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Duality for Nonlinear Multiobjective Programming with Cone Constraints

by

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Graduate School of Education

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Duality for Nonlinear Multiobjective Programming with Cone Constraints

추 제약식을 갖는 비선형 다목적 계획문제에 관한 쌍대성

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CONTENTS

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요 약

본 논문에서는 추 제약식을 가지는 다목적 계획문제의 Wolfe형과 Mond-Weir형의 쌍대문제에 관련된 네 가지 쌍대문제를 정식화하고 일반화된 인벡스티 함수 조건아래에서 약 유효해에 관한 약 쌍대관계를 정립하였다. 그리고 약 유효해에 관한 필요최적조건을 이용하여 강쌍대정리를 증명하였다. 아울러 이러한 쌍대결과로부터 얻을 수 있는 기존의 알려진 결과를 특수한 경우로서 제시하였다.

1 Introduction and Preliminaries

While studying duality under generalized convexity, Mond and Weir [15] proposed a number of different duals for various nonlinear programming problems with nonnegative variable and proved various duality theorems under appropriate pseudo-convexity and quasi-convexity assumptions. The notion of symmetric duality was developed significantly by Dantzig et al. [4], Chandra and Husain [3] and Mond and Weir [16]. Dantzig et al. [4] established symmetric duality results for convex/concave functions with non-negative orthant as the cone. Later Mond and Weir [16] , Weir and Mond [20] as well as Gulati [5] et al. generalized single objective symmetric duality to multiobjective case motivated by Hanson [6] and Hanson and Mond [7]. Nanda and Das [18] formulated a pair of symmetric dual nonlinear programming problems for pseudo-invex functions and arbitrary cones. Nanda [17] also studied symmetric dual problems assuming the functions to be invex with non-negative orthant as the cone. Kim et al. [10] formulated a pair of multiobjective symmetric dual programs for pseudo-invex functions and arbitrary cones and established duality results. Mishra [11] formulated a pair of multiobjective second order symmetric dual nonlinear programming problems under second order pseudo-invexity assumptions on the functions involved over arbitrary cones and established duality results. The model given by Mishra [12] unifies the Wolfe and Mond-Weir type second order vector symmetric dual models. Furthermore, several second order duality and self-duality theorems was also established for the pair of dual models. Recently Khurana [8] introduced cone-pseudo-invex and strongly cone-pseudo-invex functions and established

duality theorems for a pair of Mond-Weir type multiobjective symmetric dual over arbitrary cones. Suneja and Aggarwal and Davar [19] formulated a fair of symmetric dual programs over arbitrary cones and establish weak, strong, converse and self duality theorems by using cone-convexity and the objective function was optimized with respect to an arbitrary closed convex cone by assuming the function involved th be cone-convex. Kim and Song [9] also presented two pairs of nonlinear multiobjective mixed integer programs for the polars of arbitrary cones, and established the weak, strong and converse duality theorems by using the concept of efficiency.
In this dissertation, we formulated four dual problems related with Wolfe and Mond-Weir of integrated programming with a cone constraint and established dual relations about under the general invexity assumptions. By using a necessary optimality condition about solutions, we demonstrated the strong duality. Furthermore we delivered existing known results as specific cases from these duality results .

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and let $\mathbb{R}^n_+$ be its non-negative orthant. We denote the interior of $\mathbb{R}^n_+$ by int $\mathbb{R}^n_+$.

The following convention for inequalities will be used in this paper :
If $x, u \in \mathbb{R}^n$, then

$$x \leqq u \iff u - x \in \mathbb{R}^n_+ \ ;$$

$$x \leq u \iff u - x \in \mathbb{R}^n_+ \setminus \{0\} \ ;$$

$$x < u \iff u - x \in \text{int } \mathbb{R}^n_+ \ ;$$

$$x \nleq u \text{ is the negation of } x \leq u \ .$$

For $x,\ u \in \mathbb{R}$, $x \leqq u$ and $x < u$ have the usual meaning.

Definition 1.1 *A nonempty set C in $\mathbb{R}^n$ is said to be a cone with vertex zero, if $x \in C$ implies that $\lambda x \in C$ for all $\lambda \geqq 0$. If, in addition, C is convex, then C is called a convex cone.*

Consider the following multiobjective programming problem :

$$\begin{aligned} (\mathbf{MP}) \quad \text{Minimize} \quad & f(x) \\ \text{subject to} \quad & g(x) \in C_2^*, \quad x \in C_1, \end{aligned}$$

where $f : S \to \mathbb{R}^k,\ g : S \to \mathbb{R}^m$ and $C_1 \subseteq S,\ S \subseteq \mathbb{R}^n$ is open. C_2^* is a polar cone of $C_2 \subseteq \mathbb{R}^m$.

Definition 1.2 *A feasible point $\overline{x}$ is a weakly efficient solution of* (**MP**)*, if there exists no other $x \in X$ such that $f(\overline{x}) - f(x) > 0$.*

Definition 1.3 *The polar cone C^* of C is defined by*

$$C^* = \{z \in \mathbb{R}^n \quad | \quad x^T z \leqq 0 \quad \textit{for all} \quad x \in C\}.$$

Definition 1.4 *[2] Let $S \subseteq \mathbb{R}^n$ be open and $f : S \to \mathbb{R}$.*

(i)f is said to be pseudo-invex with respect to η on S, where η is a function from $S \times S$ to $\mathbb{R}^n$, if for all $x, u \in S$,

$$\eta^T(x,u)\nabla f(u) \geqq 0 \Rightarrow f(x) \geqq f(u).$$

(ii)f is said to be quasi-invex with respect to η on S, where η is a function from $S \times S$ to $\mathbb{R}^n$, if for all $x, u \in S$,

$$f(x) \leqq f(u) \Rightarrow \eta^T(x,u)\nabla f(u) \leqq 0.$$

2 Duality

We propose the following multiobjective dual problem to the primal problem (**MP**) :

$$
\begin{aligned}
(\mathbf{MD})_{\mathbf{1}} \quad \text{Maximize} \quad & f(u) + y^T g(u)e - u^T[\nabla(\lambda^T f)(u) + \nabla y^T g(u)]e \\
\text{subject to} \quad & -[\nabla(\lambda^T f)(u) + \nabla y^T g(u)] \in C_1^*, \\
& y \in C_2, \quad \lambda \geq 0,
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{MD})_{\mathbf{2}} \quad \text{Maximize} \quad & f(u) \\
\text{subject to} \quad & -[\nabla(\lambda^T f)(u) + \nabla y^T g(u)] \in C_1^*, \\
& y^T g(u) - u^T[\nabla(\lambda^T f)(u) + \nabla y^T g(u)] \geqq 0, \\
& y \in C_2, \quad \lambda \geq 0,
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{MD})_{\mathbf{3}} \quad \text{Maximize} \quad & f(u) - u^T[\nabla(\lambda^T f)(u) + \nabla y^T g(u)]e \\
\text{subject to} \quad & -[\nabla(\lambda^T f)(u) + \nabla y^T g(u)] \in C_1^*, \\
& -g(u) \in C_2^*, \quad y \in C_2, \quad \lambda \geq 0,
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{MD})_{\mathbf{4}} \quad \text{Maximize} \quad & f(u) + y^T g(u)e \\
\text{subject to} \quad & -[\nabla(\lambda^T f)(u) + \nabla y^T g(u)] \in C_1^*, \\
& u^T[\nabla(\lambda^T f)(u) + \nabla y^T g(u)] \leqq 0, \\
& y \in C_2, \quad \lambda \geq 0,
\end{aligned}
$$

where

(1) C_1 and C_2 are closed convex cones in $\mathbb{R}^n$ and $\mathbb{R}^m$ with nonempty interiors, respectively,

(2) $S \subseteq \mathbb{R}^n$ is open and $C_1 \subseteq S$,

(3) $f : S \to \mathbb{R}^k$ and $g : S \to \mathbb{R}^m$ are twice differentiable functions,

(4) C_1^* and C_2^* are polar cones of C_1 and C_2, respectively,

(5) λ and $e = (1, \cdots, 1)$ are vectors in $\mathbb{R}^k$.

Further let $\nabla f(\overline{x})$ and $\nabla^2 f(\overline{x})$, respectively denote the gradient and the Hessian matrix of f evaluated at $\overline{x}$. The symbols $\nabla g_i(\overline{x})$ and $\nabla^2 g_i(\overline{x})(i = 1, 2, \cdots, m)$ are defined similarly.

Now we establish the duality theorems for $(\mathbf{MP})$ and $(\mathbf{MD})_\mathbf{1} - (\mathbf{MD})_\mathbf{4}$.

Theorem 2.1 (Weak Duality) *Let x be feasible solution of* $(\mathbf{MP})$ *and* (x, λ, y) *be feasible for* $(\mathbf{MD})_\mathbf{1}$. *Let for all $v \in C_1^*$, $\lambda^T f + y^T g + v^T(\cdot)$ be pseudo-invex with respect to η. Then*

$$f(x) \not< f(u) + y^T g(u)e - u^T[\nabla(\lambda^T f)(u) + \nabla y^T g(u)]e.$$

Proof. Assume that

$$f(x) < f(u) + y^T g(u)e - u^T[\nabla(\lambda^T f)(u) + \nabla y^T g(u)]e.$$

Multiplying which by $\lambda \geq 0$,

$$(\lambda^T f)(x) < (\lambda^T f)(u) + y^T g(u) - u^T[\nabla(\lambda^T f)(u) + \nabla y^T g(u)]. \qquad (2.1)$$

From the first dual constraint $-[\nabla(\lambda^T f)(u)+\nabla y^T g(u)] \in C_1^*$ and there exist $v \in C_1$ such that

$$v = -[\nabla(\lambda^T f)(u) + \nabla y^T g(u)]. \tag{2.2}$$

Multiplying (2.2) by $\eta(x,u)$,

$$\eta(x,u)^T[\nabla(\lambda^T f)(u) + \nabla y^T g(u) + v] = 0.$$

By the pseudo-invexity of $\lambda^T f + y^T g + v^T(\cdot)$, it implies that

$$(\lambda^T f)(x) + y^T g(x) + v^T x \geqq (\lambda^T f)(u) + y^T g(u) + v^T u. \tag{2.3}$$

By $y \in C_2$ and $g(x) \in C_2^*$,

$$y^T g(x) \leqq 0. \tag{2.4}$$

By $v \in C_1^*$ and $x \in C_1$,

$$v^T x \leqq 0. \tag{2.5}$$

Using (2.4) and (2.5) in (2.3), we obtain

$$(\lambda^T f)(x) \geqq (\lambda^T f)(u) + y^T g(u) + v^T u,$$

i.e.,

$$(\lambda^T f)(u) + y^T g(u) - u^T[\nabla(\lambda^T f)(u) + \nabla y^T g(u)] - (\lambda^T f)(x) \leqq 0,$$

which is a contradiction to the inequality (2.1).
Therefore

$$f(x) \nleq f(u) + y^T g(u)e - u^T[\nabla((\lambda^T f))(u) + \nabla y^T g(u)]e.$$

□

Theorem 2.2 (Weak Duality) *Let x be feasible solution of* (**MP**) *and* (x, λ, y) *be feasible for* $(\mathbf{MD})_{\mathbf{2}}$. *Let f be pseudo-invex with respect to η for all $v \in C_1^*$, $\lambda^T f + y^T g + v^T(\cdot)$ be a quasi-invex with respect to same η. Then*

$$f(x) \nless f(u).$$

Proof. Assume that

$$f(x) < f(u).$$

Multiplying which by $\lambda \geq 0$,

$$(\lambda^T f)(x) < (\lambda^T f)(u). \tag{2.6}$$

From the first dual constraint,

$$-[\nabla(\lambda^T f)(u) + \nabla y^T g(u)] \in C_1^*.$$

And there exist $v \in C_1^*$ such that

$$v = -[\nabla(\lambda^T f)(u) + \nabla y^T g(u)]. \tag{2.7}$$

Multiplying (2.7) by $\eta(x, u)$

$$\eta(x, u)^T [\nabla(\lambda^T f)(u) + \nabla y^T g(u) + v] = 0.$$

By the quasi-invexity of $\lambda^T f + y^T g + v^T(\cdot)$, it implies that

$$(\lambda^T f)(x) + y^T g(x) + v^T x \geqq (\lambda^T f)(u) + y^T g(u) + v^T u. \tag{2.8}$$

From the primal and dual constraints, we have

$$y^T g(x) \leqq 0 \quad \text{and} \quad v^T x \leqq 0. \tag{2.9}$$

Using (2.9) in (2.8), we get

$$(\lambda^T f)(x) \geqq (\lambda^T f)(u),$$

which is a contradiction to the inequality (2.6). Therefore

$$f(x) \not< f(u).$$

□

Theorem 2.3 (Weak Duality) *Let x be feasible solution of* (**MP**) *and* (x, λ, y) *be feasible for* $(\mathbf{MD})_\mathbf{3}$. *Let for all $v \in C_1^*$, $\lambda^T f + v^T(\cdot)$ be pseudo-invex with respect to η and $y^T g$ be quasi-invex with respect to same η. Then*

$$f(x) \not< f(u) - \ u^T[\nabla(\lambda^T f)(u) + \nabla y^T g(u)]e.$$

Proof. Assume that

$$f(x) < f(u) - u^T[\nabla(\lambda^T f)(u) + \nabla y^T g(u)]e.$$

Multiplying which by $\lambda \geq 0$,

$$(\lambda^T f)(x) < (\lambda^T f)(u) - u^T[\nabla(\lambda^T f)(u) + \nabla y^T g(u)]. \tag{2.10}$$

From the primal and dual constraints, we get

$$y^T g(x) \leqq y^T g(u).$$

By the quasi-invexity of $y^T g$ with respect to η,

$$\eta(x, u)^T \nabla y^T g(u) \leqq 0.$$

From the first dual constraint,

$$-[\nabla(\lambda^T f)(u) + \nabla y^T g(u)] \in C_1^*.$$

And there exist $v \in C_1^*$ such that

$$v = -[\nabla(\lambda^T f)(u) + \nabla y^T g(u)].$$

So,

$$\begin{aligned}\eta(x,u)^T \nabla y^T g(u) &= \eta(x,u)^T[-v - \nabla(\lambda^T f)(u)] \\ &= -\eta(x,u)^T[\nabla(\lambda^T f)(u) + v] \\ &\leqq 0,\end{aligned}$$

i.e.,

$$\eta(x,u)^T[\nabla(\lambda^T f)(u) + v] \geqq 0.$$

By the pseudo-invexity of $\lambda^T f + v^T(\cdot)$ with respect to η,

$$(\lambda^T f)(x) + v^T x - (\lambda^T f)(u) - v^T u \geqq 0.$$

Since $v = -[\nabla(\lambda^T f)(u) + \nabla y^T g(u)] \in C_1^*$ and $x \in C_1$,

$$v^T x \leqq 0.$$

Hence

$$(\lambda^T f)(x) \geqq (\lambda^T f)(u) - \ u^T[\nabla(\lambda^T f)(u) + \nabla y^T g(u)],$$

which is a contradiction to the inequality (2.10).

Therefore

$$f(x) \nleq f(u) - u^T[\nabla(\lambda^T f)(u) + \nabla y^T g(u)]e.$$

□

Theorem 2.4 (Weak Duality) *Let x be feasible solution of* $(\mathbf{MP})$ *and* (x, λ, y) *be feasible for* $(\mathbf{MD})_{\mathbf{4}}$. *Let $\lambda^T f + y^T g$ be pseudo-invex with respect to η where η satisfies the condition $(\eta(x,u) + u) \in C_1$.*
Then

$$f(x) \nless f(u) + y^T g(u)e.$$

Proof. Assume that

$$f(x) < f(u) + y^T g(u)e.$$

Multiplying which by $\lambda \geq 0$,

$$(\lambda^T f)(x) < (\lambda^T f)(u) + y^T g(u). \tag{2.11}$$

From the constraint $-[\nabla(\lambda^T f)(u) + \nabla y^T g(u)] \in C_1^*$ and $(\eta(x,u) + u) \in C_1$,

$$(\eta(x,u) + u)^T [\nabla(\lambda^T f)(u) + \nabla y^T g(u)] \geqq 0.$$

By the dual constraint, the above inequality establishes

$$\eta(x,u)^T [\nabla(\lambda^T f)(u) + \nabla y^T g(u)] \geqq 0.$$

By the pseudo-invexity of $\lambda^T f + y^T g$ with respect to η,

$$(\lambda^T f)(x) + y^T g(x) \geqq (\lambda^T f)(u) + y^T g(u).$$

Since $y^T g(x) \leqq 0$,

$$(\lambda^T f)(x) \geqq (\lambda^T f)(u) + y^T g(u).$$

This is a contradiction to the inequality (2.11). Therefore

$$f(x) \nless f(u) + y^T g(u)e.$$

□

Lemma 2.1 *From the [1], if* $\overline{x}$ *is a weakly efficient solution of* (**MP**)*, then there exist* $\overline{\lambda} \geq 0$ *and* $\overline{y} \in C_2$ *not both zero such that*

$$[\nabla \overline{\lambda}^T f(\overline{x}) + \nabla \overline{y}^T g(\overline{x})]^T (x - \overline{x}) \geqq 0 \quad \text{for all} \quad x \in C_1,$$

$$\overline{y}^T g(\overline{x}) = 0.$$

Theorem 2.5 (Strong Duality) *Let* $\overline{x}$ *be a weakly efficient solution for* (**MP**) *at which constraint qualification be satisfied. Fix* $\lambda = \overline{\lambda}$ *in* $(\mathbf{MD})_\mathbf{1}$*. Then there exist* $\overline{\lambda} \geq 0$ *and* $\overline{y} \in C_2$ *such that* $(\overline{x}, \overline{\lambda}, \overline{y})$ *is feasible for* $(\mathbf{MD})_\mathbf{1}$ *and the objective values of* (**MP**) *and* $(\mathbf{MD})_\mathbf{1}$ *are equal. Furthermore, if the hypothesis of Theorem 2.1 is also satisfied, then* $(\overline{x}, \overline{\lambda}, \overline{y})$ *is a weakly efficient solution for* $(\mathbf{MD})_\mathbf{1}$*.*

Proof. Since $\overline{x}$ is a weakly efficient solution for (**MP**) at which constraint qualification be satisfied, by Lemma 2.1, there exist $\overline{\lambda} \geq 0$ and $\overline{y} \in C_2$ with $(\overline{\lambda}, \overline{y}) \neq 0$ such that

$$\overline{y}^T g(\overline{x}) = 0$$

and

$$[\nabla \overline{\lambda}^T f(\overline{x}) + \nabla \overline{y}^T g(\overline{x})]^T (x - \overline{x}) \geqq 0 \quad \text{for all} \quad x \in C_1. \tag{2.12}$$

Since $x \in C_1, \overline{x} \in C_1$ and C_1 is a closed convex cone, we have $x + \overline{x} \in C_1$ and thus the inequality (2.12) implies

$$[\nabla \overline{\lambda}^T f(\overline{x}) + \nabla \overline{y}^T g(\overline{x})]^T x \geqq 0 \quad \text{for all} \quad x \in C_1,$$

i.e.,

$$-[\nabla\overline{\lambda}^T f(\overline{x}) + \nabla\overline{y}^T g(\overline{x})] \in C_1^*.$$

Hence $(\overline{x}, \overline{\lambda}, \overline{y})$ is feasible for $(\mathbf{MD})_\mathbf{1}$.

Moreover, by letting $x = 0$ and x=$2\overline{x}$ in (2.12), we obtain

$$[\nabla\overline{\lambda}^T f(\overline{x}) + \nabla\overline{y}^T g(\overline{x})] = 0. \tag{2.13}$$

From $\overline{y}^T g(\overline{x}) = 0$ and (2.13)

$$f(\overline{x}) = f(\overline{x}) + \overline{y}^T g(\overline{x})e - \overline{x}^T[\nabla\overline{\lambda}^T f(\overline{x}) + \nabla\overline{y}^T g(\overline{x})]e.$$

Thus the objective values of (**MP**) and $(\mathbf{MD})_\mathbf{1}$ are equal.

We will now show that $(\overline{x}, \overline{\lambda}, \overline{y})$ is a weakly efficient solution for $(\mathbf{MD})_\mathbf{1}$, otherwise, there exists a feasible solution $(u, \overline{\lambda}, y)$ for $(\mathbf{MD})_\mathbf{1}$ such that

$$\begin{aligned}&[f(u) + y^T g(u)e - u^T[\nabla\overline{\lambda}^T f(u) + \nabla y^T g(u)]e]\\ &-[f(\overline{x}) + \overline{y}^T g(\overline{x})e - \overline{x}^T[\nabla\overline{\lambda}^T f(\overline{x}) + \nabla\overline{y}^T g(\overline{x})]e] > 0.\end{aligned}$$

Since the objective values of (**MP**) and $(\mathbf{MD})_\mathbf{1}$ are equal, it follows that

$$[f(u) + y^T g(u)e - u^T[\nabla\overline{\lambda}^T f(u) + \nabla y^T g(u)]e] - f(\overline{x}) > 0,$$

which contradicts weak duality. Hence the results hold. □

Theorem 2.6 (Strong Duality) *Let $\overline{x}$ be a weakly efficient solution for* (**MP**) *at which constraint qualification be satisfied. Fix $\lambda = \overline{\lambda}$ in* $(\mathbf{MD})_\mathbf{2}$.

Then there exist $\overline{\lambda} \geq 0$ and $\overline{y} \in C_2$ such that $(\overline{x}, \overline{\lambda}, \overline{y})$ is feasible for $(\mathbf{MD})_\mathbf{2}$ *and the objective values of* $(\mathbf{MP})$ *and* $(\mathbf{MD})_\mathbf{2}$ *are equal. Furthermore, if the hypotheses of Theorem 2.2 are also satisfied, then $(\overline{x}, \overline{\lambda}, \overline{y})$ is a weakly efficient solution for* $(\mathbf{MD})_\mathbf{2}$.

Proof. Since $\overline{x}$ is a weakly efficient solution for $(\mathbf{MP})$ at which constraint qualification be satisfied, by Lemma 2.1, there exist $\overline{\lambda} \geq 0$ and $\overline{y} \in C_2$ with $(\overline{\lambda}, \overline{y}) \neq 0$ such that

$$\overline{y}^T g(\overline{x}) = 0$$

and

$$[\nabla \overline{\lambda}^T f(\overline{x}) + \nabla \overline{y}^T g(\overline{x})]^T (x - \overline{x}) \geqq 0 \quad \text{for all} \quad x \in C_1. \tag{2.14}$$

Since $x \in C_1, \overline{x} \in C_1$ and C_1 is a closed convex cone, we have $x + \overline{x} \in C_1$ and thus the inequality (2.14) implies

$$[\nabla \overline{\lambda}^T f(\overline{x}) + \nabla \overline{y}^T g(\overline{x})]^T x \geqq 0 \quad \text{for all} \quad x \in C_1,$$

i.e.,

$$-[\nabla \overline{\lambda}^T f(\overline{x}) + \nabla \overline{y}^T g(\overline{x})] \in C_1^*.$$

By letting $x = 0$ in (2.14), we obtain

$$-[\nabla \overline{\lambda}^T f(\overline{x}) + \nabla \overline{y}^T g(\overline{x})]^T \overline{x} \geqq 0 \quad \text{for all} \quad x \in C_1.$$

From above the inequality and $\overline{y}^T g(\overline{x}) = 0$, we get

$$\overline{y}^T g(\overline{x}) - [\nabla \overline{\lambda}^T f(\overline{x}) + \nabla \overline{y}^T g(\overline{x})]^T \overline{x} \geqq 0.$$

So $(\overline{x}, \overline{\lambda}, \overline{y})$ is feasible for $(\mathbf{MD})_\mathbf{2}$. Thus $f(\overline{x}) = f(\overline{x})$.

We will now show that $(\overline{x}, \overline{\lambda}, \overline{y})$ is a weakly efficient solution for $(\mathbf{MD})_\mathbf{2}$, otherwise, there exists a feasible solution $(u, \overline{\lambda}, y)$ for $(\mathbf{MD})_\mathbf{2}$ such that

$$f(u) - f(\overline{x}) > 0.$$

Since the objective values of $(\mathbf{MP})$ and $(\mathbf{MD})_\mathbf{2}$ are equal, it follows that

$$f(u) - f(\overline{x}) > 0,$$

which contradicts weak duality. Hence the results hold. □

Theorem 2.7 (Strong Duality) *Let $\overline{x}$ be a weakly efficient solution for* $(\mathbf{MP})$ *at which constraint qualification be satisfied. Fix $\lambda = \overline{\lambda}$ in* $(\mathbf{MD})_\mathbf{3}$. *Then there exist $\overline{\lambda} \geq 0$ and $\overline{y} \in C_2$ such that $(\overline{x}, \overline{\lambda}, \overline{y})$ is feasible for* $(\mathbf{MD_3})$ *and the objective values of* $(\mathbf{MP})$ *and* $(\mathbf{MD})_\mathbf{3}$ *are equal. Furthermore, if the hypotheses of Theorem 2.3 are also satisfied, then $(\overline{x}, \overline{\lambda}, \overline{y})$ is a weakly efficient solution for* $(\mathbf{MD})_\mathbf{3}$.

Proof. Since $\overline{x}$ is a weakly efficient solution for $(\mathbf{MP})$ at which constraint qualification be satisfied, by Lemma 2.1, there exist $\overline{\lambda} \geq 0$ and $\overline{y} \in C_2$ with $(\overline{\lambda}, \overline{y}) \neq 0$ such that

$$\overline{y}^T g(\overline{x}) = 0$$

and

$$[\nabla \overline{\lambda}^T f(\overline{x}) + \nabla \overline{y}^T g(\overline{x})]^T (x - \overline{x}) \geqq 0 \quad \text{for all} \quad x \in C_1. \tag{2.15}$$

Since $x \in C_1, \overline{x} \in C_1$ and C_1 is a closed convex cone, we have $x + \overline{x} \in C_1$ and thus the inequality (2.15) implies

$$[\nabla \overline{\lambda}^T f(\overline{x}) + \nabla \overline{y}^T g(\overline{x})]^T x \geqq 0 \quad \text{for all} \quad x \in C_1,$$

i.e.,

$$-[\nabla \overline{\lambda}^T f(\overline{x}) + \nabla \overline{y}^T g(\overline{x})] \in C_1^*.$$

Since $\overline{y}^T g(\overline{x}) = 0$, we have $\overline{y}^T g(\overline{x}) \leqq 0$ and $\overline{y}^T g(\overline{x}) \geqq 0$. From $\overline{y}^T g(\overline{x}) \geqq 0$ and $\overline{y} \in C_2$, we get $-g(\overline{x}) \in C_2^*$. So, $(\overline{x}, \overline{\lambda}, \overline{y})$ is feasible for $(\mathbf{MD})_\mathbf{3}$.

Also, by letting $x = 0$ and x=$2\overline{x}$ in (2.15), we obtain

$$[\nabla \overline{\lambda}^T f(\overline{x}) + \nabla \overline{y}^T g(\overline{x})] = 0.$$

Therefore

$$f(\overline{x}) = f(\overline{x}) - \overline{x}^T [\nabla \overline{\lambda}^T f(\overline{x}) + \nabla \overline{y}^T g(\overline{x})] e.$$

We will now show that $(\overline{x}, \overline{\lambda}, \overline{y})$ is a weakly efficient solution for $(\mathbf{MD})_\mathbf{3}$, otherwise, there exists a feasible solution $(u, \overline{\lambda}, y)$ for $(\mathbf{MD})_\mathbf{3}$ such that

$$[f(u) - u^T [\nabla \overline{\lambda}^T f(u) + \nabla y^T g(u)] e] - [f(\overline{x}) - \overline{x}^T [\nabla \overline{\lambda}^T f(\overline{x}) + \nabla \overline{y}^T g(\overline{x})] e] > 0.$$

Since the objective values of $(\mathbf{MP})$ and $(\mathbf{MD})_\mathbf{3}$ are equal, it follows that

$$[f(u) - u^T [\nabla \overline{\lambda}^T f(u) + \nabla y^T g(u)] e] - f(\overline{x}) > 0,$$

which contradicts weak duality. Hence the results hold. □

Theorem 2.8 (Strong Duality) *Let $\overline{x}$ be a weakly efficient solution for* (**MP**) *at which constraint qualification be satisfied. Fix $\lambda = \overline{\lambda}$ in* $(\mathbf{MD})_\mathbf{4}$. *Then there exist $\overline{\lambda} \geq 0$ and $\overline{y} \in C_2$ such that $(\overline{x}, \overline{\lambda}, \overline{y})$ is feasible for* $(\mathbf{MD})_\mathbf{4}$ *and the objective values of* (**MP**) *and* $(\mathbf{MD})_\mathbf{4}$ *are equal. Furthermore, if the hypothesis of Theorem 2.4 is also satisfied, then $(\overline{x}, \overline{\lambda}, \overline{y})$ is a weakly efficient solution for* $(\mathbf{MD})_\mathbf{4}$.

Proof. Since $\overline{x}$ is a weakly efficient solution for (**MP**) at which constraint qualification be satisfied, by Lemma 2.1, there exist$\overline{\lambda} \geq 0$ and $\overline{y} \in C_2$ with $(\overline{\lambda}, \overline{y}) \neq 0$ such that

$$\overline{y}^T g(\overline{x}) = 0$$

and

$$[\nabla \overline{\lambda}^T f(\overline{x}) + \nabla \overline{y}^T g(\overline{x})]^T (x - \overline{x}) \geqq 0 \quad \text{for all} \quad x \in C_1. \tag{2.16}$$

And by letting $x = 0$ in (2.16), we obtain

$$[\nabla \overline{\lambda}^T f(\overline{x}) + \nabla \overline{y}^T g(\overline{x})]^T \overline{x} \leqq 0 \quad \text{for all} \quad x \in C_1,$$

i.e.,

$$-[\overline{\lambda}^T f(\overline{x}) + \nabla \overline{y}^T \nabla g(\overline{x})] \in C_1^*.$$

Hence $(\overline{x}, \overline{\lambda}, \overline{y})$ is feasible for $(\mathbf{MD})_\mathbf{4}$.

Moreover, from $\overline{y}^T g(\overline{x}) = 0$,

$$f(\overline{x}) = f(\overline{x}) + \overline{y}^T g(\overline{x}) e.$$

Thus the objective values of (**MP**) and $(\mathbf{MD})_\mathbf{4}$ are equal.

We will now show that $(\overline{x}, \overline{\lambda}, \overline{y})$ is a weakly efficient solution for $(\mathbf{MD})_\mathbf{4}$, otherwise, there exists a feasible solution $(u, \overline{\lambda}, y)$ for $(\mathbf{MD})_\mathbf{4}$ such that

$$[f(u) + y^T g(u)e] - [f(\overline{x}) + \overline{y}^T g(\overline{x})e] > 0.$$

Since the objective values of (**MP**) and $(\mathbf{MD})_\mathbf{4}$ are equal, it follows that

$$[f(u) + y^T g(u)e] - f(\overline{x}) > 0,$$

which contradicts weak duality. Hence the results hold. □

3 Special Cases

We give some special cases of our dual programming.

(1) If $k = 1$, then (**MP**) and $(\mathbf{MD})_\mathbf{1}$-$(\mathbf{MD})_\mathbf{4}$ are reduced to programs studied in S. Chandra, Abha [2].

(2) If $k = 1, \eta(x, u) \in C_1$ then (**MP**) and $(\mathbf{MD})_\mathbf{1}$-$(\mathbf{MD})_\mathbf{4}$ are reduced to programs studied in S. Nanda and L.N. Das. [18].

(3) If $k = 1,\ C_1 = R^n_+$ and $C_2 = R^m_+$, then (**MP**) and $(\mathbf{MD})_\mathbf{1}$-$(\mathbf{MD})_\mathbf{4}$ are reduced to programs considered in B. Mond and T. Weir. [16].

Remark 3.1 *If the replace $u \in R^n$ by $u \in C_1$,*

(i) Theorem 2.1 and 2.5 hold under the pseudo-invexity of $f + y^T g$,

(ii) Theorem 2.2 and 2.6 hold under the pseudo-invexity of $y^T g$,

(iii) Theorem 2.3 and 2.7 hold under the pseudo-invexity of f,

(iv) Theorem 2.4 and 2.8 hold under the pseudo-invexity of $f + y^t g$,

then the same conclusion of Theorem 2.1 and 2.8 also holds.

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