Thesis for the Degree of Doctor of Philosophy

Properties of Group Constructions Defined by Using Graphs

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그래프를 이용하여 정의된 군 구성의 성질에 대한 연구

Advisor: Prof. Young Gheel Baik

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Defined by Using Graphs

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서 유 덕

부경대학교 대학원 응용수학과

이 논문은 주어진 군들로부터 새로운 군을 만드는 방법(group construction) 중에서 군의 그래프(graphs of groups)와 군 곱(graph product)의 성질을 연구한 것이다. 이 두 개념은 논문에서 설명된 것처럼 이미 널리 연구되어 온 많은 군 구성을 아우르는 새로운 개념이다. 주어진 군들의 제 2의 호모토피 가군(homotopy module) π_2의 생성자(generator) 집합들로부터 각 군 표시(group presentation) π_2의 생성자 집합을 기하적인 도형 즉, 그림(picture)을 이용하여 결정한다. 이를 이용하여 3장에서는 군의 그래프에 관한 첨가이데알(augmentation ideal), 관계가군(relation module), 제 2의 호모토피 가군(second homotopy module)에 대한 완전열(exact sequence)들을 만든다. 이는 위의 여러 가군의 구조를 파악하는 데에 매우 중요하다. 또한 위와 같이 얻어진 완전열들 사이의 관계를 규명하였다. 또한 4장에서는 군 곱의 π_2의 생성자 집합을 구하고 이를 이용하여 호모로지군(homology group)과 코호모로지군(cohomology group)을 계산하였다. 이들의 응용으로서 군과 그 표시의 효율성(efficiency)를 연구하였다.

Notations

Let G, H, and K be groups.

$H \oplus K$: the direct sum

$H * K$: the free product

$H_u *_{H_e} H_v$: the free product of H_u and H_v with H_e, $H_{e^{-1}}$ amalgamated

$HNN(H_v, H_{e^{\pm 1}})$: the HNN extension of H_v with associated subgroups H_e and $H_{e^{-1}}$.

$G \cong H$: G is isomorphic to H

G/H : the quotient of G by H

$AutG$: the automorphism group of G

$EndG$: the endomorphism ring of G

$sgp_H\{\ \}$: the subgroup generated by a subset of H

$[a,b]$: the commutator of a and b

$\mathbb{Z}G$: the integral group ring

IG : the augmentation ideal

$- \otimes_G -$: the tensor product of $\mathbb{Z}G$-modules

$rk(G)$: the rank of the torsion-free part

$d(G)$: the least number of generators

$\nu(G) = 1 - rk(H_1(G)) + d(H_2(G))$

$H_k(G, A)$: the $\mathbf{k}$-th homology group of G with coefficients in A

$H^k(G, B)$: the $\mathbf{k}$-th cohomology group of G with coefficients in B

$H_2(G)$: the second integral homology of G

$H^2(G)$: the second integral cohomology of G

ρ_1, ρ_2 : the standard surjections

μ_1, μ_2 : the standard injections

$A \xrightarrow{\iota} B$: the inclusion of A into B

$\mathbb{Z}$: the integers

$\ker \alpha$: the kernel of α

$\mathrm{im}\alpha$: the image of α

Notation concerning presentations.

Let $\wp$ be a group presentation.

$\pi_2(\wp)$: the second homotopy module

$M(\wp)$: the relation module

$\chi(\wp)$: the Euler characteristic

Notation concerning pictures.

Let $\mathbb{P}$ be a picture.

$\partial(\mathbb{P})$: the boundary of $\mathbb{P}$

$W(\mathbb{P})$: the label of $\mathbb{P}$

$-\mathbb{P}$: the mirror image of $\mathbb{P}$

$< \mathbb{P} >$: the equivalence class containing $\mathbb{P}$

$\mathbb{P}^W$: the spherical picture obtained from a spherical picture $\mathbb{P}$ by surrounding it by a collection of concentric closed arcs with total label W

$W(\gamma)$: the label of a path γ

$W(\Delta)$: the label of a disc Δ

$exp_R(\mathbb{P})$: the exponent sum of R in $\mathbb{P}$

$exp_x(W)$: the exponent sum of x in W

Notation concerning graphs.

V : the vertex set

E : the edge set

E^+ : the orientated edge set

$\iota(e)$: the initial vertex of e

$\tau(e)$: the terminal vertex of e

H_v : the vertex group of graphs of groups

H_e : the edge group of graphs of groups

K_v : the vertex group of graph products

K_e : the edge group of graph products

Miscellaneous notation.

Let σ be a sequence of words.

$\Pi\sigma$: the product of terms of σ

$< \sigma >$: the equivalence class containing σ

$\sigma(\underline{\gamma})$: the sequence associated with $\underline{\gamma}$

Chapter 1

Introduction

Combinatorial group theory is connected with several parts of mathematics, in particular, algebra and low dimensional topology. This discipline was initiated by Poincare, Dehn, Tietze, and other topologists.

Let $\wp =< \mathbf{x} \ ; \ \mathbf{r} >$ be a finite group presentation, where $\mathbf{x}$ is a set and $\mathbf{r}$ is a set of elements of the free group F on $\mathbf{x}$. We call $\mathbf{x}$ and $\mathbf{r}$ a set of generators and a set of relators, respectively. Then we can get a group G is isomorphic to $F/N,$ where N is the normal closure of $\mathbf{r}$ in F.

We may associated $\wp$ with a 2 dimensional CW-complex C which consists of a 0-cell, 1-cells and 2-cells of which boundaries are the elements of $\mathbf{r}$. Then the above G is the fundamental group of C.

Therefore, to any topological space there is an associated group, which is the fundamental group of the space. In trying to investigate the properties of certain spaces we are led to problems in a combinatorial group theory. Conversely, some problems in group theory are solved by geometric and topological discussions of suitable fundamental groups.

A main concept in this thesis is the second homotopy module $\pi_2(\wp)$ of the given presentation $\wp$. It can be investigated by an algebraic method - identity equations or a geometric method - pictures. These are to be reviewed late in Preliminaries. Pictures have dual concepts to cancelation diagrams but more convenient to get a set of generators of $\pi_2(\wp)$.

Graphs of groups and graph products of groups are the main topics in this thesis which are group constructions from given groups. They can be defined by using graphs.

In chapter 3, we study some exact sequences concerning the graph of groups. We get exact sequence about relation modules(Th 3.2.9), augmentation ideals(Th 3.2.4), second homotopy modules(Th 3.2.13) and their relation ships(Cor 3.2.8, Th 3.2.16, Th 3.2.17). And then we can describe relation modules, augmentation ideals and second homotopy modules of graph of groups in terms of relation modules, augmentation ideals and second homotopy modules of given vertex groups.

In chapter 4, we study two extreme cases of a graph product of groups that is, the free product of groups and the direct products of the groups. Therefore a graph product shares many interesting properties of these extreme cases.

After determining a generating set of the second homotopy module of a graph product, and we calculate second homology groups and cohomology groups(Th 4.3.3) we apply it to study the efficiency(Th 4.4.4).

Chapter 2

Preliminaries

In this chapter we will introduce basic concepts concerning to the thesis.

2.1 Graphs

A graph Γ consists of two disjoint non empty sets V, E and three functions

$$\iota \,:\, E \longrightarrow V, \quad \tau \,:\, E \longrightarrow V, \quad {}^{-1} \,:\, E \longrightarrow E$$

satisfying : $\iota(e) = \tau(e^{-1}),\ \ (e^{-1})^{-1} = e,\ \ e^{-1} \neq e$ for all $e \in E$. The set V is the set of vertices and E is the set of edges. We call $\iota(e)$ and $\tau(e)$ the *initial* and the *terminal* point of $e \in E$ respectively. And an orientation E^+ of Γ consists of a choice of exactly one edge from each edge pair $e,\ e^{-1}(e \in E)$. We will refer to the pair $(V,\ E^+)$ with the functions $\iota,\ \tau$ as an oriented graph with an oriented edge set E^+. We call $e \in E$ a *loop* if $\iota(e) = \tau(e)$.

A graph Γ is *simple* if whenever $\iota(e_1) = \iota(e_2)$ and $\tau(e_1) = \tau(e_2)$, then $e_1 = e_2$.

We call Γ is *finite* if both V and E are finite. A simple graph Γ is *complete* if for any two distinct vertices u and v, there is an edge e with $\iota(e) = u, \ \tau(e) = v$.

A *path* is a finite sequence $e_1, \ e_2, \ \cdots, \ e_n$ of edges such that $\tau(e_i) = \iota(e_{i+1})$ for all $i < n$. A *closed path* is a path $e_1, \ e_2, \ \cdots, \ e_n$ with $\tau(e_n) = \iota(e_1)$. A path is *simple* if $\iota(e_i)(i = 1, \ 2, \ \cdots, \ n)$ are all distinct.
We write $e_1 e_2 \cdots e_n$ instead $e_1, \ e_2, \ \cdots, \ e_n$.

A *subgraph* Γ' of Γ consists of subsets $V'(V' \subset V)$ and $E'(E' \subset E)$ such that for any $e, \ e^{-1} \in E', \ \iota(e), \ \tau(e) \in V'$. A graph Γ is *connected* if given any two vertices of Γ there is a path joining them. And we call a connected subgraph of Γ a *component* of Γ if no connected subgraph of Γ properly contains it.

A graph T is a *tree* if T has no simple closed path. And a tree T is *maximal* if for each $u, \ v$ in V, there exists a path $e_1, \ e_2, \ \cdots, \ e_n$ such that $\iota(e_1) = u,$ and $\tau(e_n) = v$.

2.2 Second homotopy modules

Let $\wp =< \mathbf{x}\ ;\ \mathbf{r} >$ be a group presentation, where $\mathbf{x}$ is a set and $\mathbf{r}$ is a set of cyclically reduced words on $\mathbf{x} \cup \mathbf{x^{-1}}$.

Let N be the normal closure of $\mathbf{r}$ in F, where F is the free group on $\mathbf{x}$. Then the quotient group G of F by N is called the *group defined by* $\wp$.

We denote by $\mathbf{w}$ the set of all words on $\mathbf{x} \cup \mathbf{x^{-1}}$. If $\mathbf{s}$ is a subset of $\mathbf{r}$ then $\mathbf{s}^W$ is the set of all words of the form

$$WS^{\varepsilon}W^{-1}\ (W \in \mathbf{w},\ S \in \mathbf{s},\ \varepsilon = \pm 1).$$

Let σ be a finite sequence of elements of $\mathbf{r}^W$, say $\sigma = (c_1,\ c_2,\ \cdots,\ c_n)$, where $c_i \in \mathbf{r}^W$ $(i = 1,\ 2,\ \cdots,\ n)$. Then we define $\Pi\sigma$ to be the product $c_1 c_2 \cdots c_n$. If $\Pi\sigma$ is freely equal to 1, then σ is called an *identity sequence*. We define the *inverse* σ^{-1} of σ to be $(c_n^{-1},\ \cdots,\ c_2^{-1},\ c_1^{-1})$, and for $W \in \mathbf{w}$ we define the *conjugate* $W\sigma W^{-1}$ of σ by W to be $(Wc_1W^{-1},\ Wc_2W^{-1},\ \cdots,\ Wc_nW^{-1})$. We define operations on sequence as follows. Let

$$c_i = W_i R_i^{\varepsilon_i} W_i^{-1}\ (W_i \in \mathbf{w},\ R_i \in \mathbf{r},\ \varepsilon_i = \pm 1,\ i = 1,\ 2,\ \cdots,\ n).$$

(1) Replace each W_i by a word freely equal to it.

(2) Delete two consecutive terms if one is identically equal to the inverse of the other.

(3) The opposite of (2).

(4) Replace two consecutive terms $c_i,\ c_{i+1}$ by either $c_{i+1},\ {c_{i+1}}^{-1}\, c_i\, c_{i+1}$ or $c_i\, c_{i+1}\, {c_i}^{-1},\ c_i$.

Two sequences $\sigma_1,\ \sigma_2$ will be said to be *(Peiffer) equivalent* if one can be obtained from the other by a finite number of applications of the operations (1), (2), (3) and (4). The equivalence class containing σ will be denoted by $<\sigma>$.

The set of all equivalence classes forms a group under the following binary operation

$$<\sigma_1> \ + \ <\sigma_2> = <\sigma_1\sigma_2>$$

where $\sigma_1\sigma_2$ is the juxtaposition of the two sequence $\sigma_1,\ \sigma_2$.

We let $\pi_2(\wp)$ denote the subgroup of consisting of all elements $<\sigma>$ where σ is an identity sequence. We can think of an identity sequence as a relation (an identity) among relators. So $\pi_2(\wp)$ gives us a description of all relations among relators.

We note that the group of all equivalence classes is not abelian under the operation $+$ but the subgroup $\pi_2(\wp)$ is abelian ([31]).

We can also consider $\pi_2(\wp)$ as a left $\mathbb{Z}G$-module by the G-action given by

$$WN\cdot <\sigma> = <W\sigma W^{-1}> \quad (W \in F).$$

We call $\pi_2(\wp)$ the second homotopy module of $\wp$.

2.3 Pictures

In this section, we introduce the basic concepts of the pictures and the identity sequences. We also study some short exact sequences concerned about the second homotopy modules associated with amalgamated free products.

A picture $\mathbb{P}$ is a geometric configuration consisting of the following:

(a) A disc D^2 with a basepoint O on ∂D^2.

(b) Disjoint discs $\Delta_1,\ \Delta_2,\ \cdots,\ \Delta_n$ in the interior of D^2.

Each disc Δ_λ ($\lambda = 1,\ 2,\ \cdots,\ n$) has $p(\lambda)$ basepoints $O_{\lambda_1},\ O_{\lambda_2},\ \cdots,\ O_{\lambda_{p(\lambda)}}$ on $\partial\Delta_\lambda$, encounted in the order $O_{\lambda_1},\ O_{\lambda_2},\ \cdots,\ O_{\lambda_{p(\lambda)}}$ if we start at O_{λ_1} and travel once around $\partial\Delta_\lambda$ in the clockwise direction.

(c) A finite number of disjoint arcs $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_m$.

Each arc lies in the closure of $D^2 - \bigcup_{\lambda=1}^{n} \Delta_\lambda$ and is either a simple closed curve having trivial intersection with $\partial D^2 \cup \partial\Delta_1 \cup \partial\Delta_2 \cup \cdots \cup \partial\Delta_n$, or a simple non-closed curve which joins two points of $\partial D^2 \cup \partial\Delta_1 \cup \partial\Delta_2 \cup \cdots \cup \partial\Delta_n$, neither point being a basepoint. Each arc has a normal orientation, indicated by a short arrow meeting the arc transversely, and is labelled by an element of $\mathbf{x} \cup \mathbf{x^{-1}}$.

A picture $\mathbb{P}$ is called to be *connected* if

$$\bigcup\{\Delta_1,\ \Delta_2,\ \cdots,\ \Delta_n\} \cup \bigcup\{\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_m\}$$

is connected.

For each disc Δ, the *corners* of Δ are the closures of connected components of $\partial\Delta - \bigcup\{\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_m\}$, where $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_m$ are arcs of Δ. The *regions* of $\mathbb{P}$ are the closures of connected components of $D^2 - (\bigcup\{discs\} \ \cup\ \bigcup\{arcs\})$. An *inner region* of $\mathbb{P}$ is a simply connected region of $\mathbb{P}$ that does not meet ∂D^2.

We remark that when we refer to the discs of $\mathbb{P}$ we mean the discs $\Delta_1,\ \Delta_2,\ \cdots,\ \Delta_n$, but not the ambient disc D^2. We define $\partial\mathbb{P}$ to be ∂D^2.

We say that $\mathbb{P}$ is *spherical* if no arcs meet $\partial\mathbb{P}$. If $\mathbb{P}$ is spherical then we often omit $\partial\mathbb{P}$.

Definition 2.3.1. *A picture* $\mathbb{P}$ *is over a presentation* $\wp$ *if the following conditions hold:*

(1) Each arc is labelled by an element of $\mathbf{x} \cup \mathbf{x}^{-1}$.

(2) If we travel around $\partial\Delta_\lambda$ *once in the clockwise direction starting at* O_λ *and read off the labels on the arcs encountered then we obtain a word which belongs to* $\mathbf{r} \cup \mathbf{r}^{-1}$ *and we call this word the label of* $\Delta_\lambda(\lambda = 1,\ 2,\ \cdots,\ n)$.

Example 2.3.2. *Let* $\wp = < x, y : x^5, xyx^{-3}y^{-1} >$ *. Then the following picture is a spherical picture over* $\wp$.

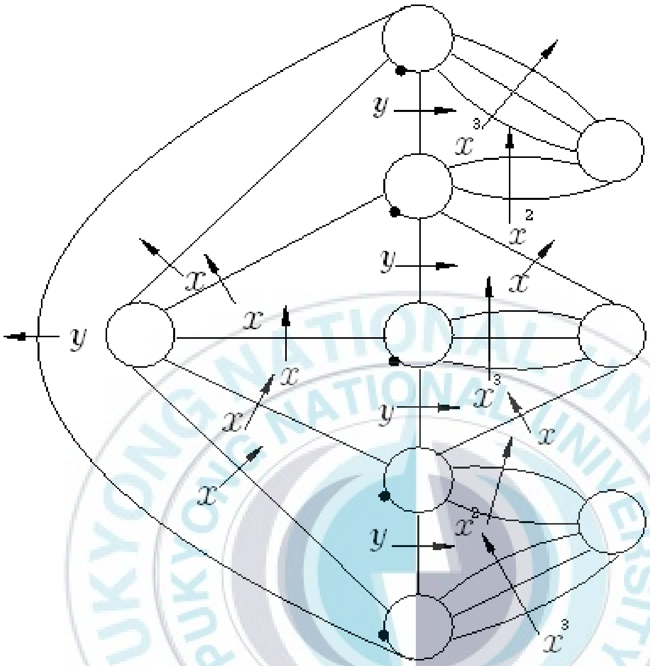

Figure 2.1.1. Spherical picture.

Let $\mathbf{y}$ and $\mathbf{s}$ be subsets of $\mathbf{x}$ and $\mathbf{r}$ respectively. An arc labelled by an element of $\mathbf{y} \cup \mathbf{y^{-1}}$ is called a $\mathbf{y}$-*arc* and a disc labelled by an element of $\mathbf{s} \cup \mathbf{s^{-1}}$ is called an $\mathbf{s}$-*disc*.

The *label* on $\mathbb{P}$ (denoted $W(\mathbb{P})$) is the word read off by traveling around ∂D^2 once in the clockwise starting at O.

Example 2.3.3. *Let* $\wp =< a, b, c : a^2, [a, b], [b, c], [a, c] >$.

Then $W(\mathbb{P}) = c^{-1}abcab^{-1}$.

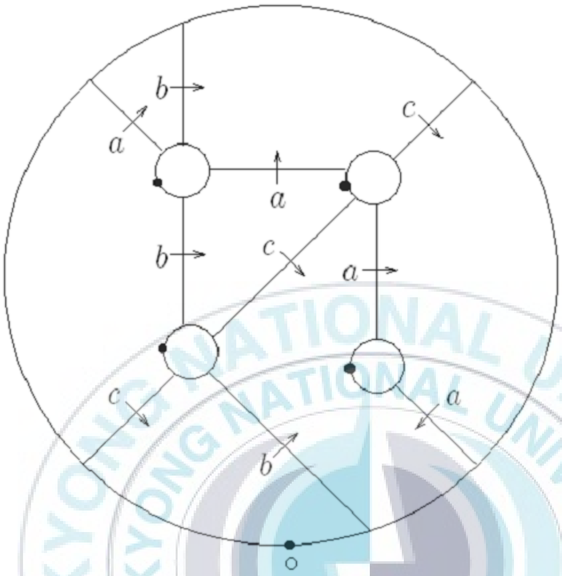

Figure 2.1.2. Label of $\mathbb{P}$

A *circle* in a picture over $\wp$ consists of a collection of distinct arcs $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_n$ and distinct $\Delta_1,\ \Delta_2,\ \cdots,\ \Delta_n$ such that α_i joins Δ_i to $\Delta_{i+1}\,(i = 1,\ 2,\ \cdots,\ n$ subscripts mod $n)$, and all $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_n$ have the same label and the same orientation. We call a circle C *minimal* if there is no circles contained in the region enclosed by C. A disc Δ in the region enclosed by C will be said to be *adjacent* to C if Δ is joined to one of the discs $\Delta_1,\ \Delta_2,\ \cdots,\ \Delta_n$ by an arc.

A *(transverse) path* in $\mathbb{P}$ is a path in the closure of $D^2 - \bigcup_{\lambda=1}^{n} \Delta_\lambda$ which intersects the arcs of $\mathbb{P}$ only finitely many times. If we travel along a path γ from its initial point to its terminal point, we will cross various arcs, and we can read off the labels on these arcs, giving a word $W(\gamma)$, the *label on* γ.

A *spray* for $\mathbb{P}$ is a sequence $\underline{\gamma} = (\gamma_1,\ \gamma_2,\ \cdots,\ \gamma_n)$ of simple paths satisfying the following; for $\lambda = 1,\ 2,\ \cdots,\ n$, γ_λ starts at O and ends at the basepoint $\mathrm{O}_{\theta(\lambda)}$ of $\Delta_{\theta(\lambda)}$, where θ is a permutation of $\{1,\ 2,\ \cdots,\ n\}$ (depending on $\underline{\gamma}$) ; for $1 \le \lambda < \mu \le n$, γ_λ and γ_μ intersect only at O; travelling around O clockwise in $\mathbb{P}$ we encounter the paths in the order $\gamma_1,\ \gamma_2,\ \cdots,\ \gamma_n$.

The sequence $\sigma(\underline{\gamma})$ associated with $\underline{\gamma}$ is

$$(W(\gamma_1)\, W(\gamma_{\theta(1)})\, W(\gamma_1)^{-1},\ \cdots,\ (W(\gamma_n)\, W(\gamma_{\theta(n)})\, W(\gamma_n)^{-1}).$$

A picture will be said to *represent* a sequence σ if there is a spray for the picture whose associated sequence is σ.

Example 2.3.4. *Let* $\sigma = (c^{-1}[a,b]c,\ c^{-1}[b,c]c,\ bc^{-1}[a,c]cb^{-1},\ ba^2b^{-1})$.

Then the following picture represents σ.

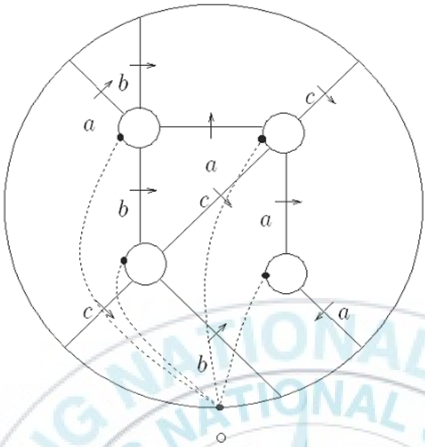

Figure 2.1.3. Spray of $\mathbb{P}$

Remark 2.3.5. *(1)* $-\mathbb{P}$ *is called the mirror image of* $\mathbb{P}$ *.*

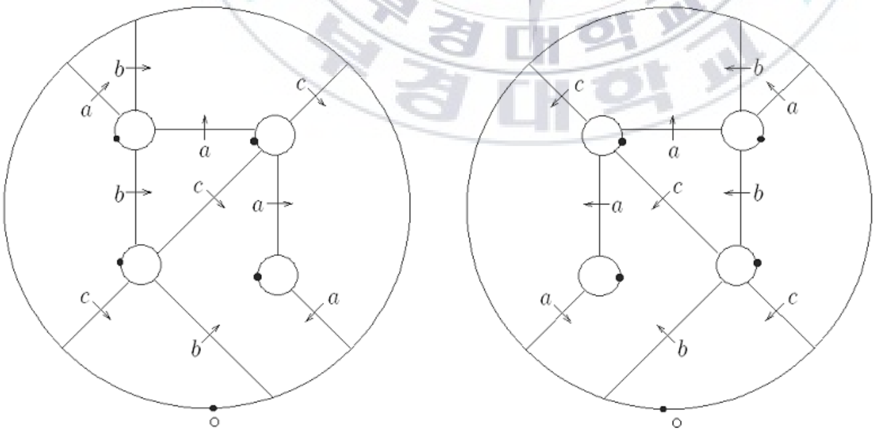

Figure 2.1.4. Mirror image of $\mathbb{P}$.

(2) $\mathbb{P}_1 + \mathbb{P}_2$ *is the sum of* $\mathbb{P}_1$ *and* $\mathbb{P}_2$.

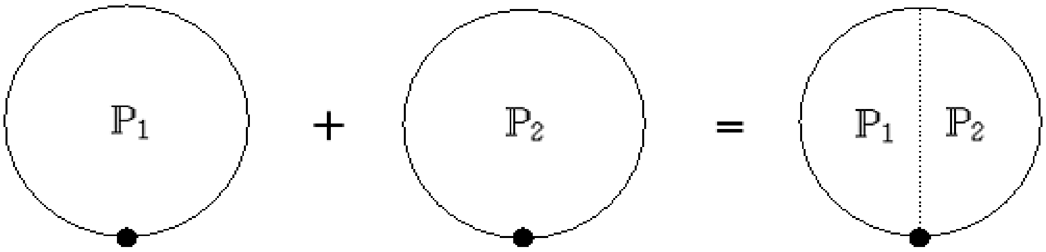

Figure 2.1.5. Sum of $\mathbb{P}_1$ *and* $\mathbb{P}_2$.

Theorem 2.3.6. ([31] Theorem 2.1(ii)) *Every identity sequence can be represented by a spherical picture.*

Now we introduce the basic operations on pictures.

(A) : Deletion of a closed arc which encircles no discs or arcs of $\mathbb{P}$ (such a closed arc is called a *floating circle*).

$(A)^{-1}$: Insertion of a floating circle.

A *cancelling pair* is a spherical picture with exactly two discs, and when their basepoint lie in the same region like Figure 2.1.6.

(B) : If there is a simple closed path β in $\mathbb{P}$ such that the part of $\mathbb{P}$ encircled by β is a cancelling pair, then remove that part of $\mathbb{P}$ encircled by β.

$(B)^{-1}$: The opposite of (B).

(C) : Bridge move.

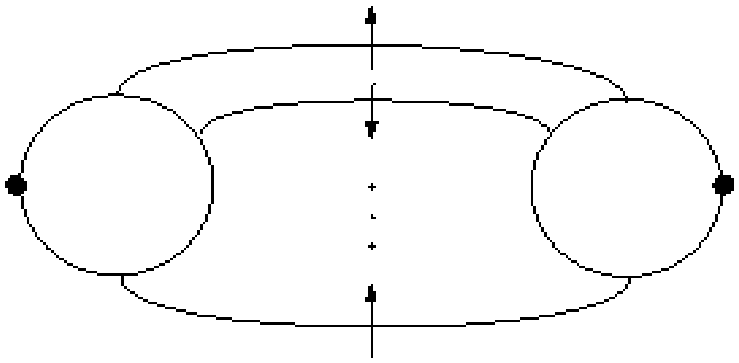

Figure 2.1.6. Cancelling pair.

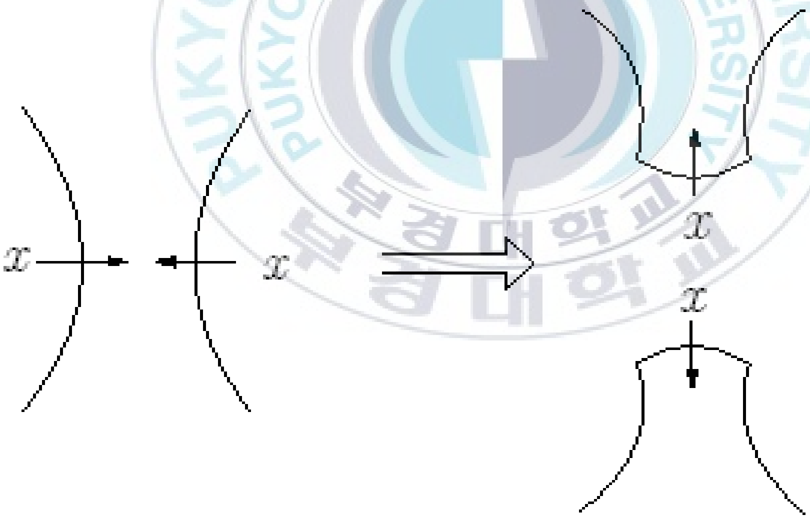

Figure 2.1.7. Bridge move.

Remark 2.3.7. *Since we allow only one basepoint on each disc, when a relator is a proper power, we need more caution. That is to say,* $\mathbb{P}_1$ *and* $\mathbb{P}_2$ *are cancelling pairs, whereas* $\mathbb{P}$ *is not. So we will only insert basepoints for discs whose labels are proper powers.*

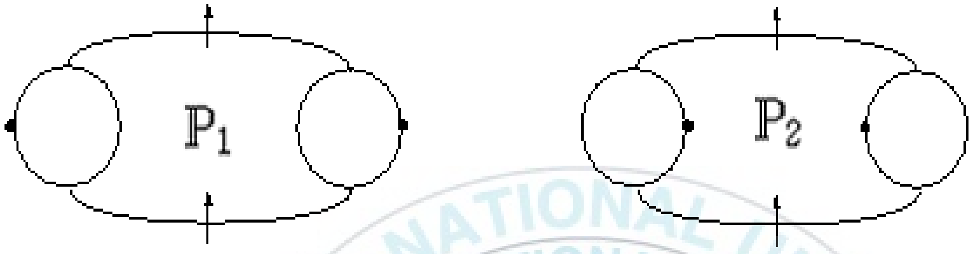

Figure 2.1.8. Cancelling pairs.

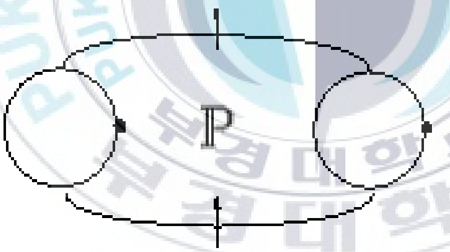

Figure 2.1.9. Non - Cancelling pair.

Two spherical pictures will be said to be *equivalent* if one can be transformed to the other by applying a finite of operations (A), $(A)^{-1}$, (B), $(B)^{-1}$, or (C).

We let $<\mathbb{P}>$ denote the equivalence class containing $\mathbb{P}$.

The set $\sum$ of all equivalence classes of all spherical pictures over $\wp$ forms a group under the following binary operation

$$<\mathbb{P}_1> + <\mathbb{P}_2> \;\; = \;\; <\mathbb{P}_1 + \mathbb{P}_2>$$

where the inverse of $<\mathbb{P}>$ is $<-\mathbb{P}>$ and the identity is the equivalence class containing the empty picture.

Let $\mathbb{P}^W$ be the spherical picture obtained from a spherical picture $\mathbb{P}$ by surrounding it with a collection of concentric closed arcs with total label W like Figure 2.1.10.

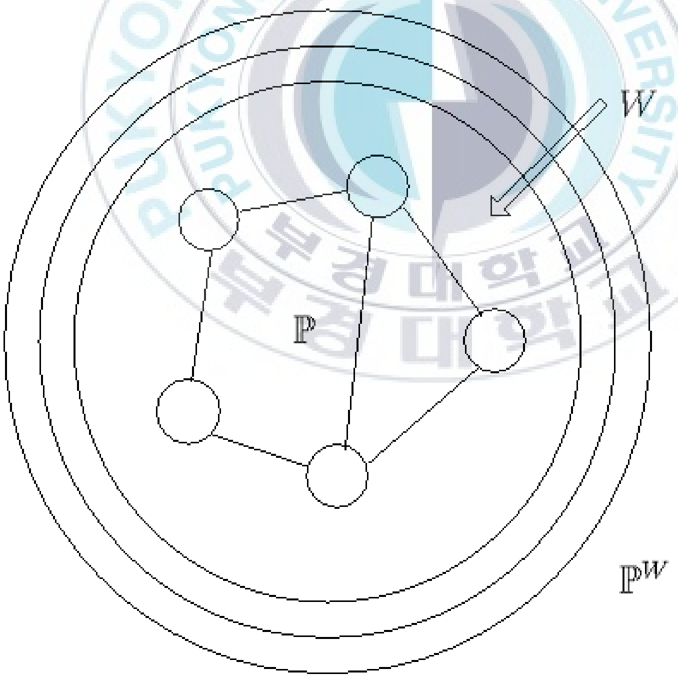

Figure 2.1.10. Spherical picture $\mathbb{P}^W$.

Remark 2.3.8. *If* $< \mathbb{P} >$ *represents* σ *then* $\mathbb{P}^W$ *represents* $W\sigma W^{-1}$.

Theorem 2.3.9. ([31] Theorem 2.5) *Let* σ *and* σ' *be sequences represented by* $\mathbb{P}$*, and* $\mathbb{P}'$ *respectively. Then* σ *and* σ' *are equivalent iff* $\mathbb{P}$ *and* $\mathbb{P}'$ *are equivalent.*

Remark 2.3.10. *The group* $\sum$ *is abelian under the operration* $+$. *Consider the following Figure 2.1.11. Then the sequences* $\sigma(\underline{\gamma})$, $\sigma(\underline{\gamma}')$ *are equivalent since they are spray for* $\mathbb{P}_1 + \mathbb{P}_2$ ([31] Theorem 2.4). *And since* $\sigma(\underline{\gamma}') = \sigma(\underline{\gamma}'')$*, by Theorem 2.3.9.* $\mathbb{P}_1 + \mathbb{P}_2$ *and* $\mathbb{P}_2 + \mathbb{P}_1$ *are equivalent.*

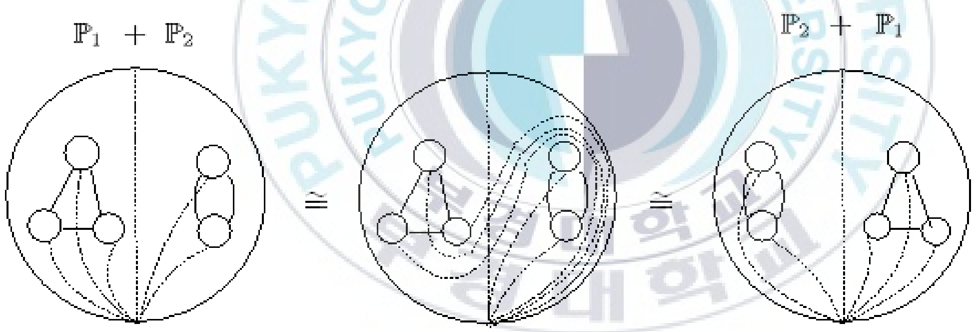

Figure 2.1.11. Equivalent spray.

We can consider $\sum$ as a left $\mathbb{Z}G$-module by the G-action given by

$$WN\cdot < \mathbb{P} > \; = \; < \mathbb{P}^W > \qquad (W \in F).$$

Then we call $\pi_2(\wp)$ the *second homotopy module of* $\wp$.

And now we define a map $\psi : \pi_2(\wp) \longrightarrow \sum$ *by* $< \sigma > \longmapsto \mathbb{P}$ where $\mathbb{P}$ is a spherical picture representing σ. By Theorem 2.3.6 ψ is well-defined and injective. By Theorem 2.3.9 ψ is surjective. And by the above Remarks 2.3.8 and 2.3.10, ψ is a module homomorpsism. From now on, we will identify $\pi_2(\wp)$ with $\sum$.

Consider a collection $\mathbf{X}$ of spherical pictures over $\wp$. We introduce two further operations on $\pi_2(\wp)$ as follows.

(D) If there is a simple closed path β in a picture such that the part of the picture enclosed by β is a copy of $\mathbb{P}$ or $-\mathbb{P}$ ($\mathbb{P} \in \mathbf{X}$), then we delete that part of the picture enclosed by β.

$(D)^{-1}$ The opposite of (D).

Two spherical picture will be said to be *equivalent* (rel $\mathbf{X}$) if one can be transformed to the other by applying a finite of operations (A), $(A)^{-1}$, (B), $(B)^{-1}$, (C), (D) or $(D)^{-1}$.

Theorem 2.3.11. ([31] theorem 2.6) *The elements* $< \mathbb{P} >$ $(\mathbb{P} \in \mathbf{X})$ *generate* $\pi_2(\wp)$ *if and only if every spherical picture is equivalent (rel* $\mathbf{X}$*) to the empty picture.*

Remark 2.3.12. *Theorem 2.6 in* [31] *actually refers to the situation where several basepoints are allowed. But it is easily modified to our situation.*

If the elements $< \mathbb{P} >$ $(\mathbb{P} \in \mathbf{X})$ generate $\pi_2(\wp)$ then we say that $\mathbf{X}$ generates $\pi_2(\wp)$.

2.4 The augmentation ideal

Let G be a group written multiplicatively. The integral group ring $\mathbb{Z}G$ of G is defined as follows. Its underlying abelian group is the free abelian group on the set of elements of G as a basis ; the product of two basis elements is given by the product in G. Thus the elements of the group ring $\mathbb{Z}G$ are sums

$$\sum_{x\in G} m(x)x$$

where m is a function from G to $\mathbb{Z}$ which takes the value zero except on a finite number of elements of G.

The multiplication is given by

$$\left(\sum_{x\in G} m(x)x\right)\cdot\left(\sum_{y\in G} m'(y)y\right)=\sum_{x,y\in G}\left(m(x)\cdot m'(y)\right)xy.$$

The group ring is characterised by the following universal property. Let $i: G \longrightarrow \mathbb{Z}G$ be the obvious embedding.

A (left) G-module is an abelian group A together with a group homomorphism $\sigma : G \longrightarrow AutA$. In other words, the group elements acts as an automorphism on A. We shall denote the image of $a \in A$ under the automorphism $\sigma(x)$, $x \in G$, by $x \cdot a$ or simply xa if this notation cannot cause any confusion.

Since $AutA \subseteq EndA$, the universal property of the group ring yields a ring homomorphism $\sigma' : \mathbb{Z}G \longrightarrow EndA$, making A into a (left) module over $\mathbb{Z}G$. Conversely, if A is a (left) module over $\mathbb{Z}G$ then A is a (left) G-module, since any ring homomorphism takes invertible elements into invertible elements, and since the group elements in $\mathbb{Z}G$ are invertible. Thus we need not retain any distinction between the concepts of G-module and $\mathbb{Z}G$-module.

We leave it to the reader to word the definition of a right G-module. A (left) G-module is called *trivial* if the structure map $\sigma : G \longrightarrow AutA$ is trivial, i.e., if every group element of G acts as the identity in A. Every abelian group may be regarded as a trivial left or right G-module for any group G.

The trivial map from G into the integer $\mathbb{Z}$, sending every $x \in G$ into $1 \in \mathbb{Z}$, gives rise to a unique ring homomorphism

$$\varepsilon : \mathbb{Z}G \longrightarrow \mathbb{Z}.$$

This map is called the *augmentation* of $\mathbb{Z}G$. If $\sum_{x\in G} m(x)x$ is an arbitrary element in $\mathbb{Z}G$ then

$$\varepsilon\left(\sum_{x\in G} m(x)x\right) = \sum_{x\in G} m(x).$$

The kernel of ε is denoted by IG and is called the *augmentation ideal* of $\mathbb{Z}G$.

Lemma 2.4.1. ([22]) (i) *An abelian group IG is free on the set*

$$W = \{x - 1 \mid 1 \neq x \in G\}.$$

(ii) *Let S be a generating set for G. Then the set $S - 1 = \{s - 1 \mid s \in S\}$ generates IG as a $\mathbb{Z}G$-module.*

2.5 Some exact sequences concerning $\pi_2(\wp)$

Let $\wp = <\mathbf{x}\ ;\ \mathbf{r}>$ and X be a generating set for $\pi_2(\wp)$.

Let G be the group defined by $\wp$, that is, $G = F/N$, where F is the free group on $\mathbf{x}$ and N is the normal closure of $\mathbf{r}$ in F.

The relation module $M(\wp)$ of $\wp$ is the abelianization N/N' of N regarded as a left $\mathbb{Z}G$-module, with G-action by

$$WN.UN' \;=\; WUW^{-1}N' \quad (W \in F,\ U \in N).$$

Let

$$P_0 = \mathbb{Z}G,\ P_1 = \bigoplus_{x\in\mathbf{x}} \mathbb{Z}Gt_x,\ P_2 = \bigoplus_{R\in\mathbf{r}} \mathbb{Z}Gt_R,\ P_3 = \bigoplus_{\mathbb{P}\in X} \mathbb{Z}Gt_{\mathbb{P}}.$$

Then we have the following exact sequence ([31]).

(1) $$0 \longrightarrow \pi_2(\wp) \xrightarrow{\mu_2} P_2 \xrightarrow{\rho_2} M(\wp) \longrightarrow 0$$

$$<\mathbb{P}> \longmapsto \sum_{i=1}^{n} \varepsilon_i W_i N t_{R_i} \quad (\mathbb{P} \in X)$$

$$t_R \longmapsto RN' \qquad (R \in \mathbf{r})$$

where $\mathbb{P}$ represents

$$\sigma = (W_1 {R_1}^{\varepsilon_1} {W_1}^{-1},\ W_2 {R_2}^{\varepsilon_2} {W_2}^{-1},\ \cdots,\ W_n {R_n}^{\varepsilon_n} {W_n}^{-1}).$$

We often write $\mu_2(\mathbb{P})$ instead of $\mu_2(<\mathbb{P}>)$.

We regard $\mathbb{Z}$ as a left $\mathbb{Z}G$-module with the trivial G-action. There is the augmentation map $\varepsilon : P_0 \longrightarrow \mathbb{Z}$ which takes each element of G to 1. The kernel of this map is called the augmentation ideal denote by IG.

(2) $$0 \longrightarrow IG \xrightarrow{incl} P_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

(3) $$0 \longrightarrow M(\wp) \xrightarrow{\mu_1} P_1 \xrightarrow{\rho_1} IG \longrightarrow 0$$

$$\begin{aligned} WN' &\longmapsto \sum_{x \in \mathbf{x}} \rho\left(\frac{\partial W}{\partial x}\right) t_x \quad && (W \in N) \\ t_R &\longmapsto xN - 1 && (x \in \mathbf{x}). \end{aligned}$$

Here $\frac{\partial}{\partial x} : \mathbb{Z}F \longrightarrow \mathbb{Z}F$ is the Fox derivation ([27] Section II.3) and $\rho : \mathbb{Z}F \longrightarrow \mathbb{Z}G$ is induced by the natural epimorphism $F \longrightarrow G$.

We call μ_1, μ_2 the standard injections and ρ_1, ρ_2 the standard surjections.

If we put the three sequences (1), (2), (3) together we get the exact sequence

(4) $$P_3 \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

where

$$\begin{aligned} &\partial_3 : t_{\mathbb{P}} \longmapsto \mu_2(\mathbb{P}) \\ &\partial_2 = \mu_1 \rho_2 \\ &\partial_1 = \rho_1. \end{aligned}$$

.

Chapter 3

Graphs of groups

3.1 Graphs of groups

In this section we introduce fundamental facts about the graphs of groups.

Definition 3.1.1. *A graph of groups,* G *consists of*

(1) an oriented graph Γ *with a vertex set* V *and an oriented edge set* E^+*;*

(2) for each $v \in V,$ *a group* H_v *(vertex group) and for each* $e \in E^+,$ *groups* $H_e,\ H_{e^{-1}}$ *(edge groups);*

(3) for each $e \in E^+,$ *there is an isomorphism*

$$\gamma_e \ : \ H_e \longrightarrow H_{e^{-1}}$$

for a subgroup H_e *of* $H_{\iota(e)}$ *and a subgroup* $H_{e^{-1}}$ *of* $H_{\tau(e)}$*.*

Let T be a maximal forest in Γ and let $F(E^+)$ be the free group with basis E^+. Let G be the quotient of $F(E^+) * (*_{v\in V} H_v)$ by the normal closure of the set $\{g_e e(\gamma_e g_e)^{-1}e^{-1} \ : \ e \in E^+, \ g_e \in H_e\} \cup \{e \ : \ e \in T\}$. Then G is called the *fundamental group of the graphs of groups* G ([12]).

Remark 3.1.2. *A Graph of groups is a general concept of the followings.*

(1) Let Γ *be a simple graph such that for each* $e \in E^+$, $\iota(e) = v_0$ *where* v_0 *is a fixed vertex, and suppose all* $H_e, H_{e^{-1}}(e \in E^+)$ *and* H_{v_0} *are trivial. Then* G *is the free product of the vertex groups* $H_v(v \neq v_0)$. *We denote it* $G = *_{v \in V} H_v$.

(2) If Γ *consists of a single vertex* v *and a single oriented edge* e *then* G *is the HNN extension of* H_v *with associated subgroups* H_e *and* $H_{e^{-1}}$. *We denote it* $G = HNN(H_v, H_{e^{\pm 1}})$.

(3) If Γ *has two vertices* u, v *and a single oriented edge* e *joining* u *and* v, *then* G *is the free product of* H_u *and* H_v *with* H_e *and* $H_{e^{-1}}$ *amalgamated. We denote it* $G = H_u *_{H_e} H_v$.

(4) If Γ *is a tree then* G *is called the tree product of the vertex groups* ([22]).

(5) If all vertex groups and edge groups are trivial then G *is the fundamental group of* Γ.

Now we will write down a presentation $\wp$ for G from presentations for vertex groups.

For $v \in V$, let H_v be a group given by a presentation

$$Q_v = < \mathbf{x}_v \ ; \ \mathbf{s}_v >$$

(so that H_v is the quotient group of the free group on $\mathbf{x}_v$ by the normal closure S_v of $\mathbf{s}_v$).

For $e \in E^+$, let $a_{i,e},\ a_{i,e}^{-1}\ (i \in I(e))$ be non empty freely reduced words on $\mathbf{x}_{\iota(e)},\ \mathbf{x}_{\tau(e)}$ respectively. We assume that the mapping

$$a_{i,e}\, S_{\iota(e)} \longmapsto a_{i,e^{-1}}\, S_{\tau(e)}\ (i \in I(e))$$

defines an isomorphism from

$$H_e = sgp\{a_{i,e}\, S_{\iota(e)}\ :\ i \in I(e)\}\ of\ H_{\iota(e)}$$

to

$$H_{e^{-1}} = sgp\{a_{i,e^{-1}}\, S_{\tau(e)}\ :\ i \in I(e)\}\ of\ H_{\tau(e)}.$$

Let T be a maximal tree in Γ and let

$$\wp = < \mathbf{x}_v\ (v \in V),\ e(e \in E^+ - T)\ ;\ \mathbf{s}_v\ (v \in V),\ r_{i,e}(e \in E^+,\ i \in I(e)) >$$

where

$$r_{i,e} = a_{i,e}\, \hat{e}\, a_{i,e^{-1}}^{-1} \hat{e}^{-1}$$

and $\hat{e}$ is the empty word if $e \in T$, and $\hat{e} = e$ if $e \notin T$.

Let G be the group defined by $\wp$. Then G is the quotient of the free group on

$$\bigcup_{v \in V} \mathbf{x}_v\ \cup\ \{e\ :\ e \in E^+ - T\}$$

by the normal closure of

$$\{\mathbf{s}_v(v \in V),\ \ r_{i,e}(e \in E^+,\ i \in I(e))\}.$$

It is well known that the natural map $H_v \to G$ $(v \in V)$ are injective. We can thus regard H_v $(v \in V)$ as $sgp_G\{xR \;:\; x \in \mathbf{x}_v\}$, and we can regard $H_{e^{\pm 1}}$ $(e \in E^+)$ as $sgp_G\{a_{i,e^{\pm 1}} R \;:\; i \in I(e)\}$. We let

$$C_e = sgp\{\hat{e}\, R\} \; (e \in E^+).$$

Then C_e is trivial if $e \in T$ and is infinite cyclic otherwise.

For $e \in E^+$, let F_e be the free group with basis

$$\mathbf{y}_e = \{y_{i,e} \;:\; i \in I(e)\}$$

and let N_e be the normal subgroup of the epimorphism

$$F_e \to H_e \;\; by \;\; y_{i,e} \mapsto a_{i,e}R$$

then N_e/N_e' is an H_e-module with action defined as follows.

Let $a \in H_e$ and let $WN_e' \in N_e/N_e'$. Now $a = U(a_{i,e})R$ for some word $U(a_{i,e})$ in the $a_{i,e}'s$ $(i \in I(e))$. Define $a \cdot WN_e' = UNU^{-1}N_e'$ where $U = U(y_{i,e})$. Then action is well-defined.

Let $X_v (v \in V)$ be the collection of all spherical pictures over $\wp_v (v \in V)$ and let $\mathbf{X} = \cup_{v \in V} X_v$. If $e \notin T$ then Y_e denotes the collection of all spherical pictures $\mathbb{P}_W$ as in Figure 3.1.1 and if $e \in T$ then Y_e denotes the collection of all spherical pictures $\mathbb{Q}_W$, as in Figure 3.1.2 where $W = W(a_i) = a_{\lambda_1}^{\varepsilon_1} a_{\lambda_2}^{\varepsilon_2} \cdots a_{\lambda_n}^{\varepsilon_n}$ is an element of N_v. And let $\mathbf{Y} = \cup_{e \in E^+} Y_e$.

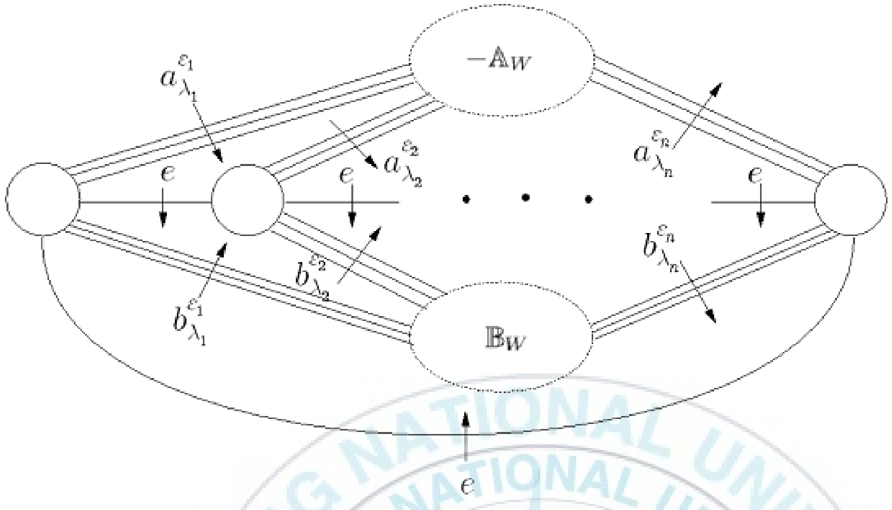

Figure 3.1.1. $\mathbb{P}_W$.

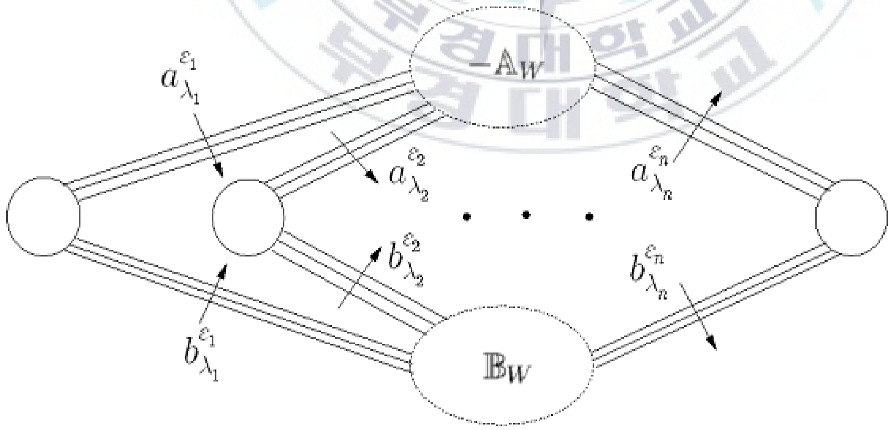

Figure 3.1.2. $\mathbb{Q}_W$.

Then we get the following.

Theorem 3.1.3. ([2]) $\mathbf{X}\cup\mathbf{Y}$ *generates* $\pi_2(\wp)$.

3.2 Exact sequences and their relationships

We study some exact sequences of graphs of groups and their relationships among them.

Theorem 3.2.1. *There is a short exact sequence.*

$$(1)\quad 0\to \oplus_{e\in E^+}(\mathbb{Z}G\otimes_{H_e}\mathbb{Z}H_e)\xrightarrow{q}(\oplus_{v\in V}(\mathbb{Z}G\otimes_{H_v}\mathbb{Z}H_v))$$
$$\oplus(\oplus_{e\in E^+-T}\mathbb{Z}G\otimes_{C_e}\mathbb{Z}C_e)\xrightarrow{p}\mathbb{Z}G\to 0$$

where q and p are defined by

$$q: ng\otimes mh\longmapsto ng\otimes_{\iota(e)}mh-(\hat{e}R\cdot ng\otimes_{\tau(e)}mh)+((\hat{e}R-1)ng\cdot mh)\otimes_{C(e)}1,$$
$$p:(kg'\otimes lh',\alpha g''\otimes(1-\hat{e}R))\longmapsto kg'\cdot lh'+\alpha g''(1-\hat{e}R).$$

Proof.

$$\begin{aligned}&pq(ng\otimes mh)\\&=p(ng\otimes_{\iota(e)}mh-\hat{e}R(ng\otimes_{\tau(e)}mh)+(\hat{e}R-1)ng\cdot mh\otimes_{C(e)}1)\\&=ng\cdot mh-\hat{e}Rng\cdot mh+(\hat{e}R-1)ng\cdot mh\\&=(1-\hat{e}R)ng\cdot mh+(\hat{e}R-1)ng\cdot mh=0.\end{aligned}$$

□

Example 3.2.2. *For* $G = H_u *_{H_e} H_v$,

$$q(ng \otimes mh) = (ng \otimes_{H_u} mh - (ng \otimes_{H_v} mh)$$

$$pq(ng \otimes mh) = p(ng \otimes_{H_u} mh - (ng \otimes_{H_v} mh))$$

$$= ng \cdot mh - ng \cdot mh = 0.$$

Example 3.2.3. *For* $G = HNN(H_v, H_{e^{\pm 1}})$,

$$q(ng \otimes mh)$$

$$= ng \otimes_{H_v} mh - \hat{e}R(ng \otimes_{H_v} mh) + (\hat{e}R - 1)ng \cdot mh \otimes 1$$

$$= (1 - \hat{e}R)(ng \otimes mh) + (\hat{e}R - 1)ng \cdot mh \otimes 1$$

$$p(kg' \otimes lh', \alpha g'' \otimes (1 - \hat{e}R)) = kg' \cdot lh' + \alpha g''(1 - \hat{e}R).$$

Thus

$$pq(ng \otimes mh)$$

$$= p((1 - \hat{e}R)(ng \otimes mh) + (\hat{e}R - 1)ng \cdot mh \otimes 1)$$

$$= (1 - \hat{e}R)ng \cdot mh + (\hat{e}R - 1)ng \cdot mh = 0.$$

Theorem 3.2.4. *There is a short exact sequence.*

(2) $0 \longrightarrow \oplus_{e \in E^+}(\mathbb{Z}G \otimes_{H_e} \mathbb{Z}) \xrightarrow{\partial} (\oplus_{v \in V}(\mathbb{Z}G \otimes_{H_v} \mathbb{Z})) \xrightarrow{\varepsilon^*} \mathbb{Z} \longrightarrow 0$

where ∂ *and* ε^* *are defined by*

$$\partial : ng \otimes m \longmapsto ng \otimes_{\iota(e)} m - \hat{e}Rng \otimes_{\tau(e)} m,$$

$$\varepsilon^* : (kg' \otimes l) \longmapsto \varepsilon(kg') \cdot l = k \cdot l$$

where $\varepsilon : \mathbb{Z}G \longrightarrow \mathbb{Z}$, *augmentation.*

Proof.

$$\begin{aligned}
&\varepsilon^*\partial(ng \otimes m) \\
&= \varepsilon^*(ng \otimes_{\iota(e)} m - \hat{e}R(ng \otimes_{\tau(e)} m) \\
&= \varepsilon^*(1 - \hat{e}R)(ng \otimes m) \\
&= \varepsilon((1 - \hat{e}R)ng) \cdot m = 0.
\end{aligned}$$

□

Example 3.2.5. *For* $G = H_u *_{H_e} H_v$,

$$\begin{aligned}
&\partial(ng \otimes m) = ng \otimes_{H_u} m - ng \otimes_{H_v} m, \\
&\varepsilon^*(kg' \otimes l) = \varepsilon(kg') \cdot l = k \cdot l.
\end{aligned}$$

Thus $\varepsilon^*\partial(ng \otimes m) = \varepsilon^*(ng \otimes_{H_u} m - ng \otimes_{H_v} m) = ng \cdot m - ng \cdot m = 0.$

Example 3.2.6. *For* $G = HNN(H_v, H_e)$,

$$\begin{aligned}
&\partial(ng \otimes m) = (ng \otimes_{H_v} m) + (-\hat{e}R)(ng \otimes_{H_v} m) = (1 - \hat{e}R)(ng \otimes_{H_v} m), \\
&\varepsilon^*(kg' \otimes l) = \varepsilon(kg') \cdot l = k \cdot l.
\end{aligned}$$

Thus $\varepsilon^*\partial(ng \otimes m) = \varepsilon((1 - \hat{e}R)ng) \cdot m = 0.$

The following is a relationship between the above Theorem 3.2.4 and Corollary 3.2.8.

Theorem 3.2.7. *$0 \longrightarrow \oplus_{e\in E^+}(\mathbb{Z}G \otimes_{H_e} IH_e) \xrightarrow{\bar{q}} (\oplus_{v\in V}(\mathbb{Z}G \otimes_{H_v} IH_v))$ $\oplus(\oplus_{e\in E^+-T}\mathbb{Z}G\otimes_{C_e}IC_e) \xrightarrow{\bar{p}} IG \longrightarrow 0$ is short exact if and only if $0 \longrightarrow \oplus_{e\in E^+}(\mathbb{Z}G\otimes_{H_e}\mathbb{Z}) \xrightarrow{\partial} (\oplus_{v\in V}(\mathbb{Z}G\otimes$ is short exact.*

Proof. we have the following commutative diagram :

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \oplus_{e\in E^+}(\mathbb{Z}G \otimes_{H_e} IH_e) \arrow[r, "\bar{q}"] \arrow[d] & \begin{matrix}(\oplus_{v\in V}(\mathbb{Z}G \otimes_{H_v} IH_v)) \\ \oplus(\oplus_{e\in E^+}\mathbb{Z}G \otimes_{HC_e} IC_e)\end{matrix} \arrow[r, "\bar{p}"] \arrow[d] & IG \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \oplus_{e\in E^+}(\mathbb{Z}G \otimes_{H_e} \mathbb{Z}H_e) \arrow[r, "q"] \arrow[d, "\varepsilon\,'"] & \begin{matrix}(\oplus_{v\in V}(\mathbb{Z}G \otimes_{H_v} \mathbb{Z}H_v)) \\ \oplus(\oplus_{e\in E^+}\mathbb{Z}G \otimes_{C_e} IC_e)\end{matrix} \arrow[r, "p"] \arrow[d, "\varepsilon^*"] & \mathbb{Z}G \arrow[r] \arrow[d, "\varepsilon"] & 0 \\
0 \arrow[r] & \oplus_{e\in E^+}(\mathbb{Z}G \otimes_{H_e} \mathbb{Z}) \arrow[r, "\partial"] \arrow[d] & \oplus_{v\in V}(\mathbb{Z}G \otimes_{H_v} \mathbb{Z}) \arrow[r, "\varepsilon^*"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where $\varepsilon^* = \{1 \otimes \varepsilon_1 + 1 \otimes \varepsilon_2\}$, $\varepsilon_1 : \mathbb{Z}H_v \longrightarrow \mathbb{Z}$, $\varepsilon_2 : \mathbb{Z}H_e \longrightarrow \mathbb{Z}$, augmentations. By [31], the third column is exact and by tensoring, the 1st and 2nd columns are exact. Middle row is from Theorem 3.2.1. Therefore by 3×3 Lemma, the proof is completed. ☐

Corollary 3.2.8. *There is a short exact sequence.*

$$(3)\quad 0 \longrightarrow \oplus_{e\in E^+}(\mathbb{Z}G \otimes_{H_e} IH_e) \xrightarrow{\bar{q}} (\oplus_{v\in V}(\mathbb{Z}G \otimes_{H_v} IH_v)) \\ \oplus(\oplus_{e\in E^+-T}\mathbb{Z}G \otimes_{C_e} IC_e) \xrightarrow{\bar{p}} IG \longrightarrow 0$$

where $\bar{q}$ *and* $\bar{p}$ *are the restriction of* q *and* p *in Theorem 3.2.1, respectively.*

We will consider the exactness of a short sequence which characterizes the relation module of the graphs of groups in terms of the relation module of free factors and vertex groups.

Theorem 3.2.9. *There is a short exact sequence.*

$$(4)\quad 0 \to \oplus_{v\in V}\mathbb{Z}G \otimes_{H_v} S_v/S_v{}' \xrightarrow{\rho} R/R' \xrightarrow{\psi} \oplus_{e\in E^+}\mathbb{Z}G \otimes_{H_e} IH_e \to 0$$

$$\rho\ (1 \otimes_s S_v{}') \ = \ sR' \ \ (s \in S_v)$$

$$\psi\ (sR') \ = \ 0$$

$$\psi\ (r_{i,e}R') \ = \ 1 \otimes \ (a_{i,e}R - 1)$$

where $a_{i,e}R$ *is a generator of* H_e*. Thus* $a_{i,e}R - 1$ *is a generator of* IH_e*.*

Then we get the following important corollaries.

Corollary 3.2.10. ([24] Theorem 1) *If* $G = H_u *_{H_e} H_v$ *then*

$$0 \to (\mathbb{Z}G \otimes_{H_u} S_u/S_u{}') \oplus (\mathbb{Z}G \otimes_{H_v} S_v/S_v{}') \to R/R' \to \mathbb{Z}G \otimes_{H_e} IH_e \to 0$$

is exact.

Corollary 3.2.11. ([24] Theorem 2) *If* $G = HNN(H_v, H_e)$ *then*

$$0 \to \mathbb{Z}G \otimes_{H_v} S_v/{S_v}' \to R/R' \to \mathbb{Z}G \otimes_{H_e} IH_e \to 0$$

is exact.

$$(g \otimes s{S_u}',\ h \otimes \bar{s}{S_u}') \xrightarrow{\rho} g \cdot sR' + h \cdot \bar{s}R' \xrightarrow{\psi} 0$$

$$\psi(g \cdot sR') = g\psi(sR') = 0 \textit{ and}$$

$$\psi(g \cdot r_{i,e}R') = g \otimes (a_{i,e}R - 1).$$

Thus ψ *is onto (*$a_{i,e}R - 1$ *: generators of* IH_e *as* $\mathbb{Z}H_e$*-module).*

Now we consider short exact sequences concerning the second homotopy modules of graphs of groups and vertex groups.

$P_v^{(2)}$ is a free $\mathbb{Z}H_v$-module with bases $b_{s,v}(s \in \mathbf{s}_v)$

i.e., $P_v^{(2)} = \oplus_{s\in\mathbf{s}_v}\mathbb{Z}H_v b_{s,v}$

$P^{(2)}$ is free $\mathbb{Z}G$-module with bases $b_s(s \in \mathbf{s}_v),\ b_{r_{i,e}}(i \in I(e),\ e \in E^+)$

$P^{(2)} = \oplus_{v\in V}(\oplus_{s\in\mathbf{s}_v})\mathbb{Z}G\overline{b_{s,v}} \oplus (\oplus_{e\in E^+}\mathbb{Z}G\overline{b_{i,e}})$

where $\overline{b_{s,v}}$ and $\overline{b_{i,e}}$ are corresponding to $b_{s,v}$ and $b_{i,e}$ respectively.

$P_e^{(1)}$ is a free $\mathbb{Z}H_e$-module with bases $b_{i,e}(i \in I(e))$.

Theorem 3.2.12. *There is a short exact sequence.*

$$(5) \quad 0 \to \oplus_{v\in V}\mathbb{Z}G \otimes_{H_v} P_v^{(2)} \xrightarrow{\alpha} P^{(2)} \xrightarrow{\beta} \oplus_{e\in E^+}\mathbb{Z}G \otimes_{H_e} P_e^{(1)} \to 0$$

$$\alpha(1 \otimes_{H_v} b_{s,v}) = \overline{b_{s,v}}.$$

$$\beta(\overline{b_{s,v}}) \;=\; 0.$$

$$\beta(\overline{b_{i,e}}) \;=\; -1 \otimes_{H_e} b_{i,e}.$$

Now we study exact sequences concerning the second homotopy modules. From Theorem 3.1.3, we need to consider two kinds of identity sequences for $\pi_2(\wp)$.

$<\sigma>_{\wp}$ means that we consider σ as an identity sequence over $\wp$.

Theorem 3.2.13. *There is a short exact sequence.*

$$(6) \quad 0 \to \oplus_{v\in V}\mathbb{Z}G \otimes_{H_v} \Pi_2(Q_v) \xrightarrow{\xi} \Pi_2(\wp) \xrightarrow{\eta} \oplus_{e\in E^+} \mathbb{Z}G \otimes_{H_e} N_e/N_e' \to 0$$

$$\xi\ (1\otimes <\sigma>_{Q_v}) = <\sigma>_{\wp}$$

$$\eta\ (<\sigma>_{\wp}) = 0 \ (\sigma \text{ is an identity sequence over } \mathrm{Q_v})$$

$$\eta\ (<\sigma_W>_{\wp}) = 1 \otimes W{N_e}' \ (W \in N_e).$$

Lemma 3.2.14. *The following diagram is commutative*

$$\begin{tikzcd}
0 \arrow[r] & \oplus_{v\in V}\mathbb{Z}G \otimes_{H_v} P_v^{(2)} \arrow[r, "\alpha"] \arrow[d, "\oplus_{v\in \mathbf{v}}(1\otimes\phi_v)"] & P^{(2)} \arrow[r, "\beta"] \arrow[d, "\phi"] & \oplus_{e\in E^+}\mathbb{Z}G \otimes_{H_e} P_e^{(1)} \arrow[r] \arrow[d, "\oplus_{e\in E^+}(1\otimes\theta_e)"] & 0 \\
0 \arrow[r] & \oplus_{v\in V}\mathbb{Z}G \otimes_{H_v} S_v/{S_v}' \arrow[r, "\rho"] & R/R' \arrow[r, "\psi"] & \oplus_{e\in E^+}\mathbb{Z}G \otimes_{H_e} IH_e \arrow[r] & 0
\end{tikzcd}$$

$$\phi_v \ : \ P_v^{(2)} \longrightarrow S_v/{S_v}'$$

$$\theta_e \ : \ P_e^{(1)} \longrightarrow IH_e.$$

Proof.

$$\phi(\alpha(1 \otimes b_{s,v})) = \phi(b_s) = sR'.$$

$$\rho(1 \otimes \phi_v)(1 \otimes b_{s,v}) = \rho(1 \otimes sS_v{}') = sR'.$$

$$(1 \otimes \theta_e)(\beta(b_{s,v})) = O.$$

$$(1 \otimes \theta_e)(\beta(b_{r_{i,e}})) = (1 \otimes \theta_e)(-1 \otimes b_{i,e}) = -1 \otimes (1 - a_{i,e}R) and$$

$$\psi\phi(b_s) = \psi(sR') = O.$$

$$\psi\phi(b_{r_{i,e}}) = \psi(r_{i,e}R') = 1 \otimes (a_{i,e}R - 1).$$

□

Lemma 3.2.15. *The following diagram is commutative*

$$\begin{tikzcd}
0 \arrow[r] & \oplus_{v\in V}\mathbb{Z}G \otimes_{H_v} \Pi_2(Q_v) \arrow[r, "\xi"] \arrow[d, "\oplus_{v\in V}(1\otimes\kappa_v)"] & \Pi_2(\wp) \arrow[r, "\eta"] \arrow[d, "\kappa"] & \oplus_{e\in E^+} \mathbb{Z}G \otimes_{H_e} N_e/N_e' \arrow[r] \arrow[d, "\oplus_{e\in E^+}(1\otimes\mu_e)"] & 0 \\
0 \arrow[r] & \oplus_{v\in V}\mathbb{Z}G \otimes_{H_v} P_v{}^{(2)} \arrow[r, "\alpha"] & P^{(2)} \arrow[r, "\beta"] & \oplus_{e\in E^+}\mathbb{Z}G \otimes_{H_e} P_e^{(1)} \arrow[r] & 0.
\end{tikzcd}$$

Proof. $\sigma = (W_j\, s_j^{\varepsilon_j}\, W_j^{-1}) :\ W_j$ is a word on $\mathbf{x}_v,\ s_j \in \mathbf{s}_v$.

$$\begin{aligned}\alpha(1\otimes\kappa_v)(1\otimes <\sigma>_{Q_v}) &= \alpha\left(1\otimes\sum_{s\in\mathbf{s}_v}\varepsilon_j\, W_j\, S_v\, b_{s,v}\right)\\ &= \alpha\left(\sum_{s\in\mathbf{s}_v}\varepsilon_j\, W_j\, S_v\,(1\otimes b_{s,v})\right) = \sum_{s\in\mathbf{s}_v}\varepsilon_j\, W_j\, R\, b_s.\end{aligned}$$

$$\kappa(\xi(1\otimes <\sigma>_{Q_v})) = \kappa(<\sigma>_{\wp}) = \sum_{s\in\mathbf{s}_v}\varepsilon_j\, W_j\, R\, b_s.$$

$$1\otimes\mu_e(\eta(<\sigma>_{\wp})) = 1\otimes\mu_e(0) = 0,\quad <\sigma>\ \in\ \Pi_2(p_v).$$

$$1\otimes\mu_e(\eta(<\sigma>_{\wp})) = (1\otimes\mu_e)(1\otimes W N_e') = 1\otimes\sum_{i\in I(e)}\rho_e\left(\frac{\partial W}{\partial y_{i,e}}\right)\overline{b_{i,e}}.$$

$$\beta\kappa(<\sigma>_{\wp}) = \beta\left(\sum_{s\in\mathbf{s}_v}\varepsilon_j\, W_j\, Rb_{s_j,v}\right) = 0.$$

$$\beta\kappa(<\sigma_W>_{\wp}) = \beta\left(\bigcup b_s + \sum_{i\in I(e)}\rho_e\left(\frac{\partial W}{\partial y_{i,e}}\right) b_{i,e}\right) = \sum_{i\in I(e)}\rho_e\left(\frac{\partial W}{\partial y_{i,e}}\right)(1\otimes\overline{b_{i,e}}).$$

□

From these, we get the following

Theorem 3.2.16. *(6) is short exact if and only if (4) is short exact.*

Proof. Consider the following commutative diagram :

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
O \arrow[r] & \oplus_{v\in V}\mathbb{Z}G\otimes_{H_v}\Pi_2(Q_v) \arrow[r, "\xi"] \arrow[d, "\oplus_{v\in V}(1\otimes\kappa_v)"] & \Pi_2(\wp) \arrow[r, "\eta"] \arrow[d, "\kappa"] & \oplus_{e\in E^+}\mathbb{Z}G\otimes_{H_e}N_e/N_e' \arrow[r] \arrow[d, "\oplus_{e\in E^+}(1\otimes\mu_e)"] & O \\
O \arrow[r] & \oplus_{v\in V}\mathbb{Z}G\otimes_{H_v}P_v{}^{(2)} \arrow[r, "\alpha"] \arrow[d, "\oplus_{v\in\mathbf{v}}(1\otimes\phi_v)"] & P^{(2)} \arrow[r, "\beta"] \arrow[d, "\phi"] & \oplus_{e\in E^+}\mathbb{Z}G\otimes_{H_e}P_e^{(1)} \arrow[r] \arrow[d, "\oplus_{e\in E^+}(1\otimes\theta_e)"] & O \\
O \arrow[r] & \oplus_{v\in V}\mathbb{Z}G\otimes_{H_v}S_v/S_v{}' \arrow[r, "\rho"] \arrow[d] & R/R' \arrow[r, "\psi"] \arrow[d] & \oplus_{e\in E^+}\mathbb{Z}G\otimes_{H_e}IH_e \arrow[r] \arrow[d] & O \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

By Section 2.5 (1), the 1st and 2nd column are exact. By Section 2.5 (3), the third column is exact. Therefore by 3×3 Lemma, the proof is completed. ☐

Theorem 3.2.17. *(4) is short exact ⇒ (2) is short exact.*

Proof. Consider the following diagram :

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[r] \arrow[d] & \oplus(\mathbb{Z}G \otimes_{H_e} IH_e) \arrow[d] & \\
0 \arrow[r] & \oplus\mathbb{Z}G \otimes_{H_v} S_v/S_v{}' \arrow[r, "\rho"] \arrow[d, "1\otimes\mu_v"] \arrow[dr, phantom, "(I)"] & R/R\,' \arrow[r, "\psi"] \arrow[d, "\mu"] & \oplus(\mathbb{Z}G \otimes_{H_e} IH_e) \arrow[r] & 0 \\
 & \begin{array}{c} \otimes(\mathbb{Z}G \otimes_{H_v} P_v^{(1)}) \\ \oplus(\oplus\mathbb{Z}G \otimes_{C_e} IC_e) \end{array} \arrow[r, "\delta"] \arrow[d, "\alpha"] \arrow[dr, phantom, "(II)"] & P^{(1)} \arrow[r] \arrow[d, "\theta"] & 0 & \\
 & \begin{array}{c} \otimes(\mathbb{Z}G \otimes_{H_v} IH_v) \\ \oplus(\oplus\mathbb{Z}G \otimes_{C_e} IC_e) \end{array} \arrow[r, "p"] \arrow[d] & IG \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

where $\alpha = (1 \otimes \theta) \oplus id.$

We consider commutativity (I):

$$\mu\rho(1 \otimes sS_{v}{}') = \mu(sR\,') = \sum(\tfrac{\partial S}{\partial x})_{H_v} t_x$$

and

$$\begin{aligned} \delta(1 \otimes \mu_v)(1 \otimes sS_{v}{}') &= \delta\left(\sum 1 \otimes (\tfrac{\partial S}{\partial x})_{H_v} t_{x,v}\right) \\ &= \delta\left(\sum(\tfrac{\partial S}{\partial x})_{H_v}(1 \otimes t_{x,v})\right) = \sum(\tfrac{\partial S}{\partial x})_{H_v} t_x. \end{aligned}$$

We consider commutativity (II):

$$\theta\delta(1 \otimes t_{x,v}) = \theta(t_{x,v}) = 1 - xR$$
$$\theta\delta(1 \otimes (1 - \hat{e}R) = \theta(t_e) = 1 - \hat{e}R$$
$$p\alpha(1 \otimes t_{x,v}) = p(1 \otimes (1 - xR)) = 1 - xR$$
$$p\alpha(1 \otimes (1 - \hat{e}R)) = p(1 \otimes (1 - \hat{e}R)) = 1 - \hat{e}R.$$

Since ψ and δ is surjective, we can define ψ^{-1} and δ^{-1} by the axiom of choice. For each generator $1 \otimes (1 - a_{i,e}R)$ of $\mathbb{Z}G \otimes_{H_e} IH_e$, $\psi^{-1}(1 \otimes (1 - a_{i,e}R)) = -r_{i,e}R'$. For $t_{x,v}$ and t_e, $\delta^{-1}(t_{x,v}) = 1 \otimes t_{x,v}$ and $\delta^{-1}(t_e) = 1 \otimes (1 - \hat{e}R)$.

Consider the mapping $\alpha\delta^{-1}\mu\psi^{-1}$.

Let $b \in \mathrm{im}(\alpha\delta^{-1}\mu\psi^{-1})$. Then there exists $a \in \oplus\mathbb{Z}G \otimes_{H_e} IH_e$ such that $\alpha\delta^{-1}\mu\psi^{-1}(a) = b$.

$$q(b) = q\alpha\delta^{-1}\mu\psi^{-1}(a) = \theta\delta\delta^{-1}\mu\psi^{-1}(a) = \theta\mu\psi^{-1}(a) = 0.$$

Therefore $\mathrm{im}(\alpha\delta^{-1}\mu\psi^{-1}) \subset \ker q$.

Suppose $b \in \ker q$ i.e., $q(b) = 0$.

Since α is onto, there is an element $c \in \oplus(\mathbb{Z}G \otimes_{H_v} \mathcal{P}_v^{(1)}) \oplus (\oplus_{\bar{E}^+}\mathbb{Z}G \otimes_{C_e} IC_e)$ such that $\alpha(c) = b$. Then $0 = q(b) = q\alpha(c) = \theta\delta(c)$.

Thus $\delta(c) \subset \ker\theta$. Since $\ker\theta = \mathrm{im}\mu$, there is an element $d \in R/R'$ such that $\mu(d) = \delta(c)$.

Take $\psi(d) = a$ then $\alpha\delta^{-1}\mu\psi^{-1}(a) = b$.

Thus $\ker q \subset \mathrm{im}(\alpha\delta^{-1}\mu\psi^{-1})$.

Thus $\ker q = \operatorname{im}(\alpha\delta^{-1}\mu\psi^{-1})$.

On the other hand, by (I), $\alpha\delta^{-1}\mu\psi^{-1} = \alpha(1 \otimes \mu_v) = 0$ and by (II) $p\alpha\delta^{-1}\mu\psi^{-1} = p\alpha\delta^{-1}\mu = \theta\mu = 0$. Therefore we get the exactness of (2).

Actually we know $\alpha\delta^{-1}\mu\psi^{-1} = q$ by the following calculations, where q is in Theorem 3.2.1.

$$q(1\otimes(1-a_{i,e}R)) = 1\otimes_{\iota(e)}(1-a_{i,e}R)+(-\hat{e}R\otimes_{\tau(e)}(1-a_{i,e}R)+(1-a_{i,e}R)\otimes(\hat{e}R-1))$$

$$\begin{aligned}
&\alpha\delta^{-1}\mu\psi^{-1}(1\otimes(1-a_{i,e}R)) = \alpha\delta^{-1}\mu(-r_{i,e}R\,')\\
&= -\alpha\delta^{-1}\Big(\sum_{x\in\mathbf{x}_{\iota(e)}}\big(\tfrac{\partial a_{i,e}}{\partial x}\big)_G t_x - (a_{i,e}\hat{e}a^{-1}_{i,e^{-1}})\sum_{x\in\mathbf{x}_{\tau(e)}}\big(\tfrac{\partial a_{i,e^{-1}}}{\partial x}\big)_G t_x\\
&\qquad\qquad +((a_{i,e})_G-1)\; t_{\hat{e}}(t_{\hat{e}} = 0 \; if \;\, e\in T)\\
&= -\alpha\Big(\sum_{v\in\mathbf{x}}\big(\tfrac{\partial a_{i,e}}{\partial x}\big)_G t_{x,\iota(e)} - \hat{e}R\sum_{x\in\mathbf{x}_{\tau e}}\big(\tfrac{\partial a_{i,e^{-1}}}{\partial x}\big)_G t_{x,v}\\
&\qquad\qquad +(a_{i,e}R-1)(1\otimes(1\hat{e}R))\\
&= -\big(\textstyle\sum\big(\tfrac{\partial a_{i,e}}{\partial x}\big)_G(1\otimes(1-xR) - \hat{e}R\sum\big(\tfrac{\partial a_{i,e^{-1}}}{\partial x}\big)_G(1\otimes(1-xR)\\
&\qquad\qquad +(a_{i,e}R-1)(1\otimes(1-\hat{e}R))\\
&= -(1\otimes_{\iota(e)}(1-a_{i,e}R) - \hat{e}R\otimes_{\tau(e)}(1-a^{-1}_{i,e^{-1}}R)\\
&\qquad\qquad +(a_{i,e}R-1)\otimes_{C_e}(1-\hat{e}R)
\end{aligned}$$

Therefore $q = \alpha\delta^{-1}\mu\psi^{-1}$ in Theorem 3.2.1. □

Chapter 4

Graph products

4.1 Graph products

In this chapter we will describe homology(cohomology) of a graph product in terms of homology(cohomology) of each vertex group and consider the efficiency of a presentation of a graph product in connection with presentation of vertex groups.

For each $v \in V$, let K_v be the group defined by a presentation $\wp_v = < \mathbf{a}_v \ ; \ \mathbf{s}_v >$ which is called a *vertex group* and for each $u\,v \in E^+$, let $\mathbf{r}_{\mathrm{u,\,v}}$ consist of all words $[a,\ b] = aba^{-1}b^{-1}(a \in \mathbf{a}_u,\ b \in \mathbf{a}_v)$.

Let $\mathbf{a} = \bigcup\limits_{v \in V} \mathbf{a}_v, \quad \mathbf{s} = \bigcup\limits_{v \in V} \mathbf{s}_v, \quad \mathbf{r} = \bigcup\limits_{u\ v \in E^+} \mathbf{r}_{\mathrm{u,\,v}}.$

Let $\wp$ be the group presentation $< \mathbf{a} \ ; \ \mathbf{s},\ \mathbf{r} >$. The group G defined by $\wp$ is called a *graph product of the groups* $K_v(v \in V)$ ([10], [18], [19] and [20]).

If all $K_v (v \in V)$ are infinite cyclic group then G is called a *graph group* ([13], [14], [15], [16] and [35]). A graph product has two extreme cases. If the edge set E^+ is empty then G is the free product of the groups K_v $(v \in V)$. If a graph Γ is complete then G is the direct product of the groups K_v $(v \in V)$. Therefore a graph product shares many of the interesting properties of these extreme cases ([18]).

We assume for the rest of this chapter that Γ is finite.

4.2 Calculation of H_2 and H^2 from Π_2

Let $\wp =< \mathbf{x} \ ; \ \mathbf{r} >$ be a finite group presentation and let G be the group defined by $\wp$. Let $\Pi_2 = \Pi_2(\wp)$ be the module of equivalence classes of identity sequences over $\wp$.

Let L be a set of identity sequences such that $\{< \sigma > \ ; \ \sigma \in L\}$ generates Π_2 (as a $\mathbb{Z}G$-module). Then there is a partial free resolution of the trivial left G-module $\mathbb{Z}$

$$\bigoplus_{\sigma \in L} \mathbb{Z}Gt_\sigma \xrightarrow{\delta_3} \bigoplus_{R \in \mathbf{r}} \mathbb{Z}Gt_R \xrightarrow{\delta_2} \bigoplus_{x \in \mathbf{x}} \mathbb{Z}Gt_x \xrightarrow{\delta_1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

(see [31]). Thus if A is any right G-module we have

$$H_2(G, A) = \frac{\ker \ 1 \otimes \delta_2}{\mathrm{im} \ 1 \otimes \delta_3} \ .$$

In particular, taking A to be the trivial right G-module $\mathbb{Z}$ we have

(1) $$H_2(G) = \ker \ \delta_2 \ / \ \mathrm{im} \ \delta_3$$

where δ_2 and δ_3 are the maps

$$\delta_2 \ : \ \bigoplus_{R\in\mathbf{r}} \mathbb{Z}t_R \longrightarrow \bigoplus_{x\in\mathbf{x}} \mathbb{Z}t_x, \qquad t_R \longmapsto \sum_{x\in\mathbf{x}} exp_x(R)t_x \tag{2}$$

$$\delta_3 \ : \ \bigoplus_{\sigma\in\mathbf{L}} \mathbb{Z}t_\sigma \longrightarrow \bigoplus_{R\in\mathbf{r}} \mathbb{Z}t_R, \qquad t_\sigma \longmapsto \sum_{R\in\mathbf{r}} exp_R(\sigma)t_R \tag{3}$$

(Here $exp_R(\sigma)$ is the *exponent sum* of R in σ, and $exp_x(R)$ is the *exponent sum* of x in R).

Similarly if B is any left $\mathbb{Z}G$-module we have

$$H^2(G,\ B) = \frac{\ker\ Hom_{\mathbb{Z}G}(\delta_3,\ 1)}{\mathrm{im}\ Hom_{\mathbb{Z}G}(\delta_2,\ 1)}.$$

In particular, taking B to be the trivial left G-module $\mathbb{Z}$ we have

$$H^2(G) = \ker\ {\delta_3}^* \ / \ \mathrm{im}\ {\delta_2}^* \tag{4}$$

where ${\delta_2}^*$ and ${\delta_3}^*$ are the maps

$${\delta_2}^* \ : \ \bigoplus_{x\in\mathbf{x}} \mathbb{Z}{t_x}^* \longrightarrow \bigoplus_{R\in\mathbf{r}} \mathbb{Z}{t_R}^*, \qquad {t_x}^* \longmapsto \sum_{R\in\mathbf{r}} exp_x(R){t_R}^* \tag{5}$$

$${\delta_3}^* \ : \ \bigoplus_{R\in\mathbf{r}} \mathbb{Z}{t_R}^* \longrightarrow \bigoplus_{\sigma\in L} \mathbb{Z}{t_\sigma}^*, \qquad {t_R}^* \longmapsto \sum_{\sigma\in L} exp_R(\sigma){t_\sigma}^*. \tag{6}$$

We let

$$Z_2 \ (\text{or } \mathrm{Z}_2(\wp)) \ = \ \ker\ \delta_2 \tag{7}$$

$$B_2 \ (\text{or } \mathrm{B}_2(\wp)) \ = \ \mathrm{im}\ \delta_3 \tag{8}$$

$$Z^2 \ (\text{or } \mathrm{Z}^2(\wp)) \ = \ \ker\ {\delta_3}^* \tag{9}$$

$$B^2 \ (\text{or } \mathrm{B}^2(\wp)) \ = \ \mathrm{im}\ {\delta_2}^* \tag{10}$$

4.3 H_2 and H^2 of a graph products

Let L and $L_v (v \in V)$ be sets of identity sequences which generate $\Pi_2(\wp)$ and $\Pi_2(\wp_v)$ $(v \in V)$ respectively.

For each triangle $\{u,\ v,\ w\}$ in Γ we have a collection of spherical pictures of the form depicted in Figure 4.3.1 with $a \in \mathbf{a}_u,\ b \in \mathbf{b}_v,\ c \in \mathbf{c}_w$.

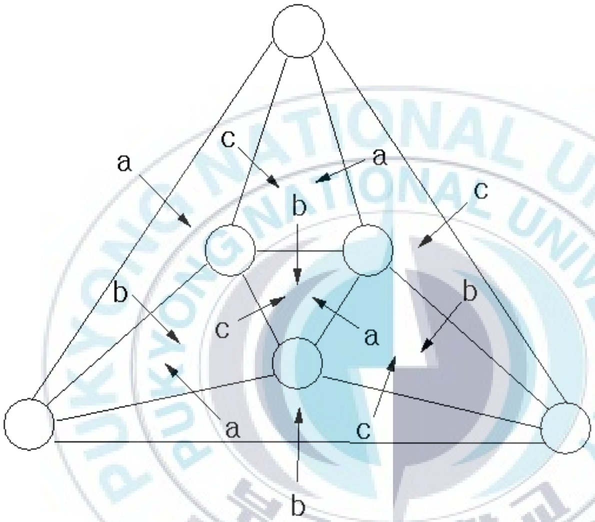

Figure 4.3.1. Triangle σ_Δ.

For each $e \in E$, with $\iota(e) = u,\ \tau(e) = v,$ let $S = x_1 x_2 \cdots x_n \in \mathbf{s}_u$. Then for each $b \in \mathbf{b}_v$, we have a spherical picture $\sigma_{S,b}$ over $\wp$ of the form depicted in Figure 4.3.2. Similarly we get $\sigma_{T,a}$ $(T = y_1 y_2 \cdots y_n \in \mathbf{s}_v, a \in \mathbf{a}_u)$.

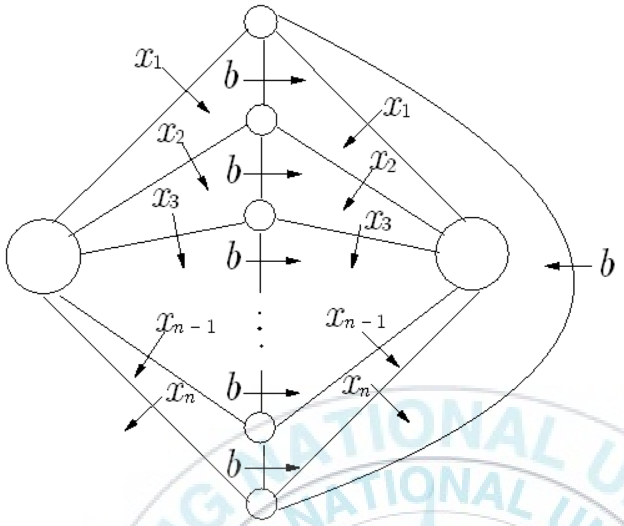

Figure 4.3.2. $\sigma_{S,b}$.

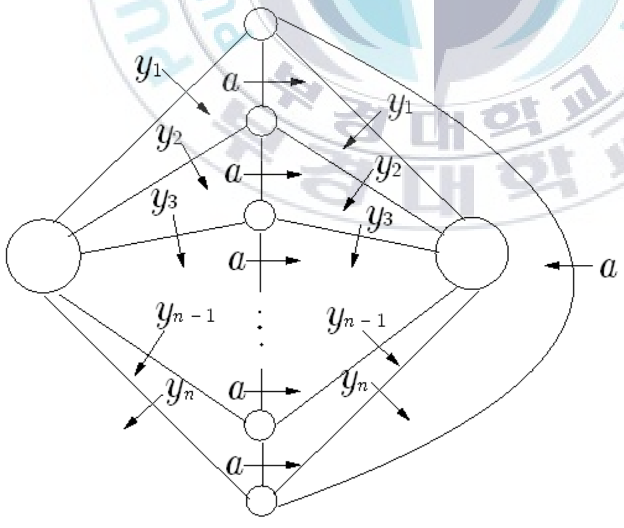

Figure 4.3.3. $\sigma_{T,a}$.

Theorem 4.3.1. ([2] Theorem 3.1.4) *$\pi_2(\wp)$ is generated by*

$$L \;=\; (\bigcup_{v\in V} L_v) \;\cup\; M \;\cup\; Y$$

where

$$M \;=\; \{\sigma_{S,b} \mid uv\in E^+,\ S\in \mathbf{s}_u,\ b\in \mathbf{a}_v\}\cup\{\sigma_{T,a} \mid uv\in E^+,\ T\in \mathbf{s}_v,\ a\in \mathbf{a}_u\}$$

and

$$Y=\{\mathbb{P}_\Delta \mid \Delta=\{u,\ v,\ w\}\ \textit{triangle in}\ \Gamma\}.$$

Remark 4.3.2. An identity sequence σ is said to be *cockroft* if $exp_R(\sigma)=0$ for all $R\in \mathbf{r}$.

Since we need to consider L_v and M only, (7), (8), (9) and (10) are given as follows.

$$\begin{aligned} Z_2(\wp) &= (\bigoplus_{v\in V} Z_2(\wp_v)) \bigoplus (\bigoplus_{R\in\mathbf{r}} \mathbb{Z}t_R) \\ B_2(\wp) &= (\bigoplus_{v\in V} B_2(\wp_v)) \bigoplus U \end{aligned}$$

where U is the image of the map

$$\bigoplus_{\sigma\in\mathbf{M}} \mathbb{Z}t_\sigma \longrightarrow \bigoplus_{R\in\mathbf{r}} \mathbb{Z}t_R \ \ by\ \ t_\sigma \longmapsto \sum_{R\in\mathbf{r}} exp_R(\sigma)t_R.$$

And

$$\begin{aligned} Z^2(\wp) &= (\bigoplus_{v\in V} Z^2(\wp_v)) \bigoplus K \\ B^2(\wp) &= \bigoplus_{v\in V} B^2(\wp_v) \end{aligned}$$

where K is the kernel of the map

$$\bigoplus_{R\in\mathbf{r}} \mathbb{Z}{t_R}^* \longrightarrow \bigoplus_{\sigma\in\mathbf{M}} \mathbb{Z}{t_\sigma}^* \ \ by \ \ {t_R}^* \longmapsto \sum_{R\in\mathbf{r}} exp_R(\sigma){t_R}^*.$$

Therefore we get following Theorem.

Theorem 4.3.3.

$$(i) \quad H_2(G) \ \cong \ (\bigoplus_{v\in V} H_2(K_v)) \ \bigoplus \ (\bigoplus_{R\in\mathbf{r}} \mathbb{Z}t_R \ / \ U)$$

$$(ii) \quad H^2(G) \ \cong \ (\bigoplus_{v\in V} H^2(K_v)) \ \bigoplus \ K.$$

Example 4.3.4. *Let* K_u *be the group defined by* $\wp_u =< a,\ b \mid a^3,\ b^2,\ abab >$, *and let* K_v *be the group defined by* $\wp_v =< x \mid x^2 >$, *i.e,* $K_u = S_3,\ K_v = \mathbb{Z}_2$. *Let* $a^3 = S_1,\ b^2 = S_2,\ abab = S_3,\ x^2 = T$. *Then* $\sigma_{S_1,\,x},\ \ \sigma_{S_2,\,x},\ \ \sigma_{S_3,\,x},\ \ \sigma_{T,\,a}$, *and* $\sigma_{T,\,b}$ *are following pictures.*

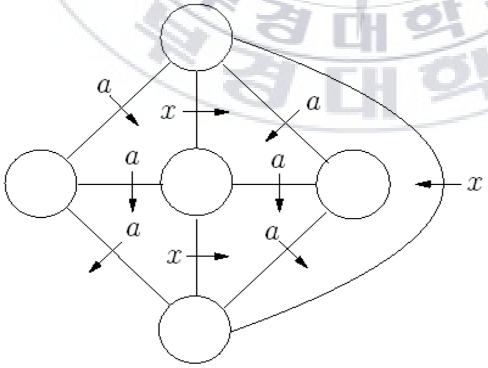

Figure 4.3.4. $\sigma_{S_1,\,x}$.

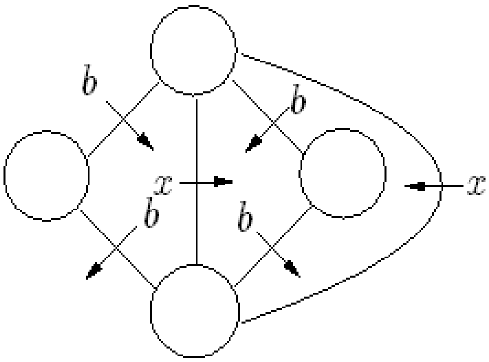

Figure 4.3.5. $\sigma_{S_2,\, x}$.

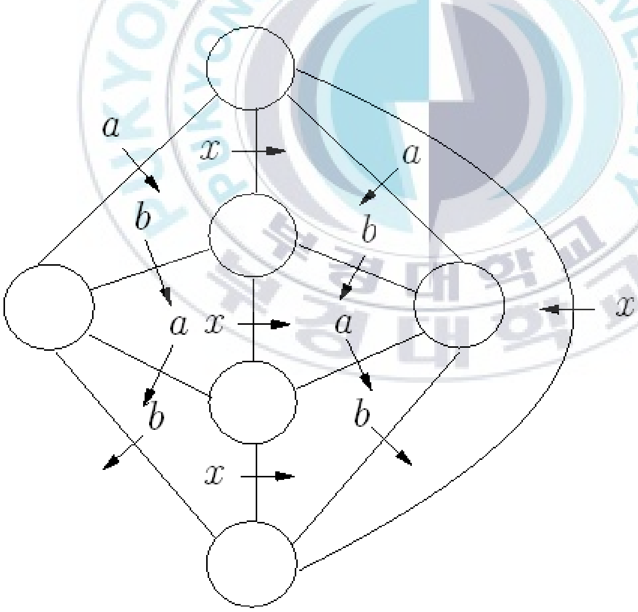

Figure 4.3.6. $\sigma_{S_3,\, x}$.

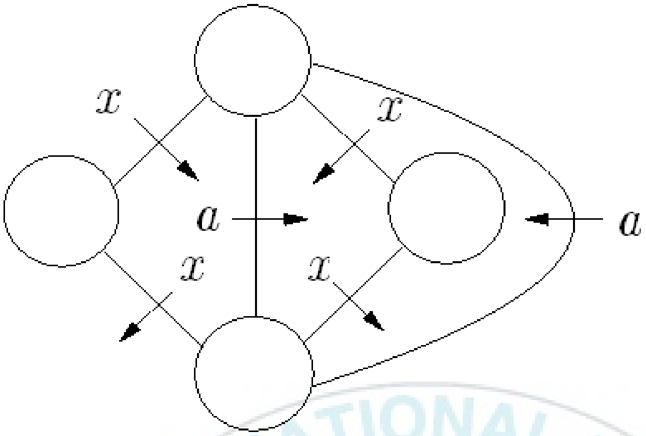

Figure 4.3.7. $\sigma_{T,\,a}$.

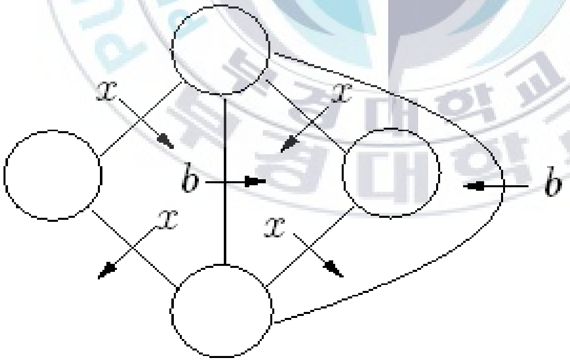

Figure 4.3.8. $\sigma_{T,\,b}$.

And

$$\delta_3(t\sigma_{S_1,\,x}) = 3tR_1, \quad R_1 = [a,\ x]$$

$$\delta_3(t\sigma_{S_2,\,x}) = 2tR_2, \quad R_2 = [b,\ x]$$

$$\delta_3(t\sigma_{S_3,\,x}) = 2tR_1 + 2tR_2$$

$$\delta_3(t\sigma_{T,\,a}) = -2tR_1$$

$$\delta_3(t\sigma_{T,\,b}) = -2tR_2.$$

Since $tR_1,\ tR_2$ *are independent, from* $2tR_1 + 2tR_2 = 0,\ \ 2tR_1 = 0,$ $2tR_2 = 0.$

$$\begin{aligned} &(\mathbb{Z}t_{R_1} \oplus \mathbb{Z}t_{R_2})/U \\ =\ &< t_{R_1}, t_{R_2} \mid 3t_{R_1},\ 2t_{R_2},\ 2t_{R_1} > \\ =\ &\mathbb{Z}t_{R_2}/2\mathbb{Z}t_{R_2} \cong \mathbb{Z}_2. \end{aligned}$$

Therefore $H_2(S_3 \oplus \mathbb{Z}_2) = H_2(S_3) \oplus \mathbb{Z}_2,$ *because* $H_2(\mathbb{Z}_2) = 0.$

Remark 4.3.5. *The second homology groups of cyclic groups are trivial, because the second homology module of infinite cyclic groups and finite cyclic groups are generated by the empty and dipoles, respectively.*

If K_u and K_v are infinite cyclic group then $\wp_u =< a \mid\ \ >,\ \wp_v =< b \mid\ \ >,$ and U is trivial. Therefore $\mathbb{Z}t_{[a,b]} \cong \mathbb{Z}.$ We has a direct summand for each edge $e \in E^+$.

Corollary 4.3.6. *If all* $K_v (v \in V)$ *are infinite cyclic groups then*

$$H_2(G) \ \cong\ \bigoplus_{|E^+|} \mathbb{Z}.$$

If $\wp_u =< a \mid \quad >$, $\wp_v =< b \mid b^n >$, then $U = gp\{nt_R\}$ where $R = [a, b]$. Therefore $\mathbb{Z}t_R/U$ has a presentation $< t_R \,|nt_R >$. Therefore $\mathbb{Z}t_R/U \;\cong\; \mathbb{Z}_n$.

Corollary 4.3.7. *If K_u are infinite cyclic group and K_v are finite cyclic group of order n then*

$$H_2(G) \;\cong\; \mathbb{Z}_n.$$

If $\wp_u =< a \mid a^m >$ and $\wp_v =< b \mid b^n >$, then $U = gp\{mt_R,\ nt_R\}$ where $R = [a, b]$. Therefore $\mathbb{Z}t_R/U$ has a presentation $< t_R \,|mt_R,\ nt_R >$. Therefore $\mathbb{Z}t_R/U \;\cong\; \mathbb{Z}_d$ where d is the greatest common divisor of $m,\ n$. Therefore we get

Corollary 4.3.8. *If K_u and K_v are finite cyclic groups of order m and n respectively then*

$$H_2(G) \;\cong\; \mathbb{Z}_m \bigoplus \mathbb{Z}_n.$$

4.4 Efficiency

We regard a finite group presentation $\wp =< \mathbf{x}\ ;\ \mathbf{r} >$ as a $2 - CW$ complex with one vertex. Then we get a chain complex

$$\oplus_{R\in\mathbf{r}}\mathbb{Z}t_R \xrightarrow{\delta_2} \oplus_{x\in\mathbf{x}}\mathbb{Z}t_x \xrightarrow{\varepsilon\delta_1} \mathbb{Z} \longrightarrow 0.$$

Let

$$\begin{aligned} H_2\wp &= \ker\delta_2 \\ H_1\wp &= \oplus_{x\in\mathbf{x}}\,\mathbb{Z}t_x \;/\; \mathrm{im}\delta_2 \\ H_0\wp &= \mathbb{Z} \end{aligned}$$

and let $\chi(\wp)$ be the Euler characteristic, i.e., $\chi(\wp) = 1 - |\mathbf{x}| + |\mathbf{r}|$.

Then we know that

$$\chi(G) = rk(H_0\wp) - rk(H_1\wp) + rk(H_2\wp) = 1 - rk(G/G') + rk(H_2\wp),$$

where rk( ) means the rank of the torsion-free part.

Let $\mu(G) = 1 - rk(H_1G) + d(H_2G)$ where $d(\ )$ means the least number of generators. It is well-known [23] that $\chi(\wp) \geq \mu(G)$.

Now we consider the efficiency of groups and presentations.

Definition 4.4.1.

(a) A presentation $\wp$ *for a group* G *is called efficient if* $\chi(\wp_0) = \mu(G)$.

(b) A group G *is called efficient if there is an efficient presentation for* G, *and if not then* G *is called non-efficient.*

(c) A presentation $\wp_0$ *is called minimal if* $\chi(\wp_0) \leq \chi(\wp)$ *for any presentation* $\wp$ *for* G.

B. H. Neumann ([29]) asked all finite groups are efficient whenever their second homology groups, the Schur multiplicators, are trivial. The example made by Swan ([36]) showed that the answer is negative. After that, many important results about efficiency have been established.

The efficency of groups and presentations are studied as follows.

Remark 4.4.2. *Examples of non-efficient groups.*

([2], [4], [26], [36]).

Remark 4.4.3. $\wp =< a, b, c, d \mid a^2, b^2, c^2, d^2, (bc)^3, (cd)^5, [a, b], [c, d], [a, d] >$
By Tietze transformations, we get another presentation

$$\wp' =< d,\ t,\ u \mid (tud)^2,\ d^2,\ (ud)^2,\ u^5 = t^3,\ u^{10} > .$$

And by Howlett's Theorem,

$$H_2(G) \ \cong\ \mathbb{Z}_2$$

so

$$\mu(G) = 1 + 1 = 2.$$

But

$$\chi(\wp') = 1 - 3 + 5 = 3$$

we suspect $\wp'$ *is minimal but can not get an efficient presentation.*

We assume that all $\wp_v$ $(v \in V)$ are efficient. Then

$$1 - |\mathbf{a}_v| + |\mathbf{s}_v| = 1 - rk(H_1 G_v) + d(H_2 G_v) \text{ for } v \in V.$$

$$-|\mathbf{a}| + |\mathbf{s}| = -\sum_{v \in V} rk(H_1 G_v) \ + \ \sum_{v \in V} d(H_2 G_v).$$

Thus

$$\chi(\wp) = 1 - |\mathbf{a}| + |\mathbf{s}| + |\mathbf{r}| = 1 - \sum_{v \in V} rk(H_1 G_v) \ + \ \sum_{v \in V} d(H_2 G_v) + |\mathbf{r}|.$$

From Theorem,

$$\chi(G) = 1 - \sum_{v \in V} rk(H_1 G_v) \ + \ \sum_{v \in V} d(H_2 G_v) + d(\oplus_{R \in \mathbf{r}} \mathbb{Z} t_R / U).$$

Therefore we get

Theorem 4.4.4. *If all* $\wp_v (v \in V)$ *are efficient then* $\wp$ *is efficient if and only if* $d(\oplus_{R \in \mathbf{r}} \mathbb{Z} t_R / U) = |\mathbf{r}|$.

Example 4.4.5. *For each* $v \in V$, $\wp_v = < x_v, y_v \mid {x_v}^2,\ {y_v}^2,\ (x_v y_v)^{\alpha_v} >$, *where* α_v *is an even positive integer. Then* $\wp_v$ *is efficient and* $U = \oplus_{R \in \mathbf{r}} 2\mathbb{Z} t_R$. *So* $\wp$ *is efficient. The special case that all* $\alpha_v (v \in V)$ *are same is a partial result of* [8](Theorem 1).

Corollary 4.4.6. *If there is a prime number* p *which divides* $exp_x(S)$ *for all* $S \in \mathbf{s}$, *for all* $x \in \mathbf{a}$, *then there is an epimorphism;*

$$\oplus_{R \in \mathbf{r}} \mathbb{Z} t_R / U \ \textit{onto} \ \oplus^l \mathbb{Z}_p = \underbrace{\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p}_{l-times}$$

where $l = d(\oplus_{R \in \mathbf{r}} \mathbb{Z} t_R / U)$.

Bibliography

[1] Baik Y. G. and Howie J. and Pride S. J., The identity problem for graph products of groups, To appear in J. Algebra.

[2] Baik Y. G.(1992), Generators of the second homotopy modules of group presentations with applications, Ph. D. Thesis, University of Glasgow.

[3] Baik Y. G. and Pride S. J.(1992), Generators of the second homotopy modules of presentations arising from group constructions, Preprint, University of Glasgow.

[4] Baik Y. G. and Pride S. J.(1997), On the efficiency of Coxeter groups, Bull. London Math. Soc. 29, 32-36

[5] Bogley W. A. and Pride S. J.(1992), Aspherical relative presentations, Proc. Edinburgh Math. Soc. 35, 1-39.

[6] Brandenburg J. and Dyer M.(1981), On J. H. C. Whitehead's aspherical question I, Comment. Math. Helvetici 56, 431-446.

[7] Brown K. S., Cohomology of groups, GTM 87, Springer-Verlag.

[8] Campbell C. M. and Robertson E. F. and Williams P. D.(1989 1990), On the efficiency of some direct powers of groups, Groups - Canberra, 106-113, LNM 1456, Springer-Verlag.

[9] Chiswell I. M.(1976), Exact sequences associated with a graph of groups, J. Pure Appl. Algebra, 63-74.

[10] Chiswell I. M.(1990), The Euler characteristic of a graph product, Preprint, Qween Mary and Westfield College London.

[11] Chiswell I. M., Collins D. J. and Huebschmann J.(1981), Aspherical group presentations. Math. Z. 178, 1-36.

[12] Cohen D. E.(1989), Combinatorial Group Theory : a topological approach, London Math. Soc. Student Texts 14, Cambridge Univ. Press, Cambridge.

[13] Droms C.(1987), Isomorphism of graph groups, Proc. Amer. Math. Soc. 100(3), 407-408.

[14] Droms C.(1987), Graph groups, coherence and three-manifolds, J. Algebra 106, 484-489.

[15] Droms C.(1987), Subgroups of graph groups, J. Algebra 110, 519-522.

[16] Droms C., Lewin J. and servatius H., Tree groups and the 4-string pure braid group, Submitted.

[17] Edjevet M., On the asphericity of one-relator relative presentations, Preprint.

[18] Greem E. R.(1990), Graph products of groups, Ph. D. Thesis, University of Leeds.

[19] Greem E. R.(1990), Graph products of groups, Preprint, University of Leeds.

[20] Greem E. R.(1990), Residual properties of graph products, Preprint, University of Leeds.

[21] Gruenberg K. W.(1970), Cohomology topics in group theory, LNM. 143.

[22] Goansu Kim(2004), On the residual finiteness of fundamental groups of graphs of certain groups, J. Korean Math. Soc. 41, 913-920.

[23] Haebich W.(1997), The multiplicator of a splitting extension, J. Algebra 44, 420-433.

[24] Hannerbauer T.(1989), Relation modules of amalgamated free products and HNN extensions, Glasgow Math. J. 31, 263-270.

[25] Johnson D. L.(1976), Presentations of groups, London Math. Soc. Lecture Note Series 22, Cambridge Univ. Press, Cambridge.

[26] Lustig M., On the rank the efficiency and the homological demension of groups: the computation of a lower bound via Fox ideals, Topology and combinatorial group theory, LNM 1440, Springer-Verlag.

[27] Lyndon R. C. and Schupp P. E.(1977), Combinatorial group theory, Springer-Verlag.

[28] Magnus W., Karrass A. and solitar D.(1996), Combinatorial group theory, 2nd edition, Dover Publication, New York.

[29] Neumann B. H.(1955), Some groups with trivial multiplicator, Publ. Math. Debrecen 4, 190-194.

[30] Pride S. J.(1987), Groups with presentations in which each defining relator involves exactly two generators, J. London Math. Soc. (20) 36, 245-256.

[31] Pride S. J.(1991), Identities among relations of group presentations, Group Theory from a geometric viewpoint, World Scientific Publishing Co. Pte. Ltd., Singapore.

[32] Pride S. J.(1992), The (co)homology of groups given by presentations in which each defining relator involves at most two types of generators, J. Austral. Math. Soc. (Series A) 52, 205-218.

[33] Pride S. J. and Stohr(1989), Relation modules of groups with presentations in which each relator involves exactly two types of generators, J. London Math. Soc. (2) 38, 99-111.

[34] Rotman J. J.(1973), The theory of groups, Allyn and Bacon, Boston.

[35] Servatius H.(1989), Automorphisms of graph groups, J. Algebra 126, 34-60.

[36] Swan R. G.(1965), Minimal resolutions for finite groups, Topology 4, 193-208.

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