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Finite Element Approximations of a Fourth Order Boundary Value Problem

by

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Finite Element Approximations of a Fourth Order Boundary Value Problem
(4계 경계값 문제의 유한 요소 근사해)

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by

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Contents

List of Tables

List of Figures

4계 경계값 문제의 유한 요소 근사해

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요 약

본 논문에서는 4계 비선형 경계값 문제의 유한 요소 근사해의 수렴성에 대해서 수치적으로 살펴보았다. 우선, 4계 비선형 경계값 문제를 연립 2계 비선형 경계값 문제로 바꾼 다음 유한 요소 근사해를 고안하였다. 또한 구분적 1차 함수(구분적 2차 함수)로 이루어진 유한 요소 공간을 이용하여 근사해를 계산하고, 오차의 L_2노름에 대한 최적성을 수치적으로 보였다.

1 Introduction

In this thesis, we consider the following fourth order boundary value problem in one space dimension:

$$\begin{aligned} &y^{(4)} - \varepsilon y'' - \frac{2}{\pi}\left(\int_0^\pi (y')^2 dx\right) y'' = p(x), \quad 0 \le x \le \pi, \\ &y(0) = y(\pi) = y''(0) = y''(\pi) = 0, \end{aligned} \tag{1.1}$$

where $\varepsilon > 0$ is a constant and $p(x)$ is a nonpositive or nonnegative continuous function on $[0, \pi]$. The problem (1.1) arises in the study of transverse vibrations of a hinged beam. For a derivation of (1.1), one is referred to [4,9].

Ohm et al.[5] established the existence and uniqueness results of the weak solution of (1.1) and proved that the error in L_2 norm is suboptimal.

The main objectives of this paper are to study the finite element method to approximate the solution of the model problem and to show numerically the optimality of the error in L_2 norm where the finite element spaces of piecewise linear(quadratic) polynomials defined on the uniform partition of $[0, \pi]$ are used to obtain the approximate solution of the model problem.

This thesis is organized as follows. In Section 2, we introduce the weak formulation of the model problem. Finite element spaces of piecewise linear(quadratic) polynomials are discussed in Section 3. Section 4 is related to numerical results and the optimality of the error in L_2 norm.

2 Weak Formulation

In the study of transverse vibrations of a hinged beam, the following fourth order boundary value problem arises:

$$
\begin{aligned}
&y^{(4)} - \varepsilon y'' - \frac{2}{\pi}\left(\int_0^\pi (y')^2 dx\right) y'' = p(x), \quad 0 \le x \le \pi, \\
&y(0) = y(\pi) = y''(0) = y''(\pi) = 0,
\end{aligned} \tag{1.1}
$$

where $\varepsilon > 0$ is a constant and $p(x)$ is a nonpositive or nonnegative continuous function on $[0, \pi]$. Throughout this paper, we will assume that $p(x)$ is a nonpositive function. For the case that $p(x)$ is a nonnegative function, a similar result can be obtained.

Letting $\phi = -y''$, we can rewrite (1.1) into the following system:

$$
\begin{aligned}
&-\phi'' + \varepsilon\phi + \frac{2}{\pi}\left(\int_0^\pi (y')^2 dx\right)\phi = p(x), \quad 0 \le x \le \pi, \\
&-y'' - \phi = 0, \quad 0 \le x \le \pi, \\
&y(0) = y(\pi) = \phi(0) = \phi(\pi) = 0.
\end{aligned} \tag{2.1}
$$

Let H_0^1 denote the Sobolev space of $L^2(0,\pi)$ functions with first derivatives in $L^2(0,\pi)$ vanishing at 0 and π. Then the weak formulation of (2.1) can be given as follows: find $(\phi, y) \in H_0^1 \times H_0^1$ such that

$$
\begin{aligned}
&(\phi', \varphi') + \varepsilon(\phi, \varphi) + \frac{2}{\pi}\left(\int_0^\pi (y')^2 dx\right)(\phi, \varphi) = (p, \varphi), \quad \forall \varphi \in H_0^1, \\
&(y', \eta') - (\phi, \eta) = 0, \quad \forall \eta \in H_0^1,
\end{aligned} \tag{2.2}
$$

where

$$(f,g) = \int_0^\pi f(x)g(x)dx.$$

To show existence of the weak solution of (2.1), we consider the following iterative scheme motivated by ones in [6,7]: for $k = 0, 1, 2, \cdots$, find $(\phi_{k+1}, y_{k+1}) \in H_0^1 \times H_0^1$ such that

$$\begin{aligned}
&(\phi'_{k+1}, \varphi') + \varepsilon(\phi_{k+1}, \varphi) + \xi_k(\phi_{k+1}, \varphi) = (p, \varphi), \quad \forall \varphi \in H_0^1, \\
&(y'_{k+1}, \eta') - (\phi_{k+1}, \eta) = 0, \quad \forall \eta \in H_0^1, \\
&\xi_{k+1} = \xi_k + \omega \left(\frac{2}{\pi} \int_0^\pi (y'_{k+1})^2 \, dx - \xi_k \right),
\end{aligned} \tag{2.3}$$

where ω is an appropriate parameter and $\xi_0 = 0$. Note that for $k = 0, 1, 2, \cdots$ there exists the unique solution $(\phi_k, y_k) \in H_0^1 \cap H_0^1$ of the system (2.3).

Theorem 2.1[5]. *Let $0 < \omega \leq \min\left[1/2, 2/(\varepsilon\|y'_1\|\|\phi_1\|)\right]$. Then, the sequence of functions (ϕ_{k+1}, y_{k+1}) generated by the above scheme converges monotonically to a solution (ϕ, y) of the weak formulation of (2.2).*

Theorem 2.2 (Uniqueness)[5]. *If there exist two solutions of the weak formulation of (2.2), then they are equal.*

For a given uniform partition of $[0, \pi]$, let S^h be the space of piecewise linear(quadratic) polynomials on the uniform partition such that $S^h \subset H_0^1$.

Then the standard finite element approximation to (2.2) can be given as follows: find $(\phi_h, y_h) \in S^h \times S^h$ such that

$$
\begin{aligned}
&(\phi_h', \varphi') + \varepsilon(\phi_h, \varphi) + \frac{2}{\pi}\left(\int_0^\pi (y_h')^2 dx\right)(\phi_h, \varphi) = (p, \varphi), \quad \forall \varphi \in S^h, \\
&(y_h', \eta') - (\phi_h, \eta) = 0, \quad \forall \eta \in S^h.
\end{aligned}
\tag{2.4}
$$

When S^h is the space of piecewise linear polynomials, the existance and uniqueness of the solution of (2.4) were proved in [8] and the error analysis was done in [5]. To show the existence of the solution of (2.4), the iterative scheme as in (2.3) was used. However, in the case of piecewise quadratic polynomials, the existance and uniqueness of the solution of (2.4) are not proved so far. And there are no known results on the error analysis.

Theorem 2.3[5]. *Let S^h be the space of piecewise linear polynomial on a uniform partition of $[0, \pi]$. If $\pi\|p\|^2/(\varepsilon + (2/\pi)^2)^3 < 1$, then we obtain the following results:*

(i) $\|(y - y_h)'\| = O(h)$,

(ii) $\|(\phi - \phi_h)\| = O(h)$,

where (y, ϕ) is a solution of (2.2) and (y_h, ϕ_h) is a solution of (2.4).

3 Finite Element Spaces

In this section, we want to describe the finite element spaces which will be used in section 4 to obtain the solution of (2.4).

3.1 Piecewise Linear Polynomial Spaces

Let $0 = x_0 < x_1 < \cdots < x_n = \pi$ be a uniform partition of $[0, \pi]$ and let S_1^h be the linear space of functions v such that

i) $v \in C^0[0, \pi]$,

ii) $v|_{[x_i, x_{i+1}]}$ is a linear polynomial for $i = 0, \cdots, n-1$, and

iii) $v(0) = v(\pi) = 0$.

Here $v|_{[x_i, x_{i+1}]}$ denotes the restriction of v to $[x_i, x_{i+1}]$. For $i = 0, 1, \cdots, n$, we define piecewise linear functions ϕ_i on $[0, \pi]$ as follows:

$$\phi_0(x) = \begin{cases} \dfrac{x_1 - x}{x_1 - x_0} & \text{for } \ x_0 \le x \le x_1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & \text{for } \ x_{i-1} \le x \le x_i \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & \text{for } \ x_i \le x \le x_{i+1} \\ 0 & \text{otherwise,} \end{cases}$$

$$\phi_n(x) = \begin{cases} \dfrac{x - x_{n-1}}{x_n - x_{n-1}} & \text{for } x_{n-1} \le x \le x_n \\ 0 & \text{otherwise.} \end{cases}$$

The graphs of ϕ_i are given in Fig. 3.1.

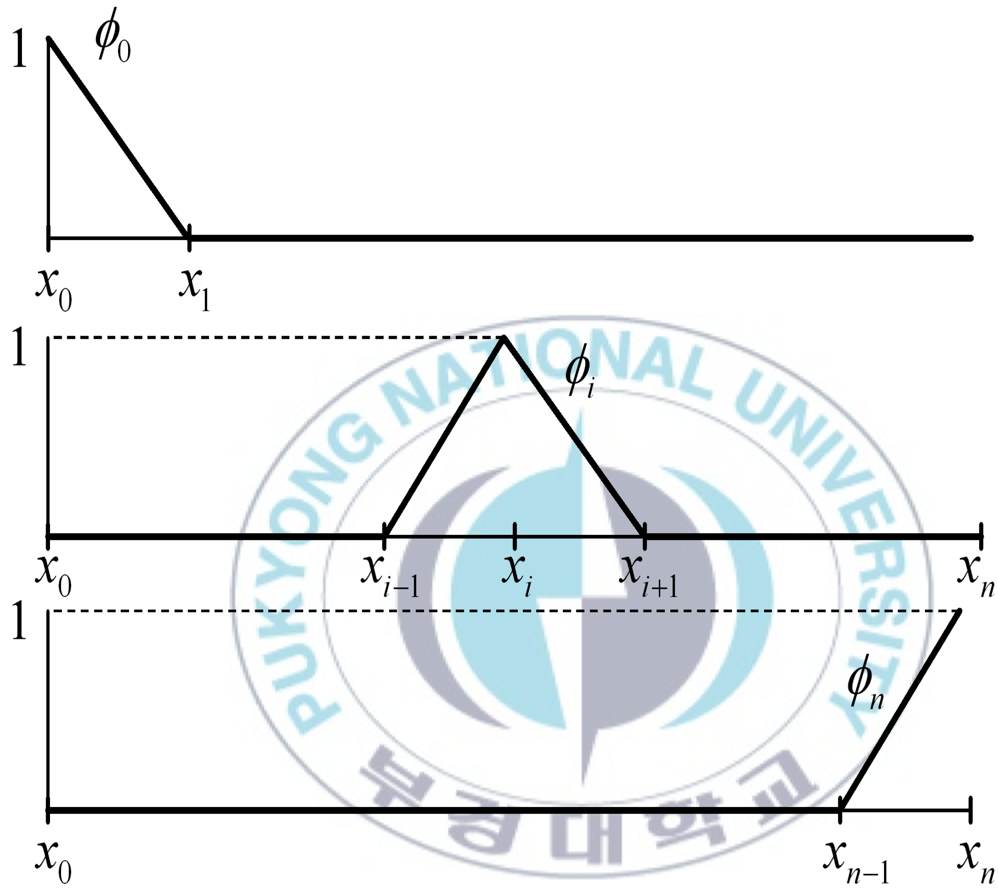

Fig. 3.1 Graphs of piecewise linear functions ϕ_i.

Theorem 3.1[1]. *The set* $\{\phi_i : 1 \le i \le n-1\}$ *is a basis for* S_1^h.

For a systematic formulation, we define linear shape functions L_1 and L_2 on $I_i = [x_i, x_{i+1}]$ as follows:

$$L_1 = \frac{x_{i+1} - x}{x_{i+1} - x_i},$$
$$L_2 = \frac{x - x_i}{x_{i+1} - x_i}.$$

The graphs of L_1 and L_2 are given in Fig. 3.2.

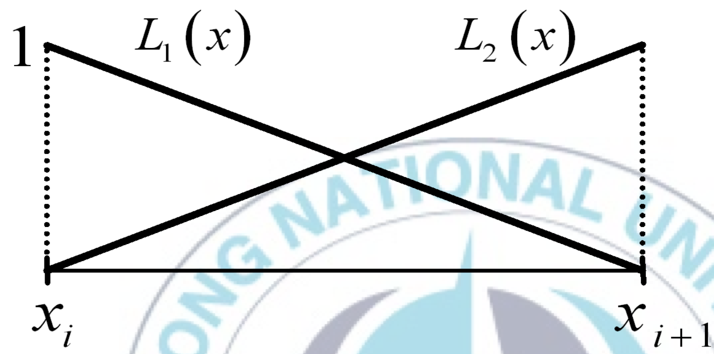

Fig. 3.2 Graphs of linear shape functions L_1 and L_2.

Theorem 3.2[1]. *Every linear function defined on* $[x_i, x_{i+1}]$ *can be expressed as a linear combination of* $L_1(x)$ *and* $L_2(x)$.

3.2 Piecewise Quadratic Polynomial Spaces

Let $0 = x_0 < x_1 < \cdots < x_n = \pi$ be a uniform partition of $[0, \pi]$ and let S_2^h be the linear space of functions v such that

i) $v \in C^0[0, \pi]$,

ii) $v|_{[x_i, x_{i+1}]}$ is a quadratic polynomial for $i = 0, \cdots, n-1$, and

iii) $v(0) = v(\pi) = 0$.

For $i = 0, 1, \cdots, n-1$ and $j = 1, 2, \cdots, n-1$, we define piecewise quadratic functions ϕ_i and ψ_j on $[0, \pi]$ as follows:

$$\phi_i(x) = \begin{cases} -\dfrac{4(x - x_i)(x - x_{i-1})}{(x_i - x_{i-1})^2} & \text{for } \; x_i \le x \le x_{i-1} \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi_0(x) = \begin{cases} \dfrac{(x - x_0)(2x - x_0 - x_1)}{(x_1 - x_0)^2} & \text{for } \; x_0 \le x \le x_1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi_j(x) = \begin{cases} \dfrac{(x - x_{j-1})(2x - x_{j-1} - x_j)}{(x_j - x_{j-1})^2} & \text{for } \; x_{j-1} \le x \le x_j \\ \dfrac{(x - x_{j+1})(2x - x_j - x_{j+1})}{(x_{j+1} - x_j)^2} & \text{for } \; x_j \le x \le x_{j+1} \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi_n(x) = \begin{cases} \dfrac{(x - x_{n-1})(2x - x_{n-1} - x_n)}{(x_n - x_{n-1})^2} & \text{for } x_{n-1} \le x \le x_n \\ 0 & \text{otherwise.} \end{cases}$$

The graphs of ϕ_i and ψ_j are given in Fig. 3.3 and Fig. 3.4, respectively.

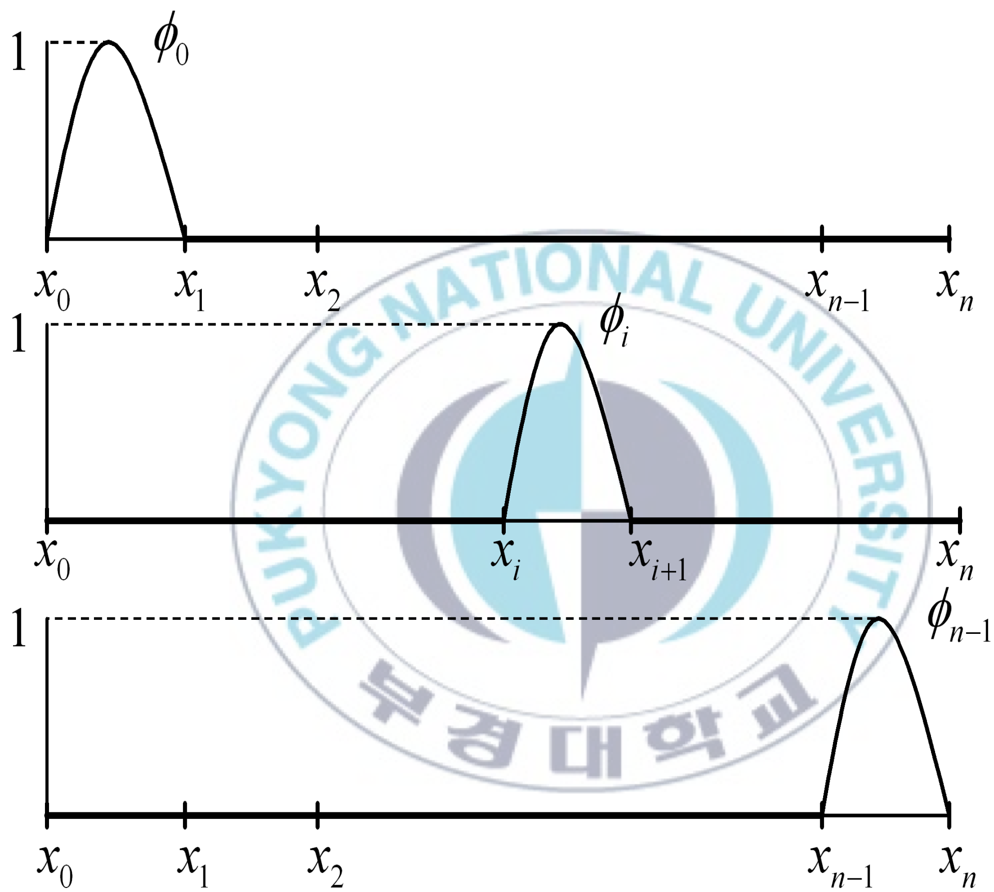

Fig. 3.3 Graphs of piecewise quadratic functions ϕ_i.

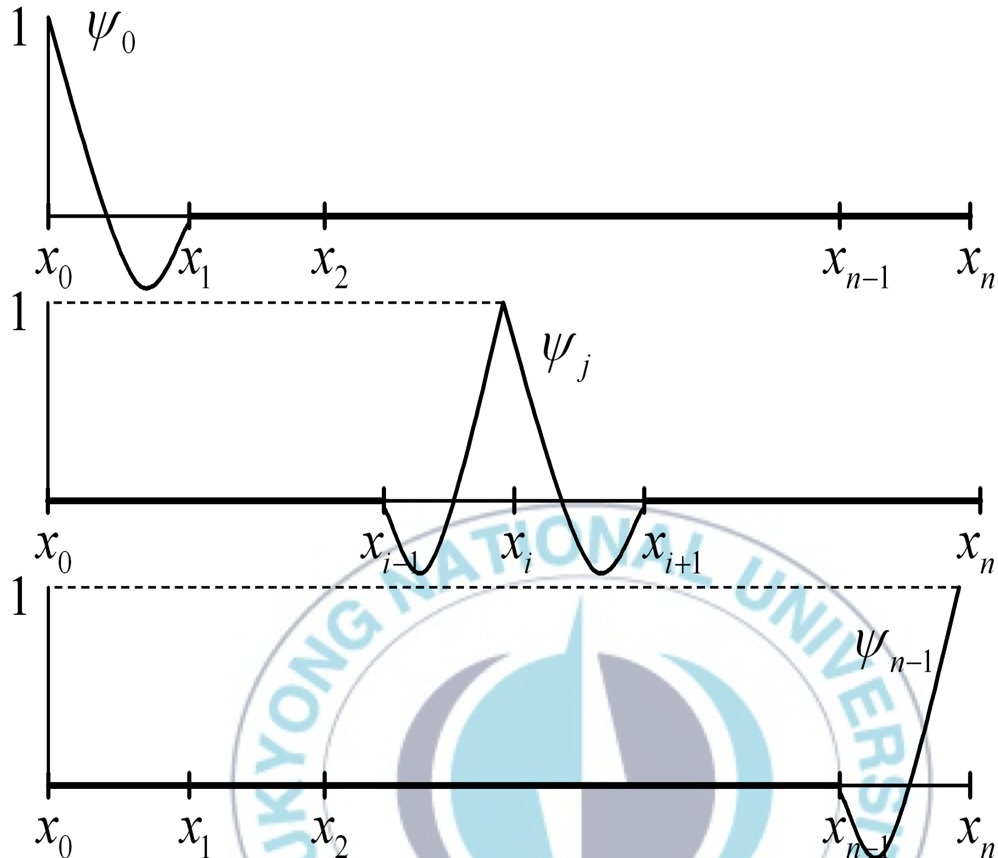

Fig. 3.4 Graphs of piecewise quadratic functions ψ_j.

Theorem 3.3[1]. *The set* $\{\phi_i, \psi_j : 0 \leq i \leq n-1,\ \ 1 \leq j \leq n-1\}$ *is a basis for* S_2^h.

For a systematic formulation, we define quadratic shape functions Q_1, Q_2, and $Q_{\frac{3}{2}}$ on $I_i = [x_i, x_{i+1}]$ as follows:

$$Q_1 = \frac{(x - x_{i+1})(2x - x_i - x_{i+1})}{(x_i - x_{i+1})(2x_i - x_i - x_{i+1})},$$

$$Q_2 = \frac{(x - x_i)(2x - x_i - x_{i+1})}{(x_{i+1} - x_i)(2x_{i+1} - x_i - x_{i+1})},$$

$$Q_{\frac{3}{2}} = -\frac{4(x - x_i)(x - x_{i+1})}{(x_{i+1} - x_i)^2}.$$

The graphs of Q_1, Q_2, and $Q_{\frac{3}{2}}$ are given in Fig. 3.5.

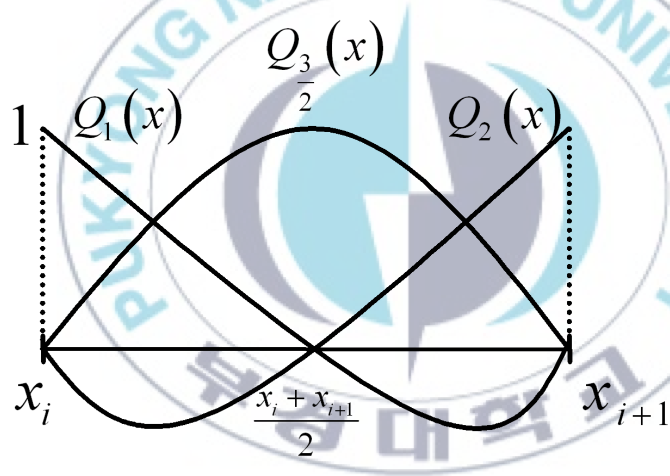

Fig. 3.5 Graphs of quadratic shape functions Q_1, Q_2, and $Q_{\frac{3}{2}}$.

Theorem 3.4[1]. *Every quadratic function defined on $[x_i, x_{i+1}]$ can be expressed as a linear combination of Q_1, Q_2, and $Q_{\frac{3}{2}}$.*

4 Numerical Experiments

The scheme discussed in Section 2 is tested when $\epsilon = 2.0$, $p(x) = -4sin\ x$ on $[0, \pi]$ and $\omega = 0.36$. Using S_1^h or S_2^h on the uniform partition of the size $h = \pi/n$ (n=5,10,20,40,80), the approximations to the solution (ϕ_h, y_h) of (2.4) are obtained. The iteration is stopped when

$$\max_{x_i} \ |\phi_h^{k+1}(x_i) - \phi_h^k(x_i)| < TOL \ \text{ and } \ \max_{x_i} \ |y_h^{k+1}(x_i) - y_h^k(x_i)| < TOL$$

where $TOL = 10^{-10}$.

We first compute approximate solution (ϕ_h, y_h) of (2.4) where the finite element spaces of piecewise linear(quadratic) polynomials defined on the uniform partition of $[0, \pi]$. The convergence of the solution (ϕ_h, y_h) to the solution $(\phi, y) = (-sin\ x, -sin\ x)$ is shown numerically and the errors in L_2 norm are computed to show the optimality numerically.

4.1 Piecewise Linear Polynomial Spaces

In this subsection, we perform numerical experiments with S_1^h. The plots of y_h^k and ϕ_h^k versus the iteration number k are given in Figs. 4.1(a)-(b), respectively when $n = 80$.

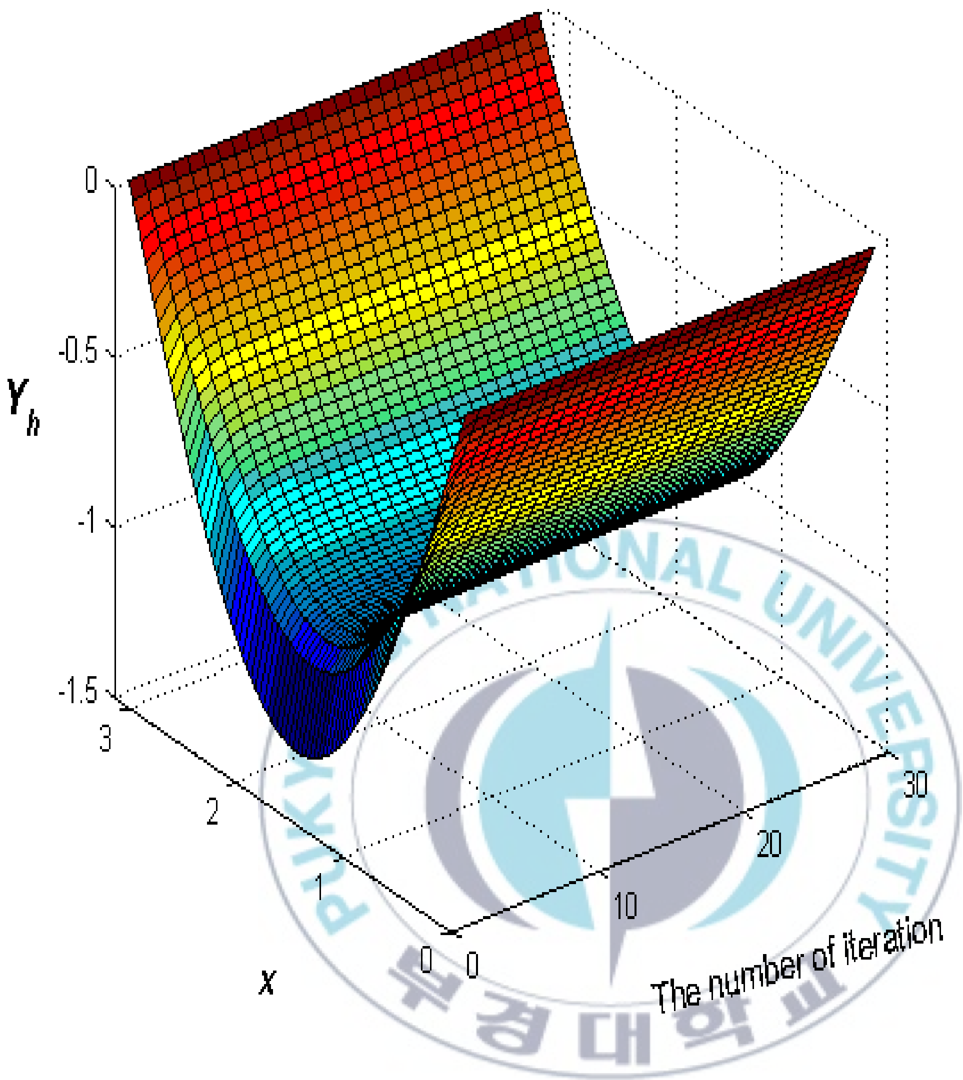

Fig. 4.1(a) Graph of y_h^k when n=80.

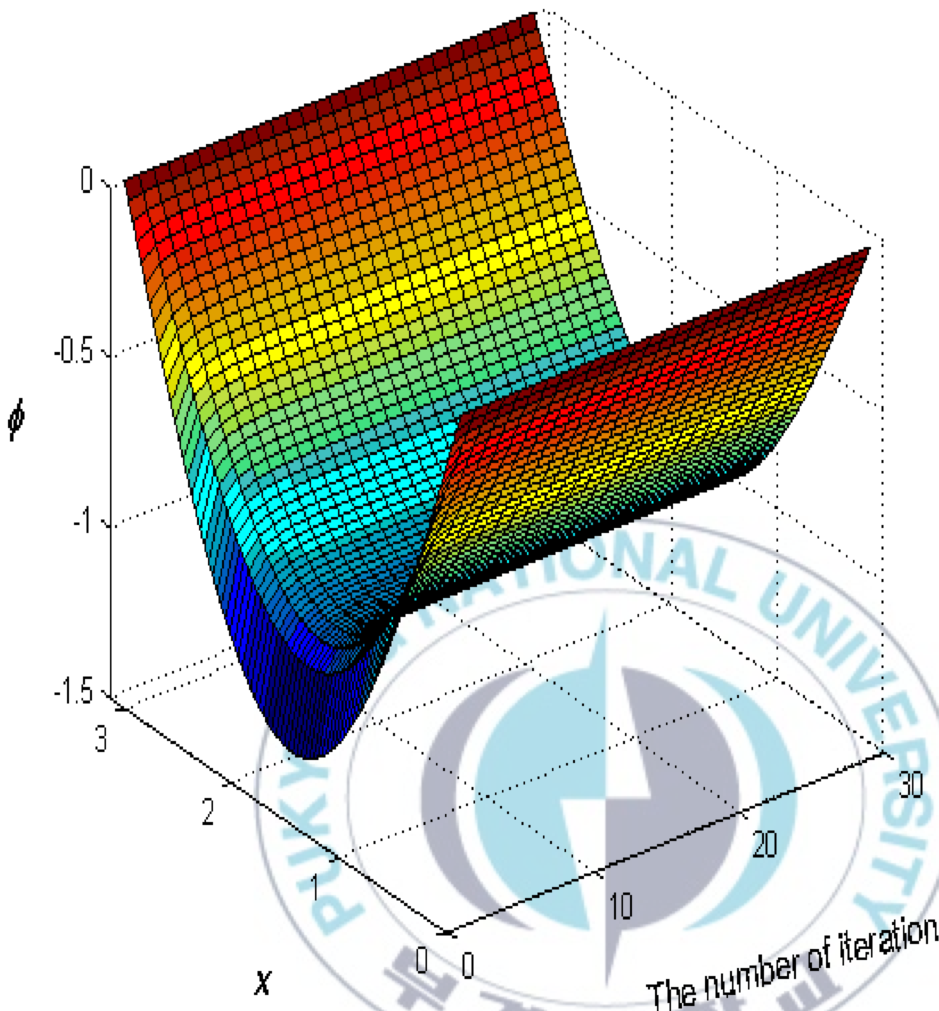

Fig. 4.1(b) Graph of $\phi_h^k(x)$ when n=80.

In Tables 4.1(a)-(b), we report the values of y_h, ϕ_h, y and ϕ for $h = \pi/n$ (n = 5,10,20,40,80) and we know from Tables 4.1(a)-(b) that the values of y_h and ϕ_h converge to ones of y and ϕ, respectively as the grid size h converges to 0. Computed errors in L_2 norm, $\|y - y_h\|$ and $\|\phi - \phi_h\|$, are given in Table 4.2 and the ratios of the computed errors in L_2 norm are given in Table 4.3. From Table 4.3, the ratios of the computed errors in L_2 norm converge to 4 as the grid size decreases by 1/2 of the previous size. We expect from this fact that $\|y - y_h\| = O(h^2)$ and $\|\phi - \phi_h\| = O(h^2)$ which are optimal. However, we have not proved these results analytically.

x	h=π/5	h=π/10	h=π/20	h=π/40	h=π/80	y(exact)
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.3141		-0.3081	-0.3088	-0.3090	-0.3090	-0.3090
0.6283	-0.5795	-0.5860	-0.5874	-0.5877	-0.5878	-0.5878
0.9424		-0.8066	-0.8085	-0.8089	-0.8090	-0.8090
1.2566	-0.9377	-0.9482	-0.9504	-0.9509	-0.9510	-0.9511
1.5707		-0.9970	-0.9993	-0.9998	-1.0000	-1.0000
1.8849	-0.9377	-0.9482	-0.9504	-0.9509	-0.9510	-0.9511
2.1991		-0.8066	-0.8085	-0.8089	-0.8090	-0.8090
2.5132	-0.5795	-0.5860	-0.5874	-0.5877	-0.5878	-0.5878
2.8274		-0.3081	-0.3088	-0.3090	-0.3090	-0.3090
3.1415	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Number of iterations	30	29	29	29	29	

Table 4.1(a) The values of y_h and y for $h = \pi/n$ (n=5,10,20,40,80).

x	h=π/5	h=π/10	h=π/20	h=π/40	h=π/80	ϕ(exact)
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.3141		-0.3055	-0.3082	-0.3088	-0.3090	-0.3090
0.6283	-0.5608	-0.5812	-0.5862	-0.5874	-0.5877	-0.5878
0.9424		-0.8000	-0.8068	-0.8085	-0.8089	-0.8090
1.2566	-0.9075	-0.9405	-0.9484	-0.9504	-0.9509	-0.9511
1.5707		-0.9889	-0.9973	-0.9993	-0.9998	-1.0000
1.8849	-0.9075	-0.9405	-0.9484	-0.9504	-0.9509	-0.9511
2.1991		-0.8000	-0.8068	-0.8085	-0.8089	-0.8090
2.5132	-0.5608	-0.5812	-0.5862	-0.5874	-0.5878	-0.5977
2.8274		-0.3055	-0.3082	-0.3088	-0.3090	-0.3090
3.1415	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Number of iterations	30	29	29	29	29	

Table 4.1(b) The values of ϕ_h and ϕ for $h = \pi/n$ (n=5,10,20,40,80).

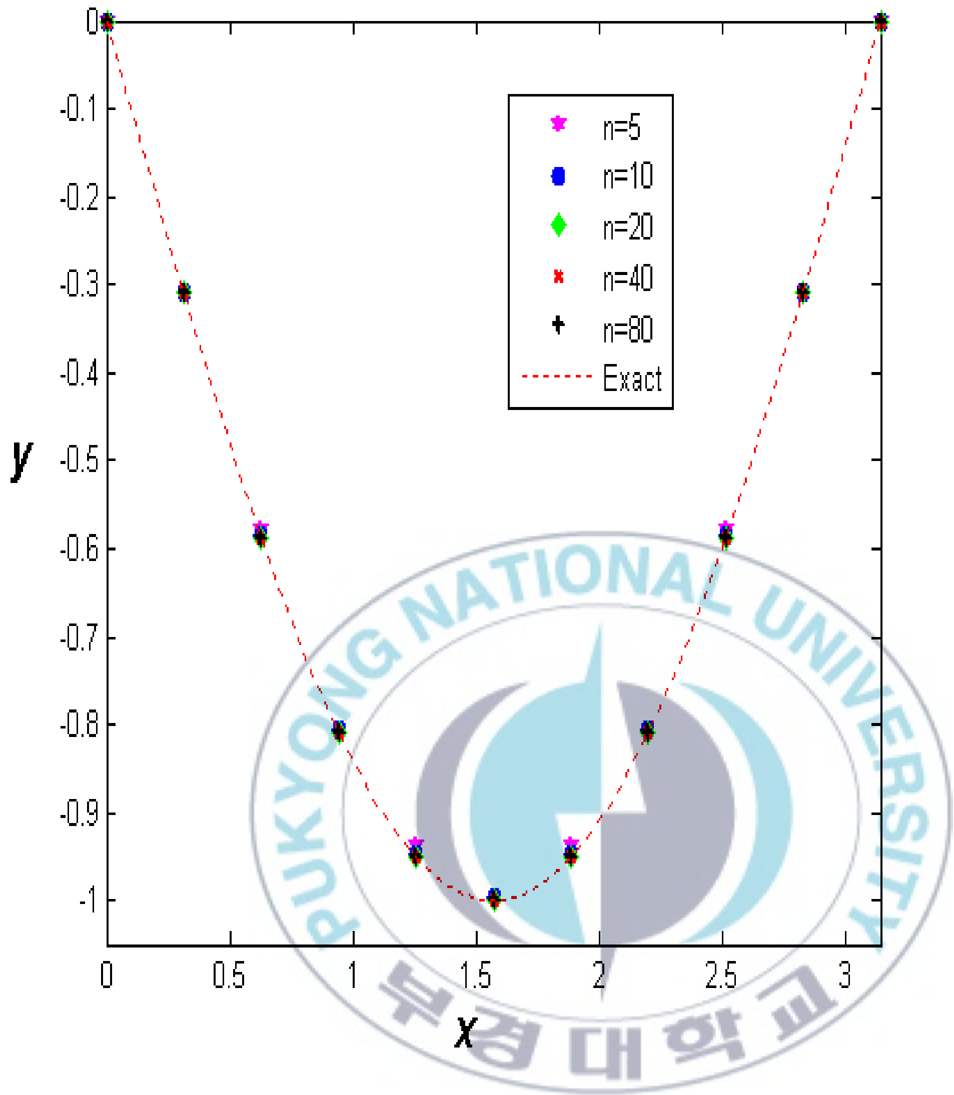

Fig. 4.2(a) Graphs of y_h and y for $h = \pi/n$ (n=5,10,20,40,80).

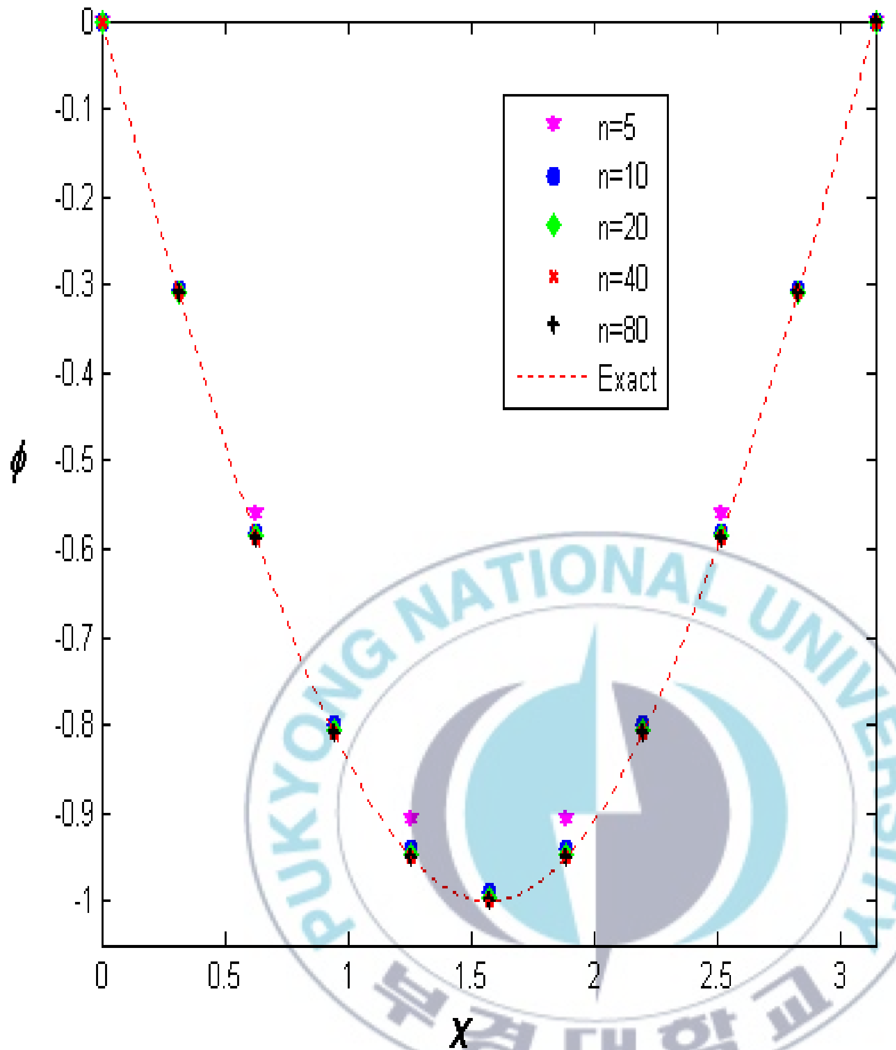

Fig. 4.2(b) Graphs of ϕ_h and ϕ for $h = \pi/n$ (n=5,10,20,40,80).

	$\|y - y_h\|$	$\|\phi - \phi_h\|$
h=$\pi/5$	0.06450729	0.10027529
h=$\pi/10$	0.02511690	0.01569763
h=$\pi/20$	0.00628213	0.00389806
h=$\pi/40$	0.00157071	0.00097287
h=$\pi/80$	0.00039269	0.00024311

Table 4.2 Error estimates of $\|y - y_h\|$ and $\|\phi - \phi_h\|$ for $h = \pi/n$ (n=5,10,20,40,80).

	$\|y - y_h\|/\|y - y_{h/2}\|$	$\|\phi - \phi_h\|/\|\phi - \phi_{h/2}\|$
h=$\pi/5$	2.568	6.387
h=$\pi/10$	3.998	4.027
h=$\pi/20$	3.999	4.006
h=$\pi/40$	3.999	4.001

Table 4.3 The ratios of $\|y - y_h\|/\|y - y_{h/2}\|$ and $\|\phi - \phi_h\|/\|\phi - \phi_{h/2}\|$ for $h = \pi/n$ (n=5,10,20,40).

4.2 Piecewise Quadratic Polynomial Spaces

In this subsection, we also perform numerical experiments with S_2^h. In Tables 4.4(a) and (b), we report the values of y_h, ϕ_h, y and ϕ for $h = \pi/n$ (n = 5,10,20,40,80) and we know from Tables 4.4(a) and (b) that the values of y_h and ϕ_h converge to ones of y and ϕ, respectively as the grid size h converges to 0. Computed errors in L_2 norm, $\|y - y_h\|$ and $\|\phi - \phi_h\|$, are given in Table 4.5 and the ratios of the computed errors in L_2 norm are given in Table 4.6. From Table 4.6, the ratios of the computed errors in L_2 norm converge to 8 as the grid size decreases by 1/2 of the previous size. We expect from this fact that $\|y - y_h\| = O(h^3)$ and $\|\phi - \phi_h\| = O(h^3)$ which are optimal. However, we have not proved these results analytically.

x	h=$\pi/5$	h=$\pi/10$	h=$\pi/20$	h=$\pi/40$	h=$\pi/80$	y(exact)
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.3141		-0.3090	-0.3090	-0.3090	-0.3090	-0.3090
0.6283	-0.5879	-0.5878	-0.5878	-0.5878	-0.5878	-0.5878
0.9424		-0.8090	-0.8090	-0.8090	-0.8090	-0.8090
1.2566	-0.9512	-0.9511	-0.9511	-0.9511	-0.9511	-0.9511
1.5707		-1.0000	-0.9993	-0.9998	-1.0000	-1.0000
1.8849	-0.9512	-0.9511	-0.9511	-0.9511	-0.9511	-0.9511
2.1991		-0.8090	-0.8090	-0.8090	-0.8090	-0.8090
2.5132	-0.5879	-0.5878	-0.5878	-0.5878	-0.5878	-0.5878
2.8274		-0.3090	-0.3090	-0.3090	-0.3090	-0.3090
3.1415	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Number of iterations	29	29	29	29	29	

Table 4.4(a) The values of y_h and y for $h = \pi/n$ (n=5,10,20,40,80).

x	h=$\pi/5$	h=$\pi/10$	h=$\pi/20$	h=$\pi/40$	h=$\pi/80$	ϕ(exact)
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.3141		-0.3090	-0.3090	-0.3090	-0.3090	-0.3090
0.6283	-0.5877	-0.5878	-0.5878	-0.5878	-0.5878	-0.5878
0.9424		-0.8090	-0.8090	-0.8090	-0.8089	-0.8090
1.2566	-0.9510	-0.9511	-0.9511	-0.9511	-0.9511	-0.9511
1.5707		-1.0000	-1.0000	-1.0000	-1.0000	-1.0000
1.8849	-0.9510	-0.9511	-0.9511	-0.9511	-0.9511	-0.9511
2.1991		-0.8090	-0.8090	-0.8090	-0.8090	-0.8090
2.5132	-0.5877	-0.5878	-0.5878	-0.5878	-0.5978	-0.5978
2.8274		-0.3090	-0.3090	-0.3098	-0.3090	-0.3090
3.1415	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Number of iterations	29	29	29	29	29	

Table 4.4(b) The values of ϕ_h and ϕ for $h = \pi/n$ (n=5,10,20,40,80).

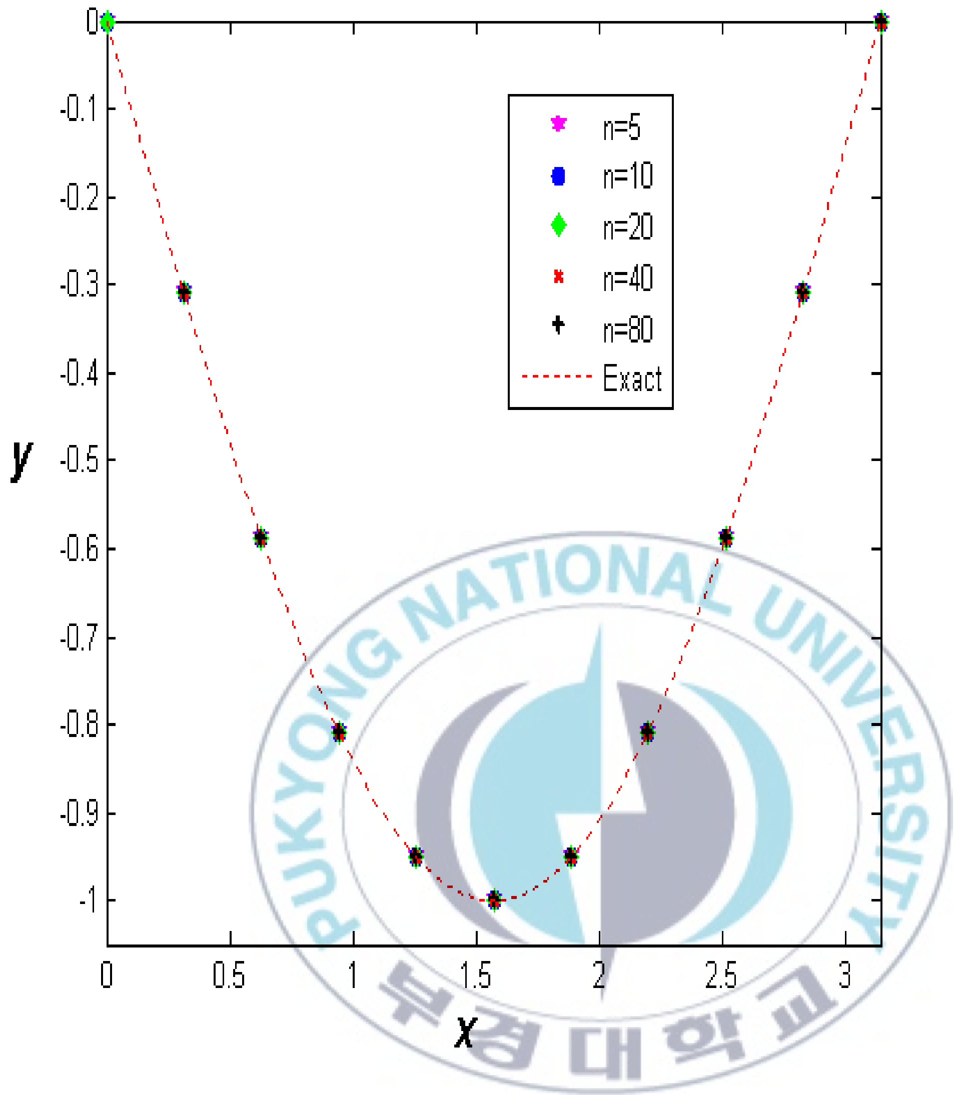

Fig. 4.3(a) Graphs of y_h and y for $h = \pi/n$ (n=5,10,20,40,80).

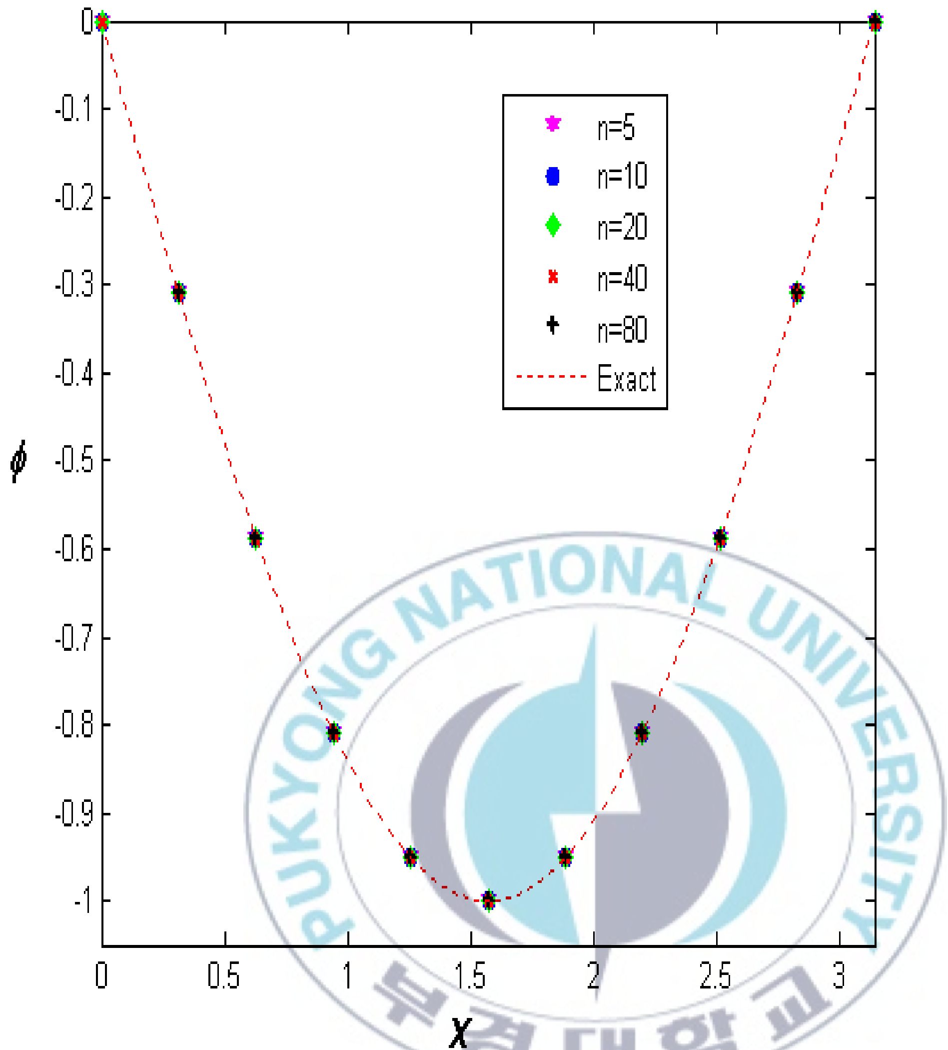

Fig. 4.3(b) Graphs of ϕ_h and ϕ for $h = \pi/n$ (n=5,10,20,40,80).

	$\|y - y_h\|$	$\|\phi - \phi_h\|$
h=$\pi/5$	0.00197235	0.00195229
h=$\pi/10$	0.00024752	0.00024697
h=$\pi/20$	0.00003097	0.00003095
h=$\pi/40$	0.00000387	0.00000387
h=$\pi/80$	0.00000048	0.00000048

Table 4.5 Error estimates for $\|y - y_h\|$ and $\|\phi - \phi_h\|$ of $h = \pi/n$ (n=5,10,20,40,80).

	$\|y - y_h\|/\|y - y_{h/2}\|$	$\|\phi - \phi_h\|/\|\phi - \phi_{h/2}\|$
h=$\pi/5$	7.968	7.904
h=$\pi/10$	7.991	7.977
h=$\pi/20$	7.997	7.994
h=$\pi/40$	7.999	7.998

Table 4.6 The ratios of $\|y - y_h\|/\|y - y_{h/2}\|$ and $\|\phi - \phi_h\|/\|\phi - \phi_{h/2}\|$ of $h = \pi/n$ (n=5,10,20,40).

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