Thesis for the Degree
Master of Education

Inclusion Properties of Certain Classes of Analytic Functions Associated with a Multiplier Transformation

by

Kyung Jin Shin

Graduate School of Education

Pukyong National University

August 2007

Inclusion Properties of Certain Classes of Analytic Functions Associated with a Multiplier Transformation
(승수변환과 관련된 해석함수들의 족들에 대한 포함성질)

Advisor : Prof. Nak Eun Cho

by
Kyung Jin Shin

A thesis submitted in partial fulfillment of the requirement
for the degree of

Master of Education

Graduate School of Education
Pukyong National University

August 2007

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Kyung Jin Shin

Approved by :

(Chairman) Tae Hwa Kim, Ph. D.

(Member) Jin Mun Jeong, Ph. D.

(Member) Nak Eun Cho, Ph. D.

August 30, 2007

CONTENTS

승수변환과 관련된 해석함수들의 족들에 대한 포함성질

신 경 진

부경대학교 교육대학원 수학교육전공

요　　약

기하함수이론은 지금까지 많은 학자들에 의하여 다양하게 연구되어 왔다. 특히, Miller와 Mocanu[7]은 미분종속이론을 소개하여 해석함수들의 여러 기하학적 성질들을 조사하였다.

본 논문에서는 Flett[3]에 의하여 정의된 승수변환과 밀접한 관련이 있고 Salagean[11]과 Uralegaddi와 Somanatha[15]에 의하여 소개된 연산자들을 확장한 새로운 승수변환을 소개하며 Miller와 Mocanu[7]에 의하여 연구된 미분 종속이론을 응용하여 해석함수들의 부분족에 대한 포함관계를 조사하였다.

또한, NunoKawa[10]의 결과를 응용하여 close-to-convex 함수들의 편각추정을 하였으며, sector상에서 적분보존성질들을 조사하여 기존에 알려진 여러 결과들을 발전시켰다.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function w in $\mathbb{U}$ such that $f(z) = g(w(z))$. We denote by $\mathcal{S}^*(\eta)$ and $\mathcal{K}(\eta)$ the subclasses of $\mathcal{A}$ consisting of all analytic functions which are, respectively, starlike and convex of order $\eta(0 \leq \eta < 1)$ in $\mathbb{U}$ (see, e.g., Srivastava and Owa [14]).

For all real numbers t, we define the multiplier transformations I_λ^t of functions $f \in \mathcal{A}$ by

$$I_\lambda^t f(z) = z + \sum_{k=2}^{\infty} \left(\frac{k+\lambda}{1+\lambda}\right)^t a_k z^k \quad (\lambda \geq 0). \tag{1.1}$$

Obviously, we have

$$I_\lambda^t (I_\lambda^s f(z)) = I_\lambda^{t+s} f(z)$$

for all real numbers s and t. For $\lambda = 1$ and nonpositive real number t, the operators I_λ^t were studied by Uralegaddi and Somanatha [15]. Also, the operators I_λ^t are closely related to the multiplier transformations studied by Flett [3] and the differential and integral operators defined by Salagean [11].

Let

$$\mathcal{S}_{t,\lambda}[A,B] = \left\{ f \in \mathcal{A} : \frac{z(I_\lambda^t f(z))'}{I_\lambda^t f(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}\ ;\ -1 \leq B < A \leq 1) \right\}$$

The class $\mathcal{S}_{0,0}[A,B]$ was studied by Janowski [4] and (more recently) by Silverman and Silvia [13]. In particular, we note that $\mathcal{S}_{0,0}[1-2\eta, -1] = \mathcal{S}^*(\eta)$ and $S_{1,0}[1-2\eta,-1] = \mathcal{K}(\eta)$.

Following Silverman and Silvia [12], for a function $f \in \mathcal{S}_{t,\lambda}[A,B]$, we have

$$\left| \frac{z(I_\lambda^t f(z))'}{I_\lambda^t f(z)} - \frac{1-AB}{1-B^2} \right| < \frac{A-B}{1-B^2} \qquad (z \in \mathbb{U}\ ;\ B \neq -1) \tag{1.2}$$

and

$$\operatorname{Re}\left\{\frac{z(I_\lambda^t f(z))'}{I_\lambda^t f(z)}\right\} > \frac{1-A}{2} \qquad (z \in \mathbb{U}\ ;\ B=-1). \tag{1.3}$$

Note that

$$\mathcal{S}_{0,0}[A,-1] = \mathcal{S}^*\left(\frac{1-A}{2}\right).$$

For any real number t, let $\mathcal{C}_{t,\lambda}(\gamma, A, B)$ be the class of functions $f \in \mathcal{A}$ satisfying the condition

$$\operatorname{Re}\left\{\frac{z(I_\lambda^t f(z))'}{I_\lambda^t g(z)}\right\} > \gamma \qquad (0 \le \gamma < 1\ ;\ z \in \mathbb{U})$$

for some $g \in \mathcal{S}_{t,\lambda}[A,B]$. We note that $\mathcal{C}_{0,0}(\gamma, 1-2\eta, -1)$ and $\mathcal{C}_{1,0}(\gamma, 1-2\eta, -1)$ are the classes of close-to-convex and quasi-convex functions of order γ and type η, respectively, studied by Silverman [12] and Noor [9].

In the present paper, we give some argument properties of analytic functions belonging to $\mathcal{A}$ which contain the basic inclusion relationship among the classes $\mathcal{C}_{t,\lambda}(\gamma, A, B)$. The integral preserving properties in connection with the operator I_λ^t defined by (1.1) are also considered. Furthermore, we obtain the previous results by Bernardi [1], Libera [6], Noor [8], Noor and Alkhorasani [9], and Nunokawa et al. [10] as special cases.

2. Main Results

In proving our main results, we need the following lemmas.

Lemma 2.1 [2]. *Let h be convex univalent in $\mathbb{U}$ with $h(0)=1$ and* Re $(\beta h(z)+\gamma) > 0 (\beta, \gamma \in \mathbb{C})$. *If p is analytic in $\mathbb{U}$ with $p(0)=1$, then*

$$p(z) + \frac{zp'(z)}{\beta p(z)+\gamma} \prec h(z) \quad (z \in \mathbb{U})$$

implies

$$p(z) \prec h(z) \quad (z \in \mathbb{U}).$$

Lemma 2.2 [7]. *Let h be convex univalent in $\mathbb{U}$ and ω be analytic in $\mathbb{U}$ with* Re $\omega(z) \ge 0$. *If p is analytic in $\mathbb{U}$ and $p(0)=h(0)$, then*

$$p(z) + \omega(z) z p'(z) \prec h(z) \quad (z \in \mathbb{U})$$

implies

$$p(z) \prec h(z) \quad (z \in \mathbb{U}).$$

Lemma 2.3 [10]. *Let p be analytic in $\mathbb{U}$ with $p(0) = 1$ and $p(z) \neq 0$ in $\mathbb{U}$. If there exist two points $z_1, z_2 \in \mathbb{U}$ such that*

$$-\frac{\pi}{2}\alpha_1 = \arg p(z_1) < \arg p(z) < \arg p(z_2) = \frac{\pi}{2}\alpha_2 \tag{2.1}$$

for some $\alpha_1, \alpha_2 (\alpha_1, \alpha_2 > 0)$ and for all $z (|z| < |z_1| = |z_2|)$, then we have

$$\frac{z_1 p'(z_1)}{p(z_1)} = -i\frac{\alpha_1 + \alpha_2}{2} m \quad \text{and} \quad \frac{z_2 p'(z_2)}{p(z_2)} = i\frac{\alpha_1 + \alpha_2}{2} m, \tag{2.2}$$

where

$$m \geq \frac{1 - |a|}{1 + |a|} \quad \text{and} \quad a = i \tan \frac{\pi}{4}\left(\frac{\alpha_2 - \alpha_1}{\alpha_1 + \alpha_2}\right). \tag{2.3}$$

At first, with the help of Lemma 2.1, we obtain the following

Proposition 2.1. *If $f \in \mathcal{S}_{t+1,\lambda}[A, B]$, then $f \in \mathcal{S}_{t,\lambda}[A, B]$.*

Proof. Let $f \in \mathcal{S}_{t+1,\lambda}[A, B]$. Then we set

$$p(z) = \frac{z(I_\lambda^t f(z))'}{I_\lambda^t f(z)},$$

where p is analytic in $\mathbb{U}$ with $p(0) = 1$. By using the equation

$$z(I_\lambda^t f(z))' = (\lambda + 1) I_\lambda^{t+1} f(z) - \lambda I_\lambda^t f(z), \tag{2.4}$$

we get

$$p(z) + \lambda = (\lambda + 1)\frac{I_\lambda^{t+1} f(z)}{I_\lambda^t f(z)}. \tag{2.5}$$

Taking logarithmic derivatives in both sides of (2.5) and multiplying by z, we have

$$\frac{z(I_\lambda^{t+1}f(z))'}{I_\lambda^{t+1}f(z)} = p(z) + \frac{zp'(z)}{p(z)+\lambda} \quad (z \in \mathbb{U}).$$

Applying Lemma 2.1, it follows that

$$p(z) \prec \frac{1+Az}{1+Bz} \ \ (z \in \mathbb{U}),$$

which means that $f \in \mathcal{S}_{t,\lambda}[A,B]$.

Proposition 2.2. *Let F be the integral operator defined by*

$$F_c(f) := F_c(f)(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \quad (c \geq 0). \tag{2.6}$$

If $f \in \mathcal{S}_{t,\lambda}[A,B]$, then $F_c(f) \in \mathcal{S}_{t,\lambda}[A,B]$.

Proof. From (2.6), we have

$$z(I_\lambda^t F_c(f)(z))' = (c+1)I_\lambda^t f(z) - cI_\lambda^t F_c(f)(z). \tag{2.7}$$

Let $f \in \mathcal{S}_{t,\lambda}[A,B]$. Then we set

$$p(z) = \frac{z(I_\lambda^t F_c(f)(z))'}{I_\lambda^t F_c(f)(z)},$$

where p is analytic in $\mathbb{U}$ with $p(0) = 1$. Then, by using (2.7), we get

$$p(z) + c = (c+1)\frac{I_\lambda^t f(z)}{I_\lambda^t F_c(f)(z)}. \tag{2.8}$$

Taking logarithmic derivatives in both sides of (2.8) and multiplying by z, we have

$$\frac{z(I_\lambda^t f(z))'}{I_\lambda^t f(z)} = p(z) + \frac{zp'(z)}{p(z)+c} \quad (z \in \mathbb{U}).$$

Therefore, by Lemma 2.1, we have that $F_c(f) \in \mathcal{S}_{t,\lambda}[A,B]$.

Now, we derive

Theorem 2.1. *Let $f \in \mathcal{A}$ and $0 < \delta_1, \delta_2 \leq 1, 0 \leq \gamma < 1$. If*

$$-\frac{\pi}{2}\delta_1 < \arg\left(\frac{z(I_\lambda^{t+1}f(z))'}{I_\lambda^{t+1}g(z)} - \gamma\right) < \frac{\pi}{2}\delta_2$$

for some $g \in \mathcal{S}_{t+1,\lambda}[A,B]$, *then*

$$-\frac{\pi}{2}\alpha_1 < \arg\left(\frac{z(I_\lambda^{t}f(z))'}{I_\lambda^{t}g(z)} - \gamma\right) < \frac{\pi}{2}\alpha_2,$$

where α_1 *and* $\alpha_2 (0 < \alpha_1, \alpha_2 \le 1)$ *are the solutions of the equations :*

$$\delta_1 = \begin{cases} \alpha_1 + \frac{2}{\pi}\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|a|)\cos\frac{\pi}{2}t_1}{2\left(\frac{1+A}{1+B}+\lambda\right)(1+|a|)+(\alpha_1+\alpha_2)(1-|a|)\sin\frac{\pi}{2}t_1}\right) & \text{for } B \neq -1, \\ \alpha_1 & \text{for } B = -1, \end{cases} \tag{2.9}$$

and

$$\delta_2 = \begin{cases} \alpha_2 + \frac{2}{\pi}\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|a|)\cos\frac{\pi}{2}t_1}{2\left(\frac{1+A}{1+B}+\lambda\right)(1+|a|)+(\alpha_1+\alpha_2)(1-|a|)\sin\frac{\pi}{2}t_1}\right) & \text{for } B \neq -1, \\ \alpha_2 & \text{for } B = -1, \end{cases} \tag{2.10}$$

when a *is given by* (2.3) *and*

$$t_1 = \frac{2}{\pi}\sin^{-1}\left(\frac{A-B}{1-AB+\lambda(1-B^2)}\right). \tag{2.11}$$

Proof. Let

$$p(z) = \frac{1}{1-\gamma}\left(\frac{z(I_\lambda^t f(z))'}{I_\lambda^t g(z)} - \gamma\right).$$

Using (2.4) and simplifying, we have

$$(1-\gamma)I_\lambda^t g(z)p(z) + \lambda I_\lambda^t f(z) = (\lambda+1)I_\lambda^{t+1}f(z) - \gamma I_\lambda^t g(z). \tag{2.12}$$

Differentiating (2.12) and multiplying by z, we obtain

$$\begin{aligned}(1-\gamma)(I_\lambda^t g(z)zp'(z) + z(I_\lambda^t g(z))'p(z)) + \lambda z(I_\lambda^t f(z))' \\ = (\lambda+1)z(I_\lambda^{t+1}f(z))' - \gamma z(I_\lambda^t g(z))'.\end{aligned} \tag{2.13}$$

Since $g \in \mathcal{S}_{t+1,\lambda}[A,B]$, by Proposition 2.1, we know that $g \in \mathcal{S}_{t,\lambda}[A,B]$. Let

$$q(z) = \frac{z(I_\lambda^t g(z))'}{I_\lambda^t g(z)}.$$

Then, using (2.4) once again, we have

$$q(z) + \lambda = (\lambda+1)\frac{I_\lambda^{t+1} g(z)}{I_\lambda^t g(z)}. \tag{2.14}$$

From (2.13) and (2.14), we obtain

$$\frac{1}{1-\gamma}\left(\frac{z(I_\lambda^{t+1} f(z))'}{I_\lambda^{t+1} g(z)} - \gamma\right) = p(z) + \frac{zp'(z)}{q(z)+\lambda}.$$

Then, by using (1.2) and (1.3), we have

$$q(z) + \lambda = \rho e^{i\frac{\pi\phi}{2}},$$

where

$$\begin{cases} \frac{1-A}{1-B} + \lambda \ < \ \rho \ < \ \frac{1+A}{1+B} + \lambda \\ -t_1 \ < \ \phi \ < \ t_1 \ \text{ for } B \neq -1, \end{cases}$$

when t_1 is given by (2.11), and

$$\begin{cases} \frac{1-A}{2} + \lambda \ < \ \rho \ < \ \infty \\ -1 \ < \ \phi \ < \ 1 \ \text{ for } B = -1. \end{cases}$$

Here, we note that p is analytic in $\mathbb{U}$ with $p(0) = 1$ and $\text{Re}\, p(z) > 0$ in $\mathbb{U}$ by applying the assumption and Lemma 2.2 with $\omega(z) = 1/(q(z)+\lambda)$. Hence $p(z) \neq 0$ in $\mathbb{U}$.

If there exist two points $z_1, z_2 \in \mathbb{U}$ such that the condition (2.1) is satisfied, then (by Lemma 2.3) we obtain (2.2) under the restriction (2.3). At first, for the case $B \neq -1$, we obtain

$$
\begin{aligned}
&\arg\left(p(z_1)+\frac{z_1p'(z_1)}{q(z_1)+\lambda}\right)\\
&=-\frac{\pi}{2}\alpha_1+\arg\left(1-i\frac{\alpha_1+\alpha_2}{2}m(\rho e^{i\frac{\pi\phi}{2}})^{-1}\right)\\
&\leq-\frac{\pi}{2}\alpha_1-\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)m\sin\frac{\pi}{2}(1-\phi)}{2\rho+(\alpha_1+\alpha_2)m\cos\frac{\pi}{2}(1-\phi)}\right)\\
&\leq-\frac{\pi}{2}\alpha_1-\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|a|)\cos\frac{\pi}{2}t_1}{2\left(\frac{1+A}{1+B}+\lambda\right)(1+|a|)+(\alpha_1+\alpha_2)(1-|a|)\sin\frac{\pi}{2}t_1}\right)\\
&=-\frac{\pi}{2}\delta_1,
\end{aligned}
$$

and

$$
\begin{aligned}
&\arg\left(p(z_2)+\frac{z_2p'(z_2)}{q(z_2)+\lambda}\right)\\
&\geq\frac{\pi}{2}\alpha_2+\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|a|)\cos\frac{\pi}{2}t_1}{2\left(\frac{1+A}{1+B}+\lambda\right)(1+|a|)+(\alpha_1+\alpha_2)(1-|a|)\sin\frac{\pi}{2}t_1}\right)\\
&=\frac{\pi}{2}\delta_2,
\end{aligned}
$$

where we have used the inequality (2.3), and $\delta_1,\ \delta_2$ and t_1 are given by (2.9), (2.10) and (2.11), respectively. Similarly, for the case $B=-1$, we have

$$
\arg\left(p(z_1)+\frac{z_1p'(z_1)}{q(z_1)+\lambda}\right)\ \leq\ -\frac{\pi}{2}\alpha_1.
$$

and

$$
\arg\left(p(z_2)+\frac{z_1p'(z_2)}{q(z_2)+\lambda}\right)\ \geq\ \frac{\pi}{2}\alpha_2.
$$

These are contradiction to the assumption of Theorem 2.1. Therefore we complete the proof of Theorem 2.1.

If we let $\delta_1=\delta_2$ in Theorem 2.1, then we see easily the following

Corollary 2.1. *The inclusion relation, $\mathcal{C}_{t+1,\lambda}(\gamma, A, B) \subset \mathcal{C}_{t,\lambda}(\gamma, A, B)$, holds for any real number t.*

Taking $t = \lambda = 0$ and $\delta_1 = \delta_2$ in Theorem 2.1, we have

Corollary 2.2. *Let $f \in \mathcal{A}$. If*

$$\left|\arg\left(\frac{(zf'(z))'}{g'(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta \ \ (0 \leq \gamma < 1;\ 0 < \delta \leq 1)$$

for some $g \in \mathcal{A}$ satisfying the condition :

$$1 + \frac{zg''(z)}{g'(z)} \ \prec\ \frac{1+Az}{1+Bz} \ \ (-1 \leq B < A \leq 1),$$

then

$$\left|\arg\left(\frac{zf'(z)}{g(z)} - \gamma\right)\right| < \frac{\pi}{2}\alpha,$$

where $\alpha (0 < \alpha \leq 1)$ is the solution of the equation :

$$\delta = \begin{cases} \alpha + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha(1+B)(1-AB)\cos\left(\sin^{-1}\left(\frac{A-B}{1-AB}\right)\right)}{(1+A)(1-AB)+\alpha(A-B)(1+B)}\right) & \text{for } B \neq -1, \\ \alpha & \text{for } B = -1. \end{cases}$$

Remark 2.1. (i). Since $\mathcal{C}_{0,\lambda}(\gamma, A, B)$ is a subclass of close-to-convex functions [5], we know from Corollary 2.1 that all functions belonging to the class $\mathcal{C}_{t,\lambda}(\gamma, A, B)$ for any nonnegative integer t are univalent.

(ii). If we put $A = 1 - 2\eta,\ B = -1$ and $\delta = 1$ in Corollary 2.2, then we see that every quasi-convex function of order γ and type η is close-to-convex function of order γ and type η, which covers the result obtained by Noor[8].

Using the same method as in the proof of Theorem 2.1, we have

Theorem 2.2. *Let $f \in \mathcal{A}$ and $0 < \delta_1,\ \delta_2 \leq 1,\ \gamma > 1$. If*

$$-\frac{\pi}{2}\delta_1 < \ \arg\left(\gamma - \frac{z(I_\lambda^{t+1} f(z))'}{I_\lambda^{t+1} g(z)}\right) \ < \ \frac{\pi}{2}\delta_2$$

for some $g \in \mathcal{S}_{t+1,\lambda}[A, B]$, then

$$-\frac{\pi}{2}\alpha_1 \ < \ \arg\ \left(\gamma - \frac{z(I_\lambda^t f(z))'}{I_\lambda^t g(z)}\right) \ < \ \frac{\pi}{2}\alpha_2.$$

where α_1 *and* $\alpha_2(0 < \alpha_1, \alpha_2 \leq 1)$ *are the solutions of the equations* (2.9) *and* (2.10).

Next, we prove

Theorem 2.3. *Let* $f \in \mathcal{A}$ *and* $0 < \delta_1,\ \delta_2 \leq 1,\ 0 \leq \gamma < 1$. *If*

$$-\frac{\pi}{2}\delta_1 < \ \arg\ \left(\frac{z(I_\lambda^t f(z))'}{I_\lambda^t g(z)} - \gamma\right) \ < \ \frac{\pi}{2}\delta_2$$

for some $g \in \mathcal{S}_{t,\lambda}[A, B]$, *then*

$$-\frac{\pi}{2}\alpha_1 \ < \ \arg\ \left(\frac{z(I_\lambda^t F_c(f)(z))'}{I_\lambda^t F_c(g)(z)} - \gamma\right) \ < \ \frac{\pi}{2}\alpha_2.$$

where F_c *is defined by* (2.6), *and* α_1 *and* $\alpha_2(0 < \alpha_1, \alpha_2 \leq 1)$ *are the solutions of the equations :*

$$\delta_1 = \begin{cases} \alpha_1 + \frac{2}{\pi}\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|a|)\cos\frac{\pi}{2}t_2}{2\left(\frac{1+A}{1+B}+c\right)(1+|a|)+(\alpha_1+\alpha_2)(1-|a|)\sin\frac{\pi}{2}t_2}\right) & \textit{for } B \neq -1, \\ \alpha_1 & \textit{for } B = -1, \end{cases}$$

and

$$\delta_2 = \begin{cases} \alpha_2 + \frac{2}{\pi}\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|a|)\cos\frac{\pi}{2}t_2}{2\left(\frac{1+A}{1+B}+c\right)(1+|a|)+(\alpha_1+\alpha_2)(1-|a|)\sin\frac{\pi}{2}t_2}\right) & \textit{for } B \neq -1, \\ \alpha_2 & \textit{for } B = -1, \end{cases}$$

when a *is given by* (2.3) *and* t_2 *is* t_1 *given by* (2.11) *with* $\lambda = c$.

Proof. Let

$$p(z) = \frac{1}{1-\gamma}\left(\frac{z(I_\lambda^t F_c(f)(z))'}{I_\lambda^t F_c(g)(z)} - \gamma\right).$$

Since $g \in \mathcal{S}_{t,\lambda}[A, B]$, we have from Proposition 2.2 that $F_c(g) \in \mathcal{S}_{t,\lambda}[A, B]$. Using (2.7) we have

$$(1-\gamma)I_\lambda^t F_c(g)(z)p(z) + cI_\lambda^t F_c(f)(z) = (c+1)I_\lambda^t f(z) - \gamma I_\lambda^t F_c(g)(z).$$

Then, by a simple calculation, we get

$$(1-\gamma)(p(z)(q(z)+c)+zp'(z)) = (c+1)\frac{z(I_\lambda^t f(z))'}{I_\lambda^t F_c(g)(z)} - \gamma(c+q(z)),$$

where

$$q(z) = \frac{z(I_\lambda^t F_c(g)(z))'}{I_\lambda^t F_c(g)(z)}.$$

Hence we have

$$\frac{1}{1-\gamma}\left(\frac{z(I_\lambda^t f(z))'}{I_\lambda^t g(z)} - \gamma\right) = p(z) + \frac{zp'(z)}{q(z)+c}.$$

The remaining part of the proof in Theorem 2.3 is similar to that of Theorem 2.1 and so we omit it.

Taking $\delta_1 = \delta_2$ in Theorem 2.3, we have

Corollary 2.3. *Let $f \in \mathcal{A}$ and $0 \le \gamma < 1, 0 < \delta \le 1$. If*

$$\left|\arg\left(\frac{z(I_\lambda^t f(z))'}{I_\lambda^t g(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta$$

for some $g \in \mathcal{S}_{t,\lambda}[A,B]$, then

$$\left|\arg\left(\frac{z(I_\lambda^t F_c(f)(z))'}{I_\lambda^t F_c(g)(z)} - \gamma\right)\right| < \frac{\pi}{2}\alpha,$$

where F_c is defined by (2.6), and $\alpha (0 < \alpha \le 1)$ is the solution of the equation :

$$\delta = \begin{cases} \alpha + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha\cos\frac{\pi}{2}t_2}{\left(\frac{1+A}{1+B}+c\right)+\alpha\sin\frac{\pi}{2}t_2}\right) & \text{for } B \neq -1, \\ \alpha & \text{for } B = -1, \end{cases}$$

when t_2 is t_1 given by (2.11) with $\lambda = c$.

From Corollary 2.3, we have the following

Corollary 2.4 *Let $f \in \mathcal{C}_{t,\lambda}(\gamma, A, B)$. Then $F_c(f) \in \mathcal{C}_{t,\lambda}(\gamma, A, B)$, where F_c is the integral operator defined by (2.6).*

Remark 2.2. If we take $t = 0$ and $t = 1$ with $\lambda = 0$, $A = 1-2\eta(0 \leq \eta < 1)$ and $B = -1$ in Corollary 2.4, respectively, then we have the corresponding results obtained by Noor and Alkhorasani [9]. Furthermore, taking $t = \lambda = \gamma = 0$, $A = 1$ and $B = -1$ in Corollary 2.4, we obtain the classical result by Bernardi [1], which implies the result studied by Libera [6].

Finally, we prove the following theorems below applying Lemma 2.3.

Theorem 2.4. *Let* $f \in \mathcal{A}$ *and* $0 < \delta_1, \ \delta_2 \leq 1$. *If*

$$-\frac{\pi}{2}\delta_1 \ < \ \arg \ \frac{I_\lambda^{t+1} f(z)}{z} \ < \ \frac{\pi}{2}\delta_2,$$

then

$$-\frac{\pi}{2}\alpha_1 \ < \ \arg \ \frac{I_\lambda^{t} f(z)}{z} \ < \ \frac{\pi}{2}\alpha_2.$$

where α_1 *and* $\alpha_2 (0 < \alpha_1, \alpha_2 \leq 1)$ *are the solutions of the equations* :

$$\delta_1 = \alpha_1 + \frac{2}{\pi}\tan^{-1}\frac{(\alpha_1+\alpha_2)(1-|a|)}{2(1+|a|)(1+\lambda)} \ \ \textit{and} \ \ \delta_2 = \alpha_2 + \frac{2}{\pi}\tan^{-1}\frac{(\alpha_1+\alpha_2)(1-|a|)}{2(1+|a|)(1+\lambda)}$$

when a *is given by* (2.3).

Proof. Let

$$p(z) = \frac{I_\lambda^{t} f(z)}{z}.$$

Using (2.4), we have

$$\frac{I_\lambda^{t+1} f(z)}{z} = p(z) + \frac{1}{1+\lambda} z p'(z).$$

Suppose that there exist two points $z_1, z_2 \in \mathbb{U}$ such that the condition (2.1) holds. Then (by Lemma 2.3) we get (2.2) with the condition (2.3). Then we have

$$\begin{aligned}\arg\ \frac{I_\lambda^{t+1}f(z_1)}{z_1} &= \arg\ \left(p(z_1)+\frac{1}{1+\lambda}z_1p'(z_1)\right)\\ &= \arg\ p(z_1)+\arg\ \left(1-i\frac{\alpha_1+\alpha_2}{2(1+\lambda)}m\right)\\ &\le -\frac{\pi}{2}\alpha_1-\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|a|)}{2(1+|a|)(1+\lambda)}\right)\\ &= -\frac{\pi}{2}\delta_1.\end{aligned}$$

and

$$\begin{aligned}\arg\ \frac{I_\lambda^{t+1}f(z_2)}{z_2} &= \arg\ \left(p(z_2)+\frac{1}{1+\lambda}z_2p'(z_2)\right)\\ &= \arg\ p(z_2)+\arg\ \left(1+i\frac{\alpha_1+\alpha_2}{2(1+\lambda)}m\right)\\ &\ge \frac{\pi}{2}\alpha_2+\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|a|)}{2(1+|a|)(1+\lambda)}\right)\\ &= \frac{\pi}{2}\delta_2,\end{aligned}$$

which contradict the conditions. Therefore we complete the proof of Theorem 2.4.

Remark 2.3. Letting $t=0$ and $\lambda=0$ in Theorem 2.4, we have the corresponding result obtained by Nunokawa et al. [10].

Theorem 2.5. *Let $f\in\mathcal{A}$ and $0<\delta_1,\delta_2\le 1$. If*

$$-\frac{\pi}{2}\delta_1\ <\ \arg\ \frac{I_\lambda^t f(z)}{z}\ <\ \frac{\pi}{2}\delta_2,$$

then

$$-\frac{\pi}{2}\alpha_1\ <\ \arg\ \frac{I_\lambda^t F(z)}{z}\ <\ \frac{\pi}{2}\alpha_2,$$

where F is defined by (2.6)*, and α_1 and $\alpha_2(0<\alpha_1,\alpha_2\le 1)$ are the solutions of the equations :*

$$\delta_1=\alpha_1+\frac{2}{\pi}\tan^{-1}\frac{(\alpha_1+\alpha_2)(1-|a|)}{2(1+|a|)(1+c)}\ \ \textit{and}\ \ \delta_2=\alpha_2+\frac{2}{\pi}\tan^{-1}\frac{(\alpha_1+\alpha_2)(1-|a|)}{2(1+|a|)(1+c)}$$

when a is given by (2.3).

Proof. Letting

$$p(z)=\frac{I_\lambda^t F(z)}{z},$$

we have

$$\frac{I_\lambda^t f(z)}{z}=p(z)+\frac{1}{c+1}zp'(z).$$

Therefore, applying the same method as in the proof of Theorem 2.4, we have Theorem 2.5.

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