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Some Remarks on Optimality Conditions for Convex Optimization Problems

by

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Graduate School of Education

Pukyong National University

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볼록 최적화 문제의 최적조건에 대한 소고

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볼록 최적화 문제의 최적조건에 대한 소고

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요 약

본 논문에서는 볼록 최적화 문제에 대한 최적조건들을 증명한다. 첫째로, 안전성 조건아래서, 볼록최적화 문제의 근사해에 대한 근사최적조건을 확립하고, 둘째로 어떠한 제약상정도 없이 성립하는 볼록-부선형(sublinear) 최적화 문제의 최적해에 대한 최적조건을 증명한다.

1 Introduction

Many authors have studied optimality conditions for several kinds of optimization problems ([2, 3, 4, 10, 11, 12, 13, 16, 14, 15, 17]). It is well known that constraint qualifications should be imposed on convex optimization problems to obtain optimality conditions for solutions. Among constraint qualifications for convex optimization problems, the Slater constraint qualification is frequently used.

In this thesis, we will prove optimality conditions for convex optimization problems. In Section 2, we give definitions, examples and preliminary results for next section. In Section 3, we give approximate optimality conditions for ϵ-approximate solutions of a convex optimization problem, which hold under appropriate stability condition. In Section 4, we establish optimality conditions for optimal solutions of convex sublinear optimization problems, which hold without any constraint qualification.

2 Preliminaries

In this section, we give definitions, examples and preliminary results. Let X be a Banach space and X^* the topological dual of X.

Definition 2.1 . *Let $g : X \to \mathbb{R}$ be a convex function.*

(1) The subdifferential of g at $a \in X$ is given by

$$\partial g(a) := \{v \in X^* \mid g(x) \geqq g(a) + v(x-a) \quad \forall x \in X\}.$$

(2) The ϵ-subdifferential of g at $a \in X$ is given by

$$\partial_\epsilon g(a) := \{v \in X^* \mid g(x) \geqq g(a) + v(x-a) - \epsilon \quad \forall x \in X\}.$$

Example 2.1 . *(1) Let $g(x) = \|x\|$, $x \in \mathbb{R}^n$. Then $\partial g(0) = B_1(0)$, where $B_1(0) = \{x \in \mathbb{R}^n \mid \|x\| \leqq 1\}$.*

(2) Let $g(x) = \sqrt{x^T A x}$, $x \in \mathbb{R}^n$, where A is a symmetric positive semidefinite $n \times n$ matrix. Then $\partial g(0) = \{A\omega \mid \omega^T A \omega \leqq 1\}$.

Definition 2.2 . *The conjugate function of a function $g : X \to \mathbb{R}$ is defined by*

$$g^*(v) = sup\{\langle v, x\rangle - g(x) \mid x \in X\}, \text{ for any } v \in X^*$$

Example 2.2 . *Let $g(x) = e^x$ for any $x \in R$. Then*

$$g^*(v) = \begin{cases} v \log v - v & \text{if} \quad v > 0, \\ 0 & \text{if} \quad v = 0, \\ +\infty & \text{if} \quad v < 0. \end{cases}$$

Definition 2.3 . *The epigraph of a function* $g : X \to \mathbb{R}$*,* epi g*, is defined by*

$$\text{epi } g := \{(x, r) \in X \times \mathbb{R} \mid g(x) \leqq r\}.$$

Proposition 2.1 *Let* $a \in X$*. If* $g : X \to \mathbb{R}$ *is sublinear (i.e., convex and positively homogeneous of degree one), then for all* $\epsilon \geqq 0$*,*

$$\partial_\epsilon g(a) = \{v \in \partial g(0) \mid g(a) - \langle v, a\rangle \leqq \epsilon\}.$$

Moreover, $\partial_\epsilon g(0) = \partial g(0)$*.*

Proof. Let $\epsilon \geqq 0$ and $v \in \partial g(0)$ and $g(a) - \langle v, a\rangle \leqq \epsilon$. Then $g(x) \geqq \langle v, x\rangle$, $\forall x \in X$, and hence

$$\begin{aligned} g(x) - g(a) &\geqq -g(a) + \langle v, x\rangle \\ &\geqq -\langle v, a\rangle - \epsilon + \langle v, x\rangle. \end{aligned}$$

So, for all $x \in X$, $g(x) \geqq g(a) + \langle v, x - a\rangle - \epsilon$. Hence $v \in \partial_\epsilon g(a)$.

Let $v \in \partial_\epsilon g(a)$. Then $g(x) \geqq g(a) + \langle v, x - a\rangle - \epsilon \ \forall x \in X$. Letting $x = 0$, $g(0) \geqq g(a) + \langle v, -a\rangle - \epsilon$. Thus $g(a) - \langle v, a\rangle \leqq \epsilon$. Moreover, $g(x + a) \geqq g(a) + \langle v, x\rangle - \epsilon \ \forall x \in \mathbb{R}^n$. Since $g(x) + g(a) \geqq g(x+a)$, $g(x) \geqq \langle v, x\rangle - \epsilon \ \forall x \in X$. Let x be fixed. Then for any $\lambda > 0$, $\lambda g(x) = g(\lambda x) \geqq \langle v, \lambda x\rangle - \epsilon$. *and so* $g(x) \geqq \langle v, x\rangle - \frac{\epsilon}{\lambda}$. Letting $\lambda \to +\infty$, $g(x) \geqq \langle v, x\rangle$. Thus $v \in \partial g(0)$. □

For the completeness, we give proofs of propositions 2.2, 2.3 and 2.4 and Lemma 2.1, 2.2, which will be appeared in this section.

Proposition 2.2 [9] *If $g : X \to \mathbb{R}$ is sublinear, then*

$$g^*(v) = \begin{cases} 0 & \text{if} \quad v \in \partial g(0) \\ +\infty & \text{if} \quad v \notin \partial g(0) \end{cases} \quad \textit{and hence } \text{epi } g^* = \partial g(0) \times \mathbb{R}_+.$$

Proof. From the definition of subdifferential,

$$\begin{aligned} v \in \partial g(0) \;\; &\Rightarrow \;\; \forall x \in X,\; g(x) - g(0) \geqq v^t(x - 0) \\ &\Rightarrow \;\; \forall x \in X,\; 0 \geqq v^t x - g(x) \\ &\Rightarrow \;\; 0 \geqq g^*(v) \\ &\Rightarrow \;\; 0 \geqq g^*(v) = \sup\{v^t x - g(x) \mid x \in \mathbb{R}^n\} \\ &\qquad\qquad\quad \geqq v^t 0 - g(0) = 0 \\ &\Rightarrow \;\; g^*(v) = 0. \end{aligned}$$

$$\begin{aligned} v \notin \partial g(0) \;\; &\Rightarrow \;\; \exists x_0 \in X \text{ such that } g(x_0) - g(0) < v^t(x_0 - 0) \\ &\Rightarrow \;\; 0 < v^t x_0 - g(x_0) \\ &\Rightarrow \;\; g^*(v) = \sup\{v^t x - g(x) \mid x \in \mathbb{R}^n\} \\ &\qquad\qquad \geqq \sup\{v^t(\lambda x_0) - g(\lambda x_0) \mid \lambda \geqq 0\} \\ &\qquad\qquad = \sup\{\lambda(v^t x_0 - g(x_0)) \mid \lambda \geqq 0\} \\ &\qquad\qquad = +\infty \\ &\Rightarrow \;\; g^*(v) = +\infty. \end{aligned}$$

Hence

$$g^*(v) = \begin{cases} 0 & \text{if} \quad v \in \partial g(0) \\ +\infty & \text{if} \quad v \notin \partial g(0) \end{cases}$$

Thus, we have

$$\begin{aligned}\text{epi } \widetilde{g}^* &= \{(v, \alpha) \in X \times \mathbb{R} \mid g^*(v) \leqq \alpha\} \\ &= \{(v, \alpha) \in X \times \mathbb{R} \mid v \in \partial g(0),\ \alpha \geqq 0\} \\ &= \partial g(0) \times \mathbb{R}_+\end{aligned}$$

□

Proposition 2.3 [9] *If* $g : X \to \mathbb{R}$ *is sublinear and if* $\widetilde{g}(x) = g(x) - k,\ for\ x \in X$ *and* $k \in \mathbb{R}$ *, then*

$$\text{epi } \widetilde{g}^* = \partial g(0) \times [k, \infty).$$

Proof. For any $v \in X^*$,

$$\begin{aligned}\widetilde{g}^*(v) &= \sup\{v(x) - \widetilde{g}(x) \mid x \in \text{dom}\widetilde{g}\} \\ &= \sup\{v(x) - g(x) + k \mid x \in \text{dom}g\} \\ &= \sup\{v(x) - g(x) \mid x \in \text{dom}g\} + k \\ &= g^*(v) + k.\end{aligned}$$

Then for any $v \in X^*$ *and* $\alpha \in \mathbb{R}$,

$$\begin{aligned}(v, \alpha) \in \text{epi } \widetilde{g}^* &\iff \widetilde{g}^*(v) \leqq \alpha \\ &\iff g^*(v) + k \leqq \alpha \\ &\iff g^*(v) \leqq \alpha - k \\ &\iff (v, \alpha - k) \in \text{epi } g^* \\ &\iff (v, \alpha) \in \text{epi } g^* + (0, k).\end{aligned}$$

Hence by Proposition 2.2,

$$\begin{aligned} \text{epi } \widetilde{g}^* &= \text{epi } g^* + (0, k) \\ &= (\partial g(0) \times \mathbb{R}_+) + (0, k) \\ &= \partial g(0) \times [k, \infty). \end{aligned}$$

□

Now we consider the following optimization problem:

$$\begin{aligned} \text{(P)} \qquad & \text{Minimize} \quad f(x) \\ & \text{subject to} \quad x \in S \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function and S is a closed convex subset of $\mathbb{R}^n$.

Definition 2.4 *Let $\epsilon \geqq 0$. Then $\bar{x} \in S$ is called an ϵ-approximate solution of (P) if*

$$f(x) + \epsilon \geqq f(\bar{x}), \ \textit{for any } x \in S.$$

Example 2.3 *Consider the following convex optimization problem:*

$$\begin{aligned} \text{(P)} \qquad & \textit{Minimize} \qquad f(x) := -x \\ & \textit{subject to} \qquad g(x) := \sqrt{x^2 + y^2} - y \leqq 0. \end{aligned}$$

The set of all ϵ-approximate solutions of (P) *is* $\{(0, y) \in \mathbb{R}^2 \mid y \geqq 0\}$.

The following lemmas are needed to prove the main results. For the completeness, we give their proofs.

Let X and Z be Banach spaces and the cone $S \subseteq Z$ be closed and convex. The (positive) polar of the cone $S \subseteq Z$ is the cone $S^+ = \{\theta \in Z^* \mid \theta(k) \geq 0 \;\; \forall k \in S\}$.

Lemma 2.1 [1] *Let $h : X \to \mathbb{R}$ be a continuous convex function and $g : X \to Z$ be a continuous S-convex function, that is, for any $x, y \in X$ and any $\lambda \in (0,1)$, $g(\lambda x + (1-\lambda)y) - \lambda g(x) - (1-\lambda)g(y) \in -S$, respectively. Suppose that $\{x \in X \mid g(x) \in -S\} \neq \emptyset$. Then the following statements are equivalent:*

(i) $\{x \in X \mid g(x) \in -S\} \subseteq \{x \in X \mid \; h(x) \geqq 0\}$

(ii) $0 \in epi\ h^* + cl \bigcup_{\lambda \in S^+} epi\ (\lambda g)^*$.

Proof. Let $Q = \{x \in X \mid g(x) \in -S\}$. Then Q $\neq \emptyset$. By proposition 2.3, epi δ_Q^* $= cl \bigcup_{\lambda \in S^+}$ epi $(\lambda h)^*$ The proof can be found in [9]), where f_c is the indicator function of Q, that is, $\delta_Q(x) = 0,\ if\ x \in Q\ and\ \delta_Q(x) = +\infty,\ if\ x \notin Q$. Here,

$$\begin{aligned}
(ii) \quad &\iff \quad 0 \in \text{epi } h^* + \text{epi } \delta_Q^* = \text{epi } (h + \delta_Q)^* \text{ (see Lemma 2.3 below)} \\
&\iff \quad (h + \delta_Q)^*(0) \leqq 0 \\
&\iff \quad (h + \delta_Q)(x) \geqq 0 \text{ for any } x \in X \\
&\iff \quad h(x) \geqq 0 \text{ for any } x \in Q \\
&\iff \quad (i).
\end{aligned}$$

Thus (i) $\iff$ (ii). □

Lemma 2.2 [7] *Let $u : X \to \mathbb{R}$ be a continuous linear mapping and let $g : X \to Z$ be a continuous S-convex mapping. Suppose that the system $g(x) \in -S$ is consistent. Let $\alpha \in \mathbb{R}$. Then the following statements are equivalent:*

(i) $\{x \in X \mid g(x) \in -S\} \subseteq \{x \in X \mid u(x) \leq \alpha\}$.

(ii) $\begin{pmatrix} u \\ \alpha \end{pmatrix} \in cl\Big(\bigcup_{\lambda \in S^+} \text{epi } (\lambda g)^*\Big)$.

Proof. Let $h(x) = \alpha - u(x)$. Then by Lemma 2.1,

$$\begin{aligned}
(i) \quad &\Longleftrightarrow \quad 0 \in \text{epi } h^* + cl\Big(\bigcup_{\lambda \in S^+} \text{epi } (\lambda g)^*\Big) \\
&\Longleftrightarrow \quad 0 \in (-u, -\alpha) + \{0\} \times \mathbb{R}_+ + cl\Big(\bigcup_{\lambda \in S^+} \text{epi } (\lambda g)^*\Big) \\
&\Longleftrightarrow \quad (u, \alpha) \in \{0\} \times \mathbb{R}_+ + cl\Big(\bigcup_{\lambda \in S^+} \text{epi } (\lambda g)^*\Big)
\end{aligned}$$

Since $cl\Big(\bigcup_{\lambda \in S^+} \text{epi } (\lambda g)^*\Big)$ is a convex cone [9] and $\{0\} \times \mathbb{R}_+ \subset cl\Big(\bigcup_{\lambda \in S^+} \text{epi } (\lambda g)^*\Big)$, we have, $\{0\} \times \mathbb{R}_+ + cl\Big(\bigcup_{\lambda \in S^+} \text{epi } (\lambda g)^*\Big) = cl\Big(\bigcup_{\lambda \in S^+} \text{epi } (\lambda g)^*\Big)$.

Thus $(i) \iff (ii)$. □

The following proposition explains the relationship between the epigraph of a conjugate function and the ϵ-subdifferential.

Proposition 2.4 [6] *If $g : X \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous convex function and if $a \in domg$, then*

$$\text{epi } g^* = \bigcup_{\epsilon \geqq 0} \{(v, v(a) + \epsilon - g(a)) \mid v \in \partial_\epsilon g(a)\},$$

Proof. Let $(v, \alpha) \in \text{epi } g^*$. Then $g^*(v) \leqq \alpha$ and hence

$$v(x) - g(x) \leqq \alpha \quad \forall x \in X. \tag{2.1}$$

In particular $v(a) - g(a) \leqq \alpha$. Let $\epsilon_0 = \alpha - v(a) + g(a)$. Then $\alpha = \epsilon_0 + v(a) - g(a)$, $\epsilon_0 \geqq 0$ and from (2.1), $v(x) - g(x) \leqq \epsilon_0 + v(a) - g(a)$ $\forall x \in X$. So, $g(x) - g(a) \geqq v(x - a) - \epsilon_0$ $\forall x \in X$. Hence $v \in \partial_{\epsilon_0} g(a)$. Thus $(v, \alpha) \in \bigcup_{\epsilon \geqq 0} \{(v, v(a) + \epsilon - g(a)) : v \in \partial_\epsilon g(a)\}$.

Conversely, let $(v, \alpha) \in \bigcup_{\epsilon \geqq 0} \{(v, v(a) + \epsilon - g(a)) : v \in \partial_\epsilon g(a)\}$. Then there exists $\epsilon \geqq 0$ such that $v \in \partial_\epsilon g(a)$. Thus

$$\begin{aligned} g(x) - g(a) \geqq v(x - a) - \epsilon &= v(x - a) + v(a) - g(a) - \alpha \\ &= v(x) - g(a) - \alpha. \end{aligned}$$

So, $g(x) \geqq v(x) - \alpha \quad \forall x \in X$. Hence $\sup\{v(x) - g(x) : x \in X\} \leqq \alpha$. Thus $g^*(v) \leqq \alpha$. Hence $(v, \alpha) \in \text{epi } g^*$. So,

$\bigcup_{\epsilon \geqq 0} \{(v, v(a) + \epsilon - g(a)) : v \in \partial_\epsilon g(a)\} \subset \text{epi } g^*$. □

Proposition 2.5 [8] *If the Slater condition for* (P) *holds, that is, there exists $x_0 \in X$ such that $g(x_0) < 0$, then $\bigcup_{\lambda \in S^+} \text{epi } (\lambda g)^*$ is closed.*

Lemma 2.3 [8] *Let* $h : X \to \mathbb{R}$ *be a continuous convex function and* $g : X \to \mathbb{R}$ *is a lower semicontinuous convex function. Then*

$$\text{epi } (h+g)^* = \text{epi } h^* + \text{epi } g^*.$$

Lemma 2.4 [5] *Let* $h_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, \cdots, m$ *be lower semicontinuous convex functions, and* $\epsilon \geqq 0$*. Then for all* $x \in \mathbb{R}^n$,

$$\partial_\epsilon (\sum_{i=1}^m h_i)(x) = \bigcup \{ \sum_{i=1}^m \partial_{\epsilon_i} h_i(x) \mid \epsilon_i \geqq 0, i = 1, \cdots, m,\ \sum_{i=1}^m \epsilon_i = \epsilon \}.$$

3 Stability and Optimality Conditions

We consider the following optimization problem:

$$\text{(P)} \qquad \text{Minimize} \quad f(x)$$
$$\text{subject to} \quad x \in S := \{x \in \mathbb{R}^n \mid g_i(x) \leqq 0, \ i = 1, \cdots, m\}$$

where $f, \ g_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \cdots, m$ are convex functions.

Let $\epsilon \geqq 0$ and $\bar{x} \in S$.

Theorem 3.1 *Suppose that* (P) *is stable, that is,* $\phi(0)$ *is finite and there exists* $r > 0$ *such that* $\phi(u) - \phi(0) \geqq -r\|u\|$ *for all* $u \in \mathbb{R}^m$*, where* $\phi(u) := \inf\{f(x) \mid g_i(x) \leqq u_i, \ i = 1, \cdots, m\}$*. Then the following are equivalent:*

(i) $\bar{x}$ *is an* ϵ*-approximate solution of* (P)*;*

(ii) there exists $M > 0$ *such that* $\bar{x}$ *is an* ϵ*-approximate solution of* (P)$'$*:*

$$\text{(P)}' \qquad \textit{Minimize} \quad f(x) + M \sum_{i=1}^{m} g_i^+(x),$$
$$\textit{subject to} \quad g_i^+(x) = \max\{0, g_i(x)\}, \quad i = 1, \cdots, m;$$

(iii) there exist $M > 0, \ \epsilon_0, \epsilon_i \geqq 0, \ \alpha_i \in [0,1], \ \delta_i \geqq 0, \ i = 1, \cdots, m$ *such that*

$$\epsilon = \epsilon_0 + \sum_{i=1}^{m} \epsilon_i,$$

$$0 \in \partial_{\epsilon_0} f(\bar{x}) + \sum_{i=1}^{m} \partial_{\delta_i}(\alpha_i M g_i)(\bar{x}), \ \textit{and}$$

$$\delta_i \leqq \alpha_i M g_i(\bar{x}) + \epsilon_i.$$

Proof. (i) $\Rightarrow$ (ii): Suppose to the contrary that (ii) does not hold. Then for each $n \in \mathbb{N}$, *there exists* $x_n \in \mathbb{R}^n$ such that

$$f(x_n) + n\sum_{i=1}^{m} g_i^+(x_n) < f(\bar{x}) + n\sum_{i=1}^{m} g_i^+(\bar{x}) - \epsilon. \tag{3.1}$$

Notice that $\sum_{i=1}^m g_i^+(x_n) = 0$ if and only if $x_n \in S$. If $\sum_{i=1}^m g_i^+(x_n) = 0$, $\bar{x}$ is not an ϵ-approximate solution of (P). Hence $\sum_{i=1}^m g_i^+(x_n) \neq 0$, i.e., $\sum_{i=1}^m g_i^+(x_n) > 0$ for all $n \in \mathbb{N}$. So, for each $n \in \mathbb{N}$, *there exists* $i \in \{1, \cdots, m\}$ such that $g_i^+(x_n) > 0$. Let $u_n^i = g_i^+(x_n),\ i = 1, \cdots, m$ and $u_n = (u_n^1, \cdots, u_n^m)$. Then $g_i(x_n) \leq u_n^i, i = 1, \cdots, m,\ u_n \neq 0$ and

$$\|u_n\| \leq \sum_{i=1}^{m} |u_n^i| = \sum_{i=1}^{m} g_i^+(x_n).$$

Thus from (3.1), we have

$$\phi(u_n) + n\|u_n\| \leq f(x_n) + n\sum_{i=1}^{m} g_i^+(x_n) < f(\bar{x}) - \epsilon. \tag{3.2}$$

Since (P) is stable, $\phi(0)$ is finite and there exists $r > 0$ such that

$$\phi(u) - \phi(0) \geqq -r\|u\| \quad \forall u \in \mathbb{R}^m. \tag{3.3}$$

Hence from (3.2) and (3.3), $(n - r)\|u_n\| + \phi(0) < f(\bar{x}) - \epsilon$ for all $n \in \mathbb{N}$. This is impossible since $u_n \neq 0$ for all $n \in \mathbb{N}$. Thus (ii) holds.

(ii) $\Rightarrow$ (iii) : Suppose that (ii) holds. Then $0 \in \partial_\epsilon(f + M\sum_{i=1}^m g_i^+)(\bar{x})$. So, by Lemma 2.4, there exist $\epsilon_0, \epsilon_i \geqq 0,\ i = 1, \cdots, m$ such that $\epsilon = \sum_{i=0}^m \epsilon_i$

and $0 \in \partial_{\epsilon_0} f(\bar{x}) + \sum_{i=1}^m \partial_{\epsilon_i}(Mg_i^+)(\bar{x})$. We know from Example 3.5.2 in [5] that,

$$\partial_{\epsilon_i}(Mg_i^+)(\bar{x}) = \bigcup \{\partial_{\delta_i}(\alpha_i M g_i)(\bar{x}) \mid \alpha_i \in [0,1],\ 0 \leqq \delta_i \leqq \alpha_i M g_i(\bar{x}) + \epsilon_i\}.$$

Thus there exist $\epsilon_0, \epsilon_i \geqq 0,\ \alpha_i \in [0,1],\ \delta_i \geqq 0,\ i = 1, \cdots, m$ such that

$$\epsilon = \epsilon_0 + \sum_{i=1}^m \epsilon_i$$

$$0 \in \partial_{\epsilon_0} f(\bar{x}) + \sum_{i=1}^m \partial_{\delta_i}(\alpha_i M g_i)(\bar{x})$$

$$\text{and } \ \delta_i \leqq \alpha_i M g_i(\bar{x}) + \epsilon_i.$$

Thus (iii) holds.

(iii) $\Rightarrow$ (i) : Suppose that (iii) holds. Then there exist $v_0 \in \partial_{\epsilon_0} f(\bar{x}),\ v_i \in \partial_{\delta_i}(\alpha_i M g_i)(\bar{x}),\ i = 1, \cdots, m$ such that

$$0 = v_0 + \sum_{i=1}^m v_i$$

$$\epsilon = \epsilon_0 + \sum_{i=1}^m \epsilon_i$$

$$\text{and } \ 0 \leqq \delta_i \leqq \alpha_i M g_i(\bar{x}) + \epsilon_i,\ i = 1, \cdots, m.$$

Then for any $x \in \mathbb{R}^n$,

$$f(x) - f(\bar{x}) \geqq \langle v_0, x - \bar{x} \rangle - \epsilon_0,$$

$$\begin{aligned}(\alpha_i M g_i)(x) - (\alpha_i M g_i)(\bar{x}) &\geqq \langle v_i, x - \bar{x}\rangle - \delta_i \\ &\geqq \langle v_i, x - \bar{x}\rangle - \alpha_i M g_i(\bar{x}) - \epsilon_i.\end{aligned}$$

Hence we have, for any $x \in \mathbb{R}^n$,

$$f(x) + \sum_{i=1}^{m}(\alpha_i M g_i)(x) - f(\bar{x}) - \sum_{i=1}^{m}(\alpha_i M g_i)(\bar{x}) \geqq -\epsilon_0 - \sum_{i=1}^{m}\alpha_i M g_i(\bar{x}) - \sum_{i=1}^{m}\epsilon_i.$$

So, for any $x \in \mathbb{R}^n$,

$$f(x) + \sum_{i=1}^{m}\alpha_i M g_i(x) - f(\bar{x}) \geqq -\epsilon.$$

Hence for any $x \in S$, $f(x) \geqq f(\bar{x}) - \epsilon$. Thus $\bar{x}$ is an ϵ-approximate solution of (P).

4 Convex-Sublinear Optimization Problems

Let X, Z be Banach spaces and S a closed convex cone in Z, which does not necessarily have nonempty interior. Let $f : X \to \mathbb{R}$ be a continuous convex function and $g : X \to Z$ a continuous S-sublinear function, where g is S-sublinear, that is, (i) $\forall x \in X \quad \forall \lambda \geqq 0,\ g(\lambda x) = \lambda g(x)$, (ii) $\forall x, y \in X,\quad g(x) + g(y) - g(x+y) \in -S$.

We consider the following optimization problem:

$$\begin{aligned} \text{(P)} \qquad & \text{Minimize} \quad f(x) \\ & \text{subject to} \quad g(x) \in -S. \end{aligned}$$

The asymptotic Lagrange conditions for (P) have been studied in ([1, 6]) when g is S-convex. Here we will present the asymptotic Lagrange conditions for (P).

And we will give an example which illustrates our asymptotic results and we know from this example that shows the condition containing

$$\text{“}0 \in \partial f(a) + cl \bigcup_{\lambda \in S^+} \partial(\lambda g)(a)\text{”}$$

may not be an asymptotic Lagrange condition for (P) even if g is S-sublinear.

Theorem 4.1 *Let $a \in X$ be such that $g(a) \in -S$. Then the following statements are equivalent:*

(i) a is an optimal solution of (P).

(ii) there exists $u \in \partial f(a)$ such that

$$-\begin{pmatrix} u \\ u(a) \end{pmatrix} \in cl\Big(\bigcup_{\lambda \in S^+} \text{epi } (\lambda g)^*\Big).$$

(iii) there exist $u \in \partial f(a)$, $\lambda_\alpha \in S^+$, $\epsilon_\alpha \geqq 0$ and $v_\alpha \in \partial_{\epsilon_\alpha}(\lambda_\alpha g)(a)$ such that

$$-u = \lim_{\alpha \to \infty} v_\alpha,$$
$$u(a) = 0,$$
$$\lim_{\alpha \to \infty} \epsilon_\alpha = 0.$$

(iv) there exists $u \in \partial f(a)$ such that

$$-u \in cl\Big(\bigcup_{\lambda \in S^+} \partial(\lambda g)(0)\Big)$$

$$u(a) = 0.$$

(v) there exists $u \in \partial f(a)$ such that

$$-u \in cl\Big(\bigcup_{\lambda \in S^+} \bigcup_{\epsilon \geq 0} \partial_\epsilon(\lambda g)(a)\Big)$$

$$u(a) = 0.$$

Proof. (i) $\iff$ (ii): (i) $\iff$ $0 \in \partial f(a) + N_A(a)$, where $A = \{x \in X \mid g(x) \in -S\}$. By Lemma 2.2,

$$u \in N_A(a) \iff -\begin{pmatrix} u \\ u(a) \end{pmatrix} \in cl\Big(\bigcup_{\lambda \in S^+} \text{epi } (\lambda g)^*\Big).$$

Hence it is true that (i) $\iff$ (ii).

(ii) $\Longleftrightarrow$ (iii):

(ii) $\Longleftrightarrow$ By Proposition 2.4, there exists $u \in \partial f(a)$ such that

$$-\begin{pmatrix} u \\ u(a) \end{pmatrix} \in cl\left(\bigcup_{\lambda \in S^+} \bigcup_{\epsilon \geqq 0} \left\{ \begin{pmatrix} v \\ v(a) + \epsilon - (\lambda g)(a) \end{pmatrix} \mid v \in \partial_\epsilon (\lambda g)(a) \right\}\right)$$

$\Longleftrightarrow$ there exists $u \in \partial f(a),\ \lambda_\alpha \in S^+,\ \epsilon_\alpha \geqq 0,\ v_n \in \partial_{\epsilon_\alpha}(\lambda_\alpha g)(a)$ such that

$$-\begin{pmatrix} u \\ u(a) \end{pmatrix} = \lim_{\alpha \to \infty} \begin{pmatrix} v_\alpha \\ v_\alpha(a) + \epsilon_\alpha - (\lambda_\alpha g)(a) \end{pmatrix}$$

$\Longleftrightarrow$ there exists $u \in \partial f(a),\ \lambda_\alpha \in S^+,\ \epsilon_\alpha \geqq 0$ and $v_\alpha \in \partial_{\epsilon_\alpha}(\lambda_\alpha g)(a)$ such that

$$-u = \lim_{\alpha \to \infty} v_\alpha,\ \lim_{\alpha \to \infty} (\lambda_\alpha g)(a) = 0 \text{ and } \lim_{\alpha \to \infty} \epsilon_\alpha = 0.$$

Since $v_\alpha \in \partial_{\epsilon_\alpha}(\lambda_\alpha g)(a)$, we have

$$(\lambda_\alpha g)(x) - (\lambda_\alpha g)(a) \geqq v_\alpha(x - a) - \epsilon_\alpha \quad \forall x \in X. \tag{4.1}$$

Taking $x = 2a$ in (4.1), it follows from (4.1) and the sublinearity of $\lambda_\alpha g$ that

$$(\lambda_\alpha g)(a) \geqq v_\alpha(a) - \epsilon_\alpha. \tag{4.2}$$

Letting $\alpha \to \infty,\ 0 = \lim_{\alpha \to \infty} (\lambda_\alpha g)(a) \geqq \lim_{\alpha \to \infty} v_\alpha(a) = -u(a)$ and hence $u(a) \geqq 0$. Taking $x = 0$ in (4.1), we get

$$-(\lambda_\alpha g)(a) \geqq -v_\alpha(a) - \epsilon_\alpha.$$

Letting $\alpha \to \infty,\ 0 = -\lim_{\alpha \to \infty} (\lambda_\alpha g)(a) \geqq u(a)$. Thus $u(a) = 0$. Hence it is true that (ii) $\Rightarrow$ (iii).

Now we prove that (iii) $\Rightarrow$ (ii). Suppose that (iii) holds. Then (4.2) holds. Since $(\lambda_\alpha g)(a) \leqq 0$, we get

$$\begin{aligned} 0 &\geqq \lim_{\alpha\to\infty} (\lambda_\alpha g)(a) \\ &\geqq \lim_{\alpha\to\infty} v_\alpha(a) - \lim_{\alpha\to\infty} \epsilon_\alpha \\ &= -u(a) \\ &= 0. \end{aligned}$$

Thus $\lim_{\alpha\to\infty} (\lambda_\alpha g)(a) = 0$. Hence it is true that (iii) $\Rightarrow$ (ii).

(ii) $\iff$ (iv) :

$$\begin{aligned} cl\Big(\bigcup_{\lambda\in S^+} \text{epi } (\lambda g)^*\Big) &= cl\Big(\bigcup_{\lambda\in S^+} [\alpha(\lambda g)(0) \times \mathbb{R}_+]\Big) \\ &= cl\Big(\bigcup_{\lambda\in S^+} (\lambda g)(0)\Big) \times \mathbb{R}_+. \end{aligned} \tag{4.3}$$

Thus it is clear that (iv) $\Rightarrow$ (ii).

Suppose that (ii) holds. Then, it follows from (4.3) that

$$-u \in cl\Big(\bigcup_{\lambda\in S^+} \alpha(\lambda g)(0)\Big) \quad \text{and} \quad \text{u}(a) \leqq 0.$$

So, there exist $\lambda_\alpha \in S^+$ and $v_\alpha \in \partial(\lambda_\alpha g)(0)$ such that

$$-u = \lim_{\alpha\to\infty} v_\alpha.$$

Since $v_\alpha \in \partial(\lambda_\alpha g)(0)$, we have

$$(\lambda_\alpha g)(x) - (\lambda_\alpha g)(0) \geqq v_\alpha(x) \quad \forall x \in X.$$

Letting $x = a$ in the above inequality,

$$0 \geqq (\lambda_\alpha g)(a) \geqq v_\alpha(a).$$

Letting $\alpha \to \infty$, we get

$$u(a) \geqq 0.$$

Thus $u(a) = 0$. So it is true that (ii)$\Rightarrow$ (iii).

(iv) $\iff$ (v) : Notice that

$$\begin{aligned} \partial_\epsilon(\lambda g)(a) &= \{v \in \partial(\lambda g)(0) \mid (\lambda g)(a) - v(a) \leqq \epsilon\} \\ &\subset \partial(\lambda g)(0). \end{aligned}$$

So, it is clear that (v) $\Rightarrow$ (iv).

Suppose that (iv) holds. Then there exist $\lambda_\alpha \in S^+$ and $v_\alpha \in \partial(\lambda_\alpha g)(0)$ such that

$$-u = \lim_{\alpha\to\infty} v_\alpha, \; u(a) = 0.$$

Since $v_\alpha \in \partial(\lambda_\alpha g)(0)$, we get

$$0 \geqq (\lambda_\alpha g)(a) \geqq v_\alpha(a).$$

Thus, $0 \geqq \lim_{\alpha\to\infty}(\lambda_\alpha g)(a) \geqq \lim_{\alpha\to\infty} v_\alpha(a) = -u(a) = 0$. Hence $\lim_{\alpha\to\infty}(\lambda_\alpha g)(a) = 0$ and $\lim_{\alpha\to\infty} v_\alpha(a) = 0$. Let $\epsilon_\alpha := (\lambda_\alpha g)(a) - v_\alpha(a)$. Then $\epsilon_\alpha \geqq 0$ and $\lim_\alpha \epsilon_\alpha = 0$. Since $\partial_{\epsilon_\alpha}(\lambda_\alpha g)(a) = \{v \in \partial(\lambda_\alpha g)(0) \mid (\lambda_\alpha g)(a) - v(a) \leqq \epsilon\}$,

$$v_\alpha \in \partial_{\epsilon_\alpha}(\lambda_\alpha g)(a).$$

Since $-u = \lim_{\alpha \to \infty} v_\alpha$, we have

$$-u \in cl \bigcup_{\lambda \in S^+} \bigcup_{\epsilon \geqq 0} (\lambda g)(a).$$

Thus (iv) holds. $\square$

Now we give an example which illustrates results of Theorem 4.1 and we show that the set "$cl\Big(\bigcup_{\lambda \in S^+} \partial(\lambda g)(0)\Big)$" in (iv) can not be replaced by "$cl\Big(\bigcup_{\lambda \in S^+} \partial(\lambda g)(a)\Big)$".

Example 4.1 *For the problem* (P), *let* $f(x, y) = x,\ g(x, y) = (x^2 + y^2)^{\frac{1}{2}} - y$ *and* $S = \mathbb{R}_+$. *Let* $a = (0, 1)$. *Then* a *is an optimal solution of* (P) *and* $\partial f(a) = \{(1, 0)\}$. *Let* $u = (1, 0)$. *Then* $u^T a = 0$,

$$\bigcup_{\lambda \in \mathbb{R}_+} \text{epi } (\lambda g)^* = \{(0, 0, \alpha) \mid \alpha \geqq 0\} \cup \{(x, y, \alpha) \mid x \in \mathbb{R},\ y < 0,\ \alpha \geqq 0\}$$

and

$$cl \bigcup_{\lambda \in \mathbb{R}^+} \text{epi } (\lambda g)^* = \{(x, y) \mid x \in \mathbb{R},\ y \leqq 0\} \times \mathbb{R}_+.$$

Thus (ii) holds. Take $\epsilon_n = \frac{1}{n}$ *and* $\lambda_n = \frac{1}{2}(n + \frac{2}{n}) + 1$. *Then* $(-1 - \frac{1}{n}, -\frac{1}{n}) \in \partial_{\epsilon_n}(\lambda_n g)(a)$. *Thus* $v_n \to -u,\ \epsilon_n \to 0$ *and* $u(a) = 0$. *So, (iii) holds.*

$$\partial(\lambda g)(0) = \{(v_1, v_2) \mid v_1^2 + (v_2 + \lambda)^2 \leqq \lambda^2\}.$$

So, $cl \bigcup_{\lambda \in S^+} \partial(\lambda g)(0) = \{(v_1, v_2) \mid v_2 \leqq 0\}$. *Thus (iv) holds. Let* $\lambda > 0$.

$$\begin{aligned}\partial_\epsilon(\lambda g)(a) &= \{(v_1, v_2) \in \partial(\lambda g)(0,0) \mid (\lambda g)(0,1) - (v_1, v_2)^T(0,1) \leqq \epsilon\} \\ &= \partial(\lambda g)(0,0) \cap \{(v_1, v_2) \mid v_2 \geqq -\epsilon\} \\ &= \{(v_1, v_2) \mid v_1^2 + (v_2 + \lambda)^2 \leqq \lambda^2\} \cap \{(v_1, v_2) \mid v_2 \geqq -\epsilon\}.\end{aligned}$$

So, $\bigcup_{\lambda \in S^+} \bigcup_{\epsilon \geqq 0} \partial_\epsilon(\lambda g)(a) = \{(v_1, v_2) \mid v_2 < 0\} \cup \{(0,0)\}$ *and* $cl\Big(\bigcup_{\lambda \in S^+} \bigcup_{\epsilon \geqq 0} \partial_\epsilon(\lambda g)(a)\Big) = \{(v_1, v_2) \mid v_2 \leqq 0\}$. *Thus (v) holds. However,* $\partial(\lambda g)(a) = \{(0,0)\}\ \forall \lambda \in S^+ (= \mathbb{R}_+)$ *and hence* $-u \notin cl\Big(\bigcup_{\lambda \in S^+} \partial(\lambda g)(a)\Big)$. *Thus the set "*$cl\Big(\bigcup_{\lambda \in S^+} \partial(\lambda g)(0)\Big)$*" in (iv) can not be replaced by "*$cl\Big(\bigcup_{\lambda \in S^+} \partial(\lambda g)(a)\Big)$*". So, the condition containing "*$0 \in \partial f(a) + cl\Big(\bigcup_{\lambda \in S^+} \partial(\lambda g)(a)\Big)$*" may not be an asymptotic Lagrange condition for (P).* □

Now we consider the following optimization problem:

$$\begin{aligned}(\mathrm{P})' \qquad & \text{Minimize} \quad f(x) \\ & \text{subject to} \quad x \in A := \{x \in X \mid g(x) - b \in -S\},\end{aligned}$$

where $f : X \to \mathbb{R}$ and $g : X \to Z$ are a continuous sublinear function and a continuous S-sublinear function, respectively.

Theorem 4.2 *Let* $\bar{x} \in A$. *Then the following statements are equivalent:*

(i) $\bar{x}$ *is an optimal solution of (P)'.*

(ii) there exists $v \in \partial f(0)$ such that $f(\bar{x}) - v(\bar{x}) \leqq 0$ and

$$-\begin{pmatrix} v \\ v(\bar{x}) \end{pmatrix}^T \in cl \bigcup_{\lambda \in S^+} \Big(\partial(\lambda \circ g)(0) \times [\lambda(b), \infty)\Big).$$

Proof. Let $\bar{x}$ be an optimal solution of (P)$'$.

$$\begin{aligned}
&\Longleftrightarrow \quad f(x) \geqq f(\bar{x}) \quad \forall x \in A \\
&\Longleftrightarrow \quad f(x) + \delta_A(x) \geqq f(\bar{x}) + \delta_A(\bar{x}) \quad \forall x \in A \\
&\Longleftrightarrow \quad 0 \in \partial(f + \delta_A)(\bar{x}) = \partial f(\bar{x}) + \partial \delta_A(\bar{x}) = \partial f(\bar{x}) + N_A(\bar{x}).
\end{aligned}$$

Since f is sublinear, $\partial f(\bar{x}) = \{v \in \partial f(0) \mid f(\bar{x}) - v(\bar{x}) \leqq 0\}$. Let $D = \{x \in X \mid g(x) - b \in -S\}$. Then $A = X \cap D$. So,

$$\begin{aligned}
-v \in N_A(\bar{x}) \quad &\Longleftrightarrow \quad -\begin{pmatrix} v \\ v(\bar{x}) \end{pmatrix}^T \in cl\Big(\bigcup_{\lambda \in S^+} \text{epi } (\lambda \circ g)^*\Big) \\
&\Longleftrightarrow \quad -\begin{pmatrix} v \\ v(\bar{x}) \end{pmatrix}^T \in cl \bigcup_{\lambda \in S^+} \Big(\partial(\lambda \circ g)(0) \times [\lambda(b), \infty)\Big).
\end{aligned}$$

□

Theorem 4.3 *Suppose that $\bigcup_{\lambda \in S^+} [\partial(\lambda \circ g)(0) - C] \times [\lambda(b), \infty)$ is closed. Then the following statements are equivalent:*

(i) $\bar{x}$ is an optimal solution of (P)$'$.

(ii) there exist $v \in \partial f(0)$ and $\lambda \in S^*$ such that $f(\bar{x}) - v(\bar{x}) \leqq 0$, $-v \in \partial(\lambda \circ g)(0) - C$ and $\lambda(b) \leqq -v(\bar{x})$.

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