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Thesis for the Degree
Master of Education

On πgb-closed Sets in Topological Spaces

by
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Graduate School of Education
Pukyong National University

August 2007

On πgb-closed Sets in Topological Spaces

위상공간상의 πgb-폐집합에 대하여

Advisor : Jin Han Park

by
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A thesis submitted in partial fulfillment of the requirement
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요약

1996년 Andrejevic은 반개집합과 전개집합보다 약하고, 반전개집합보다 강한 b-개집합을 소개하였고, 이를 이용하여 Ha는 일반화된 b-폐집합을 소개하였다. 한편, Dontchev와 Noiri는 g-폐집합보다 약한 πg-폐집합을 정의하였다. 최근 Lee, Shin과 Park은 일반화된 반폐집합(gs-폐집합)보다 약한 πgs-폐집합을 소개하여 그 성질을 조사하고, 이들을 이용하여 πgs-$T_{1/2}$공간과 πgs-연속함수를 소개하였다.

본 논문에서는 πg-폐집합과 gb-폐집합을 기초로 πgs-폐집합과 πgp-폐집합보다 약한 πgb-폐집합을 정의하여, 그 기본적인 성질을 조사하였다. 또한 πgb-폐집합을 이용하여 πgb-연속함수와 πgb-irresolute 함수를 소개하고 기존의 여러 다른 연속함수와의 관계를 조사하였다.

1 Introduction

Separation axioms, compactness, connectedness and continuity on topological space, as important and basic subjectes in the study of general topology have been researched by many mathmaticians. Recent tendency of research activities in general topology are the study of ideal topological spaces which are topological spaces having the structure of ideals, the study of operation functions on topological spaces and research of generalized closed sets in topological spaces.

Levine [16] initiated the investigation of so-called g-closed sets in topological spaces, since then many modifications of g-closed sets were defined and investigated by many authors. In 1996, Andrijević [3] introduced and studied b-open sets, which is weaker than both semi-open sets and preopen sets, and showed that the class of b-open sets generates the same topology as the class of preopen sets. Ha [14] defined gb-closed sets studied some of their properties. This notion is generalization of b-closed sets which were further studied by Andrijević [3]. On the other hand, Zaitsev [26] introduced the concept of π-closed sets and a class of topological spaces called quasi-normal spaces. Recently, Dontchev and Noiri [10] defined the concept of πg-closed sets as a weak form of g-closed sets and used this notion to obtain a characterization and some preservation theorems for quasi-normal spaces. More recently, Park, Lee and Shin [21] introduced and studied the notion of πgs-closed sets which is implied by that of πg-closed sets and implies that of gs-closed sets. The notions of πgs-open sets, πgs-$T_{1/2}$ spaces, πgs-continuity and πgs-irresoluteness are also introduced by Park, Lee and Shin [21].

In this paper, we introduce the concept of πgb-closed sets which implied by both those of πg-closed sets and gb-closed sets and study its basic properties. Further the notions of πgb-continuity and πgb-irresoluteness are introduced.

2 Preliminaries

Throughout this paper, spaces (X, τ) and (Y, σ) (simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. We denote the closure and the interior of a set A by $\mathrm{cl}(A)$ and $\mathrm{int}(A)$, respectively. A subset A is said to be regular open (resp. regular closed) if $A = \mathrm{int}(\mathrm{cl}(A))$ (resp. $A = \mathrm{cl}(\mathrm{int}(A))$). A point $x \in X$ is called a δ-cluster point [25] of A if $A \cap U \neq \emptyset$ for every regular open set containing x. The set of all δ-cluster points of A is called the δ-closure of A and is denoted by $\mathrm{cl}_\delta(A)$. If $\mathrm{cl}_\delta(A) = A$, then A is called δ-closed [25]. The complement of a δ-closed set is said to be δ-open [25]. The finite union of regular open sets is said to be π-open [26]. The complement of a π-open set is said to be π-closed [26].

A subset A is said to be α-open [19] (resp. preopen [18], semi-open [15], b-open [3], semi-preopen [2] or β-open [1]) if $A \subset \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$ (resp. $A \subset \mathrm{int}(\mathrm{cl}(A))$, $A \subset \mathrm{cl}(\mathrm{int}(A))$, $A \subset \mathrm{cl}(\mathrm{int}(A)) \cup \mathrm{int}(\mathrm{cl}(A))$, $A \subset \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$). The complement of an α-open (resp. preopen, semi-open, b-open, β-open) set is said to be α-closed (resp. preclosed, semi-closed, b-closed, β-closed). The intersection of all b-closed sets of X containing A is called the b-closure [3] of A and is denoted by $\mathrm{bcl}(A)$. The semi-closure and preclosure are similarly defined and are denoted by $\mathrm{scl}(A)$ and $\mathrm{pcl}(A)$. The union of all b-open sets of X contained in A is called b-interior [3] and is denoted by $\mathrm{bint}(A)$. Note that $\mathrm{bcl}(A) = \mathrm{scl}(A) \cap \mathrm{pcl}(A)$ and $\mathrm{bint}(A) = \mathrm{sint}(A) \cup \mathrm{pint}(A)$.

We recall the following definitions used in sequel.

Definition 2.1 A subset A of a space X is said to be:

(a) g-closed [16] if $\mathrm{cl}(A) \subset U$ whenever $A \subset U$ and U is open in X;
(b) gb-closed [14] if $\mathrm{bcl}(A) \subset U$ whenever $A \subset U$ and U is b-open in X;
(c) πg-closed [10] if $\mathrm{cl}(A) \subset U$ whenever $A \subset U$ and U is π-open in X;
(d) gs-closed [4] if $\mathrm{scl}(A) \subset U$ whenever $A \subset U$ and U is open in X;
(e) πgs-closed [21] if $\mathrm{scl}(A) \subset U$ whenever $A \subset U$ and U is π-open in X;
(f) gsp-closed [9] if $\mathrm{spcl}(A) \subset U$ whenever $A \subset U$ and U is open in X;

(g) gp-closed [17] if $\mathrm{pcl}(A) \subset U$ whenever $A \subset U$ and U is open in X;

(h) gpr-closed [12] if $\mathrm{pcl}(A) \subset U$ whenever $A \subset U$ and U is regular open in X;

(i) πgp-closed [23] if $\mathrm{pcl}(A) \subset U$ whenever $A \subset U$ and U is π-open in X;

(j) rg-closed [20] if $\mathrm{cl}(A) \subset U$ whenever $A \subset U$ and U is regular open in X;

(k) g-open (resp. gb-open, πg-open, gs-open, πgs-open, gsp-open, gp-open, gpr-open, πgp-open, rg-open) if the complement of A is g-closed (resp. gb-closed, πg-closed,gs-closed, πgs-closed, gsp-closed, gp-closed, gpr-closed, πgp-closed, rg-closed).

Lemma 2.2 (Andrijevic [3]) For a subset A of a space X, the following hold:

(a) $\mathrm{bcl}(A) = \mathrm{scl}(A) \cap \mathrm{pcl}(A)$.

(b) $\mathrm{bint}(A) = \mathrm{sint}(A) \cup \mathrm{pint}(A)$.

(c) $\mathrm{bcl}(X \setminus A) = X \setminus \mathrm{bint}(A)$.

Definition 2.3 A function $f{:}(X,\tau)\to(Y,\sigma)$ is called:

(a) πgs-continuous [21] (resp. πgp-continuous [23]) if $f^{-1}(V)$ is πgs-closed (resp. πgp-closed) in X for every closed set V of Y;

(b) b-continuous [11] (resp. b-irresolute) if $f^{-1}(V)$ is b-open in X for every open (resp. b-open) set V of Y;

(c) gb-continuous [14] (resp. gb-irresolute) if $f^{-1}(V)$ is gb-closed in X for every closed (resp. gb-closed) set V of Y;

(d) g-continuous [5] (resp. rg-continuous [20]) if $f^{-1}(V)$ is g-closed (resp. rg-closed) in X for every closed set V of Y;

(e) πg-continuous [10] (resp. almost π-continuous [10]) if $f^{-1}(V)$ is πg-closed (resp. π-closed) in X for every closed (resp. regular closed) set V of Y;

(f) pre-β-closed [14] (resp. rc-preserving [13]) if $f(V)$ is b-closed (resp. regular closed) in Y for every b-closed (resp. regular closed) set V of X.

Definition 2.4 A space (X,τ) is called:

(a) $T_{1/2}$ [16] if every g-closed set is closed;

(b) πgp-$T_{1/2}$ [23] if every πgp-closed set is preclosed.

Theorem 2.5 For a space (X, τ), the following conditions are equivalent:

(a) X is πgp-$T_{1/2}$.

(b) Every singleton of X is either π-closed or preopen.

(c) Every singleton of X is either π-closed or b-open.

(d) Every singleton of X is either π-closed or semi-preopen.

Proof It follows from Theorem 4.2 of Park [23] and Theorem 3.9 of Andrijević [3]. □

3 Basic properties of πgb-closed sets

Definition 3.1 A subset A of a space (X, τ) is said to be πgb-closed if $\mathrm{bcl}(A) \subset U$ whenever $A \subset U$ and U is π-open in X.

Remark 3.2 From definitions stated above, we have the following diagram of implications:

$$\begin{array}{ccccc} & & \pi g\text{-closed} & & \\ & \swarrow & & \searrow & \\ \pi gs\text{-closed} & & & & \pi gp\text{-closed} \\ & \searrow & & \swarrow & \\ & & \pi gb\text{-closed} & & \end{array}$$

where none of these implications is reversible as shown by Example 2.3 of Park [21, 23] and the following example shows.

Example 3.3 (a) Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Put $A = \{b\}$. Then A is πgb-closed in (X, τ) but not πgp-closed.

(b) Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, c, d\}, \{a, b, c\}\}$. Put $A = \{a\}$. Then A is πgb-closed in (X, τ) but it is neither gpr-closed nor b-closed.

(c) Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}\}$. Put $A = \{a, c\}$. Then A is πgb-closed in (X, τ) but not g-closed. Put $B = \{a\}$. Then B is

πgb-closed in (X, τ) but not gs-closed. Put $C = \{a, c, d\}$. Then C is πgb-closed in (X, τ) but not gb-closed.

(d) Let $X = \{a, b, c, d, e\}$ and $\tau = \{X, \emptyset, \{a, b\}, \{c, d\}, \{a, b, c, d\}\}$. Put $A = \{a,\ c\}$.Then A is πgb-closed in (X, τ) but not πgs-closed. Put $B = \{a, b\}$. Then B is πgb-closed in (X, τ) but not rg-closed.

(e) Let X be the real numbers with the usual topology. Let $A = (\,-\infty, 0) \cup ((\,1, 2) \cap Q^c) \cup (\,2, \infty)$ where Q stands for the set of rational numbers. Then A is πgb-closed in (X, τ) but it is neither πgp-closed nor πgs-closed.

Lemma 3.4 [8, 7] For a space (X, τ) the following hold:

(a) X is extremally disconnected if and only if $\text{cl}(A) = \text{scl}(A)$ for every semiopen set A of X.

(b) X is extremally disconnected if and only if every semiclosed set of X is α-closed.

Theorem 3.5 For a subset A of a space (X, τ) the following hold:

(a) If A is π-open and πgb-closed in X, then it is b-closed and regular open.

(b) If A is semiopen and πgb-closed in an extremally disconnected space X, then it is πgp-closed.

Proof (a) If A is π-open and πgb-closed, then $\text{bcl}(A) \subset A$ and so A is b-closed. Since π-open set is open, we have

$$A = \text{bcl}(A) = \text{bcl}(\text{int}(A)) = \text{int}(\text{cl}(\text{int}(A))) = \text{int}(\text{cl}(A)).$$

Hence A is regular open.

(b) Let $A \subset U$ where U is π-open in X. Since A is πgb-closed, $\text{bcl}(A) \subset U$. By Lemma 3.4, we have

$$\text{bcl}(A) = \text{scl}(A) \cap \text{pcl}(A) = \text{cl}(A) \cap \text{pcl}(A) = \text{pcl}(A) \subset U.$$

Hence A is πgp-closed. □

Theorem 3.6 If A is a πgb-closed subset of a space (X, τ), then $\text{bcl}(A) \setminus A$ does not contain any non-empty π-closed set.

Proof Let F be any π-closed set such that $F \subset \mathrm{bcl}(A) \setminus A$. Then $A \subset X \setminus F$. Since A is πgb-closed and $X \setminus F$ is π-open, we have $\mathrm{bcl}(A) \subset X \setminus F$, i.e. $F \subset X \setminus \mathrm{bcl}(A)$. Hence we obtain

$$F \subset (\mathrm{bcl}(A) \setminus A) \cap (X \setminus \mathrm{bcl}(A)) = \emptyset.$$

This shows that $F = \emptyset$. □

Corollary 3.7 Let A be a πgb-closed subset of a space (X, τ). Then A is b-closed if and only if $\mathrm{bcl}(A) \setminus A$ is π-closed if and only if $A = \mathrm{bint}(\mathrm{bcl}(A))$.

Remark 3.8 (a) Every finite union of πgb-closed sets may fail to be a πgb-closed set.

(b) Every finite intersection of πgb-closed sets may fail to be a πgb-closed set.

Example 3.9 (a) Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Let $A = \{a\}$ and $B = \{b\}$, then A and B are πgb-closed. But $A \cup B = \{a, b\}$ is not πgb-closed in (X, τ), since $\{a, b\}$ is π-open and $\mathrm{bcl}(A \cup B) = X$.

(b) Let $X=\{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Let $A = \{a, b, c\}$ and $B = \{a, b, d\}$. Then A and B is πgb-closed, but $A \cap B = \{a, b\}$ is not πgb-closed in (X, τ).

Recall that a space (X, τ) is called submaximal [6] if every dense subset of X is open. It is known that (X, τ) is submaximal if and only if every preopen subset of X is open [24].

Theorem 3.10 If A and B are πgb-closed sets in extremally disconnected and submaximal space (X, τ), then $A \cup B$ is πgb-closed.

Proof Let $A \cup B \subset U$ where U is π-open in X. Since A and B are πgb-closed, $\mathrm{bcl}(A) \subset U$ and $\mathrm{bcl}(B) \subset U$. Since X is extremally disconnected,

$$\mathrm{bcl}(A \cup B) = \mathrm{pcl}(A \cup B) \cap \mathrm{scl}(A \cup B) = \mathrm{pcl}(A \cup B).$$

And since X is submaximal space, hence we obtain

$$\mathrm{bcl}(A \cup B) = \mathrm{pcl}(A) \cup \mathrm{pcl}(B) \subset U.$$

This shows that $A \cup B$ is πgb-closed. □

Lemma 3.11 [22] If $A \subset Y \subset X$ and Y is α-open in X, then $\mathrm{bcl}(A) \cap Y = \mathrm{bcl}_Y(A)$, where $\mathrm{bcl}_Y(A)$ denote the b-closure of A in the subspace Y.

Lemma 3.12 [23] Let Y is open subset of a space X. Then we have:

(a) If A is π-open in Y, then there exists a π-open set B of X such that A=$B \cap Y$.

(b) If A is π-open in X, then $A \cap Y$ is π-open in Y.

Theorem 3.13 Let $A \subset Y \subset X$. Then:

(a) If Y is b-closed in X and A is πgb-closed in X, then A is πgb-closed in Y.

(b) If Y is πgb-closed and regular open in X and A is πgb-closed in Y, then A is πgb-closed in X.

Proof (a) Let $A \subset U$ where U is π-open in Y. By Lemma 3.12, $U = V \cap Y$ for some π-open V in X. Since A is πgb-closed in X, we have $\mathrm{bcl}(A) \subset V$ and by Lemma 3.11,

$$\mathrm{bcl}_Y(A) = \mathrm{bcl}(A) \cap Y \subset V \cap Y = U.$$

Hence A is πgb-closed in Y.

(b) Let $A \subset U$ where U is π-open in X. By Lemma 3.12, $U \cap Y$ is π-open in Y and since A is πgb-closed in Y, $\mathrm{bcl}_Y(A) \subset U \cap Y$. By Lemma 3.11 and Theorem 3.5 (a), we have

$$\mathrm{bcl}_X(A) = \mathrm{bcl}_Y(A) \cap Y = \mathrm{bcl}_Y(A) \subset U.$$

Hence A is πgb-closed in X. □

Corollary 3.14 If A is πgb-closed and regular open subset and B is b-closed subset of a space X, then $A \cap B$ is πgb-closed.

Proof Let $A \cap B \subset U$ where U is π-open in A. Since B is b-closed in X, $A \cap B$ is b-closed in A and thus $\mathrm{bcl}_A(A \cap B) = A \cap B$. That is $\mathrm{bcl}_A(A \cap B) \subset U$. Then $A \cap B$ is πgb-closed in the πgb-closed and regular open set A and hence by above theorem $A \cap B$ is πgb-closed in X. □

Theorem 3.15 If A is πgb-closed in a space X and $A \subset B \subset \mathrm{bcl}(A)$, then B is πgb-closed.

Proof Let $B \subset U$ where U is π-open in X. Since $A \subset U$ and A is πgb-closed, $\mathrm{bcl}(A) \subset U$ and then $\mathrm{bcl}(B) = \mathrm{bcl}(A) \subset U$. Hence B is πgb-closed. □

4 On πgb-open sets

Definition 4.1 A subset A of a space (X, τ) is called πgb-open if its complement $X \setminus A$ is πgb-closed.

Theorem 4.2 A subset A of a space X is πgb-open if and only if $F \subset \mathrm{bint}(A)$ whenever F is π-closed and $F \subset A$.

Proof Let $F \subset A$ where F be π-closed in X. Then $X \setminus A \subset X \setminus F$ and $X \setminus F$ is π-open in X. Since $X \setminus A$ is πgb-closed, $\mathrm{bcl}(X \setminus A) \subset X \setminus F$. By Lemma 2.1, we have

$$\mathrm{bcl}(X \setminus A) = X \setminus \mathrm{bint}(A) \subset X \setminus F$$

i.e. $F \subset \mathrm{bint}(A)$.

Conversely, let $X \setminus A \subset U$ where U is π-open in X. Then $X \setminus U$ is π-closed and $X \setminus U \subset A$. By hypothesis, we have $X \setminus U \subset \mathrm{bint}(A)$, i.e.,

$$\mathrm{bcl}(X \setminus A) = X \setminus \mathrm{bint}(A) \subset U.$$

This implies $X \setminus A$ is πgb-closed and thus A is πgb-open. □

Theorem 4.3 If A is a πgb-open subset of X, then $U = X$ whenever U is π-open and $\mathrm{bint}(A) \cup (X \setminus A) \subset U$.

Proof Let U be a π-open set and $\mathrm{bint}(A) \cup (X \setminus A) \subset U$. Then we have

$$X \setminus U \subset (X \setminus \mathrm{bint}(A)) \cap A$$

i.e. $X \setminus U \subset \mathrm{bcl}(X \setminus A) \cap A$. By Theorem 3.6, $X \setminus U = \emptyset$ and hence $U = X$. □

Theorem 4.4 Let $A \subset Y \subset X$ and Y be π-open and closed in X. If A is πgb-open in Y, then A is πgb-open in X.

Proof Let F be any π-closed set and $F \subset A$. Since F is π-closed in Y and A is πgb-open in Y, $F \subset \mathrm{bint}_Y(A)$, where $\mathrm{bint}_Y(A)$ is b-interior of A in subspace Y, and hence $F \subset \mathrm{bint}(A) \cap Y \subset \mathrm{bint}(A)$. This shows that A is πgb-open in X. □

Theorem 4.5 If A is πgb-open in X and $\mathrm{bint}(A) \subset B \subset A$, then B is πgb-open.

Proof Let $F \subset B$ and F be π-closed in X. Since A is πgb-open and $F \subset A$. We have $F \subset \mathrm{bint}(A)$ and thus $F \subset \mathrm{bint}(B)$. Hence B is πgb-open. □

Theorem 4.6 If A is πgb-closed in X, then $\mathrm{bcl}(A) \setminus A$ is πgb-open.

Proof Let $F \subset \mathrm{bcl}(A) \setminus A$ and F be π-closed in X. Then by Theorem 3.6, we have $F = \emptyset$ and hence $F \subset \mathrm{bint}(\mathrm{bcl}(A) \setminus A)$. This shows that $\mathrm{bcl}(A) \setminus A$ is πgb-open. □

5 πgb-continuous and πgb-irresolute functions

Definition 5.1 A function $f:(X,\tau)\to(Y,\sigma)$ is called:

(a) πgb-continuous if $f^{-1}(V)$ is πgb-closed in X for every closed set V of Y;

(b) πgb-irresolute if $f^{-1}(V)$ is πgb-closed in X for every πgb-closed set V of Y.

Remark 5.2 From Definitions 2.3 and 5.1, we obtain the following diagram:

$$\begin{array}{ccccc} & & \pi gs\text{-continuous} & & \\ & \nearrow & & \searrow & \\ \pi g\text{-continuous} & & & & \pi gb\text{-continuous} \\ & \searrow & & \nearrow & \\ & & \pi gp\text{-continuous} & & \end{array}$$

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(a) None of these implications is reversible as shown the following example.

(b) The notions of πgb-continuity and rg-continuity are independent of each other.

(c) Every πgb-irresolute function is πgb-continuous, but not conversely.

(d) The notions of πgb-irresoluteness and gb-irresoluteness are independent of each other.

Example 5.3 (a) Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$ and $\sigma = \{X, \emptyset, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Let $f{:}(X, \tau)\to(X, \sigma)$ be a function defined by $f(a) = c$, $f(b) = a$, $f(c) = d$ and $f(d) = b$. Then f is πgs-continuous (even πgp-continuous) but not πg-continuous, since $\{d\}$ is closed in (X, σ) and $f^{-1}(\{d\})$ is not πg-closed in (X, τ).

(b) Let $X = \{a, b, c, d, e\}$, $\tau = \{X, \emptyset, \{a, b\}, \{c, d\}, \{a, b, c, d\}\}$ and $\sigma = \{X, \emptyset, \{b, d, e\}\}$. Let $f{:}(X, \tau)\to(X, \sigma)$ be the identity. Then f is πgb-continuous but not πgs-continuous, since $\{a, c\}$ is closed in (X, σ) and $f^{-1}(\{a, c\})$ is not πgs-closed in (X, τ).

(c) Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{X, \emptyset, \{a\}, \{a, c\}\}$. Let $f{:}(X, \tau)\to(X, \sigma)$ be the identity. Then f is πgb-continuous but not πgp-continuous, since $\{b\}$ is closed in (X, σ) and $f^{-1}(\{b\})$ is not πgp-closed in (X, τ).

(d) Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{X, \emptyset, \{c\}, \{a, b\}\}$. Let $f{:}(X, \tau)\to(X, \sigma)$ be the identity. Then f is rg-continuous but not πgb-continuous, since $\{a, b\}$ is closed in (X, σ) and $f^{-1}(\{a, b\})$ is not πgb-closed in (X, τ).

(e) Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{X, \emptyset, \{c\}\}$. Let $f{:}(X, \tau)\to(X, \sigma)$ be the identity. Then f is πgb-continuous but not gb-continuous, since $\{a, b\}$ is closed in (X, σ) and $f^{-1}(\{a, b\})$ is not gb-closed in (X, τ).

(f) Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$ and $\sigma = \{X, \emptyset, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Let $f{:}(X, \tau)\to(X, \sigma)$ be the identity. Then f is gb-continuous but not rg-continuous, since $\{d\}$ is closed in (X, σ) and $f^{-1}(\{d\})$ is not rg-closed in (X, τ). Moreover, since X is the only nonempty regular open set in (X, σ), every subset of X is πgb-closed in (X, σ). Hence f is not πgb-irresolute, since $\{a, c\}$ is πgb-closed in (X, σ) and $f^{-1}(\{a, c\})$ is not πgb-closed in (X, τ).

Example 5.4 (a) Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a, b\}\}$ and $\sigma = \{X, \emptyset, \{c\}, \{a, b\}\}$. Let $f:(X, \tau) \to (X, \sigma)$ be the identity. Then f is πgb-irresolute but not gb-irresolute.

(b) Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{X, \emptyset, \{a\}, \{a, b\}\}$. Let $f:(X, \tau) \to (X, \sigma)$ be the identity. Then f is gb-irresolute but not πgb-irresolute.

Theorem 5.5 For a function $f:(X, \tau) \to (Y, \sigma)$, the following hold:

(a) If f is πgb-irresolute and X is πgp-$T_{1/2}$, then f is b-irresolute.

(b) If f is πgb-continuous and X is πgp-$T_{1/2}$, then f is b-continuous.

(c) If f is πgb-continuous and X is an extremally disconnected submaximal α-space, then f is πg-continuous.

Proof (a) Let V be a b-closed subset of Y. Then V is πgb-closed in Y and since f is πgb-irresolute, then $f^{-1}(V)$ is πgb-closed in X. Since X is πgp-$T_{1/2}$, $f^{-1}(V)$ is b-closed in X. Hence f is b-irresolute.

(b) Similar to (a).

(c) Let V be any closed subset of Y. Let $f^{-1}(V) \subset U$, where U is π-open in X. Then $f^{-1}(V)$ is πgb-closed in X. Since X is an extremally disconnected submaximal α-space,

$$\mathrm{cl}(f^{-1}(V)) = \mathrm{scl}(f^{-1}(V)) \cap \mathrm{pcl}(f^{-1}(V)) = \mathrm{bcl}(f^{-1}(V)) \subset U$$

i.e. $f^{-1}(V)$ is πg-closed in X. Hence f is πg-continuous. □

Theorem 5.6 If $f:(X, \tau) \to (Y, \sigma)$ is an almost π-continuous and pre-β-closed function, then $f(A)$ is πgb-closed in Y for every πgb-closed set A of X.

Proof Let A be any πgb-closed set of X. Let $f(A) \subset V$ where V is regular open in Y. Then V is π-open. Since f is an almost π-continuous, $f^{-1}(V)$ is π-open in X and $A \subset f^{-1}(V)$. Then $\mathrm{bcl}(A) \subset f^{-1}(V)$ and hence $f(\mathrm{bcl}(A)) \subset V$. Since f is pre-β-closed, $f(\mathrm{bcl}(A))$ is b-closed in Y and hence

$$\mathrm{bcl}(f(A)) \subset \mathrm{bcl}(f(\mathrm{bcl}(A))) = f(\mathrm{bcl}(A)) \subset V.$$

This shows that $f(A)$ is πgb-closed in Y. □

Theorem 5.7 Let f:$(X,\tau)\rightarrow(Y,\sigma)$ be a πgb-continuous function and H be a π-open πgb-closed subset of X. Assumed that every finite intersection of πgb-closed sets is closed. Then the restriction $f|H$:$(H,\tau|H)\rightarrow(Y,\sigma)$ is πgb-continuous.

Proof Let F be a closed subset of Y. By hypothesis and our assumption, $f^{-1}(F)\cap H = H_1$ (say) is πgb-closed in X. Since $(f|H)^{-1}(F) = H_1$, it is sufficient to show that H_1 is πgb-closed in H. Let G_1 be any π-open set of H such that $H_1 \subset G_1$. By Lemma 3.12, $G_1 = G \cap H$ for some π-open set of G in X. Since $H_1 \subset G$ and H_1 is πgb-closed in X, then $\mathrm{bcl}_X(H_1) \subset G$. By Theorem 2.13 (a), $\mathrm{bcl}_H(H_1) = \mathrm{bcl}_X(H_1) \cap H \subset G \cap H = G_1$ and so H_1 is πgb-closed in H. Hence $f|H$ is πgb-continuous. □

Theorem 5.8 Let $X = G \cup H$ be a topological space with topology τ and Y be a topological space with topology σ. Let every finite union of πgb-closed sets is closed and let f:$(G,\tau|G)\rightarrow(Y,\sigma)$ and g:$(H,\tau|H)\rightarrow(Y,\sigma)$ be πgb-continuous functions such that $f(x) = g(x)$ for every $x \in G \cap H$. Suppose that both G and H are regular open and πgb-closed in X. Then their combination $f \bigtriangledown g$:$(X,\tau)\rightarrow(X,\sigma)$ defined by $(f \bigtriangledown g)(x) = f(x)$ if $x \in G$ and $(f \bigtriangledown g)(x) = g(x)$ if $x \in H$ is πgb-continuous.

Proof Let F be any closed subset of Y. Clearly $(f \bigtriangledown g)^{-1}(F) = f^{-1}(F) \cup g^{-1}(F)$. Since $f^{-1}(F)$ is πgb-closed in G and since G is regular open πgb-closed in X, by Theorem 3.13 (b), $f^{-1}(F)$ is πgb-closed in X. Similarly, $g^{-1}(F)$ is πgb-closed in X. Then by hypothesis $(f \bigtriangledown g)^{-1}(F) = f^{-1}(F) \cup g^{-1}(F)$ is πgb-closed in X. Hence $f \bigtriangledown g$ is πgb-continuous. □

The composition of two πgb-continuous function need not be πgb-continuous. For consider the following example:

Example 5.9 Let $X = \{a,b,c,d\}$, $\tau = \{X,\emptyset,\{a\},\{b,c\},\{a,b,c\}\}$, $\sigma = \{X,\emptyset,\{a\},\{b\},\{a,b\}\}$ and $\rho = \{X,\emptyset,\{c,d\}\}$. Let f:$(X,\tau)\rightarrow(X,\sigma)$ be the identity function and g:$(X,\sigma)\rightarrow(X,\rho)$ be a function defined by $g(a) = a$, $g(b) = g(c) = b$ and

$g(d) = d$. Then f and g are πgb-continuous but the composition $g \circ f$ is not πgb-continuous. Since $\{a, b\}$ is closed in (X, ρ) and $(g \circ f)^{-1}(\{a, b\})$ is not πgb-closed in (X, τ).

However, the following theorem holds:

Theorem 5.10 Let $f{:}(X, \tau) \to (Y, \sigma)$ and $g{:}(Y, \sigma) \to (Z, \rho)$ be two functions. Then:

(a) If f is πgb-continuous and g is continuous, then $g \circ f$ is πgb-continuous.

(b) If f is πgb-irresolute and g is πgb-irresolute, then $g \circ f$ is πgb-irresolute.

(c) If f is πgb-irresolute and g is πgb-continuous, then $g \circ f$ is πgb-continuous.

(d) Let Y be a πgp-$T_{1/2}$ space. If f is irresolute and g is πgb-continuous, then $g \circ f$ is b-continuous.

Proof Obvious. □

Theorem 5.11 Let $f{:}(X, \tau) \to (Y, \sigma)$ be a πgb-irresolute and pre-β-closed surjection. If X is a πgp-$T_{1/2}$ space, then Y is also πgp-$T_{1/2}$.

Proof Let A be a πgb-closed subset of Y. Since f is πgb-irresolute, then $f(A)$ is πgb-closed in X. Since X is πgp-$T_{1/2}$, then $f^{-1}(A)$ is b-closed in X. By the rest of the assumption it follows that A is b-closed in Y. Hence Y is πgp-$T_{1/2}$. □

Definition 5.12 A function $f{:}(X, \tau) \to (Y, \sigma)$ is called pre-b-open if $f(V)$ is b-open in Y for every b-open set V of X.

Theorem 5.13 Let $f{:}(X, \tau) \to (Y, \sigma)$ be a rc-preserving and pre-b-open bijection. If X is a πgp-$T_{1/2}$ space, then Y is also πgp-$T_{1/2}$.

Proof Let $y \in Y$. Since X is πgp-$T_{1/2}$ and f is bijective, then by Theorem 2.5 for some $x \in X$ with $f(x) = y$, we have $\{x\}$ is π-closed or b-open. If $\{x\}$ is π-closed, then $\{y\} = f(\{x\})$ is π-closed, since f is rc-preserving and injective. If $\{x\}$ is b-open, then $\{y\}$ is b-open, since f is pre-b-open. Hence by Theorem 2.5, Y is πgp-$T_{1/2}$. □

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