0, 1-\alpha) \mid 0 \leqq \alpha \leqq 1\} \cup \{(\alpha, \beta, \gamma) \mid \alpha \geqq 0, \beta \geqq 0, \gamma \geqq 0, \alpha+\beta+\gamma = 1\} \\ &\cup \left\{\left(\frac{2y\xi_1 - \xi_2}{2y\xi_1 - 2x\xi_1 - 2xy\xi_1 + x\xi_2}, -\frac{2x\xi_1 - \xi_2}{2y\xi_1 - 2x\xi_1 - 2xy\xi_1 + x\xi_2},\right.\right. \\ &\qquad \left. -\frac{2xy\xi_1 - x\xi_2}{2y\xi_1 - 2x\xi_1 - 2xy\xi_1 + x\xi_2}\right) \mid x < 0, 2y\xi_1 - 2x\xi_1 - 2xy\xi_1 + x\xi_2 > 0, \\ &\qquad 2y\xi_1 - \xi_2 > 0,\ 2x\xi_1 - \xi_2 < 0\Big\}. \end{aligned}$$

Let $(\bar{x},\bar{y},\bar{\xi})$ be feasible for (VSP) and (VSD) with $\bar{y}^T\nabla_y(\bar{\xi}^T f)(\bar{x},\bar{y}) = \bar{x}^T\nabla_x(\bar{\xi}^T f)(\bar{x},\bar{y}) = 0$. We can easily check that

$$\begin{aligned} &\{(\bar{x},\bar{y},\bar{\xi}) \mid (\bar{x},\bar{y},\bar{\xi}) : \text{an efficient solution of (VSP) and (VSD)}, \\ &\qquad \bar{y}^T\nabla_y(\bar{\xi}^T f)(\bar{x},\bar{y}) = \bar{x}^T\nabla_x(\bar{\xi}^T f)(\bar{x},\bar{y}) = 0\} \\ &= \left\{\left(\frac{\xi_2}{2\xi_1}, \frac{\xi_2}{2\xi_1}, \xi_1, \xi_2\right) \mid \xi_1 > 0,\ \xi_2 > 0,\ \xi_1 + \xi_2 = 1\right\}. \end{aligned}$$

Thus,

$$\begin{aligned} &\left\{\left(\frac{\bar{x}}{1+\bar{x}+\bar{y}}, \frac{\bar{y}}{1+\bar{x}+\bar{y}}, \frac{1}{1+\bar{x}+\bar{y}}\right) \mid \bar{x} = \frac{\xi_2}{2\xi_1}, \bar{y} = \frac{\xi_2}{2\xi_1}, \xi_1 > 0, \xi_2 > 0, \xi_1 + \xi_2 = 0\right\} \\ &= \left\{\left(\frac{\xi_2}{2}, \frac{\xi_2}{2}, \xi_1\right) \mid \xi_1 > 0, \xi_2 > 0, \xi_1 + \xi_2 = 1\right\} \\ &\subset S_{(\bar{x},\bar{y})}. \end{aligned}$$

Therefore, Theorem 4.2.3 holds.

Let $(x, y) \in \mathbb{R}^2$ and $S_{(x,y)}$ be the set of vector solutions of vector matrix game $(B_1(x, y), B_2(x, y))$. Then

$$\bigcup_{(x,y)\in\mathbb{R}^2} S_{(x,y)} = \{(\alpha, 1-\alpha, 0) \mid 0 \leqq \alpha \leqq 1\} \cup \{(0, \alpha, 1-\alpha) \mid 0 \leqq \alpha \leqq 1\}$$

$$\cup\{(\alpha, 0, 1-\alpha) \mid 0 \leqq \alpha \leqq 1\} \cup \{(\alpha, \beta, \gamma) \mid \alpha \geqq 0, \beta \geqq 0, \gamma \geqq 0, \alpha + \beta + \gamma = 1\}$$

$$\cup \bigg\{ \bigg(\frac{2y\xi_1 - \xi_2}{2y\xi_1 - 2x\xi_1 - 2xy\xi_1 + x\xi_2}, -\frac{2x\xi_1 - \xi_2}{2y\xi_1 - 2x\xi_1 - 2xy\xi_1 + x\xi_2},$$

$$-\frac{2xy\xi_1 - x\xi_2}{2y\xi_1 - 2x\xi_1 - 2xy\xi_1 + x\xi_2} \bigg) \mid x < 0, 2y\xi_1 - 2x\xi_1 - 2xy\xi_1 + x\xi_2 > 0,$$

$$2y\xi_1 - \xi_2 > 0,\ 2x\xi_1 - \xi_2 < 0 \bigg\}.$$

So,

$$\left\{ \left(\frac{x^*}{z^*}, \frac{y^*}{z^*} \right) \mid z^* > 0 \text{ and } (x^*, y^*, z^*) \in S_{(\frac{x^*}{z^*}, \frac{y^*}{z^*})} \right\}$$

$$= \left\{ \left(\frac{\xi_2}{2\xi_1}, \frac{\xi_2}{2\xi_1} \right) \mid \xi_1 > 0, \xi_2 > 0, \xi_1 + \xi_2 = 1 \right\}.$$

Let F be the set of all feasible solutions of (VSD). Then we can check that $\left\{ \left(\frac{\xi_2}{2\xi_1}, \frac{\xi_2}{2\xi_1}, \xi_1, \xi_2 \right) \mid \xi_1 > 0, \xi_2 > 0, \xi_1 + \xi_2 = 1 \right\} = F$ and $(\frac{\xi_2}{2\xi_1})\nabla_y(\bar{\xi}^T f)(\frac{\xi_2}{2\xi_1}, \frac{\xi_2}{2\xi_1}) = (\frac{\xi_2}{2\xi_1})\nabla_x(\bar{\xi}^T f)(\frac{\xi_2}{2\xi_1}, \frac{\xi_2}{2\xi_1}) = 0$. Therefore, Theorem 4.2.4 holds. □

Chapter 5
Examples for Vector Matrix Game

We recall six kinds of solutions for vector matrix game which were defined in Definition 2.1.3 in Chapter 2. In this section, we intoduce examples showing that such six kinds of solutions may be different.

Example 5.1. Let

$$B_1 = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } B_2 = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Let $S_3 = \{(x_1, x_2, x_3) \mid x_1 \geqq 0,\ \ x_2 \geqq 0,\ \ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1\}$. Then we calculate the set of all the vector solution of vector matrix game (B_1, B_2). By Lemma 3.2.1, $\bar{y} \in S_3$ is a vector solution of vector matrix game (B_1, B_2) if and only if there exist $\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_1 + \lambda_2 = 1$ such that

$$\left(\lambda_1 \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \right) \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \end{pmatrix} \leqq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Longleftrightarrow$ there exist $\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_1 + \lambda_2 = 1$ such that

$$\begin{pmatrix} \bar{y}_2 - (\lambda_1 - \lambda_2)\bar{y}_3 \\ -\bar{y}_1 \\ (\lambda_1 - \lambda_2)\bar{y}_1 \end{pmatrix} \leqq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus

(i) if $\lambda_1 = \lambda_2$; $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, 0, y_3) : y_1 \geqq 0,\ y_3 \geqq 0,\ y_1 + y_3 = 1\}$, $(\bar{y}_1, \bar{y}_2, \bar{y}_3)$ is a vector solution of vector matrix game $B_i,\ i = 1, 2$.

(ii) if $\lambda_1 > \lambda_2$; $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3) : y_3 > y_2 \geqq 0,\ y_2 + y_3 = 1\}$, $(\bar{y}_1, \bar{y}_2, \bar{y}_3)$ is a vector solution of vector matrix game $B_i,\ i = 1, 2$.

(iii) if $\lambda_1 < \lambda_2$; $(1, 0, 0)$ is a vector solution of vector matrix game $B_i,\ i = 1, 2$.

Therefore, the set of all the vector solution of vector matrix game (B_1, B_2) is $sol(\text{VMG}) = \{(\bar{y}_1, 0, \bar{y}_3) : \bar{y}_1 \geqq 0,\ \bar{y}_3 \geqq 0,\ \bar{y}_1 + \bar{y}_3 = 1\} \cup \{(0, \bar{y}_2, \bar{y}_3) : \bar{y}_3 > \bar{y}_2 \geqq 0,\ \bar{y}_2 + \bar{y}_3 = 1\}$. □

Example 5.2. Consider the B_1, B_2 and S_3 described in Example 5.1. Then we calculate the set of all the weakly vector solution of vector matrix game (B_1, B_2). By Lemma 3.2.2, $\bar{y} \in S_3$ is a weakly vector solution of vector matrix game (B_1, B_2) if and only if there exist $\lambda_1 \geqq 0,\ \lambda_2 \geqq 0,\ \lambda_1 + \lambda_2 = 1$ such that

$$\left(\lambda_1 \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \right) \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \end{pmatrix} \leqq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Longleftrightarrow$ there exist $\lambda_1 \geqq 0, \lambda_2 \geqq 0,\ \ \lambda_1 + \lambda_2 = 1$ such that

$$\begin{pmatrix} \bar{y}_2 - (\lambda_1 - \lambda_2)\bar{y}_3 \\ -\bar{y}_1 \\ (\lambda_1 - \lambda_2)\bar{y}_1 \end{pmatrix} \leqq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus

(i) if $\lambda_1 = 0, \lambda_2 = 1$; $(1, 0, 0)$ is a weakly vector solution of vector matrix game (B_1, B_2).

(ii) if $\lambda_1 = 1, \lambda_2 = 0$; $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3) : y_3 \geqq y_2 \geqq 0,\ y_2 + y_3 = 1\}$, $(\bar{y}_1, \bar{y}_2, \bar{y}_3)$ is a weakly vector solution of vector matrix game (B_1, B_2).

(iii) if $\lambda_1 = \lambda_2$; $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, 0, y_3) : y_1 \geqq 0,\ y_3 \geqq 0,\ y_1 + y_3 = 1\}$, $(\bar{y}_1, \bar{y}_2, \bar{y}_3)$ is a weakly vector solution of vector matrix game (B_1, B_2).

(iv) if $\lambda_1 > \lambda_2$; $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3) : y_3 > y_2 \geqq 0,\ y_2 + y_3 = 1\}$, $(\bar{y}_1, \bar{y}_2, \bar{y}_3)$ is a weakly vector solution of vector matrix game (B_1, B_2).

(v) if $\lambda_1 < \lambda_2$; $(1, 0, 0)$ is a weakly vector solution of vector matrix game (B_1, B_2).

Therefore, the set of all the weakly vector solution of vector matrix game (B_1, B_2) is $sol(\text{WVMG}) = \{(\bar{y}_1, 0, \bar{y}_3) : \bar{y}_1 \geqq 0, \bar{y}_3 \geqq 0, \bar{y}_1 + \bar{y}_3 = 1\} \cup \{(0, \bar{y}_2, \bar{y}_3) :\ \bar{y}_3 \geqq \bar{y}_2 \geqq 0,\ \bar{y}_2 + \bar{y}_3 = 1\}$. □

Example 5.3. Consider the B_1, B_2 and S_3 described in Example 5.1. Then we calculate the set of all the scalarizing solution of vector matrix game (B_1, B_2). By Lemma 3.2.3, $(\bar{x}, \bar{y}) \in S_3 \times S_3$ is a scalarizing solution of vector matrix game (B_1, B_2) if and only if there exist $\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_1 + \lambda_2 = 1$ such that

$$\left(\lambda_1 \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\right) \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \end{pmatrix} \leqq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

and

$$\left(\lambda_1\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \lambda_2\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\right)\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} \leqq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$\Longleftrightarrow$ there exist $\lambda_1 > 0, \lambda_2 > 0, \quad \lambda_1 + \lambda_2 = 1$ such that

$$\begin{pmatrix} \bar{y}_2 - (\lambda_1 - \lambda_2)\bar{y}_3 \\ -\bar{y}_1 \\ (\lambda_1 - \lambda_2)\bar{y}_1 \end{pmatrix} \leqq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

and

$$\begin{pmatrix} \bar{x}_2 - (\lambda_1 - \lambda_2)\bar{x}_3 \\ -\bar{x}_1 \\ (\lambda_1 - \lambda_2)\bar{x}_1 \end{pmatrix} \leqq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, the set of all the scalarizing solution of vector matrix game (B_1, B_2) is $sol(\text{SVMG}) = \{(\bar{x}_1, 0, \bar{x}_3, \bar{y}_1, 0, \bar{y}_3) : \bar{x}_1 \geqq 0,\ \bar{x}_3 \geqq 0,\ \bar{x}_1 + \bar{x}_3 = 1,\ \bar{y}_1 \geqq 0,\ \bar{y}_3 \geqq 0,\ \bar{y}_1 + \bar{y}_3 = 1\} \cup \{(0, \bar{x}_2, \bar{x}_3, 0, \bar{y}_2, \bar{y}_3) : \bar{x}_3 > \bar{x}_2 \geqq 0,\ \bar{x}_2 + \bar{x}_3 = 1,\ \bar{y}_3 > \bar{y}_2 \geqq 0,\ \bar{y}_2 + \bar{y}_3 = 1\}$. □

Example 5.4. Consider the B_1, B_2 and S_3 described in Example 5.1. Then we calculate the set of all the weakly scalarizing solution of vector matrix game (B_1, B_2). By Lemma 3.2.4, $(\bar{x}, \bar{y}) \in S_3 \times S_3$ is a weakly scalarizing solution of vector matrix game (B_1, B_2) f and only if there exist $\lambda_1 \geqq 0,\ \lambda_2 \geqq$

$0,\ \lambda_1 + \lambda_2 = 1$ such that

$$\left(\lambda_1 \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\right) \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \end{pmatrix} \leqq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

and

$$\left(\lambda_1 \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\right) \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} \leqq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$\Longleftrightarrow$ there exist $\lambda_1 \geqq 0, \lambda_2 \geqq 0, \quad \lambda_1 + \lambda_2 = 1$ such that

$$\begin{pmatrix} \bar{y}_2 - (\lambda_1 - \lambda_2)\bar{y}_3 \\ -\bar{y}_1 \\ (\lambda_1 - \lambda_2)\bar{y}_1 \end{pmatrix} \leqq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

and

$$\begin{pmatrix} \bar{x}_2 - (\lambda_1 - \lambda_2)\bar{x}_3 \\ -\bar{x}_1 \\ (\lambda_1 - \lambda_2)\bar{x}_1 \end{pmatrix} \leqq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, the set of all the weakly scalarizing solution of vector matrix game (B_1, B_2) is $sol(\text{WSVMG}) = \{(\bar{x}_1, 0, \bar{x}_3, \bar{y}_1, 0, \bar{y}_3) : \bar{x}_1 \geqq 0,\ \bar{x}_3 \geqq 0,\ \bar{x}_1 + \bar{x}_3 = 1,\ \bar{y}_1 \geqq 0,\ \bar{y}_3 \geqq 0,\ \bar{y}_1 + \bar{y}_3 = 1\} \cup \{(0, \bar{x}_2, \bar{x}_3, 0, \bar{y}_2, \bar{y}_3) : \bar{x}_3 \geqq \bar{x}_2 \geqq 0,\ \bar{x}_2 + \bar{x}_3 = 1,\ \bar{y}_3 \geqq \bar{y}_2 \geqq 0,\ \bar{y}_2 + \bar{y}_3 = 1\}$. □

Example 5.5. Consider the B_1, B_2 and S_3 described in Example 5.1. Then we calculate the set of all the efficient solution of vector matrix game (B_1, B_2). By Lemma 3.2.7, $(\bar{x}, \bar{y}) \in S_3 \times S_3$ is an efficient solution of vector matrix game (B_1, B_2) if and only if there exist $\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_1 + \lambda_2 = 1,\ \mu_1 > 0,\ \mu_2 > 0,\ \mu_1 + \mu_2 = 1$ such that

$$(*) \qquad \begin{pmatrix} \bar{y}_2 - (\lambda_1 - \lambda_2)\bar{y}_3 \\ -\bar{y}_1 \\ (\lambda_1 - \lambda_2)\bar{y}_1 \end{pmatrix} \in N_{S_3} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix}$$

and

$$(**) \qquad \begin{pmatrix} \bar{x}_2 - (\mu_1 - \mu_2)\bar{x}_3 \\ -\bar{x}_1 \\ (\mu_1 - \mu_2)\bar{x}_1 \end{pmatrix} \in N_{S_3} \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \end{pmatrix}.$$

Now using the relations $(*)$ and $(**)$, we determine the set sol(EVMG) which is the set of all efficient solutions of the matrix game (B_1, B_2).

(I) the case that $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \overset{o}{S}_3$:

$(*) \iff (\bar{y}_1, \bar{y}_2, \bar{y}_3) \in S_3$ and there exist $\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_1 + \lambda_2 = 1,\ \alpha \in \mathbb{R}$ such that

$$\begin{cases} \bar{y}_2 - (\lambda_1 - \lambda_2)\bar{y}_3 &= \alpha \\ -\bar{y}_1 &= \alpha \\ (\lambda_1 - \lambda_2)\bar{y}_1 &= \alpha. \end{cases}$$

$\iff$ $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3) \ : \ y_2 = \frac{1}{2} - \frac{\lambda_2}{2\lambda_1},\ y_3 = \frac{1}{2} + \frac{\lambda_2}{2\lambda_1},\ y_2 + y_3 = 1,\ \lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_1 + \lambda_2 = 1\}.$

$\iff$ $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3) \ : \ y_3 > y_2 \geqq 0,\ y_2 + y_3 = 1\}.$

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3) \ : \ y_3 > y_2 > 0,\ y_2 + y_3 = 1\}$:

$(**)$ $\iff$ $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in S_3$ and there exist $\mu_1 > 0,\ \mu_2 > 0,\ \mu_1 + \mu_2 = 1$, $\alpha \in \mathbb{R}$ and $\beta \geqq 0$ such that

$$\begin{cases} \bar{x}_2 - (\mu_1 - \mu_2)\bar{x}_3 & = \alpha - \beta \\ -\bar{x}_1 & = \alpha \\ (\mu_1 - \mu_2)\bar{x}_1 & = \alpha. \end{cases}$$

$\iff$ $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \ : \ x_2 = \frac{1}{2} - \frac{\mu_2+\beta}{2\mu_1},\ x_3 = \frac{1}{2} + \frac{\mu_2+\beta}{2\mu_1},\ x_2 + x_3 = 1,\ \mu_1 > 0,\ \mu_2 > 0,\ \mu_1 + \mu_2 = 1,\ \beta \geqq 0\}.$

$\iff$ $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \ : \ x_3 > x_2 \geqq 0,\ x_2 + x_3 = 1\}.$

But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \overset{o}{S}_3$, so, in this case, there is no efficient solution.

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (0, 0, 1)$:

$(**)$ $\iff$ $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in S_3$ and there exist $\mu_1 > 0,\ \mu_2 > 0,\ \mu_1 + \mu_2 = 1$, $\alpha \in \mathbb{R}$, $\beta \geqq 0$ and $\gamma \geqq 0$ such that

$$\begin{cases} \bar{x}_2 - (\mu_1 - \mu_2)\bar{x}_3 & = \alpha - \beta \\ -\bar{x}_1 & = \alpha - \gamma \\ (\mu_1 - \mu_2)\bar{x}_1 & = \alpha. \end{cases}$$

$\iff$ $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) \ : \ x_1 = \frac{\gamma}{2\mu_1}, \ x_2 = \frac{1}{2} - \frac{\mu_2+\beta}{2\mu_1}, \ x_3 = \frac{1}{2} + \frac{\mu_2+\beta-\gamma}{2\mu_1}, \ x_1 + x_2 + x_3 = 1, \ \mu_1 > 0, \ \mu_2 > 0, \ \mu_1 + \mu_2 = 1, \ \beta \geqq 0, \ \gamma \geqq 0\}.$

$\iff$ $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) \ : \ x_1 \geqq 0, \ 0 \leqq x_2 < \frac{1}{2}, \ x_3 \geqq 0, \ x_1 + x_2 + x_3 = 1\}.$

By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \overset{o}{S}_3$, in this case, $\{(\bar{x}_1, \bar{x}_2, \bar{x}_3, 0, 0, 1) \ : \ \bar{x}_1 > 0, \ 0 < \bar{x}_2 < \frac{1}{2}, \ \bar{x}_3 > 0, \ \bar{x}_1 + \bar{x}_2 + \bar{x}_3 = 1\} \subset sol(\text{EVMG})$.

(II) the case that $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \ : \ x_2 > 0, \ x_3 > 0, \ x_2 + x_3 = 1\}$:

$(*)$ $\iff$ $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in S_3$ and there exist $\lambda_1 > 0, \ \lambda_2 > 0, \ \lambda_1 + \lambda_2 = 1, \ \alpha \in \mathbb{R}$ and $\beta \geqq 0$ such that

$$\begin{cases} \bar{y}_2 - (\lambda_1 - \lambda_2)\bar{y}_3 &= \alpha - \beta \\ -\bar{y}_1 &= \alpha \\ (\lambda_1 - \lambda_2)\bar{y}_1 &= \alpha. \end{cases}$$

$\iff$ $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3) \ : \ y_2 = \frac{1}{2} - \frac{\lambda_2+\beta}{2\lambda_1}, \ y_3 = \frac{1}{2} + \frac{\lambda_2+\beta}{2\lambda_1}, \ y_2 + y_3 = 1, \ \lambda_1 > 0, \ \lambda_2 > 0, \ \lambda_1 + \lambda_2 = 1, \ \beta \geqq 0\}.$

$\iff$ $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3) \mid y_3 > y_2 \geqq 0, \ y_2 + y_3 = 1\}.$

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3) \ : \ y_3 > y_2 > 0, \ y_2 + y_3 = 1\}$:

$(**)$ $\iff$ $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in S_3$ and there exists $\mu_1 > 0, \ \mu_2 > 0, \ \mu_1 + \mu_2 = 1,$

$\alpha \in \mathbb{R}$ and $\beta \geqq 0$ such that

$$\begin{cases} \bar{x}_2 - (\mu_1 - \mu_2)\bar{x}_3 & = \alpha - \beta \\ -\bar{x}_1 & = \alpha \\ (\mu_1 - \mu_2)\bar{x}_1 & = \alpha. \end{cases}$$

$\iff \quad (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \ : \ x_3 > x_2 > 0,\ x_2 + x_3 = 1\}.$

By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \ : \ x_2 > 0,\ x_3 > 0,\ x_2 + x_3 = 1\}$, in this case, $\{(0, \bar{x}_2, \bar{x}_3, 0, \bar{y}_2, \bar{y}_3) \ : \ \bar{x}_3 > \bar{x}_2 > 0,\ \bar{x}_2 + \bar{x}_3 = 1,\ \bar{y}_3 > \bar{y}_2 > 0,\ \bar{y}_2 + \bar{y}_3 = 1\} \subset sol(EVMG)$.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (0, 0, 1)$:

$(**) \iff (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \ : \ x_3 > x_2 > 0,\ x_2 + x_3 = 1\}.$

By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \ : \ x_2 > 0,\ x_3 > 0,\ x_2 + x_3 = 1\}$, in this case, $\{(0, \bar{x}_2, \bar{x}_3, 0, 0, 1) \ : \ \bar{x}_3 > \bar{x}_2 > 0, \bar{x}_2 + \bar{x}_3 = 1\} \subset sol(\text{EVMG})$.

(III) the case that $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$:

$(*) \iff (\bar{y}_1, \bar{y}_2, \bar{y}_3) \in S_3$ and there exist $\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_1 + \lambda_2 = 1,\ \alpha \in \mathbb{R}$, $\beta \geqq 0$ and $\gamma \geqq 0$ such that

$$\begin{cases} \bar{y}_2 - (\lambda_1 - \lambda_2)\bar{y}_3 & = \alpha - \beta \\ -\bar{y}_1 & = \alpha - \gamma \\ (\lambda_1 - \lambda_2)\bar{y}_1 & = \alpha. \end{cases}$$

$\iff \quad (\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, y_2, y_3) \ : \ y_1 \geqq 0,\ 0 \leqq y_2 < \frac{1}{2},\ y_3 \geqq 0,\ y_1 + y_2 + y_3 = 1\}.$

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, y_2, y_3) \ : \ y_1 > 0, \ 0 < y_2 < \frac{1}{2}, \ y_3 > 0, \ y_1 + y_2 + y_3 = 1\}$:

$(**) \iff (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in S_3$ and $\mu_1 > 0, \ \mu_2 > 0, \ \mu_1 + \mu_2 = 1$ and $\alpha \in \mathbb{R}$ such that

$$\begin{cases} \bar{x}_2 - (\mu_1 - \mu_2)\bar{x}_3 &= \alpha \\ -\bar{x}_1 &= \alpha \\ (\mu_1 - \mu_2)\bar{x}_1 &= \alpha. \end{cases}$$

$\iff (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \ : \ x_3 > x_2 \geqq 0, \ x_2 + x_3 = 1\}.$

By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$, in this case, $\{(0, 0, 1, \bar{y}_1, \bar{y}_2, \bar{y}_3) \ : \ \bar{y}_1 > 0, \ 0 < \bar{y}_2 < \frac{1}{2}, \ \bar{y}_3 > 0, \ \bar{y}_1 + \bar{y}_2 + \bar{y}_3 = 1\} \subset sol(\text{EVMG})$.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3) \ : \ y_3 > y_2 > 0, \ y_2 + y_3 = 1\}$:

$(**) \iff (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \ : \ x_3 > x_2 \geqq 0, \ x_2 + x_3 = 1\}.$

By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \{(0, 0, 1)\}$, in this case, $\{(0, 0, 1, 0, \bar{y}_2, \bar{y}_3) \ : \ \bar{y}_3 > \bar{y}_2 > 0, \ \bar{y}_2 + \bar{y}_3 = 1\} \subset sol(\text{EVMG})$.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = \{(0, 0, 1)\}$:

$(**) \iff (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) \ : \ x_1 \geqq 0, \ 0 \leqq x_2 < \frac{1}{2}, \ x_3 \geqq 0, \ x_1 + x_2 + x_3 = 1\}.$

By the condition, $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$, in this case, $\{(0, 0, 1, 0, 0, 1)\} \subset sol(\text{EVMG})$.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, 0, y_3) \ : \ y_1 > 0, \ y_3 > 0, \ y_1 + y_3 = 1\}$:

$(**)\iff(\bar{x}_1,\bar{x}_2,\bar{x}_3)\in S_3$ and there exists $\mu_1>0,\ \mu_2>0,\ \mu_1+\mu_2=1$, $\alpha\in\mathbb{R}$ and $\beta\geqq 0$ such that

$$\begin{cases}\bar{x}_2-(\mu_1-\mu_2)\bar{x}_3 &=\alpha\\ -\bar{x}_1 &=\alpha-\beta\\ (\mu_1-\mu_2)\bar{x}_1 &=\alpha.\end{cases}$$

$\iff(\bar{x}_1,\bar{x}_2,\bar{x}_3)\in\{(x_1,x_2,x_3)\ :\ x_1=\frac{\beta}{2\mu_1},\ x_2=\frac{1}{2}-\frac{\mu_2}{2\mu_1},\ x_3=\frac{1}{2}+\frac{\mu_2-\beta}{2\mu_1},\ x_1+x_2+x_3=1,\ \mu_1>0,\ \mu_2>0,\ \mu_1+\mu_2=1,\ \beta\geqq 0\}$.

$\iff(\bar{x}_1,\bar{x}_2,\bar{x}_3)\in\{(x_1,x_2,x_3)\ :\ x_1\geqq 0,\ 0\leqq x_2<\frac{1}{2},\ x_3\geqq 0,\ x_1+x_2+x_3=1\}$.

By the condition $(\bar{x}_1,\bar{x}_2,\bar{x}_3)=(0,0,1)$, in this case, $\{(0,0,1,\bar{y}_1,0,\bar{y}_3)\ :\ \bar{y}_1>0,\ \bar{y}_3>0,\ \bar{y}_1+\bar{y}_3=1\}\subset sol(\text{EVMG})$.

- in the case that $(\bar{y}_1,\bar{y}_2,\bar{y}_3)=(1,0,0)$:

$(**)\iff(\bar{x}_1,\bar{x}_2,\bar{x}_3)\in\overset{o}{S}_3$ and $\mu_1>0,\ \mu_2>0,\ \mu_1+\mu_2=1,\ \alpha\in\mathbb{R},\ \beta\geqq 0$ and $\gamma\geqq 0$ such that

$$\begin{cases}\bar{x}_2-(\mu_1-\mu_2)\bar{x}_3 &=\alpha\\ -\bar{x}_1 &=\alpha-\beta\\ (\mu_1-\mu_2)\bar{x}_1 &=\alpha-\gamma.\end{cases}$$

$\iff(\bar{x}_1,\bar{x}_2,\bar{x}_3)\in\{(x_1,x_2,x_3)\ :\ x_1=\frac{\beta-\gamma}{2\mu_1},\ x_2=\frac{1}{2}-\frac{\mu_2-\gamma}{2\mu_1},\ x_3=\frac{1}{2}+\frac{\mu_2-\beta}{2\mu_1},\ x_1+x_2+x_3=1,\ \mu_1>0,\ \mu_2>0,\ \mu_1+\mu_2=1,\ \beta\geqq 0,\ \gamma\geqq 0\}$.

$\iff$ $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) \;:\; x_1 \geqq 0,\; x_2 \geqq 0,\; x_3 \geqq 0,\; x_1+x_2+x_3 = 1\}$.

By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$, in this case, $\{(0, 0, 1, 1, 0, 0)\} \subset$ *sol*(EVMG).

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, y_2, 0) \;:\; y_1 > y_2 > 0,\; y_1 + y_2 = 1\}$:

$(**)$ $\iff$ $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in S_3$ and there exists $\mu_1 > 0,\; \mu_2 > 0,\; \mu_1 + \mu_2 = 1$, $\alpha \in \mathbb{R}$ and $\beta \geqq 0$ such that

$$\begin{cases} \bar{x}_2 - (\mu_1 - \mu_2)\bar{x}_3 & = \alpha \\ -\bar{x}_1 & = \alpha \\ (\mu_1 - \mu_2)\bar{x}_1 & = \alpha - \beta. \end{cases}$$

$\iff$ $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) \;:\; x_1 = -\frac{\beta}{2\mu_1},\; x_2 = \frac{1}{2} - \frac{\mu_2 - \beta}{2\mu_1},\; x_3 = \frac{1}{2} + \frac{\mu_2}{2\mu_1},\; x_1 \geqq 0,\; x_2 \geqq 0,\; x_3 \geqq 0,\; x_1+x_2+x_3 = 1,\; \mu_1 > 0,\; \mu_2 > 0,\; \mu_1 + \mu_2 = 1,\; \beta \geqq 0\}$.

$\iff$ $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \;:\; x_3 > x_2 \geqq 0,\; x_2 + x_3 = 1\}$.

By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$, in this case, $\{(0, 0, 1, \bar{y}_1, \bar{y}_2, 0) \;:\; \bar{y}_1 > \bar{y}_2 > 0,\; \bar{y}_1 + \bar{y}_2 = 1\} \subset$ *sol*(EVMG).

(IV) the case that $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, 0, x_3) \;:\; x_1 > 0,\; x_3 > 0,\; x_1 + x_3 = 1\}$:

$(*)$ $\iff$ $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in S_3$ and there exist $\lambda_1 > 0,\; \lambda_2 > 0,\; \lambda_1 + \lambda_2 = 1,\; \alpha \in \mathbb{R}$ and $\beta \geqq 0$ such that

$$\begin{cases} \bar{y}_2 - (\lambda_1 - \lambda_2)\bar{y}_3 & = \alpha \\ -\bar{y}_1 & = \alpha - \beta \\ (\lambda_1 - \lambda_2)\bar{y}_1 & = \alpha. \end{cases}$$

$\Longleftrightarrow$ $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, y_2, y_3) \ : \ y_1 = \frac{\beta}{2\lambda_1}, \ y_2 = \frac{1}{2} - \frac{\lambda_2}{2\lambda_1}, \ y_3 = \frac{1}{2} + \frac{\lambda_2 - \beta}{2\lambda_1}, \ y_1 + y_2 + y_3 = 1, \ \lambda_1 > 0, \ \lambda_2 > 0, \ \beta \geqq 0\}$.

$\Longleftrightarrow$ $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, y_2, y_3) \ : \ y_1 \geqq 0, \ 0 \leqq y_2 < \frac{1}{2}, \ y_3 \geqq 0, \ y_1 + y_2 + y_3 = 1\}$.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, y_2, y_3) \ : \ y_1 > 0, \ 0 < y_2 < \frac{1}{2}, \ y_3 > 0, \ y_1 + y_2 + y_3 = 1\}$:

$(**)$ $\Longleftrightarrow$ $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \ : \ x_3 > x_2 \geqq 0, \ x_2 + x_3 = 1\}$.

But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, 0, x_3) \ : \ x_1 > 0, \ x_3 > 0, \ x_1 + x_3 = 1\}$, so, in this case, there is no efficient solution.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3) \ : \ y_3 > y_2 > 0, \ y_2 + y_3 = 1\}$:

$(**)$ $\Longleftrightarrow$ $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \ : \ x_3 > x_2 \geqq 0, \ x_2 + x_3 = 1\}$.

But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, 0, x_3) \ : \ x_1 > 0, \ x_3 > 0, \ x_1 + x_3 = 1\}$, so, in this case, there is no efficient solution.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (0, 0, 1)$:

$(**)$ $\Longleftrightarrow$ $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) \ : \ x_1 \geqq 0, \ 0 \leqq x_2 < \frac{1}{2}, \ x_3 \geqq 0, \ x_1 + x_2 + x_3 = 1\}$.

By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, 0, x_3) \ : \ x_1 > 0, \ x_3 > 0, \ x_1 + x_3 = 1\}$, in this case, $\{(\bar{x}_1, 0, \bar{x}_3, 0, 0, 1) \ : \ \bar{x}_1 > 0, \ \bar{x}_3 > 0, \ \bar{x}_1 + \bar{x}_3 = 1\} \subset sol(\text{EVMG})$.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, 0, y_3) \ : \ y_1 > 0, \ y_3 > 0, \ y_1 + y_3 = 1\}$:

$(**) \iff (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) \ : \ x_1 \geqq 0, \ 0 \leqq x_2 < \frac{1}{2}, \ x_3 \geqq 0, \ x_1 + x_2 + x_3 = 1\}$.

By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, 0, x_3) \ : \ x_1 > 0, \ x_3 > 0, \ x_1 + x_3 = 1\}$, in this case, $\{(\bar{x}_1, 0, \bar{x}_3, \bar{y}_1, 0, \bar{y}_3) \ : \ \bar{x}_1 > 0, \ \bar{x}_3 > 0, \ \bar{x}_1 + \bar{x}_3 = 1, \ \bar{y}_1 > 0, \ \bar{y}_3 > 0, \ \bar{y}_1 + \bar{y}_3 = 1\} \subset sol(\text{EVMG})$.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (1, 0, 0)$:

$(**) \iff (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) \ : \ x_1 \geqq 0, x_2 \geqq 0, x_3 \geqq 0, \ x_1 + x_2 + x_3 = 1\}$.

By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, 0, x_3) \ : \ x_1 > 0, \ x_3 > 0, \ x_1 + x_3 = 1\}$, in this case, $\{(\bar{x}_1, 0, \bar{x}_3, 1, 0, 0) \ : \ \bar{x}_1 > 0, \ \bar{x}_3 > 0, \ \bar{x}_1 + \bar{x}_3 = 1\} \subset sol(\text{EVMG})$.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, y_2, 0) \ : \ y_1 > y_2 > 0, \ y_1 + y_2 = 1\}$:

$(**) \iff (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \ : \ x_3 > x_2 \geqq 0, \ x_2 + x_3 = 1\}$.

But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, 0, x_3) \ : \ x_1 > 0, \ x_3 > 0, \ x_1 + x_3 = 1\}$, so, in this case, there is no efficient solution.

(V) the case that $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$:

$(*) \iff (\bar{y}_1, \bar{y}_2, \bar{y}_3) \in S_3$ and there exist $\lambda_1 > 0, \ \lambda_2 > 0, \ \lambda_1 + \lambda_2 = 1, \ \alpha \in \mathbb{R}$,

$\beta \geqq 0$ and $\gamma \geqq 0$ such that

$$\begin{cases} \bar{y}_2 - (\lambda_1 - \lambda_2)\bar{y}_3 & = \alpha \\ -\bar{y}_1 & = \alpha - \beta \\ (\lambda_1 - \lambda_2)\bar{y}_1 & = \alpha - \gamma. \end{cases}$$

$\iff$ $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, y_2, y_3) : y_1 = \frac{\beta-\gamma}{2\lambda_1},\ y_2 = \frac{1}{2} - \frac{\lambda_2-\gamma}{2\lambda_1},\ y_3 = \frac{1}{2} + \frac{\lambda_2-\beta}{2\lambda_1},\ y_1 \geqq 0,\ y_2 \geqq 0,\ y_3 \geqq 0,\ y_1+y_2+y_3 = 1,\ \lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_1+\lambda_2 = 1,\ \beta \geqq 0,\ \gamma \geqq 0\}$.

$\iff$ $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, y_2, y_3) : y_1 \geqq 0,\ y_2 \geqq 0,\ y_3 \geqq 0,\ y_1+y_2+y_3 = 1\}$.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in S_3$:

$(**)$ $\iff$ $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) : x_3 > x_2 \geqq 0,\ x_2 + x_3 = 1\}$.

But it does not satisfy the condition$(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no efficient solution.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3) : y_2 > 0,\ y_3 > 0,\ y_2+y_3 = 1\}$:

$(**)$ $\iff$ $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) : x_3 > x_2 > 0,\ x_2 + x_3 = 1\}$.

But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no efficient solution.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (0, 0, 1)$:

$(**)$ $\iff$ $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) : x_1 \geqq 0,\ 0 \leqq x_2 < \frac{1}{2},\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1\}$.

By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, in this case, $\{(1, 0, 0, 0, 0, 1)\} \subset sol(\text{EVMG})$.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, 0, y_3) \;:\; y_1 > 0,\; y_3 > 0,\; y_1 + y_3 = 1\}$:

$(**) \iff (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) \;:\; x_1 \geqq 0,\; 0 \leqq x_2 < \frac{1}{2},\; x_3 \geqq 0,\; x_1 + x_2 + x_3 = 1\}$.

By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, in this case, $\{(1, 0, 0, \bar{y}_1, 0, \bar{y}_3) \;:\; \bar{y}_1 > 0,\; \bar{y}_3 > 0,\; \bar{y}_1 + \bar{y}_3 = 1\} \subset sol(\text{EVMG})$.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (1, 0, 0)$:

$(**) \iff (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) \;:\; x_1 \geqq 0,\; x_2 \geqq 0,\; x_3 \geqq 0,\; x_1 + x_2 + x_3 = 1\}$.

By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, in this case, $\{(1, 0, 0, 1, 0, 0)\} \subset sol(\text{EVMG})$.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, y_2, 0) \;:\; y_1 > 0,\; y_2 > 0,\; y_1 + y_2 = 1\}$:

$(**) \iff (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \;:\; x_3 > x_2 \geqq 0,\; x_2 + x_3 = 1\}$.

But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no efficient solution.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (0, 1, 0)$:

$(**) \iff (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in S_3$ and there exist $\mu_1 > 0,\; \mu_2 > 0,\; \mu_1 + \mu_2 = 1,\; \alpha \in$

$\mathbb{R}$, $\beta \geqq 0$ and $\gamma \geqq 0$ such that

$$\begin{cases} \bar{x}_2 - (\mu_1 - \mu_2)\bar{x}_3 & = \alpha - \beta \\ -\bar{x}_1 & = \alpha \\ (\mu_1 - \mu_2)\bar{x}_1 & = \alpha - \gamma. \end{cases}$$

$\iff (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) \ : \ x_1 = -\frac{\gamma}{2\mu_1}, \ x_2 = \frac{1}{2} - \frac{\beta+\mu_2-\gamma}{2\mu_1}, \ x_3 = \frac{1}{2} + \frac{\beta+\mu_2}{2\mu_1}, \ x_1 \geqq 0, \ x_2 \geqq 0, \ x_3 \geqq 0, \ x_1+x_2+x_3 = 1, \ \mu_1 > 0, \ \mu_2 > 0, \ \mu_1+\mu_2 = 1, \ \beta \geqq 0, \ \gamma \geqq 0\}$.

$\iff (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \ : \ x_3 > x_2 \geqq 0, \ x_2 + x_3 = 1\}$.

But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no efficient solution.

(VI) the case that $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, 0) \ : \ x_1 > 0, \ x_2 > 0, \ x_1 + x_2 = 1\}$:

$(*) \iff (\bar{y}_1, \bar{y}_2, \bar{y}_3) \in S_3$ and there exist $\lambda_1 > 0, \ \lambda_2 > 0, \ \lambda_1 + \lambda_2 = 1, \ \alpha \in \mathbb{R}$ and $\beta \geqq 0$ such that

$$\begin{cases} \bar{y}_2 - (\lambda_1 - \lambda_2)\bar{y}_3 & = \alpha \\ -\bar{y}_1 & = \alpha \\ (\lambda_1 - \lambda_2)\bar{y}_1 & = \alpha - \beta. \end{cases}$$

$\iff (\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3) \ : \ y_3 > y_2 \geqq 0, \ y_2 + y_3 = 1\}$.

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3) \ : \ y_3 > y_2 > 0, \ y_2 + y_3 = 1\}$:

$(**) \iff (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \ : \ x_3 > x_2 \geqq 0, \ x_2 + x_3 = 1\}.$

But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, 0) \ : \ x_1 > 0, \ x_2 > 0, \ x_1 + x_2 = 1\}$, so, in this case, there is no efficient solution.

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (0, 0, 1)$:

$(**) \iff (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) \ : \ x_1 \geqq 0, \ 0 \leqq x_2 < \frac{1}{2}, \ x_3 \geqq 0, \ x_1 + x_2 + x_3 = 1\}.$

By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, 0) \ : \ x_1 > 0, \ x_2 > 0, \ x_1 + x_2 = 1\}$, in this case, $\{(\bar{x}_1, \bar{x}_2, 0, 0, 0, 1) \ : \ \bar{x}_1 > \bar{x}_2 > 0, \ \bar{x}_1 + \bar{x}_2 = 1\} \subset sol$(EVMG).

(VII) the case that $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 1, 0)$:

$(*) \iff (\bar{y}_1, \bar{y}_2, \bar{y}_3) \in S_3$ and there exist $\lambda_1 > 0, \ \lambda_2 > 0, \ \lambda_1 + \lambda_2 = 1, \ \alpha \in \mathbb{R}$, $\beta \geqq 0$ and $\gamma \geqq 0$ such that

$$\begin{cases} \bar{y}_2 - (\lambda_1 - \lambda_2)\bar{y}_3 & = \alpha - \beta \\ -\bar{y}_1 & = \alpha \\ (\lambda_1 - \lambda_2)\bar{y}_1 & = \alpha - \gamma. \end{cases}$$

$\iff (\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3) \ : \ y_3 > y_2 \geqq 0, \ y_2 + y_3 = 1\}.$

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3) \ : \ y_3 > y_2 > 0, \ y_2 + y_3 = 1\}$:

$(**) \iff (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \ : \ x_3 > x_2 > 0, \ x_2 + x_3 = 1\}.$

But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 1, 0)$, in this case, there is no efficient solution.

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (0, 0, 1)$:

$(**) \iff (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) : x_1 \geqq 0,\ 0 \leqq x_2 < \frac{1}{2},\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1\}$.

But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 1, 0)$, in this case, there is no efficient solution.

Therefore, the set of all efficient solutions of the vector matrix game (B_1, B_2) is $sol(\text{EVMG}) = \{(\bar{x}_1, \bar{x}_2, \bar{x}_3, 0, 0, 1) : \bar{x}_1 > 0, 0 < \bar{x}_2 < \frac{1}{2}, \bar{x}_3 > 0, \bar{x}_1 + \bar{x}_2 + \bar{x}_3 = 1\} \cup \{(0, \bar{x}_2, \bar{x}_3, 0, \bar{y}_2, \bar{y}_3) : \bar{x}_3 > \bar{x}_2 > 0, \bar{x}_2 + \bar{x}_3 = 1, \bar{y}_3 > \bar{y}_2 > 0, \bar{y}_2 + \bar{y}_3 = 1\} \cup \{(0, \bar{x}_2, \bar{x}_3, 0, 0, 1) : \bar{x}_3 > \bar{x}_2 > 0, \bar{x}_2 + \bar{x}_3 = 1\} \cup \{(0, 0, 1, \bar{y}_1, \bar{y}_2, \bar{y}_3) : \bar{y}_1 > 0, 0 < \bar{y}_2 < \frac{1}{2}, \bar{y}_3 > 0, \bar{y}_1 + \bar{y}_2 + \bar{y}_3 = 1\} \cup \{(0, 0, 1, 0, \bar{y}_2, \bar{y}_3) : \bar{y}_3 > \bar{y}_2 > 0, \bar{y}_2 + \bar{y}_3 = 1\} \cup \{(0, 0, 1, 0, 0, 1)\} \cup \{(0, 0, 1, \bar{y}_1, 0, \bar{y}_3) : \bar{y}_1 > 0, \bar{y}_3 > 0, \bar{y}_1 + \bar{y}_3 = 1\} \cup \{(0, 0, 1, 1, 0, 0)\} \cup \{(0, 0, 1, \bar{y}_1, \bar{y}_2, 0) : \bar{y}_1 > \bar{y}_2 > 0, \bar{y}_1 + \bar{y}_2 = 1\} \cup \{(\bar{x}_1, 0, \bar{x}_3, 0, 0, 1) : \bar{x}_1 > 0, \bar{x}_3 > 0, \bar{x}_1 + \bar{x}_3 = 1\} \cup \{(\bar{x}_1, 0, \bar{x}_3, \bar{y}_1, 0, \bar{y}_3) : \bar{x}_1 > 0, \bar{x}_3 > 0, \bar{x}_1 + \bar{x}_3 = 1, \bar{y}_1 > 0, \bar{y}_3 > 0, \bar{y}_1 + \bar{y}_3 = 1\} \cup \{(\bar{x}_1, 0, \bar{x}_3, 1, 0, 0) : \bar{x}_1 > 0, \bar{x}_3 > 0, \bar{x}_1 + \bar{x}_3 = 1\} \cup \{(1, 0, 0, \bar{y}_1, 0, \bar{y}_3) : \bar{y}_1 > 0, \bar{y}_3 > 0, \bar{y}_1 + \bar{y}_3 = 1\} \cup \{(1, 0, 0, 1, 0, 0)\} \cup \{(\bar{x}_1, \bar{x}_2, 0, 0, 0, 1) : \bar{x}_1 > \bar{x}_2 > 0, \bar{x}_1 + \bar{x}_2 = 1\}$. □

Example 5.6. Consider the B_1, B_2 and S_3 described in Example 5.1. Then we calculate the set of all the weakly efficient solution of vector matrix game (B_1, B_2). By Lemma 3.2.8, $(\bar{x}, \bar{y}) \in S_3 \times S_3$ is a weakly efficient solution of vector matrix game (B_1, B_2) if and only if there exist $\lambda_1 \geqq 0,\ \lambda_2 \geqq 0,\ \lambda_1 + \lambda_2 = 1,\ \mu_1 \geqq 0,\ \mu_2 \geqq 0,\ \mu_1 + \mu_2 = 1$ such that

$$(*) \qquad \begin{pmatrix} \bar{y}_2 - (\lambda_1 - \lambda_2)\bar{y}_3 \\ -\bar{y}_1 \\ (\lambda_1 - \lambda_2)\bar{y}_1 \end{pmatrix} \in N_{S_3} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix}$$

and

$$(**)\qquad \begin{pmatrix} \bar{x}_2-(\mu_1-\mu_2)\bar{x}_3 \\ -\bar{x}_1 \\ (\mu_1-\mu_2)\bar{x}_1 \end{pmatrix} \in N_{S_3}\begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \end{pmatrix}.$$

Now using the relations $(*)$ and $(**)$, we determine the set sol(EVMG) which is the set of all weakly efficient solutions of matrix game (B_1, B_2). Thus

(I) the case that $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \overset{o}{S}_3$:

$(*) \iff (\bar{y}_1, \bar{y}_2, \bar{y}_3) \in S_3$ and there exist $\lambda_1 \geqq 0, \lambda_2 \geqq 0, \lambda_1+\lambda_2 = 1,\ \alpha \in \mathbb{R}$ such that

$$\begin{cases} \bar{y}_2-(\lambda_1-\lambda_2)\bar{y}_3 & = \alpha \\ -\bar{y}_1 & = \alpha \\ (\lambda_1-\lambda_2)\bar{y}_1 & = \alpha. \end{cases}$$

(i) $\lambda_1 = 0,\ \lambda_2 = 1$: In this case, there is no weakly efficient solution.

(ii) $\lambda_1 = 1,\ \lambda_2 = 0$: $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (0, \frac{1}{2}, \frac{1}{2})$.

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (0, \frac{1}{2}, \frac{1}{2})$:

$(**) \iff (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in S_3$ and there exist $\mu_1 \geqq 0,\ \mu_2 \geqq 0,\ \mu_1+\mu_2 = 1,\ \alpha \in \mathbb{R}$ and $\beta \geqq 0$ such that

$$\begin{cases} \bar{x}_2-(\mu_1-\mu_2)\bar{x}_3 & = \alpha-\beta \\ -\bar{x}_1 & = \alpha \\ (\mu_1-\mu_2)\bar{x}_1 & = \alpha. \end{cases}$$

(a) $\mu_1 = 0,\ \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3)\ :\ x_1 \geqq 0,\ x_2 \geqq 0,\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = -\beta, \beta \geqq 0\}$. Hence $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3)\ :\ x_1 \geqq 0,\ x_2 \geqq 0,\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 0\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \overset{o}{S}_3$, so, in this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3)\ :\ x_2 = \frac{1-\beta}{2},\ x_3 = \frac{1+\beta}{2},\ x_2 + x_3 = 1, \beta \geqq 0\}$. Hence $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3)\ :\ x_3 \geqq x_2 \geqq 0,\ x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \overset{o}{S}_3$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0,\ \mu_2 > 0,\ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3)\ :\ x_2 = \frac{1}{2} - \frac{\mu_2+\beta}{2\mu_1},\ x_3 = \frac{1}{2} + \frac{\mu_2+\beta}{2\mu_1},\ x_2 \geqq 0,\ x_3 \geqq 0,\ x_2 + x_3 = 1, \mu_1 > 0,\ \mu_2 > 0,\ \mu_1 + \mu_2 = 1,\ \beta \geqq 0\}$. Hence $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3)\ :\ x_3 > x_2 \geqq 0,\ x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \overset{o}{S}_3$, so, in this case, there is no weakly efficient solution.

(iii) $\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_1 + \lambda_2 = 1$: $(\bar{y}_1, \bar{y}_2, \bar{y}_2) \in \{(0, y_2, y_3)\ :\ y_3 > y_2 \geqq 0,\ y_2 + y_3 = 1\}$.

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3)\ :\ y_3 > y_2 > 0,\ y_2 + y_3 = 1\}$:

$(**) \iff (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in S_3$ and there exist $\mu_1 \geqq 0, \mu_2 \geqq 0, \mu_1 + \mu_2 = 1, \alpha \in \mathbb{R}$ and $\beta \geqq 0$ such that

$$\begin{cases} \bar{x}_2 - (\mu_1 - \mu_2)\bar{x}_3 &= \alpha - \beta \\ -\bar{x}_1 &= \alpha \\ (\mu_1 - \mu_2)\bar{x}_1 &= \alpha \end{cases}$$

(a) $\mu_1 = 0,\ \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1,\ \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3)\ :\ x_3 \geqq x_2 \geqq 0,\ x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \overset{o}{S}_3$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0,\ \mu_2 > 0,\ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3)\ :\ x_3 > x_2 \geqq 0,\ x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \overset{o}{S}_3$, so, in this case, there is no weakly efficient solution.

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (0, 0, 1)$:

$(**) \iff (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in S_3$ and there exist $\mu_1 \geqq 0, \mu_2 \geqq 0, \mu_1 + \mu_2 = 1, \alpha \in \mathbb{R}$, $\beta \geqq 0$ and $\gamma \geqq 0$ such that

$$\begin{cases} \bar{x}_2 - (\mu_1 - \mu_2)\bar{x}_3 & = \alpha - \beta \\ -\bar{x}_1 & = \alpha - \gamma \\ (\mu_1 - \mu_2)\bar{x}_1 & = \alpha \end{cases}$$

(a) $\mu_1 = 0,\ \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1,\ \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3)\ :\ x_1 = \frac{\gamma}{2},\ x_2 = \frac{1}{2} - \frac{\beta}{2},\ x_3 = \frac{1}{2} + \frac{\beta - \gamma}{2},\ x_1 \geqq 0,\ x_2 \geqq 0,\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1,\ \beta \geqq 0,\ \gamma \geqq 0\}$. Hence $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3)\ :\ x_1 \geqq 0,\ 0 <\leqq x_2 \leqq \frac{1}{2},\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \overset{o}{S}_3$, in this case, $\{(\bar{x}_1, \bar{x}_2, \bar{x}_3, 0, 0, 1)\ :\ \bar{x}_1 > 0,\ 0 < \bar{x}_2 \leqq \frac{1}{2},\ \bar{x}_3 > 0,\ \bar{x}_1 + \bar{x}_2 + \bar{x}_3 = 1\} \subset sol(\text{WEVMG})$.

(c) $\mu_1 > 0,\ \mu_2 > 0,\ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3)\ :\ x_1 = \frac{\gamma}{2\mu_1},\ x_2 = \frac{1}{2} - \frac{\mu_2 + \beta}{2\mu_1},\ x_3 = \frac{1}{2} + \frac{\mu_2 + \beta - \gamma}{2\mu_1},\ x_1 \geqq 0,\ x_2 \geqq 0,\ x_3 \geqq 0,\ x_1 + x_2 +$

$x_3 = 1, \mu_1 > 0, \mu_2 > 0, \mu_1 + \mu_2 = 1, \beta \geqq 0, \gamma \geqq 0\}$. Hence $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) \;:\; x_1 \geqq 0,\; 0 \leqq x_2 < \frac{1}{2},\; x_3 \geqq 0,\; x_1 + x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \overset{o}{S}_3$, in this case, $\{(\bar{x}_1, \bar{x}_2, \bar{x}_3, 0, 0, 1) \;:\; \bar{x}_1 > 0,\; 0 < \bar{x}_2 < \frac{1}{2},\; \bar{x}_3 > 0,\; \bar{x}_1 + \bar{x}_2 + \bar{x}_3 = 1\} \subset sol(\text{WEVMG})$.

(II) the case that $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \;:\; x_2 > 0,\; x_3 > 0,\; x_2 + x_3 = 1\}$:

$(*) \iff (\bar{y}_1, \bar{y}_2, \bar{y}_3) \in S_3$ and there exist $\lambda_1 \geqq 0,\; \lambda_2 \geqq 0,\; \lambda_1 + \lambda_2 = 1,\; \alpha \in \mathbb{R}$ and $\beta \geqq 0$ such that

$$\begin{cases} \bar{y}_2 - (\lambda_1 - \lambda_2)\bar{y}_3 & = \alpha - \beta \\ -\bar{y}_1 & = \alpha \\ (\lambda_1 - \lambda_2)\bar{y}_1 & = \alpha. \end{cases}$$

(i) $\lambda_1 = 0,\; \lambda_2 = 1$: In this case, there is no weakly efficient solution.

(ii) $\lambda_1 = 1,\; \lambda_2 = 0$: $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3) \;:\; y_2 = \frac{1}{2} - \frac{\beta}{2},\; y_3 = \frac{1}{2} + \frac{\beta}{2},\; y_2 \geqq 0,\; y_3 \geqq 0,\; y_2 + y_3 = 1, \beta \geqq 0\}$. Hence $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, \bar{y}_2, \bar{y}_3) \;:\; \bar{y}_3 \geqq \bar{y}_2 \geqq 0,\; \bar{y}_2 + \bar{y}_3 = 1\}$.

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3) \;:\; y_3 \geqq y_2 > 0,\; y_2 + y_3 = 1\}$:

$(**) \iff (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in S_3$ and there exist $\mu_1 \geqq 0,\; \mu_2 \geqq 0,\; \mu_1 + \mu_2 = 1,\; \alpha \in \mathbb{R}, \beta \geqq 0$ such that

$$\begin{cases} \bar{x}_2 - (\mu_1 - \mu_2)\bar{x}_3 & = \alpha - \beta \\ -\bar{x}_1 & = \alpha \\ (\mu_1 - \mu_2)\bar{x}_1 & = \alpha \end{cases}$$

(a) $\mu_1 = 0,\ \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1,\ \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3)\ :\ x_3 \geqq x_2 > 0,\ x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3)\ :\ x_2 > 0,\ x_3 > 0,\ x_2 + x_3 = 1\}$, in this case, $\{(0, \bar{x}_2, \bar{x}_3, 0, \bar{y}_2, \bar{y}_3)\ :\ \bar{x}_3 \geqq \bar{x}_2 > 0,\ \bar{x}_2 + \bar{x}_3 = 1,\ \bar{y}_3 \geqq \bar{y}_2 > 0,\ \bar{y}_2 + \bar{y}_3 = 1\} \subset sol(\text{WEVMG})$.

(c) $\mu_1 > 0,\ \mu_2 > 0,\ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in (0, x_2, x_3)\ :\ x_3 > x_2 > 0,\ x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3)\ :\ x_2 > 0,\ x_3 > 0,\ x_2 + x_3 = 1\}$, in this case, $\{(0, \bar{x}_2, \bar{x}_3, 0, \bar{y}_2, \bar{y}_3)\ :\ \bar{x}_3 > \bar{x}_2 > 0,\ \bar{x}_2 + \bar{x}_3 = 1, \bar{y}_3 \geqq \bar{y}_2 > 0,\ \bar{y}_2 + \bar{y}_3 = 1\} \subset sol(\text{WEVMG})$.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (0, 0, 1)$:

$(**) \iff (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in S_3$ and there exist $\mu_1 \geqq 0, \mu_2 \geqq 0, \mu_1 + \mu_2 = 1, \alpha \in \mathbb{R},\ \beta \geqq 0$ and $\gamma \geqq 0$ such that

$$\begin{cases} \bar{x}_2 - (\mu_1 - \mu_2)\bar{x}_3 & = \alpha - \beta \\ -\bar{x}_1 & = \alpha - \gamma \\ (\mu_1 - \mu_2)\bar{x}_1 & = \alpha \end{cases}$$

(a) $\mu_1 = 0,\ \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1,\ \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3)\ :\ x_1 \geqq 0,\ 0 \leqq x_2 \leqq \frac{1}{2},\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1\}$. Hence $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) :\ x_3 \geqq x_2 > 0,\ x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3)\ :\ x_2 > 0,\ x_3 > 0,\ x_2 + x_3 = 1\}$, in this case, $\{(0, \bar{x}_2, \bar{x}_3, 0, 0, 1)\ :\ \bar{x}_3 \geqq \bar{x}_2 > 0, \bar{x}_2 + \bar{x}_3 = 1\} \subset sol(\text{WEVMG})$.

(c) $\mu_1 > 0,\ \mu_2 > 0,\ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3)\ :\ x_1 \geqq 0,\ 0 \leqq x_2 < \frac{1}{2},\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1\}$. Hence $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) : x_3 > x_2 > 0,\ x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3)\ :\ x_2 > 0,\ x_3 > 0,\ x_2 + x_3 = 1\}$, in this case, $\{(0, \bar{x}_2, \bar{x}_3, 0, 0, 1)\ :\ \bar{x}_3 > \bar{x}_2 > 0,\ \bar{x}_2 + \bar{x}_3 = 1\} \subset sol$(WEVMG).

(iii) $\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_1 + \lambda_2 = 1$: $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3)\ :\ y_2 = \frac{1}{2} - \frac{\lambda_2+\beta}{2\lambda_1},\ y_3 = \frac{1}{2} + \frac{\lambda_2+\beta}{2\lambda_1},\ y_2 \geqq 0,\ y_3 \geqq 0,\ y_2 + y_3 = 1, \lambda_1 > 0, \lambda_2 > 0, \beta \geqq 0\}$. Hence $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3) :\ y_3 > y_2 \geqq 0,\ y_2 + y_3 = 1\}$.

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3)\ :\ y_3 > y_2 > 0,\ y_2 + y_3 = 1\}$:

(a) $\mu_1 = 0,\ \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1,\ \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) :\ x_3 \geqq x_2 > 0,\ x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3)\ :\ x_2 > 0,\ x_3 > 0,\ x_2 + x_3 = 1\}$, in this case, $\{(0, \bar{x}_2, \bar{x}_3, 0, \bar{y}_2, \bar{y}_3)\ :\ \bar{x}_3 \geqq \bar{x}_2 > 0,\ \bar{x}_2 + \bar{x}_3 = 1,\ \bar{y}_3 > \bar{y}_2 > 0,\ \bar{y}_2 + \bar{y}_3 = 1\} \subset sol$(WEVMG).

(c) $\mu_1 > 0,\ \mu_2 > 0$: $(\bar{x}_2, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) :\ x_3 > x_2 > 0,\ x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3)\ :\ x_2 > 0,\ x_3 > 0,\ x_2 + x_3 = 1\}$, in this case, $\{(0, \bar{x}_2, \bar{x}_3, 0, \bar{y}_2, \bar{y}_3)\ :\ \bar{x}_3 > \bar{x}_2 > 0,\ \bar{x}_2 + \bar{x}_3 = 1, \bar{y}_3 > \bar{y}_2 > 0,\ \bar{y}_2 + \bar{y}_3 = 1\} \subset sol$(WEVMG).

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (0, 0, 1)$:

(a) $\mu_1 = 0,\ \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1$, $\mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) : \ x_3 \geqq x_2 > 0, \ x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \ : \ x_2 > 0, \ x_3 > 0, \ x_2 + x_3 = 1\}$, in this case, $\{(0, \bar{x}_2, \bar{x}_3, 0, 0, 1) \ : \ \bar{x}_3 \geqq \bar{x}_2 > 0, \ \bar{x}_2 + \bar{x}_3 = 1\} \subset sol(\text{WEVMG})$.

(c) $\mu_1 > 0, \mu_2 > 0, \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) : \ x_3 > x_2 > 0, \ x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \ : \ x_2 > 0, \ x_3 > 0, \ x_2 + x_3 = 1\}$, in this case, $\{(0, \bar{x}_2, \bar{x}_3, 0, 0, 1) \ : \ \bar{x}_3 > \bar{x}_2 > 0, \ \bar{x}_2 + \bar{x}_3 = 1\} \subset sol(\text{WEVMG})$.

(III) the case that $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$:

$(*) \iff (\bar{y}_1, \bar{y}_2, \bar{y}_3) \in S_3$ and there exist $\lambda_1 \geqq 0, \ \lambda_2 \geqq 0, \ \lambda_1 + \lambda_2 = 1, \ \alpha \in \mathbb{R}$, $\beta \geqq 0$ and $\gamma \geqq 0$ such that

$$\begin{cases} \bar{y}_2 - (\lambda_1 - \lambda_2)\bar{y}_3 & = \alpha - \beta \\ -\bar{y}_1 & = \alpha - \gamma \\ (\lambda_1 - \lambda_2)\bar{y}_1 & = \alpha. \end{cases}$$

(i) $\lambda_1 = 0, \lambda_2 = 1$: In this case, there is no weakly efficient solution.

(ii) $\lambda_1 = 1, \lambda_2 = 0$: $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, y_2, y_3) : \ y_1 \geqq 0, \ 0 \leqq y_2 \leqq \frac{1}{2}, \ y_3 \geqq 0, \ y_1 + y_2 + y_3 = 1\}$.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, y_2, y_3) \ : \ y_1 > 0, \ 0 < y_2 \leqq \frac{1}{2}, \ y_3 > 0, \ y_1 + y_2 + y_3 = 1\}$:

$(**) \iff (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in S_3$ and there exist $\mu_1 \geqq 0, \mu_2 \geqq 0, \mu_1 + \mu_2 =$

$1, \alpha \in \mathbb{R}$ such that

$$\begin{cases} \bar{x}_2 - (\mu_1 - \mu_2)\bar{x}_3 & = \alpha \\ -\bar{x}_1 & = \alpha \\ (\mu_1 - \mu_2)\bar{x}_1 & = \alpha. \end{cases}$$

(a) $\mu_1 = 0,\ \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1,\ \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, \frac{1}{2}, \frac{1}{2})$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0, \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \ :\ x_3 > x_2 \geqq 0,\ x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$, in this case, $\{(0, 0, 1, \bar{y}_1, \bar{y}_2, \bar{y}_3) \ :\ \bar{y}_1 > 0,\ 0 < \bar{y}_2 \leqq \frac{1}{2},\ \bar{y}_3 > 0,\ \bar{y}_1 + \bar{y}_2 + \bar{y}_3 = 1\} \subset$ sol(WEVMG).

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3) \ :\ y_3 > y_2 > 0,\ y_2 + y_3 = 1\}$:

(a) $\mu_1 = 0,\ \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1,\ \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \ :\ x_3 \geqq x_2 \geqq 0,\ x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$, in this case, $\{(0, 0, 1, 0, \bar{y}_2, \bar{y}_3) \ :\ \bar{y}_3 > \bar{y}_2 > 0,\ \bar{y}_2 + \bar{y}_3 = 1\} \subset sol$(WEVMG).

(c) $\mu_1 > 0, \mu_2 > 0, \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \ :\ x_3 > x_2 \geqq 0,\ x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$, in this case, $\{(0, 0, 1, 0, \bar{y}_2, \bar{y}_3) \ :\ \bar{y}_3 > \bar{y}_2 > 0,\ \bar{y}_2 + \bar{y}_3 = 1\} \subset sol$(WEVMG).

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = \{(0, 0, 1)\}$:

(a) $\mu_1 = 0,\ \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1,\ \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{((x_1, x_2, x_3) :\ x_1 \geqq 0,\ 0 \leqq x_2 \leqq \frac{1}{2},\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$, in this case, $\{(0, 0, 1, 0, 0, 1)\} \subset sol(\text{WEVMG})$.

(c) $\mu_1 > 0, \mu_2 > 0, \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) :\ x_1 \geqq 0,\ 0 \leqq x_2 < \frac{1}{2},\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$, in this case, $\{(0, 0, 1, 0, 0, 1)\} \subset sol(\text{WEVMG})$.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, 0, y_3)\ :\ y_1 > 0,\ y_3 > 0,\ y_1 + y_3 = 1\}$:

$(**) \iff (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in S_3$ and there exist $\mu_1 \geqq 0,\ \mu_2 \geqq 0,\ \mu_1 + \mu_2 = 1,\ \alpha \in \mathbb{R}$ and $\beta \geqq 0$ such that

$$\begin{cases} \bar{x}_2 - (\mu_1 - \mu_2)\bar{x}_3 & = \alpha \\ -\bar{x}_1 & = \alpha - \beta \\ (\mu_1 - \mu_2)\bar{x}_1 & = \alpha. \end{cases}$$

(a) $\mu_1 = 0,\ \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1,\ \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3)\ :\ x_1 = \frac{\beta}{2},\ x_2 = \frac{1}{2},\ x_3 = \frac{1}{2} - \frac{\beta}{2},\ x_1 \geqq 0,\ x_2 \geqq 0,\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1, \beta \geqq 0\}$. Hence $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) :\ x_1 \geqq 0,\ x_2 = \frac{1}{2},\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0, \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3)\ :\ x_1 = \frac{\beta}{2\mu_1},\ x_2 = \frac{1}{2} - \frac{\mu_2}{2\mu_1},\ x_3 = \frac{1}{2} + \frac{\mu_2 - \beta}{2\mu_1},\ \ x_1 \geqq 0,\ x_2 \geqq 0,\ x_3 \geqq 0,\ x_1 + x_2 + x_3 =$

$1, \mu_1 > 0, \mu_2 > 0, \beta \geqq 0\}$. Hence $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) : x_1 \geqq 0, \ 0 \leqq x_2 < \frac{1}{2}, \ x_3 \geqq 0, \ x_1 + x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$, in this case, $\{(0, 0, 1, \bar{y}_1, 0, \bar{y}_3) : \bar{y}_1 > 0, \ \bar{y}_3 > 0, \ \bar{y}_1 + \bar{y}_3 = 1\} \subset sol(\text{WEVMG})$.

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (1, 0, 0)$:

$(**) \iff (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in S_3$ and there exist $\mu_1 \geqq 0, \mu_2 \geqq 0, \mu_1 + \mu_2 = 1, \alpha \in \mathbb{R}, \beta \geqq 0$ and $\gamma \geqq 0$ such that

$$\begin{cases} \bar{x}_2 - (\mu_1 - \mu_2)\bar{x}_3 &= \alpha \\ -\bar{x}_1 &= \alpha - \beta \\ (\mu_1 - \mu_2)\bar{x}_1 &= \alpha - \gamma. \end{cases}$$

(a) $\mu_1 = 0, \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) : x_1 = 1 - \alpha, \ x_2 + x_3 = \alpha, \ x_1 \geqq 0, \ x_2 \geqq 0, \ x_3 \geqq 0, \ x_1 + x_2 + x_3 = 1, \alpha \in \mathbb{R}\}$. Hence $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) : x_1 \geqq 0, \ x_2 \geqq 0, \ x_3 \geqq 0, \ x_1 + x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$, in this case, $\{(0, 0, 1, 1, 0, 0)\} \subset sol(\text{WEVMG})$.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) : x_1 = \frac{\beta - \gamma}{2}, \ x_2 = \frac{1+\gamma}{2}, \ x_3 = \frac{1-\beta}{2}, \ x_1 \geqq 0, \ x_2 \geqq 0, \ x_3 \geqq 0, \ x_1 + x_2 + x_3 = 1, \alpha \in \mathbb{R}, \beta \geqq 0, \gamma \geqq 0\}$. Hence $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) : x_1 \geqq 0, \ \frac{1}{2} \leqq x_2 \leqq 1, \ x_3 \geqq 0, \ x_1 + x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0, \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) : x_1 = \frac{\beta - \gamma}{2\mu_1}, \ x_2 = \frac{1}{2} - \frac{\mu_2 - \gamma}{2\mu_1}, \ x_3 = \frac{1}{2} + \frac{\mu_2 - \beta}{2\mu_1}, \ x_1 \geqq 0, \ x_2 \geqq 0, \ x_3 \geqq 0, \ x_1 + x_2 + x_3 = 1, \mu_1 > 0, \mu_2 > 0, \beta \geqq 0, \gamma \geqq 0\}$. Hence $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) : x_1 \geqq$

$0,\ x_2 \geqq 0,\ x_3 \geqq 0,\ x_1+x_2+x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0,0,1)$, in this case, $\{(0,0,1,1,0,0)\} \subset sol(\text{WEVMG})$.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, y_2, 0)\ :\ y_1 > y_2 > 0,\ y_1 + y_2 = 1\}$:

$(**) \iff (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in S_3$ and there exist $\mu_1 \geqq 0, \mu_2 \geqq 0, \mu_1 + \mu_2 = 1, \alpha \in \mathbb{R}$ and $\beta \geqq 0$ such that

$$\begin{cases} \bar{x}_2 - (\mu_1 - \mu_2)\bar{x}_3 & = \alpha \\ -\bar{x}_1 & = \alpha \\ (\mu_1 - \mu_2)\bar{x}_1 & = \alpha - \beta. \end{cases}$$

(a) $\mu_1 = 0, \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, \frac{1}{2}, \frac{1}{2})$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0,0,1)$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0, \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3)\ :\ x_1 = -\frac{\beta}{2\mu_1},\ x_2 = \frac{1}{2} - \frac{\mu_2}{2\mu_1},\ x_3 = \frac{1}{2} + \frac{\mu_2}{2\mu_1},\ x_1 \geqq 0,\ x_2 \geqq 0,\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1, \mu_1 > 0, \mu_2 > 0, \beta \geqq 0\}$. Hence $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3)\ :\ x_3 > x_2 \geqq 0,\ x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0,0,1)$, in this case, $\{(0,0,1,1,0,0)\} \subset sol(\text{WEVMG})$.

(iii) $\lambda_1 > 0, \lambda_2 > 0, \lambda_1 + \lambda_2 = 1$: $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, y_2, y_3) :\ y_1 \geqq 0,\ 0 \leqq y_2 < \frac{1}{2},\ y_3 \geqq 0,\ y_1 + y_2 + y_3 = 1\}$.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, y_2, y_3)\ :\ y_1 > 0,\ 0 < y_2 < \frac{1}{2},\ y_3 > 0,\ y_1 + y_2 + y_3 = 1\}$:

(a) $\mu_1 = 0, \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \mu_2 = 0$: In this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0, \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$, in this case, $\{(0, 0, 1, \bar{y}_1, \bar{y}_2, \bar{y}_3) \ : \ \bar{y}_1 > 0, \ 0 < \bar{y}_2 < \frac{1}{2}, \ \bar{y}_3 > 0, \ \bar{y}_1 + \bar{y}_2 + \bar{y}_3 = 1\} \subset sol(\text{WEVMG})$.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3) \ : \ y_3 > y_2 > 0, \ y_2 + y_3 = 1\}$:

(a) $\mu_1 = 0, \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$, in this case, $\{(0, 0, 1, 0, \bar{y}_2, \bar{y}_3) \ : \ \bar{y}_3 > \bar{y}_2 > 0, \ \bar{y}_2 + \bar{y}_3 = 1\} \subset sol(\text{WEVMG})$.

(c) $\mu_1 > 0, \mu_2 > 0, \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$, in this case, $\{(0, 0, 1, 0, \bar{y}_2, \bar{y}_3) \ : \ \bar{y}_3 > \bar{y}_2 > 0, \ \bar{y}_2 + \bar{y}_3 = 1\} \subset sol(\text{WEVMG})$.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (0, 0, 1)$:

(a) $\mu_1 = 0, \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$, in this case, $\{(0, 0, 1, 0, 0, 1)\} \subset sol(\text{WEVMG})$.

(c) $\mu_1 > 0, \mu_2 > 0, \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$, in this case, $\{(0, 0, 1, 0, 0, 1)\} \subset sol(\text{WEVMG})$.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, 0, y_3) \ : \ y_1 > 0, \ y_3 > 0, \ y_1 + y_3 = 1\}$:

(a) $\mu_1 = 0, \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \mu_2 = 0$: In this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0, \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$, in this case, $\{(0, 0, 1, \bar{y}_1, 0, \bar{y}_3) \ : \ \bar{y}_1 > 0, \ \bar{y}_3 > 0, \ \bar{y}_1 + \bar{y}_3 = 1\} \subset sol$(WEVMG).

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (1, 0, 0)$:

(a) $\mu_1 = 0, \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$, in this case, $\{(0, 0, 1, 1, 0, 0)\} \subset sol$(WEVMG).

(b) $\mu_1 = 1, \mu_2 = 0$: In this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0, \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$, in this case, $\{(0, 0, 1, 1, 0, 0)\} \subset sol$(WEVMG).

• in the case of $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, y_2, 0) \ : \ y_1 > y_2 > 0, \ y_1 + y_2 = 1\}$:

(a) $\mu_1 = 0, \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \mu_2 = 0$: In this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0, \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$, in this case, $\{(0, 0, 1, \bar{y}_1, \bar{y}_2, 0) \ : \ \bar{y}_1 > \bar{y}_2 > 0, \ \bar{y}_1 + \bar{y}_2 = 1)\} \subset sol$(WEVMG).

(IV) the case that $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, 0, x_3) \ : \ x_1 > 0, \ x_3 > 0, \ x_1 + x_3 = 1\}$:

$(*) \iff (\bar{y}_1, \bar{y}_2, \bar{y}_3) \in S_3$ and there exist $\lambda_1 \geqq 0, \ \lambda_2 \geqq 0, \ \lambda_1 + \lambda_2 = 1, \ \alpha \in \mathbb{R}$

and $\beta \geqq 0$ such that

$$\begin{cases} \bar{y}_2 - (\lambda_1 - \lambda_2)\bar{y}_3 & = \alpha \\ -\bar{y}_1 & = \alpha - \beta \\ (\lambda_1 - \lambda_2)\bar{y}_1 & = \alpha. \end{cases}$$

(i) $\lambda_1 = 0,\ \lambda_2 = 1$: In this case, there is no weakly efficient solution.

(ii) $\lambda_1 = 1,\ \lambda_2 = 0$: $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, y_2, y_3)\ :\ y_1 \geqq 0,\ y_2 = \frac{1}{2},\ y_3 \geqq 0,\ y_1 + y_2 + y_3 = 1\}$.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, y_2, y_3)\ :\ y_1 > 0,\ y_2 = \frac{1}{2},\ y_3 > 0,\ y_1 + y_2 + y_3 = 1\}$.

(**) $\Longleftrightarrow$ $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in S_3$ and there exist $\mu_1 \geqq 0, \mu_2 \geqq 0, \mu_1 + \mu_2 = 1, \alpha \in \mathbb{R}$ such that

$$\begin{cases} \bar{x}_2 - (\mu_1 - \mu_2)\bar{x}_3 & = \alpha \\ -\bar{x}_1 & = \alpha \\ (\mu_1 - \mu_2)\bar{x}_1 & = \alpha. \end{cases}$$

(a) $\mu_1 = 0, \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, \frac{1}{2}, \frac{1}{2})$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, 0, x_3)\ :\ x_1 > 0,\ x_3 > 0,\ x_1 + x_3 = 1\}$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0, \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3)\ :\ x_3 \geqq x_2 \geqq 0,\ x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in$

$\{(x_1, 0, x_3) \ : \ x_1 > 0, \ x_3 > 0, \ x_1 + x_3 = 1\}$, so, in this case, there is no weakly efficient solution.

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (\frac{1}{2}, \frac{1}{2}, 0)$:

$(**) \iff (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in S_3$ and there exist $\mu_1 \geqq 0, \mu_2 \geqq 0, \mu_1 + \mu_2 = 1, \alpha \in \mathbb{R}$ and $\beta \geqq 0$ such that

$$\begin{cases} \bar{x}_2 - (\mu_1 - \mu_2)\bar{x}_3 & = \alpha \\ -\bar{x}_1 & = \alpha \\ (\mu_1 - \mu_2)\bar{x}_1 & = \alpha - \beta. \end{cases}$$

(a) $\mu_1 = 0, \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, \frac{1}{2}, \frac{1}{2})$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, 0, x_3) \ : \ x_1 > 0, \ x_3 > 0, \ x_1 + x_3 = 1\}$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0, \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \ : \ x_3 > x_2 \geqq 0, \ x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, 0, x_3) \ : \ x_1 > 0, \ x_3 > 0, \ x_1 + x_3 = 1\}$, so, in this case, there is no weakly efficient solution.

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (0, \frac{1}{2}, \frac{1}{2})$:

$(**) \iff (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in S_3$ and there exist $\mu_1 \geqq 0, \mu_2 \geqq 0, \mu_1 + \mu_2 = 1, \alpha \in \mathbb{R}$ and $\beta \geqq 0$ such that

$$\begin{cases} \bar{x}_2 - (\mu_1 - \mu_2)\bar{x}_3 & = \alpha - \beta \\ -\bar{x}_1 & = \alpha \\ (\mu_1 - \mu_2)\bar{x}_1 & = \alpha. \end{cases}$$

(a) $\mu_1 = 0, \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \;:\; 0 \leqq x_2 \leqq x_3, x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, 0, x_3) \;:\; x_1 > 0,\; x_3 > 0,\; x_1 + x_3 = 1\}$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0, \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \;:\; x_3 > x_2 \geqq 0,\; x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, 0, x_3) \;:\; x_1 > 0,\; x_3 > 0,\; x_1 + x_3 = 1\}$, so, in this case, there is no weakly efficient solution.

(iii) $\lambda_1 > 0,\; \lambda_2 > 0,\; \lambda_1 + \lambda_2 = 1$: $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, y_2, y_3) \;:\; y_1 \geqq 0,\; 0 \leqq y_2 < \frac{1}{2},\; y_3 \geqq 0,\; y_1 + y_2 + y_3 = 1\}$.

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, y_2, y_3) \;:\; y_1 > 0,\; 0 < y_2 < \frac{1}{2},\; y_3 > 0,\; y_1 + y_2 + y_3 = 1\}$:

(a) $\mu_1 = 0,\; \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1,\; \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, \frac{1}{2}, \frac{1}{2})$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, 0, x_3) \;:\; x_1 > 0,\; x_3 > 0,\; x_1 + x_3 = 1\}$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0,\; \mu_2 > 0,\; \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \;:\; x_3 > x_2 \geqq 0,\; x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in$

$\{(x_1, 0, x_3) \ : \ x_1 > 0, \ x_3 > 0, \ x_1 + x_3 = 1\}$, so, in this case, there is no weakly efficient solution.

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3) \ : \ _3 > y_2 > 0, \ y_2 + y_3 = 1\}$:

(a) $\mu_1 = 0, \ \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \ \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \ : \ x_3 \geqq x_2 \geqq 0, \ x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, 0, x_3) \ : \ x_1 > 0, \ x_3 > 0, \ x_1 + x_3 = 1\}$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \ \mu_2 > 0, \ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \ : \ x_3 > x_2 \geqq 0, \ x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, 0, x_3) \ : \ x_1 > 0, \ x_3 > 0, \ x_1 + x_3 = 1\}$, so, in this case, there is no weakly efficient solution.

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (0, 0, 1)$:

(a) $\mu_1 = 0, \ \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \ \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) \ : \ x_1 \geqq 0, \ 0 \leqq x_2 \leqq \frac{1}{2}, \ x_3 \geqq 0, \ x_1 + x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, 0, x_3) \ : \ x_1 > 0, \ x_3 > 0, \ x_1 + x_3 = 1\}$, in this case, $\{(\bar{x}_1, 0, \bar{x}_3, 0, 0, 1) \ : \ \bar{x}_1 > 0, \ \bar{x}_3 > 0, \ \bar{x}_1 + \bar{x}_3 = 1\} \subset sol(\text{WEVMG})$.

(c) $\mu_1 > 0, \ \mu_2 > 0, \ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) \ : \ x_1 \geqq 0, \ 0 \leqq x_2 < \frac{1}{2}, \ x_3 \geqq 0, \ x_1 + x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, 0, x_3) \ : \ x_1 > 0, \ x_3 > 0, \ x_1 + x_3 = 1\}$, in this case, $\{(\bar{x}_1, 0, \bar{x}_3, 0, 0, 1) \ : \ \bar{x}_1 > 0, \ \bar{x}_3 > 0, \ \bar{x}_1 + \bar{x}_3 = 1\} \subset sol(\text{WEVMG})$.

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, 0, y_3) \ : \ y_1 > 0, \ y_3 > 0, \ y_1 + y_3 = 1\}$:

(a) $\mu_1 = 0,\ \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1,\ \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3)\ :\ x_1 \geqq 0,\ x_2 = \frac{1}{2},\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, 0, x_3)\ :\ x_1 > 0,\ x_3 > 0,\ x_1 + x_3 = 1\}$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0,\ \mu_2 > 0,\ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3)\ :\ x_1 \geqq 0,\ 0 \leqq x_2 < \frac{1}{2},\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, 0, x_3)\ :\ x_1 > 0,\ x_3 > 0,\ x_1 + x_3 = 1\}$, in this case, $\{(\bar{x}_1, 0, \bar{x}_3, \bar{y}_1, 0, \bar{y}_3)\ :\ \bar{x}_1 > 0,\ \bar{x}_3 > 0,\ \bar{x}_1 + \bar{x}_3 = 1,\ \bar{y}_1 > 0,\ \bar{y}_3 > 0,\ \bar{y}_1 + \bar{y}_3 = 1\} \subset sol(\text{WEVMG})$.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (1, 0, 0)$:

(a) $\mu_1 = 0,\ \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3)\ :\ x_1 \geqq 0,\ x_2 \geqq 0,\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, 0, x_3)\ :\ x_1 > 0,\ x_3 > 0,\ x_1 + x_3 = 1\}$, in this case, $\{(\bar{x}_1, 0, \bar{x}_3, 1, 0, 0)\ :\ \bar{x}_1 > 0,\ \bar{x}_3 > 0,\ \bar{x}_1 + \bar{x}_3 = 1\} \subset sol(\text{WEVMG})$.

(b) $\mu_1 = 1,\ \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3)\ :\ x_1 \geqq 0,\ x_2 \geqq x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, 0, x_3)\ :\ x_1 > 0,\ x_3 > 0,\ x_1 + x_3 = 1\}$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0,\ \mu_2 > 0,\ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3)\ :\ x_1 \geqq 0,\ x_2 \geqq 0,\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, 0, x_3)\ :\ x_1 > 0,\ x_3 > 0,\ x_1 + x_3 = 1\}$, in this case, $\{(\bar{x}_1, 0, \bar{x}_3, 1, 0, 0)\ :\ \bar{x}_1 > 0,\ \bar{x}_3 > 0,\ \bar{x}_1 + \bar{x}_3 = 1\} \subset sol(\text{WEVMG})$.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, y_2, 0) \ : \ y_1 > y_2 > 0, \ y_1 + y_2 = 1\}$:

(a) $\mu_1 = 0, \ \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \ \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, \frac{1}{2}, \frac{1}{2})$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, 0, x_3) \ : \ x_1 > 0, \ x_3 > 0, \ x_1 + x_3 = 1\}$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \ \mu_2 > 0, \ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 0, 1)$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, 0, x_3) \ : \ x_1 > 0, \ x_3 > 0, \ x_1 + x_3 = 1\}$, so, in this case, there is no weakly efficient solution.

(V) the case that $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$:

$(*) \iff (\bar{y}_1, \bar{y}_2, \bar{y}_3) \in S_3$ and there exist $\lambda_1 \geqq 0, \lambda_2 \geqq 0, \lambda_1 + \lambda_2 = 1, \alpha \in \mathbb{R}, \beta \geqq 0, \gamma \geqq 0$ such that

$$\begin{cases} \bar{y}_2 - (\lambda_1 - \lambda_2)\bar{y}_3 & = \alpha \\ -\bar{y}_1 & = \alpha - \beta \\ (\lambda_1 - \lambda_2)\bar{y}_1 & = \alpha - \gamma. \end{cases}$$

(i) $\lambda_1 = 0, \lambda_2 = 1$: $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, y_2, y_3) : \ y_1 \geqq 0, y_2 \geqq 0, y_3 \geqq 0, y_1 + y_2 + y_3 = 1\}$.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \overset{o}{S}_3$:

(a) $\mu_1 = 0, \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) : \ x_1 + x_2 + x_3 = 0\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, \frac{1}{2}, \frac{1}{2})$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0,\ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3)\ :\ x_3 > x_2 \geqq 0,\ x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no weakly efficient solution.

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3)\ :\ y_2 > 0,\ y_3 > 0,\ y_2 + y_3 = 1\}$:

(a) $\mu_1 = 0, \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3)\ :\ x_3 \geqq x_2 \geqq 0,\ x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0,\ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3)\ :\ x_3 > x_2 > 0,\ x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no weakly efficient solution.

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (0, 0, 1)$:

(a) $\mu_1 = 0, \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3)\ :\ x_1 \geqq 0,\ 0 \leqq x_2 \leqq \frac{1}{2},\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, in this case, $\{(1, 0, 0, 0, 0, 1)\} \subset sol(\text{WEVMG})$.

(c) $\mu_1 > 0, \mu_2 > 0,\ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3)\ :\ x_1 \geqq 0,\ 0 \leqq x_2 < \frac{1}{2},\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, in this case, $\{(1, 0, 0, 1, 0, 0)\} \subset sol(\text{WEVMG})$.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, 0, y_3) \ : \ y_1 > 0, \ y_3 > 0, \ y_1 + y_3 = 1\}$:

(a) $\mu_1 = 0, \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) \ : \ x_1 \geqq 0, \ x_2 = \frac{1}{2}, \ x_3 \geqq 0, \ x_1 + x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0, \ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) \ : \ x_1 \geqq 0, \ 0 \leqq x_2 < \frac{1}{2}, \ x_3 \geqq 0, \ x_1 + x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, in this case, $\{(1, 0, 0, \bar{y}_1, 0, \bar{y}_3) \ : \ y_1 > 0, \ \bar{y}_3 > 0, \ \bar{y}_1 + \bar{y}_3 = 1\} \subset sol(\text{WEVMG})$.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (1, 0, 0)$:

(a) $\mu_1 = 0, \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) \ : \ x_1 \geqq 0, \ x_2 \geqq 0, \ x_3 \geqq 0, \ x_1 + x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, in this case, $\{(1, 0, 0, 1, 0, 0)\} \subset sol(\text{WEVMG})$.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) \ : \ x_1 \geqq 0, \ x_2 \geqq x_3 \geqq 0, \ x_1 + x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0, \ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) \ : \ x_1 \geqq 0, \ x_2 \geqq 0, \ x_3 \geqq 0, \ x_1 + x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, in this case, $\{(1, 0, 0, 1, 0, 0)\} \subset sol(\text{WEVMG})$.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, y_2, 0) \ : \ y_1 > 0, \ y_2 > 0, \ y_1 + y_2 = 1\}$:

(a) $\mu_1 = 0, \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, \frac{1}{2}, \frac{1}{2})$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0,\ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3)\ :\ x_3 > x_2 \geqq 0,\ x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no weakly efficient solution.

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (0, 1, 0)$:

$(*) \iff (\bar{x}_1, \bar{x}_2, \bar{x}_3) \in S_3$ and there exist $\mu_1 \geqq 0, \mu_2 \geqq 0, \mu_1 + \mu_2 = 1, \alpha \in \mathbb{R}, \beta \geqq 0,\ \gamma \geqq 0$ such that

$$\begin{cases} \bar{x}_2 - (\mu_1 - \mu_2)\bar{x}_3 & = \alpha - \beta \\ -\bar{x}_1 & = \alpha \\ (\mu_1 - \mu_2)\bar{x}_1 & = \alpha - \gamma \end{cases}$$

(a) $\mu_1 = 0, \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3)\ :\ x_1 + x_2 + x_3 = -\beta,\ x_1 \geqq 0,\ x_2 \geqq 0,\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1,\ \beta \geqq 0\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3)\ :\ x_1 = -\frac{\gamma}{2},\ x_2 = \frac{1}{2} + \frac{\gamma - \beta}{2},\ x_3 = \frac{1}{2} + \frac{\beta}{2},\ x_1 \geqq 0,\ x_2 \geqq 0,\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1,\ \beta \geqq 0,\ \gamma \geqq 0\}$. Hence $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3)\ :\ x_3 \geqq x_2 \geqq 0,\ x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0,\ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3)\ :\ x_1 = -\frac{\gamma}{2\mu_1},\ {}_2 = \frac{1}{2} - \frac{\beta+\mu_2-\gamma}{2\mu_1},\ x_3 = \frac{1}{2} + \frac{\beta+\mu_2}{2\mu_1},\ x_1 \geqq 0,\ x_2 \geqq 0,\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1,\ \mu_1 \geqq 0, \mu_2 \geqq 0, \beta \geqq 0,\ \gamma \geqq 0\}$. Hence $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3)\ :\ x_3 > x_2 \geqq 0,\ x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no weakly efficient solution.

(ii) $\lambda_1 = 1, \lambda_2 = 0$: $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, y_2, y_3)\ :\ y_1 \geqq 0,\ y_2 \geqq y_3 \geqq 0,\ y_1 + y_2 + y_3 = 1\}$.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, y_2, y_3)\ :\ y_1 > 0,\ y_2 \geqq y_3 > 0,\ y_1 + y_2 + y_3 = 1\}$:

(a) $\mu_1 = 0, \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, \frac{1}{2}, \frac{1}{2})$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0,\ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3)\ :\ x_3 > x_2 \geqq 0,\ x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no weakly efficient solution.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, y_2, 0)\ :\ y_2 > y_1 > 0,\ y_1 + y_2 = 1\}$:

(a) $\mu_1 = 0, \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, \frac{1}{2}, \frac{1}{2})$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0,\ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3)\ :\ x_3 > x_2 \geqq 0,\ x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no weakly efficient solution.

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (0, 1, 0)$:

(a) $\mu_1 = 0, \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3)\ :\ x_3 \geqq x_2 \geqq 0,\ x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0,\ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3)\ :\ x_3 > x_2 \geqq 0,\ x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no weakly efficient solution.

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3)\ :\ y_2 > y_3 > 0,\ y_2 + y_3 = 1\}$:

(a) $\mu_1 = 0, \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3)\ :\ x_3 \geqq x_2 \geqq 0,\ x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0,\ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3)\ :\ x_3 > x_2 > 0,\ x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no weakly efficient solution.

(iii) $\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_1 + \lambda_2 = 1$: $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, y_2, y_3)\ :\ y_1 \geqq 0, y_2 \geqq 0, y_3 \geqq 0, y_1 + y_2 + y_3 = 1\}$.

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \overset{o}{S}_3$:

(a) $\mu_1 = 0, \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) \ : \ x_1 + x_2 + x_3 = 0\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, \frac{1}{2}, \frac{1}{2})$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0, \ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \ : \ x_3 > x_2 \geqq 0, x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no weakly efficient solution.

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3) \ : \ y_2 > 0, \ y_3 > 0, \ y_2 + y_3 = 1\}$:

(a) $\mu_1 = 0, \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \ : \ x_3 \geqq x_2 \geqq 0, \ x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0, \ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \ : \ x_3 > x_2 > 0, \ x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no weakly efficient solution.

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (0, 0, 1)$:

(a) $\mu_1 = 0, \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) \ : \ x_1 \geqq 0, \ 0 \leqq x_2 \leqq \frac{1}{2}, \ x_3 \geqq 0, \ x_1 + x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, in this case, $\{(1, 0, 0, 0, 0, 1)\} \subset sol(\text{WEVMG})$.

(c) $\mu_1 > 0, \mu_2 > 0,\ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3)\ :\ x_1 \geqq 0,\ 0 \leqq x_2 < \frac{1}{2},\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, in this case, $\{(1, 0, 0, 1, 0, 0)\} \subset sol(\text{WEVMG})$.

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, 0, y_3)\ :\ y_1 > 0,\ y_3 > 0,\ y_1 + y_3 = 1\}$:

(a) $\mu_1 = 0, \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3)\ :\ x_1 \geqq 0,\ x_2 = \frac{1}{2},\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0,\ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3)\ :\ x_1 \geqq 0,\ 0 \leqq x_2 < \frac{1}{2},\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, in this case, $\{(1, 0, 0, \bar{y}_1, 0, \bar{y}_3)\ :\ y_1 > 0,\ \bar{y}_3 > 0,\ \bar{y}_1 + \bar{y}_3 = 1\} \subset sol(\text{WEVMG})$.

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (1, 0, 0)$:

(a) $\mu_1 = 0, \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3)\ :\ x_1 \geqq 0,\ x_2 \geqq 0,\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, in this case, $\{(1, 0, 0, 1, 0, 0)\} \subset sol(\text{WEVMG})$.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3)\ :\ x_1 \geqq 0,\ x_2 \geqq x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0,\ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3)\ :\ x_1 \geqq 0,\ x_2 \geqq 0,\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, in this case, $\{(1, 0, 0, 1, 0, 0)\} \subset sol(\text{WEVMG})$.

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(y_1, y_2, 0)\ :\ y_1 > 0,\ y_2 > 0,\ y_1 + y_2 = 1\}$:

(a) $\mu_1 = 0, \mu_2 = 1$: In this case, there is no weakly efficientt solution.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, \frac{1}{2}, \frac{1}{2})$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0,\ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3)\ :\ x_3 > x_2 \geqq 0,\ x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no weakly efficient solution.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (0, 1, 0)$:

(a) $\mu_1 = 0, \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3)\ :\ x_1 + x_2 + x_3 = -\beta, x_1 \geqq 0,\ x_2 \geqq 0,\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1,\ \beta \geqq 0\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3)\ :\ x_1 = -\frac{\gamma}{2},\ x_2 = \frac{1}{2} + \frac{\gamma-\beta}{2},\ x_3 = \frac{1}{2} + \frac{\beta}{2},\ x_1 \geqq 0,\ x_2 \geqq 0,\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1, \beta \geqq 0,\ \gamma \geqq 0\}$. Hence $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3)\ :\ x_3 \geqq x_2 \geqq 0,\ x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0,\ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3)\ :\ x_1 = -\frac{\gamma}{2\mu_1},\ x_2 = \frac{1}{2} - \frac{\beta+\mu_2-\gamma}{2\mu_1},\ x_3 = \frac{1}{2} + \frac{\beta+\mu_2}{2\mu_1},\ x_1 \geqq 0,\ x_2 \geqq 0,\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1,\ \mu_1 \geqq 0, \mu_2 \geqq 0, \beta \geqq 0,\ \gamma \geqq 0\}$. Hence $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3)\ :\ x_3 > x_2 \geqq 0,\ x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 0, 0)$, so, in this case, there is no weakly efficient solution.

(VI) the case that $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, 0) \ : \ x_1 > 0, \ x_2 > 0, \ x_1 + x_2 = 1\}$:

$(*) \iff (\bar{y}_1, \bar{y}_2, \bar{y}_3) \in S_3$ and there exist $\lambda_1 \geqq 0, \lambda_2 \geqq 0, \lambda_1 + \lambda_2 = 1, \alpha \in \mathbb{R}, \beta \geqq 0$ such that

$$\begin{cases} \bar{y}_2 - (\lambda_1 - \lambda_2)\bar{y}_3 & = \alpha \\ -\bar{y}_1 & = \alpha \\ (\lambda_1 - \lambda_2)\bar{y}_1 & = \alpha - \beta. \end{cases}$$

(i) $\lambda_1 = 0, \ \lambda_2 = 1$: In this case, there is no weakly efficient solution.

(ii) $\lambda_1 = 1, \ \lambda_2 = 0$: $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (0, \frac{1}{2}, \frac{1}{2})$.

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (0, \frac{1}{2}, \frac{1}{2})$:

(a) $\mu_1 = 0, \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, \frac{1}{2}, \frac{1}{2})$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, 0) \ : \ x_1 > 0, \ x_2 > 0, \ x_1 + x_2 = 1\}$, ao, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0, \ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \ : \ x_3 > x_2 \geqq 0, \ x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, 0) \ : \ x_1 > 0, \ x_2 > 0, \ x_1 + x_2 = 1\}$, so, in this case, there is no weakly efficient solution.

(iii) $\lambda_1 > 0, \ \lambda_2 > 0, \ \lambda_1 + \lambda_2 = 1$: $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3) \ : \ y_3 > y_2 > 0, \ y_2 + y_3 = 1\}$.

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3) \ : \ y_3 > y_2 > 0, \ y_2 + y_3 = 1\}$:

(a) $\mu_1 = 0, \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) : x_3 \geqq x_2 \geqq 0,\ x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, 0) : x_1 > 0,\ x_2 > 0,\ x_1 + x_2 = 1\}$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0,\ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) : x_3 > x_2 \geqq 0,\ x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, 0) : x_1 > 0,\ x_2 > 0,\ x_1 + x_2 = 1\}$, so, in this case, there is no weakly efficient solution.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (0, 0, 1)$:

(a) $\mu_1 = 0, \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) : x_1 \geqq 0,\ 0 \leqq x_2 \leqq \frac{1}{2},\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, 0) : x_1 > 0,\ x_2 > 0,\ x_1 + x_2 = 1\}$, in this case, $\{(\bar{x}_1, \bar{x}_2, 0, 0, 0, 1) : \bar{x}_1 \geqq \bar{x}_2 > 0,\ \bar{x}_1 + \bar{x}_2 = 1\} \subset sol(\text{WEVMG})$.

(c) $\mu_1 > 0, \mu_2 > 0,\ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) : x_1 \geqq 0,\ 0 \leqq x_2 < \frac{1}{2},\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1\}$. By the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, 0) : x_1 > 0,\ x_2 > 0,\ x_1 + x_2 = 1\}$, in this case, $\{(\bar{x}_1, \bar{x}_2, 0, 0, 0, 1) : \bar{x}_1 > \bar{x}_2 > 0,\ \bar{x}_1 + \bar{x}_2 = 1\} \subset sol(\text{WEVMG})$.

(VII) the case that $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 1, 0)$:

$(*) \iff (\bar{y}_1, \bar{y}_2, \bar{y}_3) \in S_3$ and there exist $\lambda_1 \geqq 0, \lambda_2 \geqq 0, \lambda_1 + \lambda_2 = 1, \alpha \in \mathbb{R}, \beta \geqq 0, \gamma \geqq 0$ such that

$$\begin{cases} \bar{y}_2 - (\lambda_1 - \lambda_2)\bar{y}_3 & = \alpha - \beta \\ -\bar{y}_1 & = \alpha \\ (\lambda_1 - \lambda_2)\bar{y}_1 & = \alpha - \gamma. \end{cases}$$

(i) $\lambda_1 = 0,\ \lambda_2 = 1$: In this case, there is no weakly efficient solution.

(ii) $\lambda_1 = 1,\ \lambda_2 = 0$: $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3)\ :\ y_3 \geqq y_2 \geqq 0,\ y_2 + y_3 = 1\}$.

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3)\ :\ y_3 \geqq y_2 > 0,\ y_2 + y_3 = 1\}$:

(a) $\mu_1 = 0, \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3)\ :\ x_3 \geqq x_2 \geqq 0,\ x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 1, 0)$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0,\ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3)\ :\ x_3 > x_2 > 0,\ x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 1, 0)$, so, in this case, there is no weakly efficient solution.

- in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (0, 0, 1)$:

(a) $\mu_1 = 0, \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3)\ :\ x_1 \geqq 0,\ 0 \leqq x_2 \leqq \frac{1}{2},\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 1, 0)$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0,\ \mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3)\ :\ x_1 \geqq 0,\ 0 \leqq x_2 < \frac{1}{2},\ x_3 \geqq 0,\ x_1 + x_2 + x_3 = 1\}$. But it does not satisfy the condition

$(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 1, 0)$, so, in this case, there is no weakly efficient solution.

(iii) $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_1 + \lambda_2 = 1$: $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3) \ : \ y_3 > y_2 \geqq 0, \ y_2 + y_3 = 1\}$.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) \in \{(0, y_2, y_3) \ : \ y_3 > y_2 > 0, \ y_2 + y_3 = 1\}$:

(a) $\mu_1 = 0, \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \ : \ x_3 \geqq x_2 \geqq 0, \ x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 1, 0)$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0$, $\mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(0, x_2, x_3) \ : \ x_3 > x_2 > 0, \ x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 1, 0)$, so, in this case, there is no weakly efficient solution.

• in the case that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (0, 0, 1)$:

(a) $\mu_1 = 0, \mu_2 = 1$: In this case, there is no weakly efficient solution.

(b) $\mu_1 = 1, \mu_2 = 0$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) \ : \ x_1 \geqq 0, \ 0 \leqq x_2 \leqq \frac{1}{2}, \ x_3 \geqq 0, \ x_1 + x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 1, 0)$, so, in this case, there is no weakly efficient solution.

(c) $\mu_1 > 0, \mu_2 > 0$, $\mu_1 + \mu_2 = 1$: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) \in \{(x_1, x_2, x_3) \ : \ x_1 \geqq 0, \ 0 \leqq x_2 < \frac{1}{2}, \ x_3 \geqq 0, \ x_1 + x_2 + x_3 = 1\}$. But it does not satisfy the condition $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (0, 1, 0)$, so, in this case, there is no weakly efficient solution.

Therefore, the set of all weakly efficient solutions of the vector matrix game (B_1, B_2) is $sol(\text{WEVMG}) = \{(\bar{x}_1, \bar{x}_2, \bar{x}_3, 0, 0, 1) \ : \ \bar{x}_1 > 0, 0 < \bar{x}_2 \leqq \frac{1}{2}, \bar{x}_3 >$

$0, \bar{x}_1 + \bar{x}_2 + \bar{x}_3 = 1\} \cup \{(0, \bar{x}_2, \bar{x}_3, 0, \bar{y}_2, \bar{y}_3) : \bar{x}_3 \geqq \bar{x}_2 > 0, \bar{x}_2 + \bar{x}_3 = 1, \bar{y}_3 \geqq \bar{y}_2 > 0, \bar{y}_2 + \bar{y}_3 = 1\} \cup \{(0, \bar{x}_2, \bar{x}_3, 0, 0, 1) : \bar{x}_3 \geqq \bar{x}_2 > 0, \bar{x}_2 + \bar{x}_3 = 1\} \cup \{(0, 0, 1, \bar{y}_1, \bar{y}_2, \bar{y}_3) : \bar{y}_1 > 0, 0 < \bar{y}_2 \leqq \frac{1}{2}, \bar{y}_3 > 0, \bar{y}_1 + \bar{y}_2 + \bar{y}_3 = 1\} \cup \{(0, 0, 1, 0, \bar{y}_2, \bar{y}_3) : \bar{y}_3 > \bar{y}_2 > 0, \bar{y}_2 + \bar{y}_3 = 1\} \cup \{(0, 0, 1, 0, 0, 1)\} \cup \{(0, 0, 1, \bar{y}_1, 0, \bar{y}_3) : \bar{y}_1 > 0, \bar{y}_3 > 0, \bar{y}_1 + \bar{y}_3 = 1\} \cup \{(0, 0, 1, 1, 0, 0)\} \cup \{(0, 0, 1, \bar{y}_1, \bar{y}_2, 0) : \bar{y}_1 > \bar{y}_2 > 0, \bar{y}_1 + \bar{y}_2 = 1\} \cup \{(\bar{x}_1, 0, \bar{x}_3, 0, 0, 1) : \bar{x}_1 > 0, \bar{x}_3 > 0, \bar{x}_1 + \bar{x}_3 = 1\} \cup \{(\bar{x}_1, 0, \bar{x}_3, \bar{y}_1, 0, \bar{y}_3) : \bar{x}_1 > 0, \bar{x}_3 > 0, \bar{x}_1 + \bar{x}_3 = 1, \bar{y}_1 > 0, \bar{y}_3 > 0, \bar{y}_1 + \bar{y}_3 = 1\} \cup \{(\bar{x}_1, 0, \bar{x}_3, 1, 0, 0) : \bar{x}_1 > 0, \bar{x}_3 > 0, \bar{x}_1 + \bar{x}_3 = 1\} \cup \{(1, 0, 0, 0, 0, 1)\} \cup \{(1, 0, 0, \bar{y}_1, 0, \bar{y}_3) : \bar{y}_1 > 0, \bar{y}_3 > 0, \bar{y}_1 + \bar{y}_3 = 1\} \cup \{(1, 0, 0, 1, 0, 0)\} \cup \{(\bar{x}_1, \bar{x}_2, 0, 0, 0, 1) : \bar{x}_1 \geqq \bar{x}_2 > 0, \bar{x}_1 + \bar{x}_2 = 1\}.$
□

In Examples, it is clear from Definition 2.1.3 that the following hold:

$$\begin{array}{ccc} sol(\text{SVMG}) \subset sol(\text{EVMG}) \subset sol(\text{WEVMG}) \\ \bigcap \\ sol(\text{WSVMG}) \subset sol(\text{WEVMG}). \end{array}$$

From the above calculation, we can check that

(i) $sol(\text{WEVMG}) \setminus sol(\text{EVMG}) = \{(\bar{x}_1, \bar{x}_2, \bar{x}_3, 0, 0, 1) : \bar{x}_1 > 0,\ 0 < \bar{x}_2 \leqq \frac{1}{2},\ \bar{x}_3 > 0,\ \bar{x}_1 + \bar{x}_2 + \bar{x}_3 = 1\} \cup \{(0, \bar{x}_2, \bar{x}_3, 0, \bar{y}_2, \bar{y}_3) : \bar{x}_3 \geqq \bar{x}_2 > 0,\ \bar{x}_2 + \bar{x}_3 = 1,\ \bar{y}_3 \geqq \bar{y}_2 > 0,\ \bar{y}_2 + \bar{y}_3 = 1\} \cup \{(0, \bar{x}_2, \bar{x}_3, 0, \bar{y}_2, \bar{y}_3) : \bar{x}_3 \geqq \bar{x}_2 > 0,\ \bar{x}_2 + \bar{x}_3 = 1,\ \bar{y}_3 > \bar{y}_2 > 0,\ \bar{y}_2 + \bar{y}_3 = 1\} \cup \{(0, \bar{x}_2, \bar{x}_3, 0, 0, 1) : \bar{x}_3 \geqq \bar{x}_2 > 0, \bar{x}_2 + \bar{x}_3 = 1\} \cup \{(0, 0, 1, 0, \bar{y}_2, \bar{y}_3) : \bar{y}_1 > 0, 0 < \bar{y}_2 \leqq \frac{1}{2},\ \bar{y}_3 > 0,\ \bar{y}_1 + \bar{y}_2 + \bar{y}_3 = 1\} \cup \{(\bar{x}_1, \bar{x}_2, 0, 0, 0, 1) : \bar{x}_1 \geqq \bar{x}_2 > 0,\ \bar{x}_1 + \bar{x}_2 = 1\}.$

(ii) $sol(\text{EVMG}) \setminus sol(\text{SVMG}) = \{(\bar{x}_1, \bar{x}_2, \bar{x}_3, 0, 0, 1) : \bar{x}_1 > 0, 0 < \bar{x}_2 < \frac{1}{2}, \bar{x}_3 > 0, \bar{x}_1 + \bar{x}_2 + \bar{x}_3 = 1\} \cup \{(0, 0, 1, \bar{y}_1, \bar{y}_2, \bar{y}_3) : \bar{y}_1 > 0, 0 < \bar{y}_2 < \frac{1}{2}, \bar{y}_3 > 0, \bar{y}_1 + \bar{y}_2 + \bar{y}_3 = 1\} \cup \{(0, 0, 1, \bar{y}_1, \bar{y}_2, 0) : \bar{y}_1 > \bar{y}_2 > 0, \bar{y}_1 + \bar{y}_2 = 1\} \cup \{(\bar{x}_1, \bar{x}_2, 0, 0, 0, 1) : \bar{x}_1 > \bar{x}_2 > 0, \bar{x}_1 + \bar{x}_2 = 1\}$.

(iii) $sol(\text{WEVMG}) \setminus sol(\text{WSVMG}) = \{(\bar{x}_1, \bar{x}_2, \bar{x}_3, 0, 0, 1) : \bar{x}_1 > 0, 0 < \bar{x}_2 \leqq \frac{1}{2}, \bar{x}_3 > 0, \bar{x}_1 + \bar{x}_2 + \bar{x}_3 = 1\} \cup \{(\bar{x}_1, \bar{x}_2, \bar{x}_3, 0, 0, 1) : \bar{x}_1 > 0, 0 < \bar{x}_2 < \frac{1}{2}, \bar{x}_3 > 0, \bar{x}_1 + \bar{x}_2 + \bar{x}_3 = 1\} \cup \{(0, 0, 1, \bar{y}_1, \bar{y}_2, \bar{y}_3) : \bar{y}_1 > 0, 0 < \bar{y}_2 \leqq \frac{1}{2}, \bar{y}_3 > 0, \bar{y}_1 + \bar{y}_2 + \bar{y}_3 = 1\} \cup \{(0, 0, 1, \bar{y}_1, \bar{y}_2, \bar{y}_3) : \bar{y}_1 > 0, 0 < \bar{y}_2 < \frac{1}{2}, \bar{y}_3 > 0, \bar{y}_1 + \bar{y}_2 + \bar{y}_3 = 1\} \cup \{(0, 0, 1, \bar{y}_1, \bar{y}_2, 0) : \bar{y}_1 > \bar{y}_2 > 0, \bar{y}_1 + \bar{y}_2 = 1\} \cup \{(\bar{x}_1, \bar{x}_2, 0, 0, 0, 1) : \bar{x}_1 \geqq \bar{x}_2 > 0, \bar{x}_1 + \bar{x}_2 = 1\} \cup \{(\bar{x}_1, \bar{x}_2, 0, 0, 0, 1) : \bar{x}_1 > \bar{x}_2 > 0, \bar{x}_1 + \bar{x}_2 = 1\}$.

(iv) $sol(\text{WSVMG}) \setminus sol(\text{SVMG}) = \{(0, \bar{x}_2, \bar{x}_3, 0, \bar{y}_2, \bar{y}_3) : \bar{x}_3 \geqq \bar{x}_2 \geqq 0, \bar{x}_2 + \bar{x}_3 = 1, \bar{y}_3 \geqq \bar{y}_2 \geqq 0, \bar{y}_2 + \bar{y}_3 = 1\}$.

(v) $sol(\text{EVMG}) \setminus sol(\text{WSVMG}) = \{(\bar{x}_1, \bar{x}_2, \bar{x}_3, 0, 0, 1) : \bar{x}_1 > 0, 0 < \bar{x}_2 < \frac{1}{2}, \bar{x}_3 > 0, \bar{x}_1 + \bar{x}_2 + \bar{x}_3 = 1\} \cup \{(0, 0, 1, \bar{y}_1, \bar{y}_2, \bar{y}_3) : \bar{y}_1 > 0, 0 < \bar{y}_2 < \frac{1}{2}, \bar{y}_3 > 0, \bar{y}_1 + \bar{y}_2 + \bar{y}_3 = 1\} \cup \{(0, 0, 1, \bar{y}_1, \bar{y}_2, 0) : \bar{y}_1 > \bar{y}_2 > 0, \bar{y}_1 + \bar{y}_2 = 1\} \cup \{(\bar{x}_1, \bar{x}_2, 0, 0, 0, 1) : \bar{x}_1 > \bar{x}_2 > 0, \bar{x}_1 + \bar{x}_2 = 1\}$.

(vi) $sol(\text{WSVMG}) \setminus sol(\text{EVMG}) = \{(0, \bar{x}_2, \bar{x}_3, 0, \frac{1}{2}, \frac{1}{2}) : \bar{x}_2 \geqq \bar{x}_3 \geqq 0,\ \bar{x}_2 + \bar{x}_3 = 1\} \cup \{(0, \frac{1}{2}, \frac{1}{2}, 0, \bar{y}_2, \bar{y}_3) : \bar{y}_3 \geqq \bar{y}_2 \geqq 0, \bar{y}_2 + \bar{y}_3 = 1\}$. □

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