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Derivation of unknotting tunnels for $p(-2,3,7)$

by

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Derivation of unknotting tunnels for $p(-2,3,7)$
($p(-2,3,7)$의 비매듭터널의 유도)

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by
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A dissertation

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$p(-2,3,7)$ 비매듭 터널의 유도

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요 약

매듭 $p(-2,3,7)$의 서로 다른 두 (1,1)-분해(decomposition)에 대한 이원 분지 피복 공간의 정이면체 대칭성(dihedral symmetry of the double branched covering)을 이용하여 4개의 비매듭 터널(unknotting tunnel)을 얻는 과정을 상세히 밝힌다.

1 Introduction

In [9] using the dihedral symmetry of a Brieskorn homology sphere $\Sigma(2,3,7)$, Song showed that a pretzel knot $p(-2,3,7)$ admits two non-homeomorphic (1,1)-decompositions. Based on this result, Heath-Song([3]) classified all possible unknotting tunnels of $p(-2,3,7)$, namely τ_1, τ_2, τ_3 and τ_4 as illustrated in figure 1.

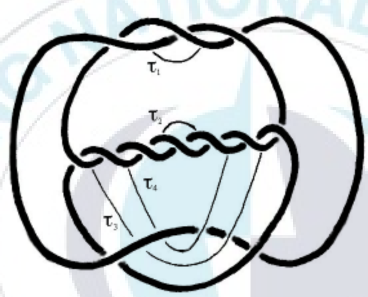

Fig. 1

The purpose of this thesis is to explicitly show how to derive the four unknotting tunnels from the two (1,1)-decompositions of $p(-2,3,7)$ which is not presented in [3].

This thesis is organized as follows. In section 2 we collect some basic concepts and definitions for study of tunnel number one knots including the dihedral branched covering space of a 2-bridge theta curve studied in [9]. In section 3 the main results of this thesis are presented . Finally we prepare section 4 as appendix which contains pictorial proofs for some claims in section 3.

2 Preliminaries

2.1 some basic concepts for study of tunnel number one knots in S^3

A smooth (or tamed) embedding of the circle $S^1 = \{(u, v) \in \mathbb{R}^2 | u^2 + v^2 = 1\}$ in the 3-sphere $S^3 = \{(x, y, z, w) \in \mathbb{R}^4 | x^2 + y^2 + z^2 + w^2 = 1\}$ is called a knot and denoted by K. By taking S^3 as the 1-point compactification of the euclidian 3-space $\mathbb{R}^3$, namely $S^3 = \mathbb{R}^3 \cup \infty$, we may assume that K is embedded in $\mathbb{R}^3$. Thus a knot K is depicted by a smooth simple closed curve in $\mathbb{R}^3$ as shown figure 2(a). Sometimes it is more convenient to describe K by a piecewise linear closed curve as shown in figure 2(b). But figure 2(c) is not consider as a knot in our concern ,which is called *a wild (or non-tamed) knot.*

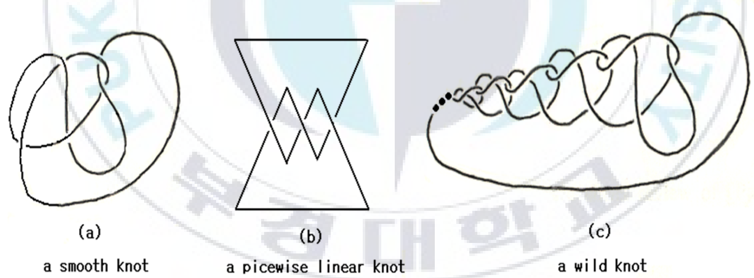

Fig. 2

We call K an unknot or a trivial knot if K bounds an embedded 2-disk D in S^3, namely $\partial D = K$ and D has no self-intersections. In figure 3, we have an example of an unknot.

Let M be a closed orientable 3-manifolds. A 3-ball with $g-$ handles is called a handlebody (of genus g). For a handlebody H of genus g, we recover a 3-ball by cutting H along disks $D_i (1 \leq i \leq g)$ properly embedded in $g-$ handles respectively. A set $\mathcal{D} = \{D_i | 1 \leq i \leq g\}$ such disks are called

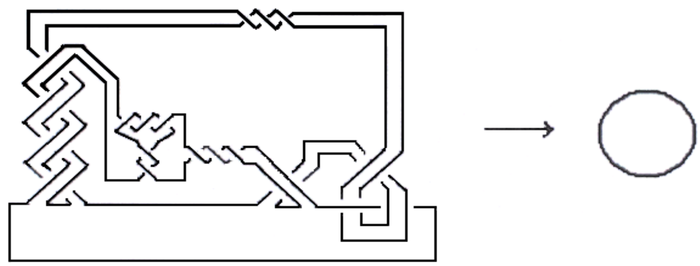

Fig. 3

a system of meridian disks of H. A closed orientable surface F in M is called *a Heegaard surface* if and only if M is decomposed of a union of two handlebodies $H_i (i = 1, 2)$ along F ; $M = H_1 \cup_F H_2$. The decomposition of M by a Heegaard surface F is called *a Heegaard splitting*. For $M = H_1 \cup_F H_2$, a Heegaard splitting of M let $\mathcal{D}_i = \{D^i_j | 1 \leq j \leq g\}$ be a system of meridian disks of H_i for each $i = 1, 2$. Then a triple $(F, \partial\mathcal{D}_i = \{\partial D^1_j | 1 \leq j \leq g\}, \partial\mathcal{D}_2 = \{\partial D^2_j | 1 \leq j \leq g\})$ is called *a Heegaard diagram* associated with the Heegaard splitting. Given a knot K in S^3, a *tunnel* is an embedded arc τ in S^3 with its endpoints on K and its interior disjoint from K. A tunnel τ is an *unknotting* tunnel if the complement of a regular neigbourhood W_1 of $K \cup \tau$ is a handlebody W_2 of genus 2. Hence an unknotting tunnel τ of K induces a Heegaard decomposition $S^3 = (W_1, K) \cup (W_2, \emptyset)$ of genus 2, which is called *a (2,0)-decomposition of* (K, τ). Any knot with an unknotting tunnel is called a *tunnel number one* , or shortly, a *tunnel-1* knot.
In general it is not easy problem to determine whether or not a given tunnel τ is an unknotting tunnel of a knot K. But using handle sliding illustrated in figure 4, if we can bring the handlebody $N(K \cup \tau)$ into the standardly embedded handlebody, then we see that τ is an unknotting tunnel of K.

An arc t in a handlebody H is said to be trivial if and only if there exists a disk D in H and an arc t' in ∂D such that $\partial t = \partial t'$ and $\partial D = t \cup t'$. Here the arc t' is said to be a projection of the trivial arc t and D is said to be

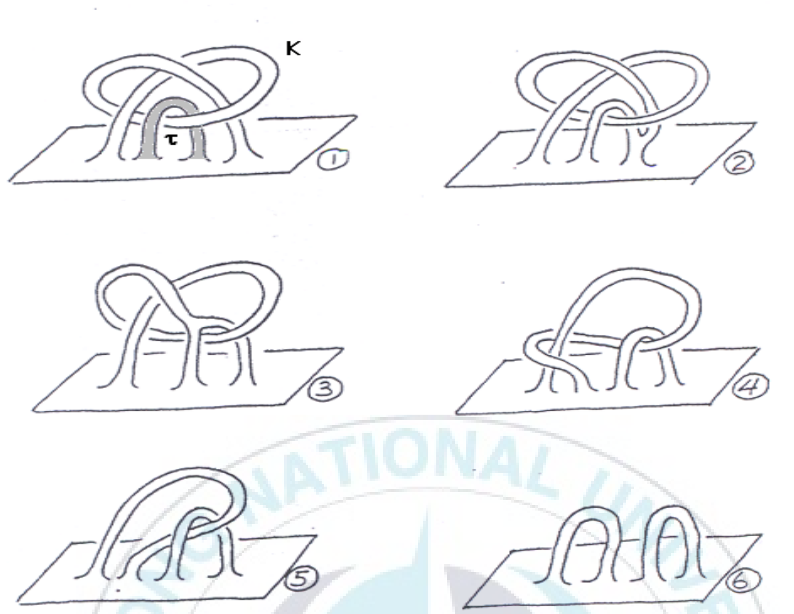

Fig. 4

a projection or spanning disk of t. A knot K in S^3 is said to admits *(g,b)-decomposition* if it has a decomposition $(S^3, K) = (H_1, \{t_1, t_2, \cdots, t_b\}) \cup (H_2, \{s_1, s_2, \cdots, s_b\})$ with properly embedded $b-$ trivial strings $t_i, 1 \leq i \leq b$ (resp. $s_i, 1 \leq i \leq b$) in a handlebody H_1 of genus g (resp.H_2).

In particular, a $(0, b)$-decomposition a knot K in S^3 stands for a $b-$ bridge decomposition of K in the usual sens. Any knot admitting a (1,1)-decomposition is called *a genus 1, 1-bridge*, or shortly, *a (1,1) knot*. It is easy to see that a (1,1)-decomposition of K induces a pair of unknotting tunnels of K called *(1,1)-tunnels* and hence all (1,1)-knots are tunnel-1 knots. Torus knots $t(p, q)$, 2-bridge knots $b(p, q)$ and certain 3-branched Montesinos' knots $M(b, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3})$ including the pretzel knot $p(-2, 3, 7)$ are all (1,1)-knots. But it is known that there are tunnel-1 knots which are not

(1,1)-knots (see [9, figure 12]).

For a $(1,1)-$ decomposition $(S^3, K) = (H_1, t_1) \cup (H_2, s_1)$ of K we may

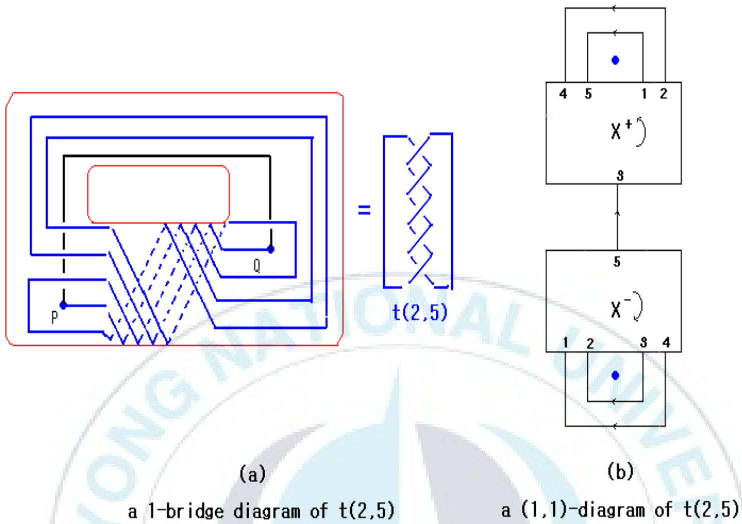

Fig. 5

associate a knot diagram $t_1 \cup s'_1$ by taking a projection s'_1 of s_1 as illustrated in figure 5(a). It is called *a 1-bridge diagram* induced by an $(1,1)-$ decomposition of K. On the other hand, figure 5(b) shows a rather abstract method of representing a $(1,1)$-knot of K, which is called *a* $(1,1)-$ *diagram* associated with $(1,1)-$ decomposition of K. It consists of a usual genus one heegaard diagram on a Heegaard torus T of S^3 with a pair of points $K \cap T$. It is not difficult to see that a given $(1,1)-$decomposition of K has the uniquely associated $(1,1)-$diagram whereas it has infinitely many associated 1-bridge diagrams.

2.2 the dihedral branched covering spaces of 2-bridge $\theta-$ curves

In this section we briefly recall some tools for study of (1,1)-knots. For more details see [7] or [9].

A knot K in S^3 is said to be *strongly invertible* if there is an involution h (called a *strong inversion*) of the pair (S^3, K) such that Fix(h) is a circle intersecting K in two points. Considering the double covering projection $p : S^3 \to S^3/h\,(\cong S^3)$ branched over a trivial knot $p(\text{Fix}(h))$, we have a θ-curve $\theta(K,h) \equiv p(\text{Fix}(h) \cup K)$ induced by the pair (K,h).

For study of tunnel-1 knots (resp. (1,1)-knots) in S^3, Morimoto-Sakuma-Yokota introduced the concept of a 3-bridge (resp. 2-bridge) decomposition of a $\theta-$curve through application of Birman-Hilden-Viro's Theorem to a (2,0)-decomposition of a tunnel-1 knot (resp. a (1,1)-decomposition of a (1,1)-knot) as follows.

A θ-curve is said to admit a 2-bridge decomposition, if and only if (S^3, θ) is a union of (B_1, t_1, a_1) and (B_2, t_2, a_2) along their boundary $S_2 = \partial B_1 = \partial B_2$, where (B_i, t_i) (respectively a_i) is a 2-strand trivial tangle (respectively a trivial arc in (B_i, t_i)) for $i = 1, 2$ as illustrated in Fig. 6(a), and it is said to admit a 3-bridge decomposition, if and only if (S^3, θ) is a union of (B_1, t_1, a) and $(B_2, t_2, \emptyset)$ along their boundary $S_3 = \partial B_1 = \partial B_2$, where (B_i, t_i) is a 3-strand trivial tangle for $i = 1, 2$ and a is a trivial arc in (B_1, t_1) as illustrated in Fig. 6(b).

In the above definition S_g is said to be a *bridge decomposing sphere* and a θ-curve admitting bridge decomposition sphere S_g is denoted by (θ, S_g).

In the sequel, we assume that $g = 2$ or 3 otherwise it is stated explicitly. The following lemma immediately follows from the definition of (θ, S_g).

Lemma 2.3.1 A θ-curve with a bridge decomposition (θ, S_g) induces those

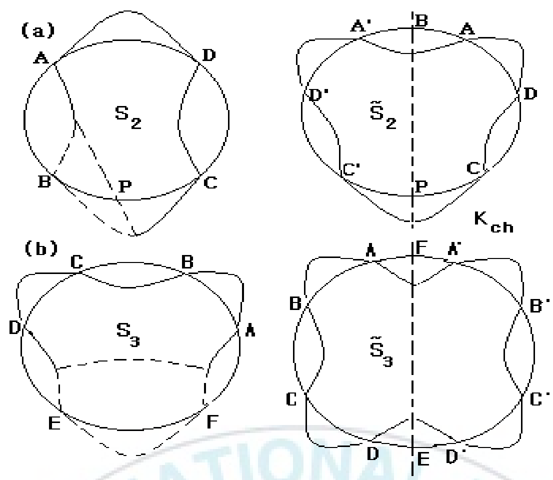

Fig. 6

of its three constituent knot $C_i = \theta - \mathrm{Int}(e_i)$ such that
(i) for $g = 2$, C_1 has a 1-bridge decomposition and C_2, C_3 have 2-bridge decompositions,
(ii) for $g = 3$, C_1, C_2 and C_3 have 1, 2 and 3-bridge decomposition, respectively.

Lemma 2.3.2 For each pair (θ, S) of a θ-curve and its g-bridge decomposing sphere $S(\equiv S_g)$, we have a triple $(K, \tilde{S}, h)$ of a knot K, its $(g+1)$-bridge decomposing sphere $\tilde{S}$ and a bridge preserving strong inversion h.

Conversely a bridge preserving strong inversion of a knot with a $(g+1)$-bridge decomposition induces a θ-curve with a g-bridge decomposition.

Proof. Consider the double covering projection $\pi : \tilde{S^3} = \tilde{B}_1 \cup \tilde{B}_2 \to S^3 = B_1 \cup B_2$ branched over a 1-bridge constituent knot C_1 where each $\tilde{B}_i$ is the 3-ball covering B_i. Since C_1 is a trivial knot, so is $\pi^{-1}(C_1)$ in the covering 3-sphere $\tilde{S^3}$. Then $\pi^{-1}(e_1)$, the lifting of the edge $e_1 = \overline{\theta - C_1}$ is a knot in $\tilde{S^3}$ with a bridge decomposition $(\tilde{S^3}, \pi^{-1}(e_1)) = (\tilde{B}_1, \pi^{-1}(e_1 \cap B_1)) \cup (\tilde{B}_2, \pi^{-1}(e_1 \cap B_2))$ where $\pi^{-1}(e_1 \cap B_i)$ consists of $g+1$ trivial arcs for each

$i = 1, 2$ as illustrated in Fig. 5. Moreover $\pi^{-1}(C_1)$ forms the fixed circle of a bridge preserving strong inversion for a pair $(K_{ch} \equiv \pi^{-1}(e_1),\ \tilde{S} \equiv \pi^{-1}(S))$.

By tracing the above argument backwards, we have the converse. $\square$

We call the knot K in Lemma 2.3.2 *the characteristic knot* of (θ, S_g) and denote it by K_{ch}.

In [7] they observed that if a strong inversion h is induced by an unknotting tunnel of K, then the set of constituent knots of $\theta(K, h)$ consists of a pair of trivial knots and a knot with a 2-bridge decomposition. Later on Song([9]) clarified geometric meanings behind their observations. It will be the main subject that we recall in this section.

Denote the dihedral group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ by D_2. It is well known that for any θ-curve in S^3, we have the D_2 covering projection $\pi_{D_2} : M \to S^3$ branched over θ which is induced by a monodromy map from the fundamental group of θ to D_2.

If a θ-curve admits a bridge decomposing sphere S_g, then we shall see that the branch set upstairs $\pi_{D_2}^{-1}(\theta)$ can be realized by fixed point circles of three (orientation preserving) involutions of M which preserve each handlebody in a Heegaard decomposition of M with genus g. Hence restriction of π_{D_2} on the associated Heegaard surface F_g induces the covering projection $\pi_{D_2}|_{F_g} : F_g \to S_g$ branched over $\theta \cap S_g$.

Let S^1_ϵ, S^1_σ and S^1_ρ be a triple of circles in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ each pair of which meets orthogonally at $S^1_\epsilon \cap S^1_\sigma \cap S^1_\rho$. Then π-rotation with respect to S^1_ϵ, S^1_σ and S^1_ρ induce involutions ϵ, σ and ρ of S^3 respectively such that $\rho = \epsilon \circ \sigma = \sigma \circ \epsilon$. Let $D_2 = \langle \epsilon, \sigma : \epsilon \circ \sigma = \sigma \circ \epsilon \rangle$. Then S^3 has D_2 symmetry with a pair of global fixed points $S^1_\epsilon \cap S^1_\sigma \cap S^1_\rho$.

Now, we consider a handlebody H_g of genus g standardly embedded in S^3 so that the D_2-action of S^3 can be restricted on H_g. Here we assume that one of involutions in D_2, say ϵ, is always taken as the standard involution

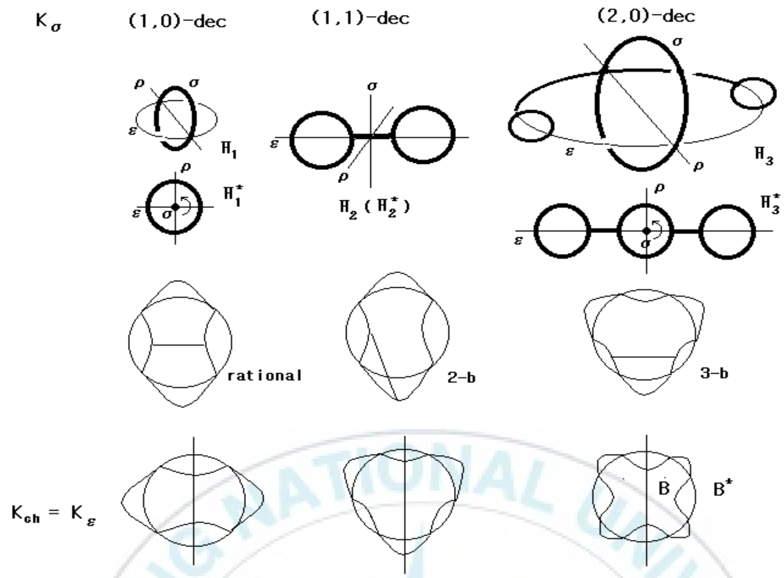

Fig. 7

of H_g so that $S^1_\epsilon \cap H_g$ may consist of $(g+1)$-arcs.

Then we have a set $\mathcal{M}_g$ of $(g+1)$-meridian discs of H_g with the following action of a non-trivial involution $I \in D_2$ on each meridian disc $D \in \mathcal{M}_g$:

(1) if $\mathrm{Fix}(I) \cap D = \emptyset$, then $I(D)$ is another meridian disc $E \in \mathcal{M}_g$;

(2) if $\mathrm{Fix}(I) \cap D$ is a single point (and hence a global fixed point of D_2), then $I(D) = D$ and I preserves the orientation of D;

(3) if $\mathrm{Fix}(I) \cap D$ is an arc, then $I(D) = D$ and I reverses the orientation of D.

We call $\mathcal{M}_g$ a system of D_2-equivariant meridian discs of H_g. Let $\mathcal{M}^*_g$ be a system of D_2-equivariant meridian discs of $H^*_g = S^3 - Int(H_g)$. Since a non-trivial involution $I \in D_2$, $I \neq \epsilon$, has the action of type (1) on each meridian disc in $\mathcal{M}_g$ or $\mathcal{M}^*_g$ which does not contain any global fixed point of D_2, we have:

Case $g = 2$. Each global fixed point of D_2 should lie on each handlebody H_2 and H_2^*, respectively. For a meridian disc D_f in $\mathcal{M}_2$ (respectively D_{f^*} in $\mathcal{M}_2^*$) containing a global fixed point f (respectively f^*), two involutions which have the action of type (2) on D_f and D_{f^*} must be the same. We take such an involution as σ. Then σ transposes the two meridian discs of $\mathcal{M}_2 - \{D_f\}$ and those of $\mathcal{M}_2^* - \{D_{f^*}\}$ as illustrated in Fig. 7.

Case $g = 3$. Both global fixed points f_1, f_2 of D_2 should lie on one of the two handlebodies, say H_3. And, two involutions which have the action of type (2) on D_{f_1} and D_{f_2} must be the same. We take such an involution as σ. Then S^1_σ forms a core of H_3 transversely meeting the meridian discs D_{f_1} and D_{f_2}, and σ transposes the two meridian discs of $\mathcal{M}_3 - \{D_{f_1}, D_{f_2}\}$. On the other hand, σ acts freely on H_3^* and pairwise transposes two meridian discs of $\mathcal{M}_3^*$ as illustrated in Fig. 7.

Since an orientation-preserving involution of S^3 is conjugate to an orthogonal transformation, we see that a D_2-symmetry of H_g with its standard involution in D_2 is uniquely determined.

If we can choose a gluing homeomorphism ψ of the two handlebodies H_g and H_g^* so that it may be compatible with ϵ and σ, i.e., $\epsilon \circ \psi = \psi \circ \epsilon$ and $\sigma \circ \psi = \psi \circ \sigma$, then we have a 3-manifold with a Heegaard decomposition $M_g = H_g \cup_\psi H_g^*$ on which the dihedral group D_2 acts so that it may preserve each handlebody. We call such a Heegaard decomposition of a 3-manifold *D_2-symmetric*.

Further we assume that the gluing homeomorphism ψ is chosen so that M may be a $\mathbb{Z}_2$-homology 3-sphere, which is necessary for M to be the double branched covering of a knot K in S^3 or the D_2-branched covering of a θ-curve in S^3. Then by classification of a D_2 action on a $\mathbb{Z}_2$-homology 3-sphere, it is guaranteed that the fixed point sets of all three involutions of M form three circles intersecting in exactly two points.

If we denote the fixed point set of each involution $I \in \{\epsilon, \sigma, \rho\}$ of M by $\mathrm{Fix}(I)$ and the union of them by $\mathrm{Fix}(\mathcal{I})$, then we have the D_2-covering projection $\pi_{D_2} : M \to M/D_2 (\cong S^3)$ branched over a θ-curve $\pi_{D_2}(\mathrm{Fix}(\mathcal{I}))$ with a bridge decomposing sphere $\pi_{D_2}(F_g)$ where F_g is a Heegaard surface associated with the Heegaard decomposition of M. And, for each $I \in \{\epsilon, \sigma, \rho\}$ we have the double covering projection $\pi_I : M \to M/I$ branched over a knot $K_I = \pi_I(\mathrm{Fix}(I))$ in M/I whose Heegaard decomposition of genus g^*, $M/I = H_g/I \cup_{\tilde{\psi}} H_g^*/I$, naturally induces a (g^*, b)-decomposition of K_I in M/I where $\tilde{\psi} = \pi_I \circ \psi \circ (\pi_I)^{-1}$.

Details of such decomposition of K_I is given in the following proposition which can be easily read off given the D_2-action on the handlebodies.

Proposition 2.3.3 Let M be a $\mathbb{Z}_2$-homology 3-sphere admitting a D_2-symmetric Heegaard decomposition of genus g. Then we have:

Case $g = 2$.

(i) K_ϵ is a knot in S^3 with a 3-bridge decomposition and with a bridge preserving strong inversion h_ϵ such that $\mathrm{Fix}(h_\epsilon) = \pi_\epsilon(\mathrm{Fix}(\sigma) \cup \mathrm{Fix}(\rho))$.

(ii) K_σ (respectively K_ρ) is a (1,1)-knot in a lens space M/σ (respectively M/ρ). And $\pi_\sigma(\mathrm{Fix}(\epsilon) \cup \mathrm{Fix}(\rho))$ (respectively $\pi_\rho(\mathrm{Fix}(\epsilon) \cup \mathrm{Fix}(\sigma))$) form the fixed point set of the standard involution of the lens space M/σ (respectively M/ρ) intersecting each unknotted string once in the (1,1)-decomposition of K_σ (respectively K_ρ).

Case $g = 3$.

(i) K_ϵ is a knot in S^3 with a 4-bridge decomposition and with a bridge preserving strong inversion h_ϵ such that $\mathrm{Fix}(h_\epsilon) = \pi_\epsilon(\mathrm{Fix}(\sigma) \cup \mathrm{Fix}(\rho))$.

(ii) K_σ admits a (2,0)-decomposition in M/σ;

$$(M/\sigma, K_\sigma) = (H_3/\sigma, K_\sigma) \cup_{\psi_\sigma} (H_3^*/\sigma, \emptyset).$$

Further, $\pi_\sigma(\mathrm{Fix}(\epsilon) \cup \mathrm{Fix}(\rho))$ form the fixed point set of the standard involution of M/σ intersecting K_σ twice.

(iii) K_ρ admits a (1,2)-decomposition in a lens space M/ρ. And $\pi_\rho(\mathrm{Fix}(\epsilon) \cup \mathrm{Fix}(\sigma))$ form the fixed point set of the standard involution of the lens space M/ρ intersecting each of two unknotted strings once on one side of a solid torus of the (1,2)-decomposition of K_ρ.

Remark. All lens spaces (including S^3) in Proposition 2.3.3 must be of odd type, i.e., $L(p,q)$, $p \equiv 1 \pmod 2$ because they are the double branched coverings of constituent knots of the θ-curve with 2-bridge decompositions.

Conversely we have:

Theorem 2.3.4([9, Theorem 4]) Let (K, S_{g+1}, h) be a triple of knot K with a $(g+1)$-bridge decomposing sphere S_{g+1} and a bridge-preserving strong inversion h. Then the double branched covering space of (S^3, K) admits a D_2-symmetric Heegaard decomposition of genus g.

Proof. Taking a gluing homeomorphism ψ of the two handlebodies H_g and H_g^* provided by the $(g+1)$-bridge decomposition of K through the method in [?], we have the double covering projection $\pi : M = H_g \cup_\psi H_g^* \to S^3$ branched over K. Thus we have a set $\mathcal{M}_g$ (respectively $\mathcal{M}_g^*$) of $(g+1)$-meridian discs of H_g (respectively H_g^*) such that they may doubly cover the spanning discs of $(g+1)$-trivial arcs in the bridge decomposition of K. And, we have an involution ϵ of M with $\pi^{-1}(K)$, the lifting of K as the fixed circle. Since h is a bridge-preserving strong inversion of K, there are a pair of involutions $\tilde{h}_1, \tilde{h}_2$ of M such that $h \circ \pi = \pi \circ \tilde{h}_i$ $(i = 1, 2)$, $\pi^{-1}(\mathrm{Fix}(h)) = \mathrm{Fix}(\tilde{h}_1) \cup \mathrm{Fix}(\tilde{h}_2)$ and $\pi^{-1}(K \cap \mathrm{Fix}(h)) = \mathrm{Fix}(\tilde{h}_1) \cap \mathrm{Fix}(\tilde{h}_2)$.

In the case of $g = 3$, both points of $\pi^{-1}(K \cap \mathrm{Fix}(h))$ lie on one side of the two handlebodies, say H_3. Then one of the two circles, say $\mathrm{Fix}(\tilde{h}_1)$, in $\pi^{-1}(\mathrm{Fix}(h))$ transversely meets a pair of meridian discs in $\mathcal{M}_3$ which are determined by the two spanning discs of the trivial arcs containing $K \cap \mathrm{Fix}(h)$.

Thus, $\mathrm{Fix}(\tilde{h}_1)$ forms a core of H_3 and $\tilde{h}_1$ is equivalent to σ. In the case of $g = 2$, $\pi^{-1}(K \cap \mathrm{Fix}(h))$ consists of a pair of points $\{p, p^*\}$ such that $p \in H_2$ and $p^* \in H_2^*$, respectively. Then, one of the two circles, say $\mathrm{Fix}(\tilde{h}_1)$, in $\pi^{-1}(\mathrm{Fix}(h))$ transversely meets a meridian disc in $\mathcal{M}_2$ (respectively $\mathcal{M}_2^*$) which is determined by the spanning disc of the trivial arc containing p (respectively p^*) in the given bridge decomposition of K. Thus $\tilde{h}_1$ is equivalent to σ. □

By Lemma 2.3.2 and Theorem 2.3.4, we have:

Corollary 2.3.5 Let (θ, S_g) be a θ-curve with a bridge decomposing sphere S_g. Then the D_2-branched covering of (θ, S_g) admits a D_2-symmetric Heegaard decomposition of genus g such that the associated Heegaard surface covers S_g.

By considering the D_2-branched covering of (θ, S_g), we have a refinement of the Morimoto–Sakuma–Yokota's method of studying tunnel 1 knots.

Theorem 2.3.6([7, Theorem 1.2 (1) and (2)]) A knot K in S^3 is a (1,1)-knot (respectively a tunnel-1 knot), if and only if there exists a strong inversion h of K such that
(i) θ-curve $\theta(K, h)$ admits a 2 (respectively 3)-bridge decomposing sphere S_2 (respectively S_3) and
(ii) $p(\mathrm{Fix}(h))$ forms a trivial constituent knot of $(\theta(K, h), S_2)$ (respectively $(\theta(K, h),\ S_3)$) with a 2-bridge (respectively 3-bridge) decomposition where p is the projection $S^3 \to S^3/h$.

3 Main Results

Theorem3.1. The Heegaard diagrams Figure8(a) and (b) represent the vertical and horizontal Heegaard splitting of a Brieskorn homology sphere $\Sigma(2,3,7)$ respectively.

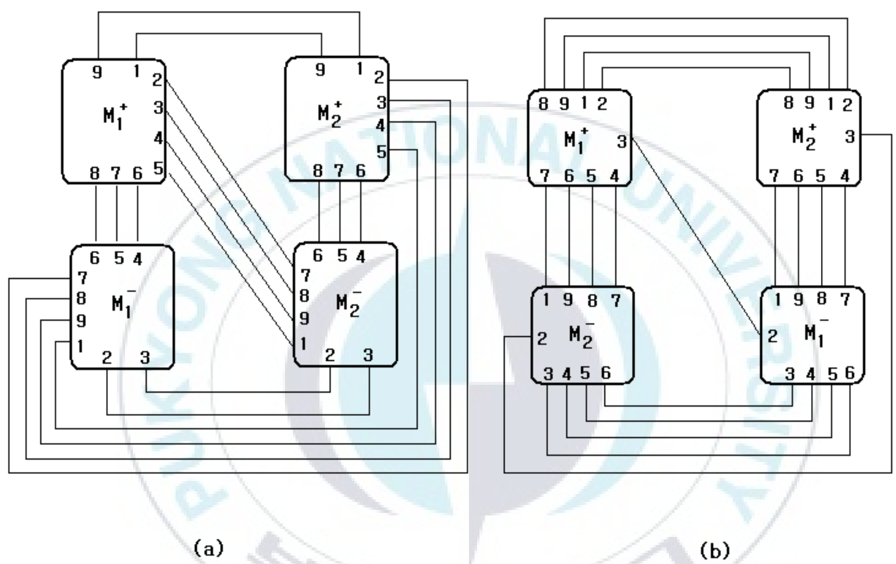

Fig. 8

Proof It is well known that the standard involution ϵ of the vertical (resp. horizontal) Heegaard decomposition of $\Sigma(2,3,7)$ induces a Montesinos knot $p(-2,3,7)$ (resp. a torus knot t(3,7)) with a 3-bridge decomposition (c.f. [2]). Thus we may verify the claim of the theorem by explicitly deriving K_ϵ from the given Heegaard diagrams.

In the given Heegaard in figure 8(a), we may recognize $Fix(\epsilon)$ by detecting 6- fixed points of ϵ, namely intersection points of $Fix(\epsilon)$ with the Heegaard surface as shown figure 9(a). For more details about detection

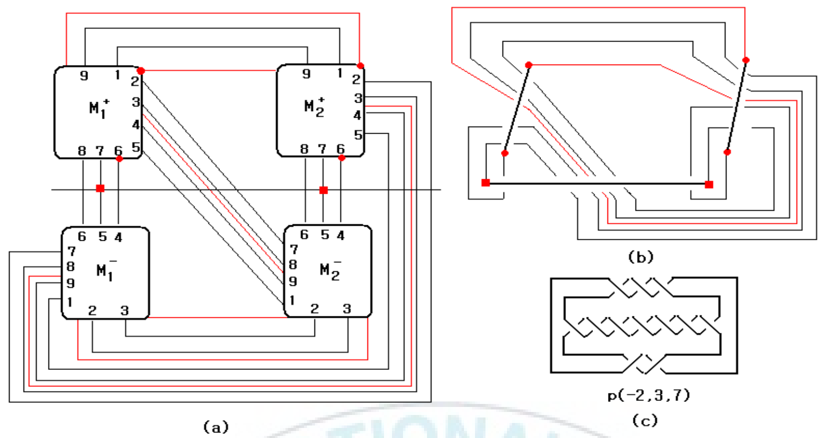

Fig. 9

of those fixed points of $Fix(\epsilon)$, see appendix. Then taking quotient of the diagram in figure 9(a) with respect to ϵ, we have a 3-bridge decomposition of K_ϵ in figure 9(b). In figure D1 and D2 of the appendix , we explicitly show that the knot in figure 9(b) is indeed $p(-2, 3, 7)$. Likewise we have $K_\epsilon = t(3, 7)$ from figure 8(b)(c.f. E1 and E2 of the appendix).

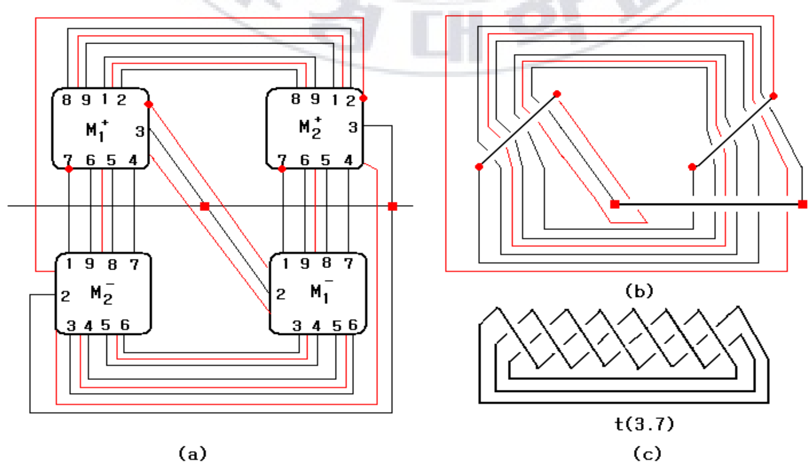

Fig. 10

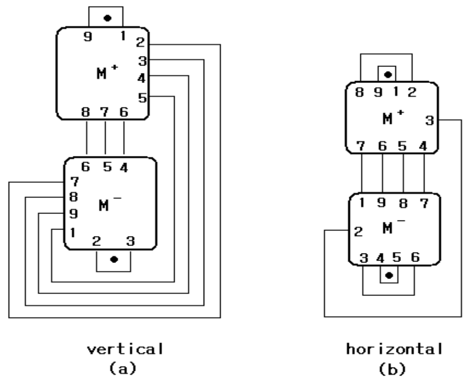

Fig. 11

Corollary3.2. The two (1,1)-decompositions of $p(-2,3,7)$ in Figure 11(a) and (b) represent the vertical and horizontal respectively.

Proof The involutions σ of the two Heegaard diagrams of theorem 00 with dihedral symmetry give rise to the (1,1)-decompositions of $p(-2,3,7)$ by [9, Theorem 9].

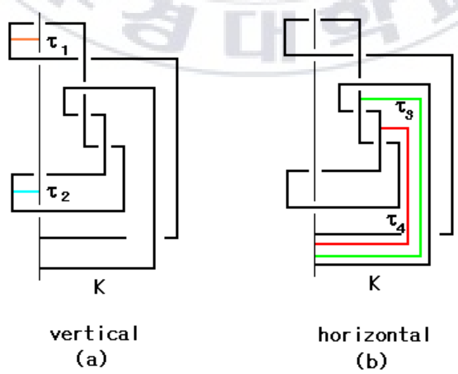

Fig. 12

Let h be a strong inversion of a knot K in S^3 induced by a unknotting tunnel τ. For the quotient map $\pi_h : (S^3, K) \rightarrow (S^3, \theta(K,h))$ we have an arc $\pi_h(\tau)$ with two end points in the theta curve $\theta(K, h)$ which is called an unknotting arc.

Theorem3.3. The vertical and horizontal (1,1)-decompositions of $p(-2, 3, 7)$ induce the two 2-bridge $\theta-$ curves with pairs of unknotting arcs in Figure 12(a) and (b) respectively.

Proof A sequence of figures from (a) to (d) in Figure 13 show how to get a 2-bridge theta-curve with a pair of unknotting arcs for the vertical (1,1)-diagram of $p(-2, 3, 7)$. We see that 2-bridge theta-curve with a pair of unknotting arcs in figure (d) is isotopic to that in figure(e). For more details, see appendix 1 and 2.

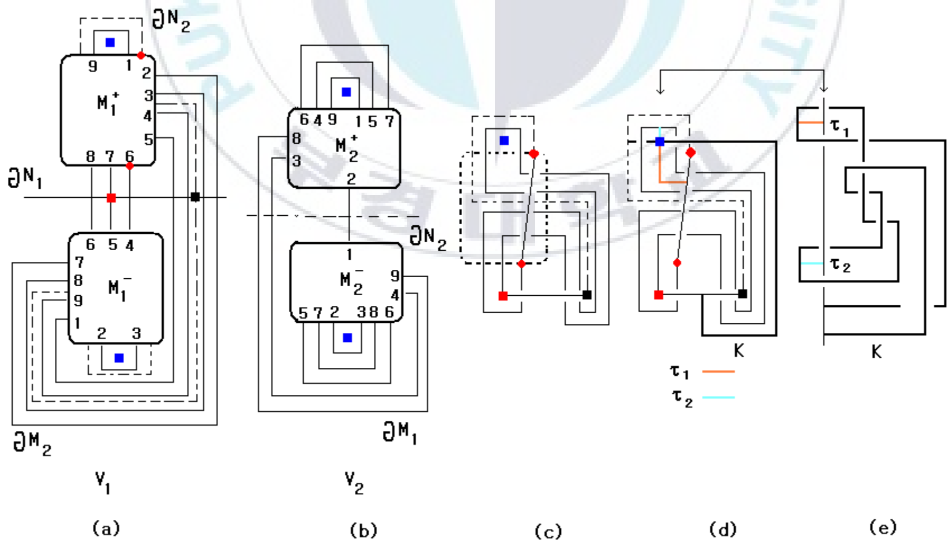

Fig. 13

Likewise the horizontal (1,1)-diagram of $p(-2, 3, 7)$ induces a 2-bridge theta-curve with a pair of unknotting arcs as shown in figure 14(e).

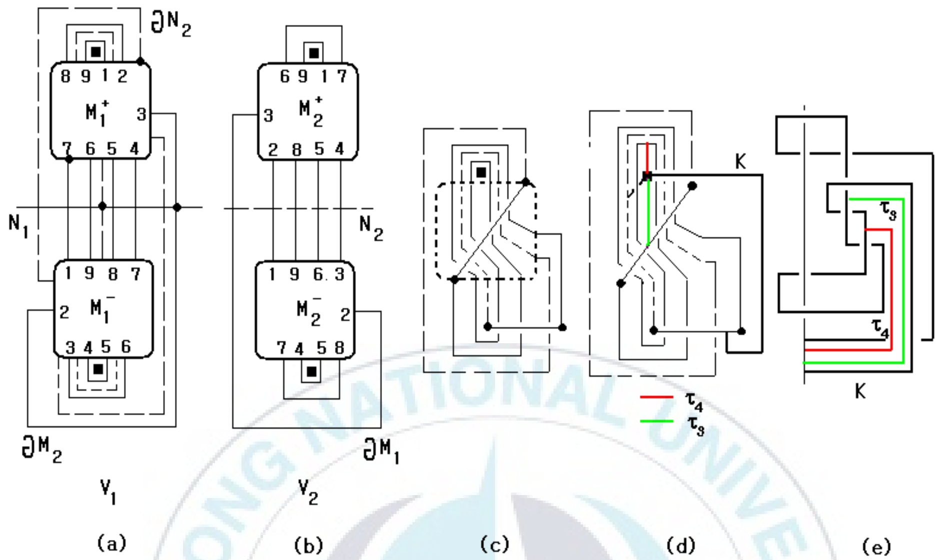

Fig. 14

Thus we have;

Corollary3.4. The pretzel knot $\Sigma(2, 3, 7)$ admits four unknotting tunnels described in figure 1.

4 Appendix

In this section we exhibit application of the theorems in section 2 as well as verification of the various unproved claims in section 3.

4.1 explanation for process from (a) to (d) in figure 13 or 14

Once we have a Heegaard decomposition or equivalently Heegaard diagram of genus 2 with dihedral symmetry, we have a triple of knots K_ϵ, K_σ and K_ρ , which are led to the same 2-bridge theta-curve as illustrated in figure A1 below.

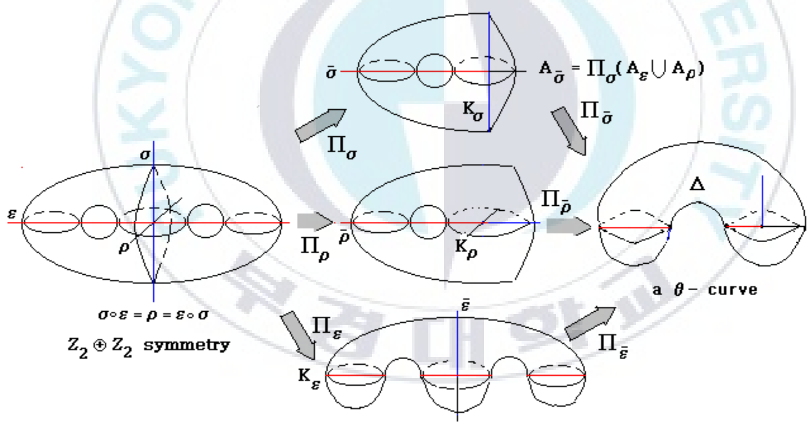

Fig. A1

Since the involution σ can be easily detected in a given Heegaard diagram of genus 2 with dihedral symmetry, it is more convenient to deal with K_σ and its strong inversion h inducing the 2-bridge theta curve $\theta(K_\sigma$ in the first instance. Note that the strong inversion h of K_σ can be thought of as the standard involution of its (1,1)-decomposing solid tori V_i $(i = 1, 2)$ such that $Fix(h)$ meets a trivial string $t_i = K_\sigma \cap V_i$ at single point for each

$i = 1, 2$(c.f. figure A2(a)).

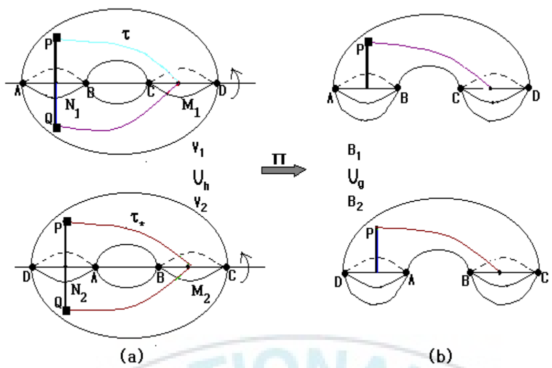

Fig. A2

Thus for each solid torus V_i, we may take a pair of meridian disks $\{M_i, N_i\}$ such that

(1) M_i is disjoint from a trivial string t_i,

(2) N_i meets t_i at a single point $Fix(h) \cap t_i$ and

(3) $Fix(h) \cap V_i$ consists of a pair of arcs $\{M_i \cap Fix(h), N_i \cap Fix(h)\}$.

Taking quotient of V_i by the involution h, we have a 3-ball B_i and a double covering projection $\pi : V_i \to B_i$ branched over a pair trivial arcs whose spanning(projection) disks are lifted to M_i and N_i respectively. Finally we choose a properly embedded arc τ_i in V_i so that

(1)$\partial\tau_i = \partial t_i$,

(2)τ_i meets M_i at a single point in $Fix(h)$,

(3)$\tau_i \cap N_i = \emptyset$ and

(4)$\tau_i \cup t_i$ is isotopic to a core of V_i. Then we can come up with a 2-bridge theta curve in S^3 with a pair of unknotting arcs

$$(B_1, \pi(\{Fix(h) \cap V_1\} \cup t_1 \cup \tau_1)) \cup (B_2, \pi(\{Fix(h) \cap V_2\} \cup t_2 \cup \tau_2))$$

.

In practice, the above discussion is carried out by detecting the 4 fixed points of the standard involution h on $T = \partial V_1 = \partial V_2$,a Heegaard torus holding the (1,1)-diagram of K_σ. Note that by our choices of meridian disks of V_i, those 4 fixed points belong to $\partial M_1 \cap \partial M_2$, $\partial M_1 \cap \partial N_2$, $\partial N_1 \cap \partial M_2$ and $\partial N_1 \cap \partial N_2$ respectively. Thus in the given (1,1)- diagram $\mathcal{H} = (T, (\{\partial M_1\}, \{\partial M_2\}), \{K_\sigma \cap T\})$, inserting ∂N_1 and ∂N_2 we have an extended (1,1)-diagram $\mathcal{H}_E = (T, (\{\partial M_1, \partial N_1\}, \{\partial M_2, \partial N_2\}), \{K_\sigma \cap T\})$.

From the extended (1,1)-diagram $\mathcal{H}_E$ we can easily detect the 4-fixed points of the involution h. Now taking the quotient of $(T, \mathcal{H}_E)$ with respect to h, we have a 2-bridge diagram of $\pi(Fix(h))$ on the 2-sphere $S^2 = \pi(T)$ with a marked point covered by the pair of points $\{K_\sigma \cap T\})$. The 2-bridge diagram can be converted into a 2-bridge position of $\pi(Fix(h))$ with with respect to S^2 via isotopic move of $\{\pi(\partial M_1)$ and $\pi(\partial N_1)$ into a pair of over arcs m and n in B_1 respectively. Finally joining a pair of arcs $\{\pi(t_1)$ and $\pi(\tau_1)$ to n and m through B_1 respectively and $\{\pi(t_2)$ and $\pi(\tau_2)$ to $\pi(\partial N_2$ and $\pi(\partial M_2)$ through B_2 respectively, we get the desired 2-bridge theta curve with a pair of unknotting arcs.

4.2 derivation of (d) from (e) in figure 13

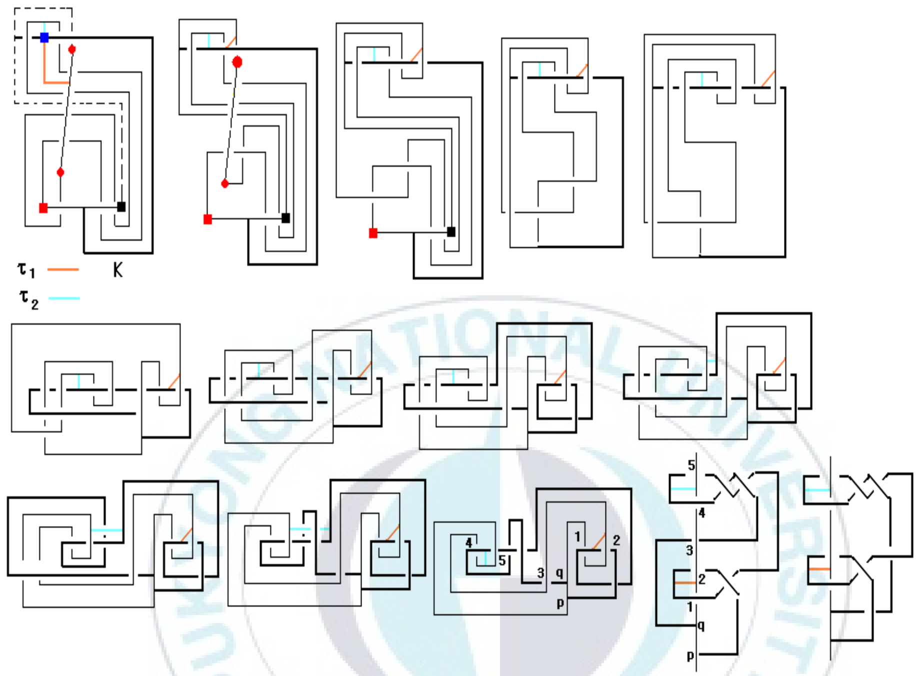

Fig. B

4.3 a quick method of detecting the fixed points of the involutions ϵ, σ and ρ

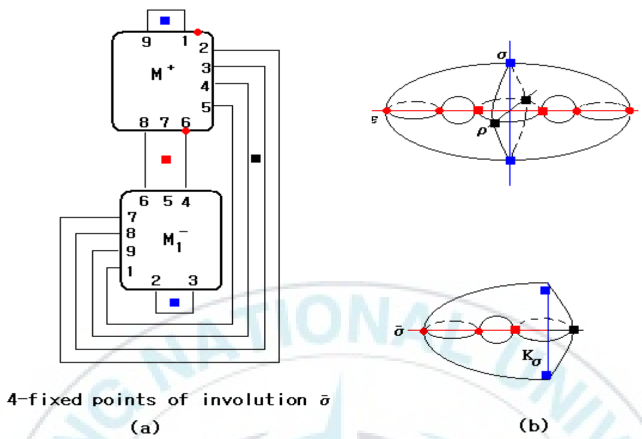

Fig. C1

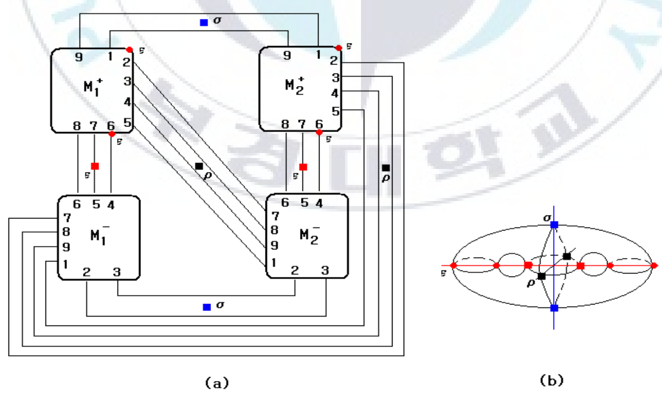

Fig. C2

4.4 derivation of the Montesinos rational tangle decomposition of $p(-2,3,7)$ from its 3-bridge position in figure9(b)

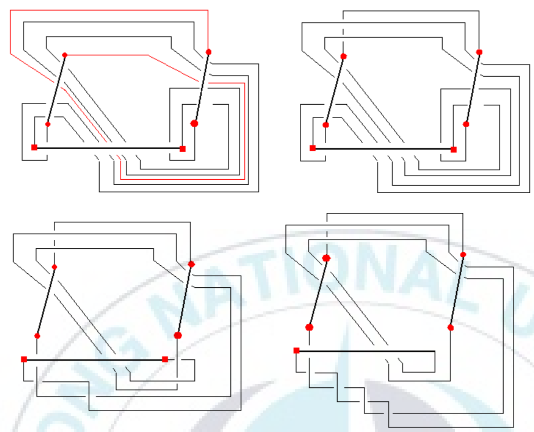

Fig. D1

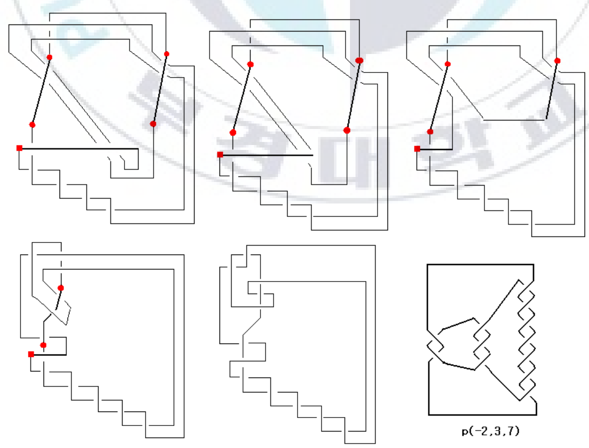

Fig. D2

4.5 derivation of the torus braiding of $t(3,7)$ from its 3-bridge position in figure10(b)

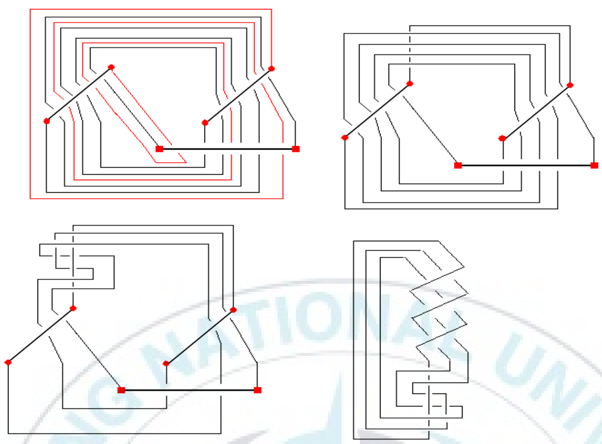

Fig. E1

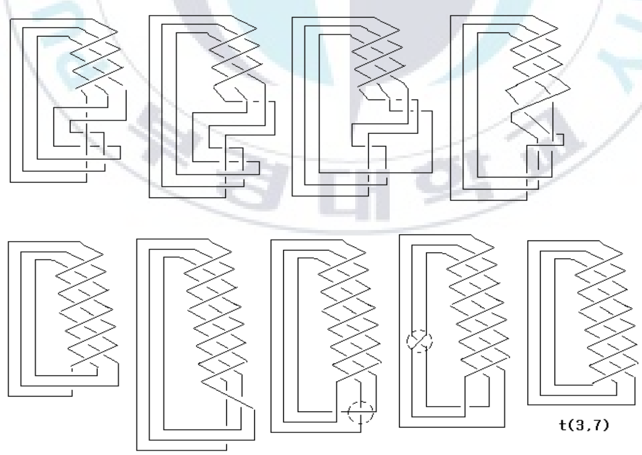

Fig. E2

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