Thesis for the Degree of Doctor of Philosophy

Second and Higher Order Duality in Multiobjective Programming Problems

by

Hun Suk Kang

Department of Applied Mathematics

The Graduate School

Pukyong National University

February 2008

Second and Higher Order Duality in Multiobjective Programming Problems
(다목적 계획문제에 대한 이계와 고계 쌍대성)

Advisor : Prof. Do Sang Kim

by

Hun Suk Kang

A thesis submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

in Department of Applied Mathematics, The Graduate School, Pukyong National University

February 2008

강훈숙의 이학박사 학위논문을 인준함

2008년 2월 28일

주	심	이학박사	정 종 수	인
위	원	이학박사	이 규 명	인
위	원	이학박사	정 진 문	인
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A dissertation

by

Hun Suk Kang

Approved by :

(Chairman) Jong Soo Jung

(Member) Gue Myung Lee

(Member) Jin-Mun Jeong

(Member) Jun Yong Shin

(Member) Do Sang Kim

February 28, 2008

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요 약

본 학위논문에서는 다목적 분수 계획문제에 대한 쌍대문제, 고계 쌍대문제 그리고 미분 불가능한 지지함수를 포함하는 대칭쌍대문제와 2계 대칭 쌍대문제를 제시하였다.

먼저 기존에 알려진 Wolfe형과 Mond-Weir형 문제를 혼합한 변수 분리된 이계 다목적 분수 계획문제에 대하여 여러 가지 invex 함수조건아래에서 유효해에 관한 쌍대관계를 밝히고, 추 제약식을 가지는 다목적 고계 쌍대문제에서 필요최적조건을 제시하고 유효해에 관한 약쌍대정리, 강쌍대정리와 역쌍대정리를 정립하였다. 그리고 특수한 Fritz-John 고계 쌍대문제에 대해서도 유사한 쌍대정리들을 증명하였다.

또한, 추 제약식을 가지는 미분 불가능한 지지함수를 포함하는 다목적 대칭 쌍대문제를 정형화하고 이계에 대한 유사 invex 함수 조건 아래에서 약 유효해와 필요최적조건을 이용하여 쌍대 관계를 정립하고 일반화된 미분 가능한 이계 대칭 쌍대를 정식화하고 Fritz John 최적조건을 밝히고 볼록함수조건아래에서 쌍대관계를 정립하였다.

Chapter 1
Introduction and Preliminaries

1

Multiobjective programming problems arise when more than one objective function is to be optimized over a given feasible region. Their optimums are the concept of solution that appears to be the natural extension of the optimization for a single objective to one of multiple objectives. In economic analysis [7], game [21] and system science, optimums are effective for treating such a multiplicity of values.

Khan and Hanson [38] have used the concept of ratio invexity to characterize optimality and duality results in a fractional programming. This concept seems to be new and it introduces a modified kind of characterization in sufficient optimality with invexity conditions. Slightly away from this but introducing ρ-invex condition, Suneja and Lalitha [75] have also characterized multiobjective fractional programming problem for duality results. In the ensuring paragraph we present on account of the fractional programming problem as depicted in Khan and Hanson [38].

Consider the nonlinear fractional programming problem:

$$\textbf{(FP)} \quad \text{Minimize} \quad \frac{f(x)}{g(x)}$$

$$\text{subject to} \quad h(x) \leqq 0, \;\; x \in X_0,$$

where X_0 is a subset of $\mathbb{R}^n$, f and g are real valued functions defined on X_0 and h is an m-dimensional vector valued functions also defined on X_0. We let $\Delta = \{x \in X_0,\ \ h(x) \leqq 0\}$ be the set of all feasible solutions. Assume that $f(x) \geqq 0$ for all $x \in \Delta$, $g(x) > 0$ for all $x \in \Delta$, and the functions f, g and h satisfy

$$x, a \in \Delta \quad \Rightarrow \quad \begin{cases} f(x) - f(a) - \nabla f(a)\eta(x,a) \geqq 0, \\ -g(x) + g(a) + \nabla g(a)\eta(x,a) \geqq 0, \\ h(x) - h(a) - \nabla h(a)\eta(x,a) \geqq 0 \end{cases}$$

with respect to $\eta : X_0 \times X_0 \to \mathbb{R}^n$.

These are called invex functions. In 1981, Hanson [30] introduced the concept of the invex function which is a generalization of the convex function. Many authors [8, 35, 53] have studied properties of invex functions and single objective(i.e., scalar) optimization problems with these functions.

$$\textbf{(P)} \quad \text{Minimize} \quad f(x)$$
$$\text{subject to} \quad h(x) \leqq 0, \quad x \in X_0.$$

The problem **(P)** is characterized as an invex problem, as was quoted in Craven [17]. The problem **(FP)** as introduced above is said to be a convex-concave problem if f is convex, g is concave and h is convex. It is then transformed into an invex functions. Most of the references like Israel and Mond [11], Reiland [71], and Khan [37] have discussed invex problems and their generalizations for the multiobjective case. The paper of Khan and Hanson [38] could be thought of as a beginning of some investigation for invex fractional programming problems.

Symmetric duality in mathematical programming was introduced by Dorn [20], who defined a program and its dual to be symmetric if the dual of the dual is the original problem. The notion of symmetric duality was developed significantly by Dantzig et al. [19], Chandra and Husain [13] and Mond and Weir [61]. Dantzig et al. [19] formulated a pair of symmetric dual programs and established duality results for convex/concave functions by taking non-negative orthant as the cone. The same result was generalized to arbitrary cones by Bazaraa and Goode [6].

Later Mond and Weir [61] presented two pairs of symmetric dual multiobjective programming problems for efficient solutions and obtained appropriate symmetric duality results concerning pseudo-convex/pseudo-concave or convex/concave functions with the non-negative orthant as the cone. Nanda and Das [68] formulated a pair of symmetric dual nonlinear programming problems for pseudo-invex functions and arbitrary cones. Nanda [67] also studied symmetric dual problems by assuming the functions to be invex with non-negative orthant as the cone. Kim et al. [48] formulated a pair of multiobjective symmetric dual programs for pseudo-invex functions and arbitrary cones and established duality results. Mishra [54] formulated a pair of second order multiobjective symmetric dual nonlinear programming problems under second order pseudo-invexity assumptions on the functions involved over arbitrary cones and established duality results. Mishra [55] also studied second order symmetric duality under second order F-convexity, F-concavity, F-pseudo-convexity and F-pseudo-concavity for second order Wolfe and Mond-Weir type models, respectively.

Suneja et al. [77] formulated a pair of symmetric dual multiobjective programs of Wolfe type over arbitrary cones in which the objective function is optimized with respect to an arbitrary closed convex cone by assuming the involved function to be cone-convex. Very recently Khurana [40] formulated a pair of differentiable multiobjective symmetric dual programs of Mond-Weir type over arbitrary cones in which the objective function is optimized with respect to an arbitrary closed convex cone by assuming the involved functions to be cone-pseudoinvex and strongly cone-pseudoinvex. Mishra and K. K. Lai [56] introduced the concept of cone-second order pseudoinvex and strongly cone-second order pseudoinvex functions and formulated a pair of Mond-Weir type multiobjective second order symmetric dual programs over arbitrary cones.

Let $\mathbb{R}^n$ be the $n-$dimensional Euclidean space and $\mathbb{R}^n_+$ its nonnegative orthant.

We consider the following multiobjective programming problem:

$$\textbf{(MP)} \quad \text{Minimize} \quad f(x) = (f_1(x), \cdots, f_k(x))$$

$$\text{subject to} \quad g_j(x) \leqq 0, \ \ j \in P, \ \ x \in \mathbb{R}^n$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$, $i \in K = \{1, \cdots, k\}$ and $g_j : \mathbb{R}^n \to \mathbb{R}$, $j \in P = \{1, \cdots, m\}$ are differentiable functions. For simplicity, we rewrite **(MP)** as follows:

$$\textbf{(MP)} \quad \text{Minimize} \quad f(x)$$

$$\text{subject to} \quad x \in S = \{x \in \mathbb{R}^n : g(x) \leqq 0\}.$$

Throughout this dissertation the following notations in $\mathbb{R}^n$ will be used:

$$x = y \quad \text{if and only if} \quad x_i = y_i, \quad i = 1, 2, \cdots, n,$$

$$x \leqq y \quad \text{if and only if} \quad x_i \leqq y_i, \quad i = 1, 2, \cdots, n,$$

$$x \leq y \quad \text{if and only if} \quad x_i \leqq y_i, \quad \text{but} \quad x \neq y,$$

$$x < y \quad \text{if and only if} \quad x_i < y_i, \quad i = 1, 2, \cdots, n,$$

$$x \nleq y \quad \text{is the negation of} \quad x \leq y,$$

$$x \nless y \quad \text{is the negation of} \quad x < y.$$

Now, we discuss the concepts of solutions of the problem **(MP)**.

The problem **(MP)** is also called a vector optimization. For multiobjective optimization problems, there are three kinds of solutions. We call them properly efficient, efficient and weakly efficient solution. The most fundamental solution is an efficient solution (also called a Pareto optimal solution or noninferior solution) with respect to the domination structure of the decision maker.

Optimization of **(MP)** is to find (properly, weakly) efficient solutions defined as follows:

Definition 1.1 *A point $\overline{x} \in S$ is said to be an efficient solution(or Pareto optimal solution) of* **(MP)** *if there exists no other $x \in S$ such that for some $i \in I = \{1, 2, \cdots, k\}$, $f_i(x) < f_i(\overline{x})$ and for all $j \in I$, $f_j(x) \leqq f_j(\overline{x})$.*

Definition 1.2 *A feasible point* $\overline{x} \in S$ *is said to be a properly efficient solution of* **(MP)** *if it is an efficient solution of* **(MP)** *and if there exists a scalar* $M > 0$ *such that for each* $i = 1, \cdots, k$ *and* $x \in S$ *satisfying* $f_i(x) < f_i(\overline{x})$*, we have*

$$\frac{f_i(\overline{x}) - f_i(x)}{f_j(x) - f_j(\overline{x})} \leqq M,$$

for some j *such that* $f_j(x) > f_j(\overline{x})$*.*

The quantity $\frac{f_i(\overline{x})-f_i(x)}{f_j(x)-f_j(\overline{x})}$ may be interpreted as the marginal trade-off for objective functions f_i and f_j between x and $\overline{x}$. Geoffrion [25] considered the concept of the proper efficiency to eliminate the unbounded trade-off between objective functions of **(MP)**.

Definition 1.3 *A point* $\overline{x} \in S$ *is said to be a weakly efficient solution of* **(MP)** *if there does not exist any feasible* x *such that* $f_i(x) < f_i(\overline{x})$*.*

We shall use the concepts of efficient and weakly efficient solutions.

The purpose of this dissertation is to establish duality theorems for multiobjective programming problems under various generalized convexity conditions involving differentiable or nondifferentiable functions. The weak, strong and converse or strictly converse duality hold between primal problems and dual problems.

This dissertation is organized as follows:

In Chapter 2, we consider the following multiobjective programming problem:

$$\textbf{(MFP)} \quad \text{Minimize} \quad \frac{f(x)}{g(x)} = \left(\frac{f_1(x)}{g(x)}, \cdots, \frac{f_k(x)}{g(x)}\right)$$

$$\text{subject to} \quad h_j(x) \leqq 0, \ \ j \in P, \ \ x \in X$$

where $\frac{f}{g} := (\frac{f_1}{g}, \cdots, \frac{f_k}{g}) : \mathbb{R}^n \to \mathbb{R}^k$ and $h := (h_1, \cdots, h_m) : \mathbb{R}^n \to \mathbb{R}^m$. Using separated variables, we formulate generalized second order multiobjective fractional dual programs for **(MFP)**. For these programs, we proved the weak, strong and strictly converse duality theorems under suitable generalized convexity assumptions on the basis of the efficiency of solutions. We established a Wolfe type dual as well as a Mond-Weir type dual programming problems as special cases. For each dual we derive weak, strong, and converse duality theorems under second order invexity assumption.

In Chapter 3, we consider the following multiobjective programming problem:

$$(\textbf{MCP}) \quad \text{Minimize} \quad f(x)$$

$$\text{subject to} \quad -g(x) \in C_2^*, \ \ x \in C_1,$$

where f and g are twice differentiable functions from $\mathbb{R}^n \to \mathbb{R}^l$ and $\mathbb{R}^n \to \mathbb{R}^m$, respectively, C_1 is a closed convex cone in $\mathbb{R}^n$, and C_2^* is the polar cone of C_2.

We construct a higher order dual of **(MCP)** and establish weak, strong and converse duality theorems for an efficient solution of **(MCP)** by using higher order generalized invexity conditions. As special cases of our duality relations, we give some known duality results.

In addition, we consider the following nonlinear programming problem:

$$\textbf{(FCP)} \quad \text{Minimize} \quad f(x)$$
$$\text{subject to} \quad g(x) \in C_2^*, \ \ x \in C_1,$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are twice differentiable functions.

We construct a Fritz John higher order dual of **(FCP)** using Fritz John necessary optimality conditions [51] instead of Karush-Kuhn-Tucker [51], and establish weak, strong and converse duality theorems under suitable higher order generalized invexity assumptions. Thus, the requirement of a constraint qualification can be eliminated.

In Chapter 4, we formulate Mond-Weir and Wolfe type non-differentiable second order multiobjective symmetric dual problems with cone constraints over arbitrary closed convex cones. Subsequently, weak, strong and converse duality theorems are established under the assumptions of second order pseudo-invex functions. And we introduced some special cases of our duality results.

In Chapter 5, we formulate a pair of generalized second order symmetric programs in multiobjective nonlinear programming. For these programs, we establish weak, strong and converse duality theorems under suitable convexity assumptions on the basis of the efficiency of solutions. These results are the extension of second order symmetric duality relations due to Kim et al. [47]. And we present some special cases of our duality results.

Chapter 2

Generalized Second Order Duality for Multiobjective Fractional Programs

2

2.1 Introduction

In 1961, Wolfe [81] considered a dual program as convex program with nonlinear constraints and apart from others proved weak and direct duality theorems under suitable assumptions. Afterward, a number of dual problems distinct from the Wolfe dual problem are proposed for the nonlinear programs by Mond and Weir [62]. Duality relations for single objective fractional programming problems with a (generalized) convexity condition, were given by many authors [8, 18, 34, 39, 53, 64, 72, 73].

Wolfe's dual problem [81] is not useful for the fractional problem. Various examples showing the unsuitability of the Wolfe dual for fractional programs have been given by Mangasarian [51] and Schaible [72, 73]. Later on, Bector [8] introduced slightly different fractional programming. Mond and Weir [62] consider the fractional programming problem as follows;

$$\textbf{(FP)} \quad \text{Minimize} \quad \frac{f(x)}{g(x)}$$

$$\text{subject to} \quad h(x) \leqq 0, \;\; x \in X_0,$$

where X_0 is a subset of $\mathbb{R}^n$, f and g are real-valued functions defined on X_0, h be an m-dimensional vector valued function also defined on X_0. Under the assumptions that f is convex and nonnegative, g is concave and positive and h is convex, a number of duality results can be obtained.

As a generalization of differentiable convex function, Hanson [30] introduced the weak convex function, where it is shown that the Kuhn-Tucker conditions are sufficient for optimality of nonlinear programming problems under the condition of a weak convex function. A weak convex function was called an invex function by Craven [17]. Most of the references like Israel and Mond [11], Reiland [71], and Khan [37] have discussed invex problems and their generalizations for the multiobjective case. Afterwards, the second order invexity was introduced by Egudo and Hanson [24] called binvexity by Bector and Bector [9].

Many authors [8, 35, 53, 43, 9, 44, 41] have studied properties of invex functions and nonlinear programming problems with these functions. Khan and Hanson [38] extended the nonlinear fractional programming problem with invex functions, that is, the ratio invexity. They gave sufficient conditions for optimality and established duality results by assuming that f and $-g$ are invex with respect to a scale function $\eta(x, u)$ and h is invex with respect to $\frac{g(u)}{g(x)}\eta(x, u)$. Reddy and Mukherjee [70] applied a generalized ratio invexity to single objective fractional programming problems. Very recently Liang et al. [50] establish sufficient conditions and duality theorems for an efficient solution of multiobjective fractional programming problems under (F, α, ρ, d)-convexity assumptions.

In this chapter, using separated variables, we formulate generalized second order multiobjective fractional programs. For these programs, we proved the weak, strong, and strictly converse duality theorems under suitable generalized convexity assumptions on the basis of efficiency of solutions. And we introduced Mond-Weir and Wolfe type second order multiobjective fractional programs, as special cases. We obtained the weak, strong, and converse duality theorems for Mond-Weir and Wolfe type second order multiobjective fractional programs under second order invexity assumptions.

2.2 Notations and Preliminaries

Let f be a twice differentiable function from $\mathbb{R}^n$ into $\mathbb{R}^k$ and $M = \{1, 2, \cdots, m\}$, $I \subset M$, and $M \setminus I = J$. Note that I or J can be empty. We rearrange y as $y = (y_I, y_J)$.

Next definition is introduced by Bector and Bector [9], and Egudo and Hanson [24].

Definition 2.1 *A twice differentiable function $f : \mathbb{R}^n \to \mathbb{R}^k$ is said to be second order invex with respect to η if for all $i = 1, \cdots, k$, there exist a vector valued function η defined on $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that,*

$$f_i(x) - f_i(u) \geqq \eta(x,u)^T \nabla f_i(u) + \eta(x,u)^T \nabla^2 f_i(u) p - \frac{1}{2} p^T \nabla^2 f_i(u) p,$$

where $p \in \mathbb{R}^n$, ∇ denotes the gradient vetor and ∇^2 is the $n \times n$ Hessian matrix of second order partial derivatives.

We recall the following definitions defined by Aghezzaf [1].

Definition 2.2 *A functional* $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *is sublinear if for any* $x, u \in \mathbb{R}^n$,

$$F(x, u; a_1 + a_2) \leqq F(x, u; a_1) + F(x, u; a_2), \quad \text{for all } a_1, a_2 \in \mathbb{R}^n$$

and

$$F(x, u; \alpha a) = \alpha F(x, u; a) \text{ for all } \alpha \in \mathbb{R}, \alpha \geqq 0, \text{ and } \alpha \in \mathbb{R}^n.$$

Let $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a sublinear functional, the function $f = (f_1, \cdots, f_k) : \mathbb{R}^n \to \mathbb{R}^k$ a twice differentiable at $u \in \mathbb{R}^n$, $\rho = (\rho_1, \cdots \rho_k) \in \mathbb{R}^k$ and $d(\cdot, \cdot)$ a metric on $\mathbb{R}^n$.

Definition 2.3 *The function* f_i *is said to be second order* (F, ρ_i)*-convex at* u *and* p*, if for all* $x \in \mathbb{R}^n$ *we have*

$$F(x, u; \nabla f_i(u) + \nabla^2 f_i(u)p) + \rho_i d(x, u) \leqq f_i(x) - f_i(u) + \frac{1}{2} p \nabla^2 f_i(u)p.$$

The vector valued function $f : \mathbb{R}^n \to \mathbb{R}^k$ is second order (F, ρ_i)-convex at u and p if each of its components f_i is second order (F, ρ_i)-convex at u and p.

Definition 2.4 *The function* f_i *is second order* (F, ρ_i)*-quasiconvex at* u *and* p*, if for all* $x \in \mathbb{R}^n$*, we have*

$$f_i(x) \leqq f_i(u) - \frac{1}{2} p^T \nabla^2 f_i(u)p \;\Rightarrow\; F(x, u; \nabla f_i(u) + \nabla^2 f_i(u)p) \leqq -\rho_i d(x, u).$$

We say that f is second order (F, ρ_i)-quasiconvex at u and p if each of its components f_i is second order (F, ρ_i)-quasiconvex at u and p.

Definition 2.5 *The function f_i is second order (F, ρ_i)-pseudoconvex at u and p, if for all $x \in \mathbb{R}^n$, we have*

$$f_i(x) < f_i(u) - \frac{1}{2}p^T\nabla^2 f_i(u)p \ \Rightarrow\ F(x, u; \nabla f_i(u) + \nabla^2 f_i(u)p) < -\rho_i d(x, u).$$

The function f is second order (F, ρ_i)-pseudoconvex at u and p if each of its components f_i is second order (F, ρ_i)-pseudoconvex at u and p.

Definition 2.6 *The function f_i is strong second order (F, ρ_i)-pseudoconvex at u and p, if for all $x \in \mathbb{R}^n$ we have*

$$f_i(x) \leq f_i(u) - \frac{1}{2}p^T\nabla^2 f_i(u)p \ \Rightarrow\ F(x, u; \nabla f_i(u) + \nabla^2 f_i(u)p) \leq -\rho_i d(x, u).$$

The class of strong second order (F, ρ_i)-pseudoconvex functions does not contain the class of second order (F, ρ_i)-pseudoconvex functions, but does contain the class of second order (F, ρ_i)-convex.

Lemma 2.1 *If $f, -g$ are second order invex at $\overline{x}$ with respect to η and (i) $\nabla g(\overline{x}) = 0$, (ii) $\nabla^2 \frac{f(\overline{x})}{g(\overline{x})}$ is negative semidefinite and positive semidefinite, then $\frac{f}{g}$ is second order invex at $\overline{x}$ with respect to $\overline{\eta}(x, \overline{x}) = \frac{g(\overline{x})}{g(x)}\eta(x, \overline{x})$.*

Proof. By differential calculus

$$\frac{\partial^2}{\partial {x_i}^2}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\nabla^2 f(x) - f(x)\nabla^2 g(x)}{g(x)^2} - \frac{2\nabla g(x)\{g(x)\nabla f(x) - f(x)\nabla g(x)\}}{g(x)^3}$$

and

$$\frac{f(x)}{g(x)} - \frac{f(\overline{x})}{g(\overline{x})} = \frac{f(x) - f(\overline{x})}{g(x)} - \frac{f(\overline{x})\{g(x) - g(\overline{x})\}}{g(x)g(\overline{x})}.$$

Since f and $-g$ are second order invex functions with respect to $\eta(x, \overline{x})$, above equation implies that

$$\begin{aligned}
&\frac{f(x)}{g(x)} - \frac{f(\overline{x})}{g(\overline{x})} = \frac{f(x) - f(\overline{x})}{g(x)} - \frac{f(\overline{x})}{g(x)g(\overline{x})}(g(x) - g(\overline{x})) \\
&\geqq \frac{1}{g(x)}\{\eta(x,\overline{x})^T\nabla f(\overline{x}) + \eta(x,\overline{x})^T\nabla^2 f(\overline{x})p - \frac{1}{2}p^T\nabla^2 f(\overline{x})p\} \\
&\quad - \frac{f(\overline{x})}{g(x)g(\overline{x})}\{\eta(x,\overline{x})^T\nabla g(\overline{x}) + \eta(x,\overline{x})^T\nabla^2 g(\overline{x})p - \frac{1}{2}p^T\nabla^2 g(\overline{x})p\} \\
&= \frac{1}{g(x)}\eta(x,\overline{x})^T\nabla f(\overline{x}) - \frac{f(\overline{x})}{g(x)g(\overline{x})}\eta(x,\overline{x})^T\nabla g(\overline{x}) \\
&\quad + \frac{1}{g(x)}\eta(x,\overline{x})^T\nabla^2 f(\overline{x})p - \frac{f(\overline{x})}{g(x)g(\overline{x})}\eta(x,\overline{x})^T\nabla^2 g(\overline{x})p \\
&\quad - \frac{1}{2}\frac{1}{g(x)}p^T\nabla^2 f(\overline{x})p - \frac{1}{2}\frac{f(\overline{x})}{g(x)g(\overline{x})}p^T\nabla^2 g(\overline{x})p \\
&= \frac{g(\overline{x})}{g(x)}\eta(x,\overline{x})^T\nabla\frac{f(\overline{x})}{g(\overline{x})} + \frac{g(\overline{x})}{g(x)}\eta(x,\overline{x})^T\nabla^2\frac{f(\overline{x})}{g(\overline{x})}p - \frac{1}{2}p^T\nabla^2\frac{f(\overline{x})}{g(\overline{x})}p
\end{aligned}$$

(by assumption).

Therefore $\frac{f}{g}$ is second order invex with respect to $\overline{\eta}(x, \overline{x}) = \frac{g(\overline{x})}{g(x)}\eta(x, \overline{x})$. □

2.3 Duality Theorems

We propose the following multiobjective fractional programming problem:

$$\textbf{(MFP)} \quad \text{Minimize} \quad \frac{f(x)}{g(x)} = \left(\frac{f_1(x)}{g(x)}, \cdots, \frac{f_k(x)}{g(x)}\right)$$

$$\text{subject to} \quad h_j(x) \leqq 0, \;\; j \in P, \;\; x \in X$$

and its generalized second order fractional dual

$$\begin{aligned} \textbf{(GMFD)} \quad \text{Maximize} \quad & \frac{f(u)}{g(u)} + (y_I^T h_I(u))e \\ & - \frac{1}{2}\left[p^T\left(\nabla^2\lambda^T\frac{f(u)}{g(u)} + \nabla^2 y_I^T h_I(u)\right)p\right]e \\ \text{subject to} \quad & \nabla y^T h(u) + \left(\nabla^2 y^T h(u)\right)p \\ & \quad + \nabla\lambda^T\frac{f(u)}{g(u)} + \left(\nabla^2\lambda^T\frac{f(x)}{g(x)}\right)p = 0, \\ & y_J^T h_J(u) - \frac{1}{2}p^T\left(\nabla^2 y_J^T h_J(u)\right)p \geqq 0, \\ & y \geqq 0, \\ & \lambda > 0, \quad \lambda^T e = 1, \end{aligned}$$

where $x \in \mathbb{R}^n, y \in \mathbb{R}^m, p \in \mathbb{R}^n, \lambda \in \mathbb{R}^k,\; e = (1, \cdots, 1)^T \in \mathbb{R}^k,\; \frac{f}{g} := (\frac{f_1}{g}, \cdots, \frac{f_k}{g}) : \mathbb{R}^n \to \mathbb{R}^k$ and $h := (h_1, \cdots, h_m) : \mathbb{R}^n \to \mathbb{R}^m$.

Now we establish the duality theorems for **(MFP)** and **(GMFD)**.

Theorem 2.1 (Weak Duality) *Let x satisfy the constraints of* **(MFP)** *and (u, y, λ, p) satisfy the constraints of* **(GMFD)**. *If f and $-g$ are second* *order invex with respect to η and h is second order invex with respect to $\overline{\eta}$, then*

$$\frac{f(x)}{g(x)} \nleq \frac{f(u)}{g(u)} + (y_I^T h_I(u))e - \frac{1}{2}\left[p^T\left(\nabla^2\lambda^T\frac{f(u)}{g(u)} + \nabla^2 y_I^T h_I(u)\right)p\right]e.$$

Proof. Assume to the contrary that

$$\frac{f(x)}{g(x)} \leq \frac{f(u)}{g(u)} + (y_I^T h_I(u))e - \frac{1}{2}\left[p^T\big(\nabla^2\lambda^T\frac{f(u)}{g(u)} + \nabla^2 y_I^T h_I(u)\big)p\right]e.$$

Then, since $\lambda > 0$,

$$\lambda^T\frac{f(x)}{g(x)} < \lambda^T\frac{f(u)}{g(u)} + y_I^T h_I(u) - \frac{1}{2}\left[p^T\big(\nabla^2\lambda^T\frac{f(u)}{g(u)} + \nabla^2 y_I^T h_I(u)\big)p\right]. \quad (2.1)$$

Since $\lambda_i > 0$ $(i = 1, \cdots, k)$ and $y_j \geqq 0$ $(j = 1, \cdots, m)$, the assumptions of invexity become

$$\lambda^T\frac{f(x)}{g(x)} - \lambda^T\frac{f(u)}{g(u)} \geqq \overline{\eta}(x,u)^T\left(\nabla\lambda^T\frac{f(u)}{g(u)} + \nabla^2\lambda^T\frac{f(u)}{g(u)}p\right) - \frac{1}{2}p^T\nabla^2\lambda^T\frac{f(u)}{g(u)}p \quad (2.2)$$

and

$$y^T h(x) - y^T h(u) \geqq \overline{\eta}(x,u)^T\left(\nabla y^T h(u) + \nabla^2 y^T h(u)p\right) - \frac{1}{2}p^T\nabla^2 y^T h(u)p. \quad (2.3)$$

Adding (2.2) and (2.3), and rearranging yield

$$\begin{aligned}
&\lambda^T \frac{f(x)}{g(x)} - \lambda^T \frac{f(u)}{g(u)} \\
&\geqq y_I^T h_I(u) + y_J^T h_J(u) - y_I^T h_I(x) - y_J^T h_J(x) \\
&\quad + \overline{\eta}(x,u)^T \left(\nabla y^T h(u) + \nabla^2 y^T h(u) p + \nabla \lambda^T \frac{f(u)}{g(u)} + \nabla^2 \lambda^T \frac{f(u)}{g(u)} p \right) \\
&\quad - \frac{1}{2} p^T \left(\nabla^2 \lambda^T \frac{f(u)}{g(u)} + \nabla^2 y_I^T h_I(u) + \nabla^2 y_J^T h_J(u) p \right) \\
&\geqq y_I^T h_I(u) - y^T h(x) - \frac{1}{2} p^T \left(\nabla^2 \lambda^T \frac{f(u)}{g(u)} + \nabla^2 y^T h(u) \right) p \\
&\geqq y_I^T h_I(u) - \frac{1}{2} p^T \left(\nabla^2 \frac{f(u)}{g(u)} + \nabla^2 y^T h(u) \right) p.
\end{aligned}$$

Thus,

$$\lambda^T \frac{f(x)}{g(x)} - \lambda^T \frac{f(u)}{g(u)} \geqq y_I^T h_I(u) - \frac{1}{2} p^T \left(\nabla^2 \lambda^T \frac{f(u)}{g(u)} + \nabla^2 y_I^T h_I(u) \right) p.$$

This contradicts (2.1). □

Next theorem is a generalization of the result of Zhang and Mond [86].

Theorem 2.2 (Weak Duality) *Assume that for all feasible* x *for* **(MFP)** *and all feasible* (u, y, λ, p) *for* **(GMFD)**,

(a) $y_J^T h_J(\cdot)$ *is second order* (F, α)*-quasiconvex at* u *and* p*, and assume that one of the following conditions holds;*

(b) $\frac{f}{g}(\cdot)+y_I^T h_I(\cdot)e$ *is strong second order* (F,ρ)*-pseudoconvex at* u *and* p *with* $\alpha+\lambda\rho \geqq 0$,

(c) $\lambda^T\frac{f}{g}(\cdot)+y_I^T h_I(\cdot)$ *is second order* (F,β)*-pseudoconvex at* u *and* p *with* $\alpha+\beta \geqq 0$.

Then the following cannot hold:

$$\frac{f(x)}{g(x)} \leq \frac{f(u)}{g(u)} + (y_I^T h_I(u))e - \frac{1}{2}\left[p^T\big(\nabla^2\lambda^T\frac{f(u)}{g(u)} + \nabla^2 y_I^T h_I(u)\big)p\right]e. \tag{2.4}$$

Proof. Let x be feasible for **(MFP)** and let (u,y,λ,p) be feasible for **(GMFD)**. Then we have

$$y_J^T h_J(x) \leqq y_J^T h_J(u) - \frac{1}{2}\left(\nabla^2 y_J^T h_J(u)\right)p. \tag{2.5}$$

From (2.5) and hypothesis (a) we obtain

$$F\left(x,u;\nabla y_J^T h_J(u) + \nabla^2 y_J^T h_J(u)p\right) \leqq -\alpha d(x,u). \tag{2.6}$$

By the feasibility of (u,y,λ,p) and the sublinearity of F, we have

$$\begin{aligned} &F(x,u;\nabla\lambda^T\frac{f(u)}{g(u)} + \nabla^2\lambda^T\frac{f(u)}{g(u)}p + \nabla y_I^T h_I(u) + \nabla^2 y_I^T h_I(u)p) \\ &\quad + F(x,u;\nabla y_J^T h_J(u) + \nabla^2 y_J^T h_J(u)p) \\ &\geqq F(x,u;\nabla\lambda^T\frac{f(u)}{g(u)} + \nabla^2\lambda^T\frac{f(u)}{g(u)}p + \nabla y^T h(u) + \nabla^2 y^T h(u)p) = 0. \end{aligned} \tag{2.7}$$

Relation (2.7) together with (2.6) yields

$$F(x,u;\nabla\lambda^T\frac{f(u)}{g(u)} + \nabla^2\lambda^T\frac{f(u)}{g(u)}p + \nabla y_I^T h_I(u) + \nabla^2 y_I^T h_I(u)p) \geqq \alpha d(x,u). \tag{2.8}$$

On the other hand, suppose contrary to the result that (2.4) holds. Since x is feasible of **(MFP)** and $y \geqq 0$, (2.4) implies

$$\frac{f(x)}{g(x)}+y_I^T h_I(x)e \leq \frac{f(u)}{g(u)}+(y_I^T h_I(u))e-\frac{1}{2}\left[p^T\big(\nabla^2\lambda^T\frac{f(u)}{g(u)}+\nabla^2 y_I^T h_I(u)\big)p\right]e. \tag{2.9}$$

Multiplying (2.9) by λ, we get

$$\lambda^T\frac{f(x)}{g(x)}+y_I^T h_I(x) < \lambda^T\frac{f(u)}{g(u)}+(y_I^T h_I(u))-\frac{1}{2}\left[p^T\big(\nabla^2\lambda^T\frac{f(u)}{g(u)}+\nabla^2 y_I^T h_I(u)\big)p\right]. \tag{2.10}$$

By the hypothesis (b) and (2.9), we have

$$F(x,u;\nabla\left[\frac{f(u)}{g(u)}+y_I^T h_I(u)e\right]+\nabla^2\left[\frac{f(u)}{g(u)}+y_I^T h_I(u)e\right]p) \leq -\rho d(x,u). \tag{2.11}$$

Multiplying (2.11) by λ, we obtain

$$\begin{aligned} &F(x,u;\nabla\left[\lambda^T\frac{f(u)}{g(u)}+y_I^T h_I(u)\right]+\nabla^2\left[\lambda^T\frac{f(u)}{g(u)}+y_I^T h_I(u)\right]p) \\ &< -\lambda\rho d(x,u) \leqq \alpha d(x,u), \end{aligned} \tag{2.12}$$

which contradicts (2.8). When the hypothesis (c) holds, (2.10) implies

$$\begin{aligned} F(x,u;\nabla\left[\lambda^T\frac{f(u)}{g(u)}+y_I^T h_I(u)\right]+\nabla^2\left[\lambda^T\frac{f(u)}{g(u)}+y_I^T h_I(u)\right]p) \\ < -\beta d(x,u) \leqq \alpha d(x,u), \end{aligned}$$

which contradicts (2.8). Therefore the proof is completed. □

Corollary 2.1 *Let* $(\overline{u}, \overline{y}, \overline{\lambda}, \overline{p})$ *be a feasible solution for* **(GMFD)** *such that*

$$\overline{y}_I^T h_I(\overline{u}) - \frac{1}{2}\overline{p}^T \nabla^2 \left[\overline{\lambda}^T \frac{f(\overline{u})}{g(\overline{u})} + \overline{y}_I^T h_I(\overline{u}) \right] \overline{p} = 0, \tag{2.13}$$

and assume that $\overline{u}$ *is feasible for* **(MFP)**. *If weak duality holds between* **(MFP)** *and* **(GMFD)**, *then* $\overline{u}$ *is efficient for* **(MFP)** *and* $(\overline{u}, \overline{y}, \overline{\lambda}, \overline{p})$ *is efficient for* **(GMFD)**.

Proof. Suppose that $\overline{u}$ is not efficient for **(MFP)**, then there exists a feasible x for **(MFP)** such that

$$\frac{f(x)}{g(x)} \leq \frac{f(\overline{u})}{g(\overline{u})} \tag{2.14}$$

and since

$$\overline{y}_I^T h_I(\overline{u}) - \frac{1}{2}\overline{p}^T \nabla^2 \left[\overline{\lambda}^T \frac{f(\overline{u})}{g(\overline{u})} + \overline{y}_I^T h_I(\overline{u}) \right] \overline{p} = 0.$$

So (2.14) can be written as

$$\frac{f(x)}{g(x)} \leq \frac{f(\overline{u})}{g(\overline{u})} + \overline{y}_I^T h_I(\overline{u}) e - \frac{1}{2}\overline{p}^T \nabla^2 \left[\overline{\lambda}^T \frac{f(\overline{u})}{g(\overline{u})} + \overline{y}_I^T h_I(\overline{u}) \right] \overline{p} e.$$

Since $(\overline{u}, \overline{y}, \overline{\lambda}, \overline{p})$ is feasible for **(GMFD)** and x is feasible for **(MFP)**, this inequality contradicts weak duality. Also suppose that $(\overline{u}, \overline{y}, \overline{\lambda}, \overline{p})$ is not efficient for **(GMFD)**. Then there exists a feasible (u, y, λ, p) for **(GMFD)** such that

$$\frac{f(u)}{g(u)} + y_I^T h_I(u) e - \frac{1}{2} p^T \nabla^2 \left[\lambda^T \frac{f(u)}{g(u)} + \overline{y}_I^T h_I(u) \right] pe$$

$$\geq \frac{f(\overline{u})}{g(\overline{u})} + \overline{y}_I^T h_I(\overline{u}) e - \frac{1}{2}\overline{p}^T \nabla^2 \left[\overline{\lambda}^T \frac{f(\overline{u})}{g(\overline{u})} + \overline{y}_I^T h_I(\overline{u}) \right] \overline{p} e. \tag{2.15}$$

Then from (2.13), (2.15) reduces to

$$\frac{f(u)}{g(u)} + y_I^T h_I(u)e - \frac{1}{2}p^T\nabla^2\left[\lambda^T\frac{f(u)}{g(u)} + \overline{y}_I^T h_I(u)\right]pe \geq \frac{f(\overline{u})}{g(\overline{u})}.$$

Since $\overline{u}$ is feasible for **(MFP)**, this inequality contradicts the weak duality. Therefore $\overline{u}$ and $(\overline{u}, \overline{y}, \overline{\lambda}, \overline{p})$ are efficient for their respective programs. □

Theorem 2.3 (Strong Duality) *Let $\overline{x}$ be an efficient solution of* **(MFP)** *at which a constraint qualification is satisfied. Then there exist $\overline{y} \in \mathbb{R}^m, \overline{\lambda} \in \mathbb{R}^k$ and $\overline{p} \in \mathbb{R}^n$ such that $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{p} = 0)$ is a feasible solution of* **(GMFD)**, *with*

$$\overline{y}_I^T h_I(\overline{u}) - \frac{1}{2}\overline{p}^T\nabla^2\left[\overline{\lambda}^T\frac{f(\overline{u})}{g(\overline{u})} + \overline{y}_I^T h_I(\overline{u})\right]\overline{p} = 0.$$

If Theorem 2.1 or Theorem 2.2 also holds between **(MFP)** *and* **(GMFD)**, *then $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{p} = 0)$ is an efficient solution for* **(GMFD)**.

Proof. Since $\overline{x}$ be an efficient solution of **(MFP)**, then there exist $\overline{\lambda} \in \mathbb{R}^k(\overline{\lambda} > 0, \quad \overline{\lambda}^T e = 1)$ and $\overline{y} \in \mathbb{R}^m$ that satisfy the following Kuhn-Tucker conditions [51]:

$$\nabla\overline{\lambda}^T\frac{f(\overline{x})}{g(\overline{x})} + \nabla\overline{y}^T h(\overline{x}) = 0,$$

$$\overline{y}^T h(\overline{x}) = 0,$$

$$\overline{y} \geq 0.$$

Clearly $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{p} = 0)$ is feasible for **(GMFD)**, and the objective values of **(MFP)** and **(GMFD)** are equal. If the assumptions of Theorem 2.1 or Theorem 2.2 also holds, then $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{p} = 0)$ is an efficient solution of **(GMFD)**. □

We now turn our attention to the strict converse duality.

Theorem 2.4 (Strict Converse Duality) *Let $\overline{x}$ be an efficient solution for* **(MFP)** *and $(\overline{u}, \overline{y}, \overline{\lambda}, \overline{p})$ be an efficient solution for* **(GMFD)** *such that*

$$\overline{\lambda}^T \frac{f(\overline{x})}{g(\overline{x})} = \overline{\lambda}^T \frac{f(\overline{u})}{g(\overline{u})} + \overline{y}_I^T h_I(\overline{u}) - \frac{1}{2}\overline{p}^T \nabla^2 \left[\overline{\lambda}^T \frac{f(\overline{u})}{g(\overline{u})} + \overline{y}_I^T h_I(\overline{u})\right] \overline{p}. \tag{2.16}$$

Assume that

(i) $\overline{\lambda}^T \frac{f}{g}(\cdot) + \overline{y}_I^T h_I(\cdot)$ is strictly invex or

(ii) Condition (i) of Theorem 2.2 is satisfied and $\overline{\lambda}^T \frac{f}{g}(\cdot) + \overline{y}_I^T h_I(\cdot)$ is strictly second order (F, β)-pseudoconvex with $\alpha + \beta \geqq 0$.

Then $\overline{x} = \overline{u}$ and $\overline{u}$ is an efficient solution for **(MFP)**.

Proof. (i) We assume $\overline{x} \neq \overline{u}$ and exhibit a contradiction. From the fact $\overline{\lambda}^T \frac{f}{g}(\cdot) + \overline{y}_I^T h_I(\cdot)$ is strictly invex, we obtain

$$\begin{aligned}
&\overline{\lambda}^T \frac{f(\overline{x})}{g(\overline{x})} + \overline{y}_I^T h_I(\overline{x}) - \overline{\lambda}^T \frac{f(u)}{g(u)} - \overline{y}_I^T h_I(u) \\
&\quad > \overline{\eta}(\overline{x}, \overline{u}) \left[\nabla \left(\overline{\lambda}^T \frac{f(\overline{u})}{g(\overline{u})} + \overline{y}_I^T h(\overline{u})\right) + \nabla^2 \left(\overline{\lambda}^T \frac{f(\overline{u})}{g(\overline{u})} + \overline{y}_I^T h_I(\overline{u})\right) p\right] \\
&\qquad\qquad - \frac{1}{2}\overline{p}^T \left(\nabla^2 \overline{\lambda}^T \frac{f(\overline{u})}{g(\overline{u})} + \nabla^2 \overline{y}_I^T h_I(\overline{u})\right) \overline{p}.
\end{aligned}$$

Since $\overline{x}$ and $(\overline{u}, \overline{y}, \overline{\lambda}, \overline{p})$ are feasible for **(MFP)** and **(GMFD)**, the inequality above implies

$$\overline{\lambda}^T \frac{f(\overline{x})}{g(\overline{x})} > \overline{\lambda}^T \frac{f(\overline{u})}{g(\overline{u})} + \overline{y}_I^T h_I(\overline{u}) - \frac{1}{2}\overline{p}^T \nabla^2 \left(\overline{\lambda}^T \frac{f(\overline{u})}{g(\overline{u})} + \overline{y}_I^T h_I(\overline{u}) \right) \overline{p},$$

which contradicts (2.16).

(ii) We assume $\overline{x} \neq \overline{u}$ and exhibit a contradiction. Since $\overline{x}$ and $(\overline{u}, \overline{y}, \overline{\lambda}, \overline{p})$ are feasible for **(MFP)** and **(GMFD)**, respectively, then $\overline{y} \geqq 0$, $\;h(\overline{x}) \leqq 0$, and

$$\overline{y}_J^T h_J(\overline{x}) \leqq \overline{y}_J^T h_J(\overline{u}) - \frac{1}{2}\overline{p}^T \nabla^2 \overline{y}_J^T h_J(\overline{u})\overline{p}. \tag{2.17}$$

By the hypothesis (ii), (2.17) implies that

$$F(\overline{x}, \overline{u}; \nabla \overline{y}_J^T h_J(\overline{u}) + \nabla^2 \overline{y}_J^T h_J(\overline{u})) \leqq -\alpha d(\overline{x}, \overline{u}). \tag{2.18}$$

By the feasibility of $(\overline{u}, \overline{y}, \overline{\lambda}, \overline{p})$ and the sublinearity of F, we have

$$F(\overline{x}, \overline{u}; \nabla \overline{\lambda}^T \frac{f(\overline{u})}{g(\overline{u})} + \nabla^2 \overline{\lambda}^T \frac{f(\overline{u})}{g(\overline{u})}\overline{p} + \nabla \overline{y}_I^T h_I(\overline{u}) + \nabla^2 \overline{y}_I^T h_I(\overline{u})\overline{p})$$

$$+F(\overline{x}, \overline{u}; \nabla \overline{y}_J^T h_J(\overline{u}) + \nabla^2 \overline{y}_J^T h_J(\overline{u})\overline{p})$$

$$\geqq F(\overline{x}, \overline{u}; \nabla \overline{\lambda}^T \frac{f(\overline{u})}{g(\overline{u})} + \nabla^2 \overline{\lambda}^T \frac{f(\overline{u})}{g(\overline{u})}\overline{p} + \nabla \overline{y}^T h_I(\overline{u}) + \nabla^2 \overline{y}^T h_I(\overline{u})\overline{p}) = 0. \tag{2.19}$$

Relation (2.19) together with (2.18) yields

$$F(\overline{x}, \overline{u}; \nabla \overline{\lambda}^T \frac{f(\overline{u})}{g(\overline{u})} + \nabla^2 \overline{\lambda}^T \frac{f(\overline{u})}{g(\overline{u})}\overline{p} + \nabla \overline{y}_I^T h_I(\overline{u}) + \nabla^2 \overline{y}_I^T h_I(\overline{u})\overline{p})$$

$$\geqq \alpha d(\overline{x}, \overline{u}) \geqq -\beta d(\overline{x}, \overline{u}).$$

Since $\overline{\lambda}^T \frac{f}{g}(\cdot) + \overline{y}_I^T h_I(\cdot)$ is strictly second order (F, β)-pseudoconvex, it follow that

$$\overline{\lambda}^T \frac{f(\overline{x})}{g(\overline{x})} + y_I^T h_I(\overline{x}) > \overline{\lambda}^T \frac{f(\overline{u})}{g(\overline{u})} + y_I^T h_I(\overline{u}) - \frac{1}{2}\overline{p}^T \nabla^2 \left[\overline{\lambda}^T \frac{f(\overline{u})}{g(\overline{u})} + \overline{y}_I^T h_I(\overline{u})\right] \overline{p}.$$

Hence by $\overline{y} \geqq 0$ and $h(\overline{x}) \leqq 0$, the inequality above implies

$$\overline{\lambda}^T \frac{f(\overline{x})}{g(\overline{x})} > \overline{\lambda}^T \frac{f(\overline{u})}{g(\overline{u})} + y_I^T h_I(\overline{u}) - \frac{1}{2}\overline{p}^T \nabla^2 \left[\overline{\lambda}^T \frac{f(\overline{u})}{g(\overline{u})} + \overline{y}_I^T h_I(\overline{u})\right] \overline{p},$$

which contradicts (2.16). Therefore the result holds. □

2.4 Special Cases

As special cases of our duality between **(MFP)** and **(GMFD)**, we give a Wolfe type duality theorem. If $J = \emptyset$ and $I = \{1, 2, \cdots, n\}$, then **(GMFD)** reduced to the Wolfe type dual of the problem **(WMFD)** :

(WMFD) Maximize
$$\frac{f(u)}{g(u)} + (y^T h(u))e - \frac{1}{2}\left[p^T \left(\nabla^2 \lambda^T \frac{f(u)}{g(u)} + \nabla^2 y^T h(u)\right) p\right]$$

subject to
$$\nabla y^T h(u) + (\nabla^2 y^T h(u))p + \nabla \lambda^T \frac{f(x)}{g(x)} + \left(\nabla^2 \lambda^T \frac{f(x)}{g(x)}\right) = 0,$$

$$y \geq 0, \quad \lambda > 0, \quad \lambda^T e = 1,$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $\lambda \in \mathbb{R}^k$, $p \in \mathbb{R}^n$ and $e = (1, \cdots, 1)^T \in \mathbb{R}^k$, $\frac{f}{g}:\mathbb{R}^n \to \mathbb{R}^k$ and $h : \mathbb{R}^n \to \mathbb{R}^m$ are twice differentiable functions.

We can prove the following weak, strong and converse duality theorems between **(MFP)** and **(WMFD)** under second order invex assumptions.

Theorem 2.5 (Weak Duality) *Let x satisfy the constraints of* **(MFP)** *and (u, y, λ, p) satisfy the constraints of* **(WMFD)**. *If f and $-g$ are second order invex with respect to η and h is second order invex with respect to $\overline{\eta}$, then*

$$\frac{f(x)}{g(x)} \nleq \frac{f(u)}{g(u)} + [y^T h(u)]e - \frac{1}{2}\left[p^T\left(\nabla^2 y^T h(u) + \nabla^2 \frac{f(u)}{g(u)}\right)p\right]e.$$

Proof. The proof follows along the lines of Theorem 2.1. □

Theorem 2.6 (Strong Duality) *Let $\overline{x}$ be an efficient solution of* **(MFP)**. *Then $(\overline{x}, y, \lambda, p = 0)$ is a feasible solution for* **(WMFD)**, *and the objective values of* **(MFP)** *and* **(WMFD)** *are equal. Assume that the assumption of Theorem 2.5 hold, then $(\overline{x}, y, \lambda, p = 0)$ is an efficient solution of* **(WMFD)**.

Proof. Since $\overline{x}$ is an efficient solution of **(MFP)**, then there exist $\lambda \in \mathbb{R}^k(\lambda > 0, \quad \lambda^T e = 1)$ and $y \in \mathbb{R}^m$ that satisfy the following Kuhn-Tucker conditions

[51]:

$$\nabla \lambda^T \frac{f(x_0)}{g(x_0)} + \nabla y^T h(x_0) = 0,$$

$$y^T h(x_0) = 0,$$

$$y \geq 0.$$

Clearly $(\overline{x}, y, \lambda, p = 0)$ is feasible for **(WMFD)**, and the objective values of **(MFP)** and **(WMFD)** are equal. If the assumptions of Theorem 2.5 also hold, $(\overline{x}, y, \lambda, p = 0)$ is an efficient solution of **(WMFD)**. □

Theorem 2.7 (Converse Duality) *Let f, g and h are three times differentiable and let $(\overline{x}, y, \lambda, p)$ be an efficient solution of (WMFD). Suppose that*
(i) the vectors $[\nabla^2 \overline{\lambda}^T \frac{f(\overline{x})}{g(\overline{x})}]_j$, $[\nabla^2 \overline{y}^T h(\overline{x})]_j$, $j = 1, \cdots, n$ are linearly independent, where $[\nabla^2 \overline{\lambda}^T \frac{f(\overline{x})}{g(\overline{x})}]_j$ is the jth row of $[\nabla^2 \overline{\lambda}^T \frac{f(\overline{x})}{g(\overline{x})}]$
and $[\nabla^2 \overline{y}^T h(\overline{x})]_j$ is the jth row of $[\nabla^2 \overline{y}^T h(\overline{x})]$, and
(ii) the $n \times n$ Hessian matrix $\nabla \left[\nabla^2 \overline{\lambda}^T \frac{f(\overline{x})}{g(\overline{x})} + \nabla^2 \overline{y}^T h(\overline{x})\right] \overline{p}$ is positive or negative definite.
Then $\overline{x}$ is satisfied the Kuhn-Tucker conditions for **(MFP)**, *that is*

$$\nabla \lambda^T \frac{f(\overline{x})}{g(\overline{x})} + \nabla y^T h(\overline{x}) = 0, \ \ y^T h(\overline{x}) = 0, \ \ y \geq 0,$$

and the corresponding values of **(MFP)** *and* **(WMFD)** *are equal. If the assumptions of Theorem 2.5 are satisfied, then $\overline{x}$ is an efficient solution for* **(MFP)**.

Proof. Since $(\overline{x}, y, \lambda, p)$ is an efficient solution for **(WMFD)**, there exist $\alpha \in \mathbb{R}^k$, $\beta \in \mathbb{R}^m$, $\rho \in \mathbb{R}^m$ and $\omega\mathbb{R}^k$, not identically zero, such that the following Fritz John conditions are satisfied [51]:

$$\alpha^T \nabla \left[\frac{f(\overline{x})}{g(\overline{x})} + (y^T h(\overline{x}))e - \frac{1}{2}\left(p^T \left(\nabla^2 \lambda^T \frac{f(\overline{x})}{g(\overline{x})} + \nabla^2 y^T h(\overline{x}) \right) p \right) e \right]$$
$$+\beta^T \left[\nabla^2 y^T h(\overline{x}) + \nabla \left(\nabla^2 y^T h(\overline{x}) \right) p \right]$$
$$+\beta^T \left[\nabla^2 \lambda^T \frac{f(\overline{x})}{g(\overline{x})} + \nabla \left(\nabla^2 \lambda^T \frac{f(\overline{x})}{g(\overline{x})} \right) p \right] = 0, \tag{2.20}$$

$$\alpha^T \left[h(\overline{x})e - \frac{1}{2}(p^T \nabla^2 h(\overline{x}) p) e \right] + \beta^T \left[\nabla h(\overline{x}) + \nabla^2 h(\overline{x}) p \right] - \rho = 0, \tag{2.21}$$

$$\alpha^T \left[-\frac{1}{2}\left(p^T \nabla^2 \frac{f(\overline{x})}{g(\overline{x})} p \right) e \right] + \beta^T \left[\nabla \frac{f(\overline{x})}{g(\overline{x})} + \nabla^2 \frac{f(\overline{x})}{g(\overline{x})} \right] - \omega = 0, \tag{2.22}$$

$$\alpha^T \left[-\left(\nabla^2 \lambda^T \frac{f(\overline{x})}{g(\overline{x})} + \nabla^2 y^T h(\overline{x}) \right) pe \right]$$
$$+\beta^T \left[\nabla^2 \lambda^T \frac{f(\overline{x})}{g(\overline{x})} + \nabla^2 y^T h(\overline{x}) \right] = 0, \tag{2.23}$$

$$\left[\nabla y^T h(\overline{x}) + (\nabla^2 y^T h(\overline{x}))p + \nabla \lambda^T \frac{f(\overline{x})}{g(\overline{x})} + \left(\nabla^2 \lambda^T \frac{f(\overline{x})}{g(\overline{x})} \right) p \right] \beta = 0, \tag{2.24}$$

$$\rho^T y = 0, \tag{2.25}$$

$$\omega^T \lambda = 0, \tag{2.26}$$

$$y \geq 0, \tag{2.27}$$

$$(\alpha, \beta, \rho, \omega) \geq 0, \tag{2.28}$$

$$(\alpha, \beta, \rho, \omega) \neq 0. \tag{2.29}$$

Since $\nabla^2 \lambda^T \frac{f(x_0)}{g(x_0)} + \nabla^2 y^T h(x_0)$ is nonsingular, (2.26) gives

$$\beta = (\alpha^T e)p. \tag{2.30}$$

It follows that

$$\alpha \neq 0 \ \text{(i.e. } \alpha \geq 0). \tag{2.31}$$

For if $\alpha = 0$, (2.33) gives $\beta = 0$, which along with (2.24) and (2.25) implies $\rho = 0$ and $\omega = 0$, respectively. Therefore, we see that $\alpha = 0$ implies $\beta = 0, \rho = 0, \omega = 0$ which contradicts (2.32). Hence (2.34) holds. Substituting (2.33) in (2.23) gives

$$(\alpha^T e) \left(\nabla \lambda^T \frac{f(x_0)}{g(x_0)} + \nabla^2 \frac{f(x_0)}{g(x_0)} p + \nabla y^T h(x_0) + \nabla^2 y^T h(x_0) p \right)$$
$$+ \frac{1}{2} (\alpha^T e) p^T \nabla \left(\nabla^2 \lambda^T \frac{f(x_0)}{g(x_0)} + \nabla^2 y^T h(x_0) \right) p = 0,$$

which in view of (2.27) gives

$$\frac{1}{2} (\alpha^T e) p^T \nabla \left(\nabla^2 \lambda^T \frac{f(\overline{x})}{g(\overline{x})} + \nabla^2 y^T h(\overline{x}) \right) p = 0. \tag{2.32}$$

Using the hypothesis that $\frac{\partial}{\partial x_i} \left(\nabla^2 \lambda^T \frac{f(\overline{x})}{g(\overline{x})} + \nabla^2 y^T h(\overline{x}) \right)_{i=1,\cdots,n}$ is positive or negative definite,

$$p = 0 \quad \text{and} \quad \beta = (\alpha^T e)p = 0. \tag{2.33}$$

From (2.24), (2.33) and (2.36) and taking (2.34) into account gives

$$y^T h(\overline{x}) = 0. \tag{2.34}$$

From (2.23), we have

$$\nabla \lambda^T \frac{f(\overline{x})}{g(\overline{x})} + \nabla y^T h(\overline{x}) = 0. \tag{2.35}$$

Substituting (2.36) in (2.24) and taking (2.31) and (2.34) into account gives

$$h(\overline{x}) \leqq 0. \tag{2.36}$$

Conditions (2.38), (2.34), (2.39) and (2.30) are the Kuhn-Tucker conditions for **(MFP)**. The corresponding values of **(MFP)** and **(WMFD)** are equal because $y^T h(\overline{x}) = 0$ and $p = 0$. If f and $-g$ are second order invex with respect to η and h is second order invex with respect to $\overline{\eta}$, then by Theorem 2.5, $\overline{x}$ is an efficient solution for **(MFP)**. □

If $I = \emptyset$ and $J = \{1, 2, \cdots, m\}$, then **(GMFD)** is reduced to the Mond-Weir type dual **(MMFD)**:

$$\textbf{(MMFD)} \quad \text{Maximize} \quad \frac{f(u)}{g(u)} - \frac{1}{2}\left[p^T \big(\nabla^2 \lambda^T \frac{f(u)}{g(u)} + \nabla^2 y^T h(u)\big) p\right] e$$

$$\text{subject to} \quad \nabla y^T h(u) + \big(\nabla^2 y^T h(u)\big) p + \nabla \lambda^T \frac{f(u)}{g(u)} + \left(\nabla^2 \lambda^T \frac{f(x)}{g(x)}\right) p = 0,$$

$$y \geq 0,$$

$$\lambda > 0, \quad \lambda^T e = 1,$$

where $p \in \mathbb{R}^n$.

And its parametric dual program:

$$\textbf{(MFP)}_\lambda \quad \text{Minimize} \quad \lambda^T \frac{f(x)}{g(x)}$$

$$\text{subject to} \quad h(x) \leqq 0,$$

$$\textbf{(MMFD)}_\lambda \quad \text{Maximize} \quad \lambda^T \frac{f(u)}{g(u)} - \frac{1}{2}\left[p^T\big(\nabla^2 \lambda^T \frac{f(u)}{g(u)} + \nabla^2 y^T h(u)\big)p\right] e$$

$$\text{subject to} \quad \nabla y^T h(u) + \big(\nabla^2 y^T h(u)\big)p + \nabla \lambda^T \frac{f(u)}{g(u)} + \left(\nabla^2 \lambda^T \frac{f(x)}{g(x)}\right) p = 0,$$

$$y \geq 0,$$

$$\lambda > 0, \quad \lambda^T e = 1.$$

We can obtain weak, strong, converse duality theorems between **(MFP)** and **(MMFD)**$_\lambda$ or **(MFP)** and **(MMFD)**.

Theorem 2.8 (Weak Duality) *Let x be feasibel for* **(MFP)**$_\lambda$ *and (u, y, λ, p) be feasible for* **(MMFD)**$_\lambda$*. If f and $-g$ are second order invex with respect to η and h is second order invex with respect to $\overline{\eta}$, then*

$$\lambda^T \frac{f(x)}{g(x)} \geqq \lambda^T \frac{f(u)}{g(u)} - \frac{1}{2}\left[p^T\big(\nabla^2 \lambda^T \frac{f(u)}{g(u)} + \nabla^2 y^T h(u)\big)p\right].$$

Proof. Since f and $-g$ are second order invex with respect to η and h is second order invex with respect to $\overline{\eta}$,

$$\begin{aligned}
&\lambda^T \frac{f(x)}{g(x)} - \lambda^T \frac{f(u)}{g(u)} \\
&\geqq \overline{\eta}(x,u)^T \nabla \lambda^T \frac{f(u)}{g(u)} + \overline{\eta}(x,u)^T \nabla^2 \lambda^T \frac{f(u)}{g(u)} p - \frac{1}{2} \nabla^2 \lambda^T \frac{f(u)}{g(u)} p \\
&= \overline{\eta}(x,u)^T \left(-\nabla y^T h(u) - \nabla^2 y^T h(u) p \right) - \frac{1}{2} p^T \nabla^2 \lambda^T \frac{f(u)}{g(u)} p \\
&\geqq -y^T h(x) + y^T h(u) - \frac{1}{2} p^T \nabla^2 y^T h(u) p - \frac{1}{2} p^T \nabla^2 \lambda^T \frac{f(u)}{g(u)} p \\
&\geqq -y^T h(x) - \frac{1}{2} p^T \nabla^2 \lambda^T \frac{f(u)}{g(u)} p \\
&\geqq -\frac{1}{2} p^T \nabla^2 \lambda^T \frac{f(u)}{g(u)} p.
\end{aligned}$$

(The last inequality follows from the constraints of $(\mathrm{MFP})_\lambda$ and $(\mathrm{MMFD})_\lambda$.) Therefore the result hold. □

Theorem 2.9 (Weak Duality) *Let x be feasible for* **(MFP)** *and (u, y, λ, p) be feasible* **(MMFD)**. *If f and $-g$ are second order invex with respect to η and h is second order invex with respect to $\overline{\eta}$, then*

$$\frac{f(x)}{g(x)} \nleq \frac{f(u)}{g(u)} - \frac{1}{2} \left[p^T \nabla^2 \lambda^T \frac{f(u)}{g(u)} \right] e.$$

Proof. It follows on the lines of Theorem 2.1. □

Theorem 2.10 (Strong Duality) *If* $\overline{x}$ *is an efficient solution of* **(MFP)**, *then there exist* $y \in \mathbb{R}^m$ *and* $\lambda \in \mathbb{R}^k$ *such that* $(\overline{x}, y, \lambda, p = 0)$ *is feasible for* **(MMFD)**, *and the corresponding values of* **(MFP)** *and* **(MMFD)** *are equal. If* f *and* g *are second order invex with respect to* η *and* h *is second order invex with respect to* $\overline{\eta}$, *then* $(\overline{x}, y, \lambda, p)$ *is an efficient solution of* **(MMFD)**.

Proof. It follows on the lines of Theorem 2.6. □

Theorem 2.11 (Converse Duality) *Let* f, g *and* h *are three times differentiable and let* $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{p})$ *be a solution of (MMFD)*$_\lambda$. *Suppose that*

(i) the $n \times n$ *Hessian matrix* $\nabla \left[\nabla^2 \overline{\lambda}^T \frac{f(\overline{x})}{g(\overline{x})} + \nabla^2 \overline{y}^T h(\overline{x})\right] \overline{p}$ *is positive or negative definite,*

(ii)$\nabla \overline{y}^T h(\overline{x}) + \nabla^2 \overline{y}^T h(\overline{x}) \overline{p} \neq 0$ *and*

(iii) the vectors

$$[\nabla^2 \overline{\lambda}^T \frac{f(\overline{x})}{g(\overline{x})}]_j, \ \ [\nabla^2 \overline{y}^T h(\overline{x})]_j, \quad j = 1, \cdots, n$$

are linearly independent, where $[\nabla^2 \overline{\lambda}^T \frac{f(\overline{x})}{g(\overline{x})}]_j$ *is the* j*th row of* $[\nabla^2 \overline{\lambda}^T \frac{f(\overline{x})}{g(\overline{x})}]$ *and* $[\nabla^2 \overline{y}^T h(\overline{x})]_j$ *is the* j*th row of* $[\nabla^2 \overline{y}^T h(\overline{x})]$.

Then $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{p} = 0)$ *is an feasible solution of (MMFD)*$_\lambda$, *and the objective values of* **(MFP)**$_\lambda$ *and* **(MMFD)**$_\lambda$ *are equal there. If also,* f *and* $-g$ *are second order invex with respect to* η *and* h *is second order invex with respect to* $\overline{\eta}$ *then* $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{p} = 0)$ *is an efficient solution of* **(MMFD)**$_\lambda$.

Proof. Since $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{p})$ is a solution of $\mathbf{(MMFD)}_\lambda$, by the Fritz John necessary conditions [51], there exist $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^n$, $\gamma \in \mathbb{R}$, $\rho \in \mathbb{R}^m$ and $\omega \in \mathbb{R}^k$ such that

$$\begin{aligned}
&-\alpha\left[\nabla\overline{\lambda}^T\frac{f(\overline{x})}{g(\overline{x})} - \frac{1}{2}\overline{p}^T\nabla\left(\nabla^2\overline{\lambda}^T\frac{f(\overline{x})}{g(\overline{x})}\right)\overline{p}\right]\\
&+\beta^T\left[\nabla^2\overline{y}^T h(\overline{x}) + \nabla\big(\nabla^2\overline{y}^T h(\overline{x})\big)\overline{p} + \nabla^2\overline{\lambda}^T\frac{f(\overline{x})}{g(\overline{x})} + \nabla\left(\nabla^2\overline{\lambda}^T\frac{f(\overline{x})}{g(\overline{x})}\right)\overline{p}\right]\\
&\qquad -\gamma\left[\nabla\overline{y}^T h(\overline{x}) - \frac{1}{2}\overline{p}^T\nabla\big(\nabla^2\overline{y}^T h(\overline{x})\big)\overline{p}\right] = 0,
\end{aligned} \tag{2.37}$$

$$\beta^T\left[\nabla h(\overline{x}) + \nabla^2 h(\overline{x})\overline{p}\right] - \gamma\left[h(\overline{x}) - \frac{1}{2}\overline{p}^T\nabla^2 h(\overline{x})\overline{p}\right] - \rho = 0, \tag{2.38}$$

$$-\alpha\left[\frac{f(\overline{x})}{g(\overline{x})} - \frac{1}{2}\overline{p}^T\left(\nabla^2\frac{f(\overline{x})}{g(\overline{x})}\right)\overline{p}\right] + \beta^T\left[\nabla\frac{f(\overline{x})}{g(\overline{x})} + \nabla^2\frac{f(\overline{x})}{g(\overline{x})}\overline{p}\right] - \omega = 0, \tag{2.39}$$

$$\begin{aligned}
&-\alpha\left[-\nabla^2\overline{\lambda}^T\frac{f(\overline{x})}{g(\overline{x})}\overline{p}\right] + \beta^T\left[\nabla^2\overline{y}^T h(\overline{x}) + \nabla^2\overline{\lambda}^T\frac{f(\overline{x})}{g(\overline{x})}\right]\\
&\qquad -\gamma\left[-\nabla^2\overline{y}^T h(\overline{x})\overline{p}\right] = 0,
\end{aligned} \tag{2.40}$$

$$\beta^T\left[\nabla\overline{y}^T h(\overline{x}) + \nabla^2\overline{y}^T h(\overline{x})\overline{p} + \nabla\overline{\lambda}^T\frac{f(\overline{x})}{g(\overline{x})} + \nabla^2\overline{\lambda}^T\frac{f(\overline{x})}{g(\overline{x})}\overline{p}\right] = 0, \tag{2.41}$$

$$\gamma\left[\overline{y}^T h(\overline{x}) - \frac{1}{2}\overline{p}^T\nabla^2\overline{y}^T h(\overline{x})\overline{p}\right] = 0, \tag{2.42}$$

$$\rho^T\overline{y} = 0, \tag{2.43}$$

$$\omega^T\overline{\lambda} = 0, \tag{2.44}$$

$$(\alpha, \beta, \gamma, \rho, \omega) \geqq 0, \tag{2.45}$$

$$(\alpha, \beta, \gamma, \rho, \omega) \neq 0. \tag{2.46}$$

Since $\left\{ \left[\nabla^2 \overline{\lambda}^T \frac{f(\overline{x})}{g(\overline{x})} \right]_j, \left[\nabla^2 \overline{y}^T h(\overline{x}) \right]_j, j = 1, \cdots, m \right\}$ are linearly independent at $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{p})$, then (2.44)gives

$$\alpha \overline{p} + \beta = 0 \quad \text{and} \quad \gamma \overline{p} + \beta = 0. \tag{2.47}$$

Multiplying (2.42) by $\overline{y}^T$ and then using (2.46) and (2.47), we have

$$\beta^T \left[\nabla \overline{y}^T h(\overline{x}) + \nabla^2 \overline{y}^T h(\overline{x}) \overline{p} \right] = 0. \tag{2.48}$$

Using constraints in (2.41), we have

$$\begin{aligned}
(\alpha \overline{p} + \beta)^T & \left[\nabla^2 \overline{\lambda}^T \frac{f(\overline{x})}{g(\overline{x})} + \nabla \left(\nabla^2 \overline{\lambda}^T \frac{f(\overline{x})}{g(\overline{x})} \right) \overline{p} \right] \\
& + (\gamma \overline{p} + \beta)^T \left[\nabla^2 \overline{y}^T h(\overline{x}) + \nabla \left(\nabla^2 \overline{y}^T h(\overline{x}) \right) \overline{p} \right] \\
& + (\alpha - \gamma) \left[\nabla \overline{y}^T h(\overline{x}) + (\nabla^2 \overline{y}^T h(\overline{x}) \overline{p}) \right] \\
& + \frac{1}{2} \beta^T \left[\nabla \left(\nabla^2 \overline{\lambda}^T \frac{f(\overline{x})}{g(\overline{x})} \right) \overline{p} + \nabla \left(\nabla^2 \overline{y}^T h(\overline{x}) \right) \overline{p} \right] = 0. \qquad (2.49)
\end{aligned}$$

Using (2.51), (2.53) gives

$$(\alpha - \gamma) \left[\nabla \overline{y}^T h(\overline{x}) + (\nabla^2 \overline{y}^T h(\overline{x}) \overline{p}) \right] + \frac{1}{2} \beta^T \left[\nabla \left(\nabla^2 \overline{\lambda}^T \frac{f(\overline{x})}{g(\overline{x})} \right) \overline{p} + \nabla \left(\nabla^2 \overline{y}^T h(\overline{x}) \right) \overline{p} \right] = 0. \tag{2.50}$$

Multiplying (2.54) by β^T and using (2.52), we have

$$\beta^T \left[\nabla \left(\nabla^2 \overline{\lambda}^T \frac{f(\overline{x})}{g(\overline{x})} + \nabla^2 \overline{y}^T h(\overline{x}) \right) \overline{p} \right] \beta = 0.$$

$\nabla \left[\nabla^2 \overline{\lambda}^T \frac{f(\overline{x})}{g(\overline{x})} + \nabla^2 \overline{y}^T h(\overline{x}) \right] \overline{p}$ is positive or negative definite, it follows that

$$\beta = 0.$$

Using $\beta = 0$ in (2.54), we have

$$(\alpha - \gamma) \left[\nabla \overline{y}^T h(\overline{x}) + (\nabla^2 \overline{y}^T h(\overline{x}) \overline{p}) \right] = 0. \tag{2.51}$$

Because of the assumption (ii), this gives

$$\alpha = \gamma.$$

If $\alpha = 0$ then $\gamma = 0$ and so from (2.42) and (2.43) and $\beta = 0$, it follows that $\rho = \omega = 0$. Therefore $(\alpha, \beta, \gamma, \rho, \omega) = 0$ which contradicts (2.50). Hence $\alpha > 0$ and from (2.55), $\gamma > 0$. Using $\gamma > 0$, $\alpha > 0$ and $\beta = 0$, (2.51) yield

$$\overline{p} = 0.$$

This gives

$$\overline{\lambda}^T \frac{f(\overline{x})}{g(\overline{x})} = \overline{\lambda}^T \frac{f(\overline{x})}{g(\overline{x})} - \frac{1}{2} \overline{p}^T \nabla^2 \overline{\lambda}^T \frac{f(\overline{x})}{g(\overline{x})} \overline{p}.$$

Using $\gamma > 0, \beta = 0$ and $\overline{p} = 0$, (2.42) gives

$$h(\overline{x}) \leq 0.$$

Thus $\overline{x}$ is feasible for $(\mathbf{MFP})_\lambda$ and the object functions of $(\mathbf{MFP})_\lambda$ and $(\mathbf{MMFD})_\lambda$ are equal. f and $-g$ are second order invex with respect to η and h is second order invex with respect to $\overline{\eta}$, then by Theorem 2.9, $\overline{x}$ is an efficient solution for $(\mathbf{MFP})_\lambda$. □

Chapter 3

Higher Order Duality in Nonlinear Programming with Cone Constraints

3

3.1 Introduction

We consider the following nonlinear programming problem:

$$\begin{aligned} (\mathbf{P}) \quad \text{Minimize} \quad & f(x) \\ \text{subject to} \quad & g(x) \geq 0, \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are twice differentiable functions.

The first order Wolfe dual problem [81] is

$$\begin{aligned} (\mathbf{D1}) \quad \text{Maximize} \quad & f(u) - y^T g(u) \\ \text{subject to} \quad & \nabla f(u) - \nabla y^T g(u) = 0, \\ & y \geq 0. \end{aligned}$$

The Mangasarian second order dual [52] is

$$\begin{aligned} (\mathbf{D2}) \quad \text{Maximize} \quad & f(u) - y^T g(u) - \frac{1}{2} p^T \nabla^2 [f(u) - y^T g(u)] p \\ \text{subject to} \quad & \nabla [f(u) - y^T g(u)] + \nabla^2 [f(u) - y^T g(u)] p = 0, \\ & y \geq 0. \end{aligned}$$

Several approaches to duality for (P) may be found in the literature. These include the use of the first order dual [12, 16, 22, 23, 28, 33, 36, 78, 79] and second order dual [24, 66] to establish duality theorems.

Higher order duality in nonlinear programming has been studied by many researchers [52, 57, 58, 65, 85]. By introducing two differentiable functions $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$, Mangasarian [52] formulated the higher order dual

$$
\begin{aligned}
(\mathbf{HD1}) \quad \text{Maximize} \quad & f(u) + h(u,p) - y^T g(u) - y^T k(u,p) \\
\text{subject to} \quad & \nabla_p h(u,p) = \nabla_p (y^T k(u,p)), \\
& y \geq 0,
\end{aligned}
$$

where $\nabla_p h(u,p)$ denotes the $n \times 1$ gradient of h with respect to p and $\nabla_p(y^T k(u,p))$ denotes the $n \times 1$ gradient of $y^T k$ with respect to p.

Mangasarian, however, did not prove a weak duality theorem for **(P)** and **(HD1)** and only gave a limited version of strong duality. In [63], Mond and Weir gave the conditions for which duality holds between **(P)** and **(HD1)**. They also consider other higher order dual to **(P)**:

$$
\begin{aligned}
(\mathbf{HD}) \quad \text{Maximize} \quad & f(u) + h(u,p) - p^T \nabla_p h(u,p) \\
\text{subject to} \quad & \nabla_p h(u,p) = \nabla_p (y^T k(u,p)), \\
& y^T g(u) + y^T k(u,p) - p^T \nabla_p (y^T k(u,p)) \leq 0, \\
& y \geq 0.
\end{aligned}
$$

Mond and Zhang [65] obtained duality results for various higher order dual programming problems under higher order invexity assumptions. Later on, under more general invexity-type assumptions, such as higher order type-I, higher order pseudo-type-I or higher order quasi-type-I conditions, Mishra and Rueda [57, 58] gave various duality results, which included Mangasarian higher order duality [52] and Mond-Weir higher order duality [63] as special cases. Chen [15] also discussed the duality theorems under the higher order F-convexity (F-pseudoconvexity, F-quasiconvexity) for a pair of nondifferentiable programs.

In this chapter, we present Mond-Weir and Wolfe type higher order programming problems with cone constraints and prove weak, strong and converse duality theorems under generalized convexity and invexity assumptions. These results are the extension of higher order duality relations due to Zhang [85]. And we formulate a Fritz John higher order programming problem by using Fritz John [51] necessary optimality condition instead of Karush-Kuhn-Tucker one [51] and establish weak, strong, and converse duality theorems. Thus, the requirement of a constraint qualification can be eliminated.

3.2 Notations and Preliminaries

We consider the following multiobjective programming problem:

$$\textbf{(KP)} \quad \text{Minimize} \quad f(x)$$
$$\text{subject to} \quad -g(x) \in Q, \quad x \in C,$$

where $f : \mathbb{R}^n \to \mathbb{R}^k$, $g : \mathbb{R}^n \to \mathbb{R}^m$ and $C \subset \mathbb{R}^n$, Q is a closed convex cone with nonempty interior in $\mathbb{R}^m$.

We shall denote the feasible set of **(KP)** by $S = \{x| - g(x) \in Q, x \in C\}$.

Definition 3.1 *The polar cone K^* of K is defined by*

$$K^* = \{z \in \mathbb{R}^k \quad | \quad x^T z \leqq 0 \quad \text{for all} \quad x \in K\}.$$

The following definitions are due to Preda [69] and Mond and Zhang [65].

Definition 3.2 *Let $C \subseteq \mathbb{R}^n$ be open, $f : C \to \mathbb{R}$ be a differentiable function.*
(i) f is said to be higher order invex if there exists a function $\eta : C \times C \to C$, for all $x, u \in C$,

$$f(x) - f(u) \geq \eta(x,u)^T \nabla_p h(u,p) + h(u,p) - p^T \nabla_p h(u,p).$$

(ii) f is said to be higher order pseudo-invex, if there exists a function $\eta : C \times C \to C$, for all $x, u \in C$,

$$\eta(x,u)^T \nabla_p h(u,p) \leq 0 \Rightarrow f(x) - f(u) - h(u,p) + p^T \nabla_p h(u,p) \geq 0.$$

Let $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a sublinear functional, the function $f = (f_1, \cdots, f_k) : \mathbb{R}^n \to \mathbb{R}^k$ a twice differentiable at $u \in \mathbb{R}^n$, $\rho = (\rho_1, \cdots \rho_k) \in \mathbb{R}^k$ and $d(\cdot, \cdot)$ a metric on $\mathbb{R}^n$.

Definition 3.3 *The function f is said to be (F, ρ)-convex at u, if for all $x \in S$,*

$$f(x) - f(u) \geqq F(x, u; \nabla f(u)) + \rho d(x,u).$$

This function f is said to be strongly F-convex, F-convex or weakly F-convex at u according to $\rho > 0$, $\rho = 0$, or $\rho < 0$.

Definition 3.4 *The function f is (F, ρ)-quasiconvex at u for all $x \in S$,*

$$f(x) \leqq f(u) \Rightarrow F(x, u; \nabla f(u)) \leqq -\rho d(x, u).$$

This function f is said to be strongly F-quasiconvex, F-quasiconvex or weakly F-quasiconvex at u according to $\rho > 0$, $\rho = 0$, or $\rho < 0$.

Definition 3.5 *The function f is said to be second order (F, ρ)-convex at u and p, if for all $x \in S$*

$$f(x) - f(u) + \frac{1}{2}p^T \nabla^2 f(u)p \geqq F(x, u; \nabla f(u) + \nabla^2 f(u)p) + \rho d(x, u).$$

This function f is said to be strongly second order F-convex, second order F-convex, ora weakly second order F-convex at u and p, according to $\rho > 0$, $\rho = 0$ or $\rho < 0$.

Definition 3.6 *The function f is said to be second order (F, ρ)-quasiconvex at u and p, if for all $x \in S$*

$$f(x) \leqq f(u) - \frac{1}{2}p^T \nabla^2 f(u)p \Rightarrow F(x, u; \nabla f(u) + \nabla^2 f(u)p) \leqq -\rho d(x, u).$$

This function f is said to be strongly second order F-quasiconvex, second order F-quasiconvex, or weakly second order F-quasiconvex at u and p, according to $\rho > 0$, $\rho = 0$ or $\rho < 0$.

Definition 3.7 *The function f is said to be second order (F,ρ)-pseudonvex at u and p, if for all $x \in S$*

$$F(x,u;\nabla f(u)+\nabla^2 f(u)p) \geqq -\rho d(x,u) \Rightarrow f(x) \geqq f(u) - \frac{1}{2}p^T\nabla^2 f(u)p.$$

This function f is said to be strongly second order F-pseudonvex, second order F-pseudonvex, or weakly second order F-pseudonvex at u and p, according to $\rho > 0,\ \ \rho = 0$ or $\rho < 0$.

Note that second order (F,ρ)-convexity, second order (F,ρ)-quasiconvexity and second order (F,ρ)-pseudonvexity imply, respectively, first order (F,ρ)-convexity, (F,ρ)-quasiconvexity and (F,ρ)-pseudonvexity since the respective inequalities must hold for $p = 0$.

3.3 Mond-Weir Type Higher Order Duality

In this section, we propose the following higher order multiobjective programming problem,

$$\textbf{(MCP)} \quad \text{Minimize} \quad f(x)$$
$$\text{subject to} \quad -g(x) \in C_2^*,\ \ x \in C_1,$$

and the Mond-Weir higher order multiobjective dual

$$\textbf{(MMCD)} \quad \text{Maximize} \quad f(u) + (\lambda^T h(u,p))e - p^T\nabla_p(\lambda^T h(u,p))e$$
$$\text{subject to} \quad \nabla_p(\lambda^T h(u,p)) = \nabla_p(y^T k(u,p)), \tag{3.1}$$
$$g(u) + k(u,p) - p^T\nabla_p k(u,p) \in C_2^*,$$
$$y \in C_2,\ \ \lambda > 0,\ \ \lambda^T e = 1,$$

where

(1) $f : \mathbb{R}^n \to \mathbb{R}^l$, $g : \mathbb{R}^n \to \mathbb{R}^m$ are differentiable functions,

(2) C_1 and C_2 are closed convex cones in $\mathbb{R}^n$ and $\mathbb{R}^m$ with nonempty interiors, respectively,

(3) C_1^* and C_2^* are polar cones of C_1 and C_2, respectively,

(4) $e = (1, \cdots, 1)^T$ is vector in $\mathbb{R}^l$,

(5) $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^l$ and $k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ are differentiable functions; $\nabla_p(h_j(u,p))$ denotes the $n \times 1$ gradient of h_j with respect to p, and $\nabla_p(y^T k(u,p))$ denotes the $n \times 1$ gradient of $y^T k$ with respect to p.

Now we establish the duality theorems for **(MCP)** and **(MMCD)**.

Theorem 3.1 (Weak Duality) *Let x be feasible solutions of* **(MCP)** *and (u, y, λ, p) feasible for* **(MMCD)**. *Assume that*

(i) $\eta(x,u)^T(\nabla_p(\lambda^T h(u,p))) \geqq 0$

$$\Rightarrow \lambda^T f(x) \geqq \lambda^T f(u) + (\lambda^T h(u,p)) - p^T \nabla_p(\lambda^T h(u,p)) \tag{3.2}$$

$-\eta(x,u)^T(\nabla_p(y^T k(u,p))) \geqq 0$

$$\Rightarrow -y^T g(x) > -y^T g(u) - (y^T k(u,p)) + p^T \nabla_p(y^T k(u,p)); \textit{or} \tag{3.3}$$

(ii) $f_j(x) - f_j(u) - h_j(u,p) + p^T \nabla_p h_j(u,p)$

$$\geqq F(x,u; \nabla_p h_j(u,p)) + \rho_{1j} d(x,u), \quad j = 1, \cdots, l \quad \textit{and} \tag{3.4}$$

$-g_i(x) - g_i(u) + k_i(u,p) - p^T \nabla_p k_i(u,p)$

$$\geqq F(x,u; \nabla_p k_i(u,p)) + \rho_{2i} d(x,u), \quad i = 1, \cdots, m \tag{3.5}$$

(iii) $\lambda^T f(x) - y^T g(x) - [\lambda^T f(u) - y^T g(u)]$

$$-[\lambda^T h(u,p) - y^T k(u,p)] + p^T[\nabla_p(\lambda^T h(u,p)) - \nabla_p(y^T k(u,p))]$$

$$\geqq F(x,u;\nabla_p(\lambda^T h(u,p)) - \nabla_p(y^T k(u,p))) + \rho d(x,u) \quad (3.6)$$

such that $\rho \geqq 0$; *or*

(iv) $F(x,u;\nabla_p(\lambda^T h(u,p))) \geqq -\rho_1 d(x,u)$

$$\Rightarrow \lambda^T f(x) - \lambda^T f(u) - (\lambda^T h(u,p)) + p^T \nabla_p(\nabla_p \lambda^T h(u,p)) \geqq 0 \text{ and} \quad (3.7)$$

$$-[y^T g(x) - y^T g(u) - y^T k(u,p) + p^T \nabla_p(y^T k(u,p))] \leqq 0$$

$$\Rightarrow F(x,u;-\nabla_p(y^T k(u,p))) \leqq -\rho_0 d(x,u) \quad (3.8)$$

such that $\rho_0 + \rho_1 \geqq 0$

for all feasible (x,u,y,λ,p), *then*

$$f(x) \nleq f(u) + (\lambda^T h(u,p))e - p^T \nabla_p(\lambda^T h(u,p))e.$$

Proof. (i) Assume to the contrary that

$$f(x) \leq f(u) + (\lambda^T h(u,p))e - p^T \nabla_p(\lambda^T h(u,p))e.$$

Since $\lambda_i > 0$ $(i = 1, \cdots l)$,

$$\lambda^T f(x) < \lambda^T f(u) + (\lambda^T h(u,p)) - p^T \nabla_p(\lambda^T h(u,p)). \quad (3.9)$$

This in view of (3.2),

$$\eta(x,u)^T(\nabla_p(\lambda^T h(u,p))) < 0. \quad (3.10)$$

From the constraints of **(MCP)** and **(MMCD)**,

$$y^T g(x) \geqq y^T g(u) + (y^T k(u,p)) - p^T \nabla_p(y^T k(u,p)). \quad (3.11)$$

By condition (3.3),

$$\eta(x,u)^T(\nabla_p(y^Tk(u,p))) > 0. \tag{3.12}$$

Combining (3.10) and (3.12), we have

$$\eta(x,u)^T(\nabla_p(\lambda^T h(u,p)) - \nabla_p(y^Tk(u,p))) < 0,$$

which contradicts (3.1). Hence

$$f(x) \nleq f(u) + (\lambda^T h(u,p))e - p^T\nabla_p(\lambda^T h(u,p))e.$$

(ii) Since x and (u, y, λ, p) are feasible for **(MCP)** and **(MMCD)**, respectively. Subtracting (3.11) from (3.9) and rearranging yields

$$\begin{aligned}&\lambda^T f(x) - \lambda^T f(u) - (\lambda^T h(u,p)) + p^T\nabla_p(\lambda^T h(u,p))\\ &\quad -[y^Tg(x) - y^Tg(u) - (y^Tk(u,p)) + p^T\nabla_p(y^Tk(u,p))] < 0\end{aligned}$$

By multiplying (3.4) by $\lambda_j > 0$, (3.5) by $y_i \in C_2$, then

$$\begin{aligned}&\lambda^T f(x) - \lambda^T f(u) - (\lambda^T h(u,p)) + p^T\nabla_p(\lambda^T h(u,p))\\ &\qquad \geqq F(x,u;\nabla_p(\lambda^T h(u,p))) + \textstyle\sum_{j=1}^{l}\lambda_j\rho_{1j}d(x,u)\end{aligned} \tag{3.13}$$

$$\begin{aligned}&-y^Tg(x) + y^Tg(u) + y^Tk(u,p) - p^T\nabla_p(y^Tk(u,p))\\ &\quad \geqq F(x,u;-\nabla_p(y^Tk(u,p))) + \textstyle\sum_{i=1}^{m} y_i\rho_{2i}d(x,u)\end{aligned} \tag{3.14}$$

Summing (3.13) and (3.14), and using sublinearity of $F(x, u; \cdot)$, we have

$$\begin{aligned} 0 &> \left[\lambda^T f(x) - \lambda^T f(u) - \lambda^T h(u,p) + p^T \nabla_p(\lambda^T h(u,p))\right] \\ &\qquad -\left[y^T g(x) - y^T g(u) 9 y^T k(u,p) + p^T \nabla_p(y^T k(u,p))\right] \\ &\geqq F(x, u; \nabla_p \lambda^T h(u,p) - \nabla_p(y^T k(u,p))) + \Big(\sum_{j=1}^{l} \lambda_j \rho_{1j} + \sum_{i=1}^{m} y_i \rho_{2i}\Big) d(x,u), \end{aligned}$$

which is a contradiction since(3.1), $F(x, u; 0) = 0$ and $\left(\sum_{j=1}^{l} \lambda_j \rho_{1j} + \sum_{i=1}^{m} y_i \rho_{2i}\right) \geqq 0$.

(iii) Subtracting (3.11) from (3.9), then yield

$$\begin{aligned} \lambda^T f(x) - y^T g(x) &< \lambda^T f(u) - y^T g(u) + \lambda^T h(u,p) - y^T k(u,p) \\ &\qquad -p^T[\nabla_p(\lambda^T h(u,p)) - \nabla_p(y^T k(u,p))]. \end{aligned}$$

From the assumption (iii),

$$\begin{aligned} 0 &> \lambda^T f(x) - y^T g(x) - [\lambda^T f(u) - y^T g(u)] - [\lambda^T h(u,p) - y^T k(u,p)] \\ &\qquad -p^T[\nabla_p(\lambda^T h(u,p)) - \nabla_p(y^T k(u,p))] \\ &\geqq F(x, u; \nabla_p(\lambda^T h(u,p)) - \nabla_p(y^T k(u,p))) + \rho d(x,u) \end{aligned}$$

such that $\rho \geqq 0$.

It follows that

$$F(x, u; \nabla_p(\lambda^T h(u,p)) - \nabla_p(y^T k(u,p))) < -\rho d(x,u). \tag{3.15}$$

Hence $F(x, u; 0) = 0$ and (3.15) imply that $\rho d(x,u) < 0$, which contradicts $\rho \geqq 0$.

(iv) From $\lambda_i > 0$, we have (3.9), then the assumption of (3.7) gives

$$F(x,u;\nabla_p(\lambda^T h(u,p))) < -\rho_1 d(x,u). \tag{3.16}$$

And (3.11) gives

$$F(x,u;-\nabla_p(y^T k(u,p))) \leqq -\rho_0 d(x,u). \tag{3.17}$$

Hence (3.16), (3.17), the sublinearity of F and $\rho_1 + \rho_0 \geqq 0$ then imply

$$F(x,u;\nabla_p(\lambda^T h(u,p)) - \nabla_p(y^T k(u,p))) < 0,$$

which is a contradiction since $F(x,u;0) = 0$. □

Lemma 3.1 **[45]** *If $\bar{x}$ is an efficient solution of* **(MCP)**, *then there exist $\alpha \geqq 0$ and $y \in C_2$ not both zero such that*

$$[\nabla \alpha^T f(\overline{x}) + \beta^T \nabla g(\overline{x})]^T (x - \overline{x}) \geqq 0, \quad \textit{for all} \quad x \in C_1,$$

$$\beta^T g(\overline{x}) = 0.$$

Equivalently, there exist $\alpha \in K^*$, $\beta \in Q^*$ and $\beta_1 \in C^*, (\lambda, \beta, \beta_1) \neq 0$ such that

$$\alpha^T \nabla f(\overline{x}) + \beta^T \nabla g(\overline{x}) - \beta_1^T I = 0,$$

$$\beta^T g(\overline{x}) = 0,$$

$$\beta_1^T \overline{x} = 0.$$

Proof. (Sufficiency) Substituting $x = 0$ and $x = 2\overline{x}$, we get

$$(\alpha^T \nabla f(\overline{x}) + \beta^T \nabla g(\overline{x}))\overline{x} = 0.$$

Since $\alpha^T \nabla f(\overline{x}) + \beta^T \nabla g(\overline{x}) \in C^*$, let $\beta_1 = \alpha^T \nabla f(\overline{x}) + \beta^T \nabla g(\overline{x})$.

$$\alpha^T \nabla f(\overline{x}) + \beta^T \nabla g(\overline{x}) - \beta_1^T I = 0,$$

$$\beta^T g(\overline{x}) = 0,$$

$$\beta_1^T \overline{x} = 0.$$

(Necessity) Since $\alpha^T \nabla f(\overline{x}) + \beta^T \nabla g(\overline{x}) = \beta_1 \in C^*$, we get

$$(\alpha^T \nabla f(\overline{x}) + \beta^T \nabla g(\overline{x}))x \geqq 0, \quad \text{for all} \quad x \in C,$$

and

$$\beta_1^T \overline{x} = (\alpha^T \nabla f(\overline{x}) + \beta^T \nabla g(\overline{x}))\overline{x} = 0.$$

Therefore,

$$(\alpha^T \nabla f(\overline{x}) + \beta^T \nabla g(\overline{x}))(x - \overline{x}) \geqq 0, \quad \text{for all} \quad x \in C,$$
$$\beta^T g(\overline{x}) = 0.$$

□

Theorem 3.2 (Strong Duality) *Let $\overline{x}$ be an efficient solution for* **(MCP)** *and let*

$$h(\overline{x}, 0) = 0, \;\; k(\overline{x}, 0) = 0, \;\; \nabla_p h(\overline{x}, 0) = \nabla f(\overline{x}), \;\; \nabla_p k(\overline{x}, 0) = \nabla g(\overline{x}). \quad (3.18)$$

Then there exist $\lambda \geqq 0$ and $y \in C_2$ not both zero such that $(\overline{x}, y, \lambda, p = 0)$ is feasible for **(MCD)** *and the corresponding values of* **(MCP)** *and* **(MCD)** *are equal. If for all feasible $(\overline{x}, u, y, \lambda, p)$, the assumptions of Theorem 3.1 are satisfied, then $(\overline{x}, y, \lambda, p = 0)$ is efficient for* **(MCD)**.

Proof. Since $\overline{x}$ is an efficient solution of **(MCP)**, then there exist $\lambda \geqq 0$ and $y \in C_2$ such that

$$[\nabla \lambda^T f(\overline{x}) + y^T \nabla g(\overline{x})]^T (x - \overline{x}) \geqq 0, \quad \text{for all} \quad x \in C_1 \tag{3.19}$$

and

$$y^T g(\overline{x}) = 0. \tag{3.20}$$

Since $x \in C_1$, $\overline{x} \in C_1$ and C_1 is a closed convex cone, we have $x + \overline{x} \in C_1$ and thus the inequality (3.19) implies

$$[\nabla \lambda^T f(\overline{x}) + y^T \nabla g(\overline{x})]^T x \geqq 0 \quad \text{for all} \;\; x \in C_1.$$

By letting $x = 0$ and $x = 2\overline{x}$ in (3.19), we obtain

$$[\nabla \lambda^T f(\overline{x}) + y^T \nabla g(\overline{x})] = 0.$$

And (3.20) implies $y^T g(\overline{x}) \geqq 0$, then

$$-g(\overline{x}) \in C_2^*.$$

Clearly $(\overline{x}, y, \lambda, p = 0)$ is feasible for **(MCD)** and corresponding values of **(MCP)** and **(MCD)** are equal. If the assumptions of Theorem 3.1 are satisfied, then $(\overline{x}, y, \lambda, p = 0)$ must be efficient solution for **(MCD)**. □

Theorem 3.3 (Converse Duality) *Let $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{p})$ be an efficient solution of* **(MMCD)**. *Let the condition of (3.18) be satisfied. Assume that*

(i) the matrix

$$\nabla_p \left[\nabla \overline{\lambda}^T f(\overline{x}) + \nabla(\overline{\lambda}^T h(\overline{x}, \overline{p})) + \nabla \overline{y}^T g(\overline{x}) + \nabla(\overline{y}^T k(\overline{x}, \overline{p})) \right]$$

is positive or negative definite and

(ii) the vectors

$$\left\{\nabla_p^2 \overline{\lambda}_i h_i(\overline{x},\overline{p})\right\}_{i=1,\cdots,l} \quad \textit{and} \quad \left\{\nabla_p^2 k_j(\overline{x},\overline{p})\right\}_{j=1,\cdots,m}$$

are linearly independent.

If the conditions of Theorem 3.1 hold, then $\overline{x}$ is an efficient solution for **(MCP)**.

Proof. Since $(\overline{x},\overline{y},\overline{\lambda},\overline{p})$ is an efficient solution for **(MMCD)**, by lemma 3.1, there exist $\alpha \in \mathbb{R}^l,\ \beta \in C_1,\ \gamma \in C_2,\ \delta \in C_2^*$ and $\xi \in \mathbb{R}^l$ such that

$$\begin{aligned}
&-\nabla f(\overline{x})\alpha - (\alpha^T e)[\nabla(\overline{\lambda}^T h(\overline{x},\overline{p})) - \nabla(\nabla_p(\overline{\lambda}^T h(\overline{x},\overline{p})))p] \\
&\quad -[\nabla(\nabla_p(\overline{\lambda}^T h(\overline{x},\overline{p}))) + \nabla(\nabla_p(\overline{y}^T k(\overline{x},\overline{p})))]\beta \\
&\qquad -\gamma^T[\nabla g(\overline{x}) + \nabla k(\overline{x},\overline{p}) - \overline{p}^T \nabla(\nabla_p k(\overline{x},\overline{p}))] = 0, \quad (3.21)
\end{aligned}$$

$$-\beta \nabla_p k(\overline{x},\overline{p}) - \delta = 0, \quad (3.22)$$

$$((\alpha^T e)\overline{p} - \beta)^T \nabla_p^2(\overline{\lambda}^T h(\overline{x},\overline{p})) + (\gamma^T \overline{p} - \beta^T \overline{y})^T \nabla_p^2 k(\overline{x},\overline{p}) = 0, \quad (3.23)$$

$$-(\alpha^T e)[h(\overline{x},\overline{p}) - \overline{p}^T \nabla_p h(\overline{x},\overline{p})] - \beta^T[\nabla_p h(\overline{x},\overline{p})] - \xi = 0, \quad (3.24)$$

$$\beta^T[\nabla_p(\overline{\lambda}^T h(\overline{x},\overline{p})) + \nabla_p(\overline{y}^T k(\overline{x},\overline{p}))] = 0, \quad (3.25)$$

$$\gamma^T[g(\overline{x}) + k(\overline{x},\overline{p}) - \overline{p}^T \nabla_p k(\overline{x},\overline{p})] = 0, \quad (3.26)$$

$$\delta^T \overline{y} = 0, \quad (3.27)$$

$$\xi^T \overline{\lambda} = 0, \quad (3.28)$$

$$(\alpha,\beta,\gamma,\delta,\xi) \leq 0, \quad (3.29)$$

$$(\alpha,\beta,\gamma,\delta,\xi) \neq 0. \quad (3.30)$$

$\{\nabla_p^2 \lambda_i^T h_i(\overline{x},\overline{p})\}_{i=1,\cdots,l}$ and $\{\nabla_p^2 k_j(\overline{x},\overline{p})\}_{j=1,\cdots,m}$ are linearly independent, then (3.23) gives

$$(\alpha^T e)\overline{p} - \beta = 0 \text{ and } \gamma\overline{p} - \beta\overline{y} = 0. \tag{3.31}$$

Multiplying (3.22) by $\overline{y}^T$ and using (3.27)

$$-\beta^T \nabla_p(\overline{y}^T k(\overline{x},\overline{p})) = 0. \tag{3.32}$$

Using (3.31) in (3.21), we have

$$-\alpha^T[\nabla f(\overline{x}) + \nabla h(\overline{a}x,\overline{p})] - \gamma^T[\nabla g(\overline{x}) + \nabla k(\overline{x},\overline{p})] = 0. \tag{3.33}$$

Multiplying (3.33) by $\overline{p}$ and using (3.31) gives

$$-(\alpha^T e)\overline{p}^T[\nabla \lambda^T f(\overline{x}) + \nabla(\overline{\lambda}^T h(\overline{x},\overline{p}))] + \beta^T[\nabla \overline{y}^T g(\overline{x}) + \overline{y}^T k(\overline{x},\overline{p})] = 0,$$

that is

$$\beta^T[\nabla\overline{\lambda}^T f(\overline{x}) + \nabla(\overline{\lambda}^T h(\overline{x},\overline{p})) + \nabla\overline{y}^T g(\overline{x}) + \nabla\overline{y}^T k(\overline{x},\overline{p})] = 0. \tag{3.34}$$

Differentiating (3.32) with respect to p yields

$$\beta^T\nabla_p[\nabla\overline{\lambda}^T f(\overline{x}) + \nabla(\overline{\lambda}^T h(\overline{x},\overline{p})) + \nabla\overline{y}^T g(\overline{x}) + \nabla\overline{y}^T k(\overline{x},\overline{p})] = 0. \tag{3.35}$$

Multiplying (3.35) by β, we get

$$\beta^T\nabla_p[\nabla\overline{\lambda}^T f(\overline{x}) + \nabla(\overline{\lambda}^T h(\overline{x},\overline{p})) + \nabla\overline{y}^T g(\overline{x}) + \nabla(\overline{y}^T k(\overline{x},\overline{p}))]\beta = 0.$$

$\nabla\overline{\lambda}^T f(\overline{x}) + \nabla(\overline{\lambda}^T h(\overline{x},\overline{p})) + \nabla\overline{y}^T g(\overline{x}) + \nabla(\overline{y}^T k(\overline{x},\overline{p}))$ is positive or negative definite,

$$\beta = 0. \tag{3.36}$$

Hence (3.31) yields

$$(\alpha^T e)\overline{p} = 0 \text{ and } \gamma\overline{p} = 0.$$

If $\alpha = 0$ and $\gamma = 0$, then (3.22) and (3.24) gives $\xi = 0$ and $\delta = 0$, which contradicts (3.28). Hence

$$\overline{p} = 0. \tag{3.37}$$

Using (3.36) and (3.37), (3.26) yields

$$\gamma^T[g(\overline{x}) + k(\overline{x}, 0) - \overline{p}^T \nabla_p k(\overline{x}, 0)] = 0.$$

From (3.18), $\gamma^T g(\overline{x}) = 0$ implies $\gamma^T g(\overline{x}) \geqq 0$. Since $\gamma \in C_2$ then $-g(\overline{x}) \in C_2^*$. The corresponding value of **(MCP)** and **(MMCD)** are equal because $p = 0$ and (3.18). If the conditions of Theorem 3.1 are satisfied, then $\overline{x}$ is an efficient solution for **(MCP)**. □

3.4 Wolfe Type Higher Order Duality

In this section, we propose the following Wolfe type higher order multi-objective dual problem to the primal problem **(MCP)**:

(**MWCD**) Maximize $f(u) - y^T g(u)e + \left(\lambda^T h(u,p) - y^T k(u,p)\right) e$

$-p^T \left(\nabla_p \lambda^T h(u,p)) - \nabla_p y^T k(u,p)\right) e$

subject to $\nabla_p(\lambda^T h(u,p)) = \nabla_p(y^T k(u,p)),$ (3.38)

$y \in C_2, \ \lambda \geqq 0,$

Now we establish the duality theorems for **(MCP)** and **(MWCD)**.

Theorem 3.4 (Weak Duality) *Let x be feasible solutions of* **(MCP)** *and (u, y, λ, p) feasible for* **(MWCD)** *and $\lambda > 0(\lambda^T e = 1)$. Assume that*

$$(i)\ \lambda^T f(x) - \lambda^T f(u)$$

$$\geqq \eta(x,u)^T \nabla_p(\lambda^T h(u,p)) + \lambda^T h(u,p) - p^T \nabla_p(\lambda^T h(u,p)) \tag{3.39}$$

$$y^T g(x) - y^T g(u)$$

$$\leqq \eta(x,u)^T \nabla_p(y^T k(u,p)) + y^T k(u,p) - p^T \nabla_p(\nabla_p y^T k(u,p));\ or \tag{3.40}$$

$$(ii)\ \eta(x,u)^T(\nabla_p(\lambda^T h(u,p) - y^T k(u,p)) \geqq 0$$

$$\Rightarrow \lambda^T f(x) - y^T g(x)$$

$$\geqq \lambda^T f(u) - y^T g(u) + (\lambda^T h(u,p) - y^T k(u,p))$$

$$-p^T \left[\nabla_p(\lambda^T h(u,p)) - \nabla_p(y^T k(u,p))\right];\ or \tag{3.41}$$

$$(iii)\ \lambda^T f(x) - y^T g(x) - \left(\lambda^T f(u) - y^T g(u)\right) - \left(\lambda^T h(u,p) - y^T k(u,p)\right)$$

$$+p^T \left[\nabla_p(\lambda^T h(u,p)) - \nabla_p(y^T k(u,p))\right]$$

$$\geqq F(x,u; \nabla_p(\lambda^T h(u,p)) - \nabla_p(y^T k(u,p))) \tag{3.42}$$

such that $\rho \geqq 0$,

then

$$f(x) \nleq\ f(u) - y^T g(u)e + \left(\lambda^T h(u,p) - y^T k(u,p)\right)e$$

$$-p^T \left(\nabla_p \lambda^T h(u,p) - \nabla_p y^T k(u,p)\right)e.$$

Proof. (i) Assume to the contrary that

$$f(x) \leq \; f(u) - y^T g(u)e + \left(\lambda^T h(u,p) - y^T k(u,p)\right) e$$
$$-p^T \left(\nabla_p \lambda^T h(u,p)) - \nabla_p y^T k(u,p)\right) e.$$

Since $\lambda_i > 0 \;\; (i = 1, \cdots l)$,

$$\lambda^T f(x) < \; \lambda^T f(u) - y^T g(u) + \left(\lambda^T h(u,p) - y^T k(u,p)\right)$$
$$-p^T \left(\nabla_p \lambda^T h(u,p)) - \nabla_p y^T k(u,p)\right). \tag{3.43}$$

By the conditions (3.39) and (3.40),

$$\lambda^T f(x) - y^T g(x) - \lambda^T f(u) + y^T g(u)$$
$$\geqq \eta(x,u)^T \left(\nabla_p(\lambda^T h(u,p)) - \nabla_p(y^T k(u,p))\right)$$
$$+\lambda^T h(u,p) - y^T k(u,p) - p^T(\nabla_p(\lambda^T h(u,p)) - \nabla_p(y^T k(u,p)).$$

From the constraints of **(MCP)** and **(MWCD)**,

$$\lambda^T f(x) \geqq \lambda^T f(u) - y^T g(u) + \lambda^T h(u,p) - y^T k(u,p)$$
$$-p^T \left(\nabla_p(\lambda^T h(u,p)) - \nabla_p(y^T k(u,p))\right),$$

which contradicts (3.43).

(ii) From (3.43), and the constraints of **(MCP)** and **(MWCD)**,

$$\lambda^T f(x) - y^T g(x) < \lambda^T f(u) - y^T g(u) + \lambda^T h(u,p) - y^T k(u,p)$$
$$-p^T \left(\nabla_p(\lambda^T h(u,p)) - \nabla_p(y^T k(u,p))\right). \tag{3.44}$$

From (3.41), we obtain

$$\eta(x,u)^T \left(\nabla_p(\lambda^T h(u,p)) - \nabla y^T k(u,p)\right) < 0,$$

which contradicts (3.43). Hence the result hold. (iii) By (3.44)

$$\begin{aligned}0 > \lambda^T f(x) - y^T g(x) - \lambda^T f(u) + y^T g(u) - \left(\lambda^T h(u,p) - y^T k(u,p)\right)\\ +p^T \left(\nabla_p(\lambda^T h(u,p)) - \nabla_p(y^T k(u,p))\right).\\ \geqq F\left(x,u;\nabla_p(\lambda^T h(u,p)) - \nabla_p(y^T k(u,p))\right) + \rho d(x,u)\end{aligned}$$

such that $\rho \geqq 0$. It follows that

$$F\left(x,u;\nabla_p(\lambda^T h(u,p)) - \nabla_p(y^T k(u,p))\right) < -\rho d(x,u). \tag{3.45}$$

Hence $F(x,u;0) = 0$ and (3.45) imply that $\rho d(x,u) < 0$, which contradicts $\rho \geqq 0$.

Theorem 3.5 (Strong Duality) *Let $\overline{x}$ be an efficient solution for* **(MCP)** *and (3.18) is satisfied.*
Then there exist $\lambda \geqq 0$ and $y \in C_2$ not both zero such that $(\overline{x}, y, \lambda, p = 0)$ is feasible for **(MWCD)** *and the corresponding values of* **(MCP)** *and* **(MWCD)** *are equal. If for all feasible $(\overline{x}, u, y, \lambda, p)$, the assumptions of Theorem 3.4 are satisfied, then $(\overline{x}, y, \lambda, p = 0)$ is efficient for* **(MWCD)**.

Proof. It follows on the lines of Theorem 3.2. □

Theorem 3.6 (Converse Duality) *Let $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{p})$ be an efficient solution of* **(MWCD)** *and let the condition of (3.18) be satisfied. Assume that*

(i) the matrix

$$\nabla_p \left[\nabla \overline{\lambda}^T f(\overline{x}) + \nabla(\overline{\lambda}^T h(\overline{x}, \overline{p})) - \nabla \overline{y}^T g(\overline{x}) - \nabla(\overline{y}^T k(\overline{x}, \overline{p})) \right]$$

is positive or negative definite and

(ii) the vectors

$$\left\{ \nabla_p^2 \overline{\lambda}_i h_i(\overline{x}, \overline{p}) \right\}_{i=1,\cdots,l} \textit{and} \left\{ \nabla_p^2 k_j(\overline{x}, \overline{p}) \right\}_{j=1,\cdots,m}$$

are linearly independent.

If the conditions of Theorem 3.4 hold, then $\overline{x}$ is and efficient solution for **(MCP)**.

Proof. Since $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{p})$ is an efficient solution for **(MWCD)**, by lemma 3.1, there exist $\alpha \in \mathbb{R}^l$, $\beta \in \mathbb{R}^n$, $\gamma \in C_2^*$ and $\delta \in \mathbb{R}^l$ such that

$$-\nabla f(\overline{x})\alpha + (\alpha^T e)\nabla \overline{y}^T g(x) - (\alpha^T e)\nabla \left(\overline{\lambda}^T h(\overline{x}, \overline{p}) - \overline{y}^T k(\overline{x}, \overline{p}) \right)$$

$$+((\alpha^T e)p + \beta)^T \left[\nabla \left(\nabla_p \overline{\lambda}^T h(\overline{x}, \overline{p}) - \nabla_p \overline{y}^T k(\overline{x}, \overline{p}) \right) \right] = 0, \tag{3.46}$$

$$(\alpha^T e)(g(x) + k(\overline{x}, \overline{p})) - ((\alpha^T e)p + \beta)^T \nabla_p k(\overline{x}, \overline{p}) - \gamma = 0, \tag{3.47}$$

$$((\alpha^T e)p + \beta)^T \left(\nabla_p^2 (\overline{\lambda}^T h(\overline{x}, \overline{p})) - \nabla_p^2 (\overline{y}^T k(\overline{x}, \overline{p})) \right) = 0, \tag{3.48}$$

$$-(\alpha^T e)h(\overline{x}, \overline{p}) + ((\alpha^T e)p + \beta)^T \nabla_p h(\overline{x}, \overline{p}) - \delta = 0, \tag{3.49}$$

$$\beta^T \left(\nabla_p (\overline{\lambda}^T h(\overline{x}, \overline{p})) - \nabla_p (\overline{y}^T k(\overline{x}, \overline{p})) \right) = 0, \tag{3.50}$$

$$\gamma^T \overline{y} = 0, \tag{3.51}$$

$$\delta^T \overline{\lambda} = 0, \tag{3.52}$$

$$(\alpha, \beta, \gamma, \delta) \geq 0, \tag{3.53}$$

$$(\alpha, \beta, \gamma, \delta) \neq 0. \tag{3.54}$$

$\left\{\nabla_p^2 \overline{\lambda}_i h_i(\overline{x}, \overline{p})\right\}_{i=1,\cdots,l}$ and $\left\{\nabla_p^2 k_j(\overline{x}, \overline{p})\right\}_{j=1,\cdots,m}$ are linearly independent, then (3.48) gives

$$(\alpha^T e)\overline{p} + \beta = 0. \tag{3.55}$$

Multiplying (3.47) by $\overline{y}^T$ and using (3.51) and (3.55)

$$(\alpha^T e)\overline{y}^T g(x) + \overline{y}^T k(\overline{x}, \overline{p}) = 0. \tag{3.56}$$

Using (3.55) in (3.46), we have

$$-\nabla f(\overline{x})\alpha + (\alpha^T e)\nabla \overline{y}^T g(\overline{x}) - (\alpha^T e)\left(\nabla(\overline{\lambda}^T h(\overline{x}, \overline{p})) - \nabla(\overline{y}^T k(\overline{x}, \overline{p}))\right) = 0. \tag{3.57}$$

Multiplying (3.57) by $\overline{p}$ and using (3.55),

$$-(\alpha^T e)\overline{p}^T[\nabla \overline{\lambda}^T f(\overline{x}) + (\alpha^T e)p\nabla \overline{y}^g(\overline{x}) - (\alpha^T e)p^T\left(\nabla \overline{\lambda}^T h(\overline{x}, \overline{p}) - \nabla \overline{y}^T k(\overline{x}, \overline{p})\right) = 0,$$

that is

$$\beta^T[\nabla \overline{\lambda}^T f(\overline{x}) + \nabla(\overline{\lambda}^T h(\overline{x}, \overline{p})) - \nabla \overline{y}^T g(\overline{x}) - \nabla \overline{y}^T k(\overline{x}, \overline{p})] = 0. \tag{3.58}$$

Differentiating (3.58) with respect to p yields

$$\beta^T \nabla_p[\nabla \overline{\lambda}^T f(\overline{x}) + \nabla(\overline{\lambda}^T h(\overline{x}, \overline{p})) - \nabla \overline{y}^T g(\overline{x}) - \nabla \overline{y}^T k(\overline{x}, \overline{p})] = 0. \tag{3.59}$$

Multiplying (3.59) by β, we obtain

$$\beta^T \nabla_p [\nabla \overline{\lambda}^T f(\overline{x}) + \nabla (\overline{\lambda}^T h(\overline{x}, \overline{p})) - \nabla \overline{y}^T g(\overline{x}) - \nabla \overline{y}^T k(\overline{x}, \overline{p})] \beta = 0.$$

$\nabla_p [\nabla \overline{\lambda}^T f(\overline{x}) + \nabla (\overline{\lambda}^T h(\overline{x}, \overline{p})) - \nabla \overline{y}^T g(\overline{x}) - \nabla \overline{y}^T k(\overline{x}, \overline{p})]$ is positive or negative definite,

$$\beta = 0. \tag{3.60}$$

Hence (3.55) yields

$$(\alpha^T e)\overline{p} = 0.$$

If $\alpha = 0$, then from (3.47) and (4.49), we get $\gamma = 0$ and $\delta = 0$, which contradict (3.54). Hence

$$\overline{p} = 0. \tag{3.61}$$

Using (3.61), (3.56) yields

$$(\alpha^T e) \left(\overline{y}^T g(\overline{x}) + \overline{y}^T k(\overline{x}, 0) \right) = 0.$$

From (3.18), $\overline{y}^T g(\overline{x}) = 0$ implies $\overline{y}^T g(\overline{x}) \geqq 0$. Since $\overline{y} \in C_2$ then $-g(\overline{x}) \in C_2^*$. Since $p = 0$ and (3.18), the corresponding value of **(MCP)** and **(MWCD)** are equal. If the conditions of Theorem 3.4 are satisfied, then $\overline{x}$ is an efficient solution of **(MCP)**. □

Remark 3.1 *If* $C_1 = \mathbb{R}^n$ *and* $C_2 = \mathbb{R}^m_+$,

(i) $h(u,p) = p^T \nabla_p f(u)$, $k(u,p) = p^T \nabla g(u)$, *then our higher order dual programs become first order dual programs [81], and*

(ii) $h(u,p) = p^T \nabla_p f(u) + \frac{1}{2} p^T \nabla^2 f(u) p$, $k(u,p) = p^T \nabla g(u) + \frac{1}{2} p^T \nabla^2 g(u) p$, *then we obtain second order dual programs in [52].*

3.5 Fritz John Higher Order Duality with Cone Constraints

In this section, we consider the Fritz John higher order programming problem and establish weak, strong and converse duality theorems using Fritz John [51] necessary optimality conditions instead of Karush Kuhn-Tucker [51].

We propose the following nonlinear programming problem,

$$
\begin{aligned}
(\mathbf{FCP}) \quad & \text{Minimize} \quad && f(x) \\
& \text{subject to} \quad && g(x) \in C_2^*, \ \ x \in C_1,
\end{aligned}
$$

and its Fritz John higher order dual

$$
\begin{aligned}
(\mathbf{FCD}) \quad & \text{Maximize} \quad && f(u) + h(u,p) - p^T \nabla_p h(u,p) \\
& \text{subject to} \quad && \gamma \nabla_p h(u,p) + \nabla_p (y^T k(u,p)) = 0, \qquad (3.62) \\
& && -(g(u) + k(u,p) - p^T \nabla_p k(u,p)) \in C_2^*, \\
& && y \in C_2, \ \ \gamma \in \mathbb{R}, \ \ (\gamma, y) \neq 0,
\end{aligned}
$$

where

(1) $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$are differentiable functions,

(2) C_1 and C_2 are closed convex cones in $\mathbb{R}^n$ and $\mathbb{R}^m$ with nonempty interiors, respectively,

(3) C_1^* and C_2^* are polar cones of C_1 and C_2, respectively,

(4) $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are differentiable functions;

$\nabla_p(h_j(u,p))$ denotes the $n \times 1$ gradient of h_j with respect to p, and $\nabla_p(y^T k(u,p))$ denotes the $n \times 1$ gradient of $y^T k$ with respect to p, respectively.

Now we establish the duality theorems for **(FCP)** and **(FCD)**.

Theorem 3.7 (Weak Duality) *Let x be feasible solutions of* **(FCP)** *and (u, y, λ, p) feasible for* **(FCD)**. *Assume that f be an higher order pseudo-invex and $y^T g$ be an strictly higher order quasi-invex with respect to same η for all feasible $(\gamma, x, u, y, \lambda, p)$, then*

$$\begin{aligned} f(x) \geq \;& f(u) + y^T g(u) + \left(\lambda^T h(u,p) + y^T k(u,p)\right) \\ & - p^T \left(\nabla_p \lambda^T h(u,p)) + \nabla_p y^T k(u,p)\right). \end{aligned}$$

Proof. Suppose that

$$f(x) < f(u) + h(u,p) - p^T \nabla_p h(u,p). \tag{3.63}$$

This in view of higher order pseudo-invexity of $f(\cdot)$ yields

$$\eta(x,u)^T \nabla_p h(u,p) < 0.$$

Thus

$$\eta(x,u)^T \gamma \nabla_p h(u,p) \leqq 0. \tag{3.64}$$

From the constraints of **(FCP)** and **(FCD)**,

$$y^T g(x) \leqq y^T g(u) + y^T k(u,p) - p^T \nabla_p (y^T k(u,p)).$$

By strictly higher order quasi-invex of $y^T g$ assumption,

$$\eta(x,u)^T \nabla_p (y^T k(u,p)) < 0. \tag{3.65}$$

Combining (3.64) and (3.65), we have

$$\eta(x,u)^T [\gamma \nabla_p h(u,p) + \nabla_p (y^T k(u,p))] < 0$$

which contradicts (3.62). Hence

$$f(x) \geq f(u) + h(u,p) - p^T \nabla_p h(u,p).$$

That is

$$inf(\mathbf{FCP}) \geq sup(\mathbf{FCD}).$$

□

Theorem 3.8 (Strong Duality) *If $\overline{x}$ is an optimal solution for* **(FCP)** *and let*

$$h(\overline{x},0) = 0, \;\; k(\overline{x},0) = 0, \;\; \nabla_p h(\overline{x},0) = \nabla f(\overline{x}), \;\; \nabla_p k(\overline{x},0) = \nabla g(\overline{x}). \tag{3.66}$$

Then there exist $\gamma \in \mathbb{R}_+$ and $y \in C_2$ such that $(\gamma, \overline{x}, y, p = 0)$ is feasible for (FCD) and the corresponding values of **(FCP)** *and* **(FCD)** *are equal. If the assumptions of Theorem 3.7 is satisfied, then $(\gamma, \overline{x}, y, p = 0)$ is optimal solution for* **(FCD)**.

Proof. Since $\overline{x}$ is an optimal solution for **(FCP)**, by lemma 3.1, there exist $\gamma \in \mathbb{R}_+$ and $y \in C_2$ with $(\gamma, y) \neq 0$ such that

$$y^T g(\overline{x}) = 0$$

and

$$[\gamma\nabla f(\overline{x}) + \nabla y^T g(\overline{x})]^T(x - \overline{x}) \geqq 0 \quad \text{for all} \quad x \in C_1. \tag{3.67}$$

Since $x \in C_1$ and C_1 is a closed convex cone, we have $x + \overline{x} \in C_1$ and thus the inequality (3.67) implies

$$[\gamma\nabla f(\overline{x}) + y^T g(\overline{x})]^T x \geqq 0. \tag{3.68}$$

By letting $x = 0$ and $x = 2\overline{x}$ in (3.68), we obtain

$$[\gamma\nabla f(\overline{x}) + y^T g(\overline{x})] = 0.$$

By (3.66),

$$\gamma\nabla_p h(\overline{x}, p) + \nabla_p(y^T k(\overline{x}, p)) = 0.$$

From $y^T g(\overline{x}) = 0$ implies $y^T g(\overline{x}) \leqq 0$,

$$g(\overline{x}) \in C_2^*$$

Thus, $(\gamma, \overline{x}, y, p = 0)$ is feasible for **(FCD)**, and corresponding values of **(FCP)** and **(FCD)** are equal. If assumptions of Theorem 3.7 are satisfied, then $(\gamma, \overline{x}, y, p = 0)$ must be an optimal solution for **(FCD)**. □

Theorem 3.9 (Converse Duality) *Let* $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{p})$ *be an optimal solution of* **(FCD)**. *Let the condition of (3.66) is satisfied. Assume that*

(i) the matrix

$$\gamma\nabla_p(\nabla h(\overline{x}, \overline{p}) + \nabla y^T k(\overline{x}, \overline{p})) - \nabla_p^2(\gamma h(\overline{x}, \overline{p}) + y^T k(\overline{x}, \overline{p}))$$

is positive or negative definite and

(ii) the vectors

$$\{\nabla_p^2 \overline{\lambda}_i^T h_i(\overline{x}, \overline{p})\}_{i=1,\cdots,l} \text{ and } \{\nabla_p^2 y^T k_j(\overline{x}, \overline{p})\}_{i=1,\cdots,m}$$

are linearly independent.

If the conditions of Theorem 3.7 hold, then $\overline{x}$ is an optimal solution for **(FCP)**.

Proof. Since $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{p})$ is an optimal solution for **(FCD)**, by lemma 3.1, there exist $\alpha \in \mathbb{R}_+$, $\beta \in \mathbb{R}_+^n$, $\rho \in C_2$, $\delta \in C_2^*$ and $\omega \in \mathbb{R}_+$ such that

$$\begin{aligned} &-\alpha \left[\nabla f(\overline{x}) + \nabla h(\overline{x}, \overline{p}) - p^T \nabla(\nabla_p h(\overline{x}, \overline{p}))\right] \\ &\quad +\beta^T \left[\gamma \nabla_p h(\overline{x}, \overline{p}) + \nabla_p 9 y^T k(\overline{x}, \overline{p})\right] \\ &\qquad -\rho \left[\nabla g(x) + \nabla k(\overline{x}, \overline{p}) - p^T \nabla(\nabla_p k(\overline{x}, \overline{p}))\right] = 0, \end{aligned} \tag{3.69}$$

$$\beta^T(\nabla_p k(\overline{x}, \overline{p})) - \delta = 0, \tag{3.70}$$

$$(\alpha p + \gamma \beta)^T \nabla_p^2 h(\overline{x}, \overline{p}) + (\rho p) + \beta^T y \nabla_p^2 k(\overline{x}, \overline{p}) = 0, \tag{3.71}$$

$$\beta^T \left[\nabla_p h(\overline{x}, \overline{p})\right] - \omega = 0, \tag{3.72}$$

$$\beta^T \left[\gamma \nabla_p h(\overline{x}, \overline{p}) + \nabla_p y^T k(\overline{x}, \overline{p})\right] = 0, \tag{3.73}$$

$$-\rho \left[g(x) + k(\overline{x}, \overline{p}) - p_p^n k(\overline{x}, \overline{p})\right] = 0, \tag{3.74}$$

$$\delta^T y = 0, \tag{3.75}$$

$$\omega \gamma = 0, \tag{3.76}$$

$$(\alpha, \beta, \rho, \delta, \omega) \neq 0. \tag{3.77}$$

Since $\nabla_p^2 h(\overline{x},\overline{p}),\ \ \nabla_p^2 k(\overline{x},\overline{p})$ are linearly independent, then (3.71) gives

$$\alpha p + \gamma\beta = 0 \ \text{ and } \ \rho p + \beta^T y = 0. \tag{3.78}$$

Multiplying (3.70) by y^T and then using (3.75),

$$\beta^T \left[\nabla_p (y^T k(\overline{x},\overline{p}))\right] = 0. \tag{3.79}$$

Using (3.78) in (3.69), we have

$$-\alpha \left[\nabla f(\overline{x}) + \nabla h(\overline{x},\overline{p})\right] - \rho^T \left[g(\overline{x}) + \nabla k(\overline{x},\overline{p})\right] = 0. \tag{3.80}$$

Multiplying (3.80) by p and using (3.78), we obtain

$$\beta^T \left[\gamma\nabla f(\overline{x}) + \gamma\nabla h(\overline{x},\overline{p}) + \nabla y^T g(\overline{x}) + \nabla y^T k(\overline{x},\overline{p})\right] = 0,$$

that is

$$\begin{aligned}\beta^T [\gamma\nabla f(\overline{x}) &+ \gamma\nabla h(\overline{x},\overline{p}) - \gamma\nabla_p h(\overline{x},\overline{p}) \\ &- \nabla_p y^T k(\overline{x},\overline{p}) + \nabla y^T g(\overline{x}) + \nabla y^T k(\overline{x},\overline{p})\big] = 0.\end{aligned} \tag{3.81}$$

Differentiation (3.81) with respect to p yields

$$\beta^T \left[\gamma\nabla_p (\nabla h(\overline{x},\overline{p}) + \nabla y^T k(\overline{x},\overline{p})) - \nabla_p^2 (\gamma h(\overline{x},\overline{p}) + y^T k(\overline{x},\overline{p}))\right] = 0. \tag{3.82}$$

Multiplying (3.82) by β, we get

$$\beta^T \left[\gamma\nabla_p (\nabla h(\overline{x},\overline{p}) + \nabla y^T k(\overline{x},\overline{p})) - \nabla_p^2 (\gamma h(\overline{x},\overline{p}) + y^T k(\overline{x},\overline{p}))\right] \beta = 0.$$

Assuming that condition (i), it follows that

$$\beta = 0. \tag{3.83}$$

Hence (3.78) yields

$$\alpha\overline{p} = 0 \text{ and } \rho\overline{p} = 0.$$

If $\alpha = 0$ and $\rho = 0$ then (3.70) and (3.72) give $\delta = 0$ and $\omega = 0$. Hence $(\alpha, \beta, \rho, \delta, \omega) = 0$, which contradicts (3.77). Hence

$$\overline{p} = 0. \tag{3.84}$$

Using (3.83) and (3.74) yield

$$\rho\left[g(\overline{x}) + k(\overline{x}, 0) - p^T \nabla_p k(\overline{x}, 0)\right] = 0.$$

$\rho g(\overline{x}) = 0$ implies $\rho g(\overline{x}) \leqq 0$. Since $\rho \in C_2$, then

$$g(\overline{x}) \in C_2^*.$$

Hence $\overline{x}$ is feasible for **(FCP)** and since $\overline{p} = 0$ and $h(\overline{x}, 0) = 0$, the objective values of **(FCP)** and **(FCD)** are equal. If assumptions of Theorem 3,7 hold, then $\overline{x}$ is an optimal solution of **(FCP)**. □

Remark 3.2 *If $C_1 = \mathbb{R}^n$, $C_2 = \mathbb{R}^m_+$, $h(u,p)p^T\nabla_p f(u) + \frac{1}{2}p^T\nabla^2 f(u)p$ and $k(u,p) = p^T\nabla g(u) + \frac{1}{2}p^T\nabla^2 g(u)p$, then we get Fritz John second order dual programs studied by Husain et al. [32].*

Chapter 4

Second Order Non-Differentiable Symmetric Duality for Multiobjective Programming Programs with Cone Constraints

4

4.1 Introduction

In the literature of mathematical programming there are a large number of papers discussing duality theory for a problem involving the square root of a positive semidefinite quadratic function, $\sqrt{x^T Bx}$. The square root of a positive semidefinite quadratic form is one of the few cases of a non-differentiable function for which one can write down the sub or quasi differentials explicitly. Mond and Schechter [60] replace $\sqrt{x^T Bx}$ by a somewhat more general function, namely the support function of a compact convex set, for which the subdifferential may be simply expressed.

Suneja et al. [74] formulated a pair of multiobjective symmetric dual programs of Wolfe type over arbitrary cones in which the objective function was optimized with respect to an arbitrary closed convex cone by assuming the involved function to be cone-convex. Recently, Khurana [49] introduced cone-pseudo-invex and strongly cone-pseudo-invex functions and established duality theorems for a pair of Mond-Weir type multiobjective symmetric dual over arbitrary cones. Very recently, Kim and Kim [45] studied two pairs of non-differentiable multiobjective symmetric dual problems with cone constraints over arbitrary closed convex cones, which are Wolfe type and Mond-Weir type.

In the second order case, Mishra [54] formulated a pair of multiobjective second order symmetric dual nonlinear programming problems under second order pseudo-invexity assumptions on the involved functions over arbitrary cones and established duality results. Mishra and Lai [56] introduced the concept of cone-second order pseudo-invex and strongly cone-second order pseudo-invex functions and formulated a pair of Mond-Weir type multiobjective second order symmetric dual programs over arbitrary cones.

In this chapter, we formulate Mond-Weir and Wolfe type non-differentiable multiobjective second order symmetric dual problems with cone constraints over arbitrary closed convex cones. Subsequently, weak, strong, and converse duality theorems for weakly efficient solutions are establish under the assumptions of second order pseudo-invex functions. And we introduce some special cases of our results.

4.2 Notations and Preliminaries

Now we will give some definitions and preliminary results needed in next sections.

Definition 4.1 *A nonempty set K in $\mathbb{R}^k$ is said to be a cone with vertex zero if $x \in K$ implies that $\lambda x \in K$ for all $\lambda \geqq 0$. If, in addition, K is convex, then K is called a convex cone.*

Definition 4.2 *A feasible point $\overline{x}$ is a weakly efficient solution of (KP) if there exists no other $x \in X$ such that $f(\overline{x}) - f(x) \in intK$.*

Definition 4.3 **[54]** *Let $f : C_1 \times C_2 \to \mathbb{R}$ be a twice differentiable function.*

(i) f is said to be second order pseudo-invex in the first variable at $u \in C_1$ for fixed $v \in C_2$ if there exists a function $\eta_1 : C_1 \times C_1 \to C_1$ such that for $r \in C_1$,

$$\eta_1^T(x,u)[\nabla_x f(u,v) + \nabla_{xx} f(u,v)r] \geqq 0$$
$$\Rightarrow f(x,v) - f(u,v) + \frac{1}{2} r^T \nabla_{xx} f(u,v) r \geqq 0.$$

(ii) f is said to be second order pseudo-invex in the second variable at $v \in C_2$ for fixed $u \in C_1$ if there exists a function $\eta_2 : C_2 \times C_2 \to C_2$ such that for $p \in C_2$,

$$\eta_2^T(y,v)[\nabla_y f(u,v) + \nabla_{yy} f(u,v)p] \geqq 0$$
$$\Rightarrow f(u,y) - f(u,v) + \frac{1}{2} p^T \nabla_{yy} f(u,v) p \geqq 0,$$

for all $x, u \in C_1$ and $y, v \in C_2$.

f is second order pseudo-incave at $u \in C_1$ with respect to $r \in C_1$, if $-f$ is second order pseudo-invex at $u \in C_1$ with respect to $r \in C_1$.

Definition 4.4 [60] *Let B be a compact convex set in $\mathbb{R}^n$. The support function $s(x|B)$ of B is defined by*

$$s(x|B) := max\{x^T y : y \in B\}.$$

The support function $s(x|B)$, being convex and everywhere finite, has a sub-differential, that is, there exists z such that

$$s(y|B) \geq s(x|B) + z^T(y - x) \ \text{ for all } \ y \in B.$$

Equivalently,

$$z^T x = s(x|B).$$

The subdifferential of $s(x|B)$ is given by

$$\partial s(x|B) := \{z \in B : z^T x = s(x|B)\}.$$

For any set $S \subset \mathbb{R}^n$, the normal cone to S at a point $x \in S$ is defined by

$$N_S(x) := \{y \in \mathbb{R}^n : y^T(z - x) \leq 0 \ \text{ for all } \ z \in S\}.$$

It is readily verified that for a compact convex set B, y is in $N_B(x)$ if and only if $s(y|B) = x^T y$, or equivalently, x is in the subdifferential of s at y.

4.3 Mond-Weir Type Symmetric Duality

We consider the following pair of second order Mond-Weir type non-differentiable multiobjective programming problem:

$$\textbf{(NMP)}\quad \text{Minimize}\quad K(x,y,\lambda,w,p)$$

$$= f(x,y) + s(x|D) - (y^T w)e - \frac{1}{2}[p^T \nabla_{yy}(\lambda^T f)(x,y)p]e$$

$$\text{subject to}\quad -[\nabla_y(\lambda^T f)(x,y) - w + \nabla_{yy}(\lambda^T f)(x,y)p \in C_2^*, \quad (4.1)$$

$$y^T[\nabla_y(\lambda^T f)(x,y) - w + \nabla_{yy}(\lambda^T f)(x,y)p] \geqq 0, \quad (4.2)$$

$$x \in C_1, \quad w \in E_i, \quad \lambda \in K^*, \quad \lambda^T e = 1, \quad e \in intK,$$

$$\textbf{(NMD)}\quad \text{Maximize}\quad G(u,v,\lambda,z,r)$$

$$= f(u,v) - s(v|E) + (u^T z)e - \frac{1}{2}[r^T \nabla_{xx}(\lambda^T f)(u,v)r]e$$

$$\text{subject to}\quad \nabla_x(\lambda^T f)(u,v) + z + \nabla_{xx}(\lambda^T f)(u,v)r \in C_1^*, \quad (4.3)$$

$$u^T[\nabla_x(\lambda^T f)(u,v) + z + \nabla_{xx}(\lambda^T f)(u,v)r] \leqq 0, \quad (4.4)$$

$$v \in C_2, \quad z \in D_i, \quad \lambda \in K^*, \quad \lambda^T e = 1, \quad e \in intK,$$

where

(1) $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$ is a three times differentiable function,

(2) C_1 and C_2 are closed convex cones in $\mathbb{R}^n$ and $\mathbb{R}^m$ with nonempty interiors, respectively,

(3) C_1^* and C_2^* are positive polar cones of C_1 and C_2, respectively,

(4) K is a closed convex cone in $\mathbb{R}^k$ such that $intK \neq \emptyset$ and $\mathbb{R}^k_+ \subset K$,

(5) r, z are vectors in $\mathbb{R}^n$, p, w are vectors in $\mathbb{R}^m$,

(6) $e = (1, \cdots, 1)^T$ is vector in $\mathbb{R}^k$,

(7) D_i and $E_i (i = 1, \cdots, k)$ are compact convex sets in $\mathbb{R}^n$ and R^m, respectively.

Let $\nabla_x(\lambda^T f)(x, y)$ and $\nabla_y(\lambda^T f)(x, y)$ are gradients of $(\lambda^T f)(x, y)$ with respect to x and y. Similarly, $\nabla_{xx}(\lambda^T f)(x, y)$ and $\nabla_{yy}(\lambda^T f)(x, y)$ are the Hessian matrices of $(\lambda^T f)(x, y)$ with respect to x and y, respectively.

Now we establish the symmetric duality theorems for **(NMP)** and **(NMD)**.

Theorem 4.1 (Weak Duality) *Let (x, y, λ, w, p) and (u, v, λ, z, r) be feasible solutions of* **(NMP)** *and* **(NMD)**, *respectively. Assume that,*
(i) $(\lambda^T f)(\cdot, y) + (\cdot)^T z$ is second order pseudo-invex in the first variable for fixed y with respect to η_1,
(ii) $-(\lambda^T f)(x, \cdot) + (\cdot)^T w$ is second order pseudo-invex in the second variable for fixed x with respect to η_2,
(iii) $\eta_1(x, u) + u \in C_1$ and $\eta_2(v, y) + y \in C_2$, then

$$G(u, v, \lambda, z, r) - K(x, y, \lambda, w, p) \notin intK.$$

Proof. From (4.3) and $\eta_1(x, u) + u \in C_1$,

$$[\eta_1(x, u) + u]^T \left[\nabla_x(\lambda^T f)(u, v) + z + \nabla_{xx}(\lambda^T f)(u, v) r\right] \geqq 0.$$

From (4.4), it yields

$$\eta_1(x, u)^T \left[\nabla_x(\lambda^T f)(u, v) + z + \nabla_{xx}(\lambda^T f)(u, v) r\right] \geqq 0.$$

By the second order pseudo-invexity of $(\lambda^T f)(\cdot, y) + (\cdot)^T z$, we have

$$(\lambda^T f)(x, v) + x^T z - (\lambda^T f)(u, v) - u^T z + \frac{1}{2} r^T \nabla_{xx}(\lambda^T f)(u, v) r \geqq 0. \quad (4.5)$$

From (4.1) and $\eta_2(v, y) + y \in C_2$,

$$-[\eta_2(v, y) + y]^T [\nabla_y (\lambda^T f)(x, y) - w + \nabla_{yy}(\lambda^T f)(x, y) p] \geqq 0.$$

From (4.2), it yields

$$\eta_2(v, y)^T [\nabla_y (\lambda^T f)(x, y) - w + \nabla_{yy}(\lambda^T f)(x, y) p] \leqq 0.$$

By the second order pseudo-invexity of $-(\lambda^T f)(x, \cdot) + (\cdot)^T w$, we obtain

$$(\lambda^T f)(x, v) - v^T w - (\lambda^T f)(x, y) + y^T w + \frac{1}{2} p^T \nabla_{yy}(\lambda^T f)(x, y) p \leqq 0. \quad (4.6)$$

From (4.5) and (4.6), we get

$$\begin{aligned} &(\lambda^T f)(u, v) - x^T z + u^T z - \frac{1}{2} r^T \nabla_{xx}(\lambda^T f)(u, v) r \\ &\leqq (\lambda^T f)(x, y) + v^T w - y^T w - \frac{1}{2} p^T \nabla_{yy}(\lambda^T f)(x, y) p. \quad (4.7) \end{aligned}$$

Using the fact that $x^T z \leqq \lambda^T s(x|D_i)$ and $v^T w \leqq s(v|E_i)$ for $i = 1, \cdots, k$, we get

$$x^T z \leqq \lambda^T s(x|D) \ \ and \ \ v^T w \leqq \lambda^T s(v|E).$$

Finally, using these, we obtain

$$\begin{aligned} &(\lambda^T f)(x, y) + \lambda^T s(x|D) - y^T w - \frac{1}{2} p^T \nabla_{yy}(\lambda^T f)(x, y) p \\ &\geqq (\lambda^T f)(u, v) - \lambda^T s(v|E) + u^T z - \frac{1}{2} r^T \nabla_{xx}(\lambda^T f)(u, v) r. \quad (4.8) \end{aligned}$$

But suppose that $G(u, v, \lambda, z, r) - K(x, y, \lambda, w, p) \in intK$. Since $\lambda \in K^*$, it yields

$$[(\lambda^T f)(u, v) - \lambda^T s(v|E) + u^T z - \frac{1}{2} r^T \nabla_{xx}(\lambda^T f)(u, v) r]$$
$$-[(\lambda^T f)(x, y) + \lambda^T s(x|D) - y^T w - \frac{1}{2} p^T \nabla_{yy}(\lambda^T f)(x, y) p] > 0.$$

which is a contradiction to the inequality (4.8). □

In order to prove the strong duality theorem, we need the necessary optimality conditions for a point to be a weak minimum of (KP) in Lemma 3.1.

Theorem 4.2 (Strong Duality) *Let $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p})$ be a K-weakly efficient solution for* **(NMP)**. *Fix $\lambda = \overline{\lambda}$ in* **(NMD)**. *Assume that*

(i) $\nabla_{yy}(\overline{\lambda}^T f)$ *is positive definite and* $\overline{p}^T[\nabla_y(\overline{\lambda}^T f) - \overline{w}] \geqq 0$ *or*

$\nabla_{yy}\overline{\lambda}^T f$ *is negative definite and* $\overline{p}^T[\nabla_y(\overline{\lambda}^T f) - \overline{w}] \leqq 0$,

(ii) $\nabla_y \overline{\lambda}^T f - \overline{w} + \nabla_{yy}(\overline{\lambda}^T f)\overline{p} \neq 0$,

(iii) the set $\{\nabla_y f_1, \nabla_y f_2, \cdots, \overline{w}\}$ *is linearly independent where* $f = f(\overline{x}, \overline{y})$.

Then there exists $\overline{z} \in D_i (i = 1, \cdots, k)$ such that $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r} = 0)$ is a feasible solution for **(NMD)** *and objective values of* **(NMP)** *and* **(NMD)** *are equal. Furthermore, under the assumptions of Theorem 4.1, $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r} = 0)$ is a K-weakly efficient solution for* **(NMD)**.

Proof. Since $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p})$ is a K-weakly efficient solution for **(NMP)**, by Lemma 3.1, there exist $\alpha \in K^*, \beta \in C_2, \mu \in \beta R_+, \delta \in C_1^*$, and $\rho \in K$ such that

$$\alpha^T[\nabla_x f + ze] + (\beta - \mu\overline{y})^T \nabla_{yx}(\lambda^T f)$$
$$+(\beta - \mu\overline{y} - \frac{1}{2}(\alpha^T e)\overline{p})^T \nabla_x(\nabla_{yy}(\lambda^T f))\overline{p} - \delta = 0, \tag{4.9}$$

$$(\alpha - \mu\overline{\lambda})^T \nabla_y f - (\alpha^T e - \mu)^T \overline{w} + (\beta - \mu\overline{y} - \mu\overline{p})^T \nabla_{yy}(\overline{\lambda}^T f)$$
$$+(\beta - \mu\overline{y} - \frac{1}{2}(\alpha^T e)\overline{p})^T \nabla_y(\nabla_{yy}(\overline{\lambda}^T f)\overline{p}) = 0, \tag{4.10}$$

$$-\frac{1}{2}(\alpha^T e)\overline{p}^T \nabla_{yy} f\overline{p} + (\beta - \mu\overline{y})^T[\nabla_y f + \nabla_{yy} f\overline{p}] - \rho = 0, \tag{4.11}$$

$$(\alpha^T e)\overline{y} - (\beta - \mu\overline{y}) \in N_{E_i}(\overline{w}), \tag{4.12}$$

$$(\beta - \alpha^T e\overline{p} - \mu\overline{y})^T \nabla_{yy}(\overline{\lambda}^T f) = 0, \tag{4.13}$$

$$\beta^T[\nabla_y(\overline{\lambda}^T f) - \overline{w} + \nabla_{yy}(\overline{\lambda}^T f)\overline{p}] = 0, \tag{4.14}$$

$$\mu\overline{y}^T[\nabla_y(\overline{\lambda}^T f) - \overline{w} + \nabla_{yy}(\overline{\lambda}^T f)\overline{p}] = 0, \tag{4.15}$$

$$\delta^T \overline{x} = 0, \tag{4.16}$$

$$\rho^T \overline{\lambda} = 0, \tag{4.17}$$

$$z \in D_i, \;\; z^T\overline{x} = s(\overline{x}|D_i), \;\; i = 1, \cdots, k, \tag{4.18}$$

$$(\alpha, \beta, \mu, \delta, \rho) \neq 0. \tag{4.19}$$

As $\nabla_{yy}(\overline{\lambda}^T f)$ is positive or negative definite, (4.13) yields

$$\beta = (\alpha^T e)\overline{p} + \mu\overline{y}. \tag{4.20}$$

If $\alpha = 0$, then the above equality becomes

$$\beta = \mu\overline{y}. \tag{4.21}$$

From (4.10), we obtain

$$\mu[\nabla_y(\overline{\lambda}^T f) - \overline{w} + \overline{p}^T \nabla_{yy}(\overline{\lambda}^T f)] = 0. \tag{4.22}$$

By the assumption (ii), we have $\mu = 0$. Also, from (4.9), (4.11) and (4.21), we get $\delta = 0$, $\rho = 0$ and $\beta = 0$, respectively. This contradicts (4.19). So, $\alpha > 0$. From (4.14) and (4.15), we obtain

$$(\beta - \mu\overline{y})^T[\nabla_y(\overline{\lambda}^T f) - \overline{w} + \nabla_{yy}(\overline{\lambda}^T f)\overline{p}] = 0.$$

Using (4.20), it follows that

$$\overline{p}^T[\nabla_y(\overline{\lambda}^T f) - \overline{w}] + \overline{p}^T \nabla_{yy}(\overline{\lambda}^T f)\overline{p} = 0. \tag{4.23}$$

We now prove that $\overline{p} = 0$. Otherwise, the assumption (i) implies that

$$\overline{p}^T[\nabla_y(\overline{\lambda}^T f) - \overline{w}] + \overline{p}^T \nabla_{yy}(\overline{\lambda}^T f)\overline{p} \neq 0,$$

which contradicts (4.23). Hence $\overline{p} = 0$. From (4.20), we have

$$\beta = \mu\overline{y}. \tag{4.24}$$

Using (4.24) and $\overline{p} = 0$ in (4.10), we obtain

$$(\alpha - \mu\overline{\lambda})^T \nabla_y f - (\alpha^T e - \mu)\overline{w} = 0.$$

By the assumption (iii), we get

$$\alpha = \mu\overline{\lambda} \text{ and } \alpha^T e = \mu. \tag{4.25}$$

Therefore, $\mu > 0$, it follows that

$$\nabla_x(\overline{\lambda}^T f) + z \in C_1^*.$$

Multiplying (4.26) by $\overline{x}$ and using equation (4.16), we get

$$\overline{x}^T[\nabla_x(\overline{\lambda}^T f) + z] = 0.$$

Taking $\overline{z} := z \in D_i (i = 1, \cdots, k)$, we find that $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r} = 0)$ is feasible for **(NMD)**. Moreover from (12), we get $\overline{y} \in N_{E_i}(\overline{w})$ for $i = 1, \cdots, k$, so that

$$\overline{y}^T\overline{w} = s(\overline{y}|E_i) for i = 1, \cdots, k i.e., (\overline{y}^T\overline{w})e = s(\overline{y}|E).$$

Consequently, using (4.18),

$$\begin{aligned} K(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p} = 0) &= f(\overline{x}, \overline{y}) + s(\overline{x}|D) - (\overline{y}^T\overline{w})e \\ &= f(\overline{x}, \overline{y}) - s(\overline{y}|E) + (\overline{z}^T\overline{x})e \\ &= G(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r} = 0). \end{aligned}$$

Thus objective values of **(NMP)** and **(NMD)** are equal. We will now show that $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r} = 0)$ is a K-weakly efficient solution for **(NMD)**, otherwise there exists a feasible solution $(u, v, \overline{\lambda}, z, r = 0)$ for **(NMD)** such that

$$G(u, v, \overline{\lambda}, z, r = 0) - G(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r} = 0) \in intK$$

Since objective values of **(NMP)** and **(NMD)** are equal.

$$G(u, v, \overline{\lambda}, z, r = 0) - K(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p} = 0) \in intK,$$

which contradits weak duality theorem. Hence the result hold. □

Theorem 4.3 **(Converse Duality)** *Let* $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{z}, \overline{r})$ *be a K-weakly efficient solution for* **(NMD)**. *Fix* $\lambda = \overline{\lambda}$ *in* **(NMP)**. *Assume that*

(i) $\nabla_{yy}(\overline{\lambda}^T f)$ *is positive definite and* $\overline{p}^T[\nabla_y(\overline{\lambda}^T f) - \overline{w}] \geqq 0$ *or*

$$\nabla_{yy}\overline{\lambda}^T f \text{ is negative definite and } \overline{p}^T[\nabla_y(\overline{\lambda}^T f) - \overline{w}] \leqq 0,$$

(ii) $\nabla_y \overline{\lambda}^T f - \overline{w} + \nabla_{yy}(\overline{\lambda}^T f)\overline{p} \neq 0,$

(iii) the set $\{\nabla_y f_1, \nabla_y f_2, \cdots, \overline{w}\}$ *is linearly independent where* $f = f(\overline{x}, \overline{y})$.

Then there exists $\overline{w} \in E_i (i = 1, \cdots, k)$ *such that* $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{w}, \overline{p} = 0)$ *is a feasible solution for* **(NMP)** *and objective values of* **(NMP)** *and* **(NMD)** *are equal. Furthermore, under the assumptions of Theorem 4.1,* $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{w}, \overline{p} = 0)$ *is a weakly efficient solution for* **(NMP)**.

Proof. It follows on the lines of Theorem 4.2. □

4.4 Wolfe Type Symmetric Duality

We consider the following pair of second order Wolfe type non-differentiable multiobjective programming problem:

$$
\begin{aligned}
(\textbf{NWP}) \quad \text{Minimize} \quad & K(x, y, \lambda, w, p) \\
& = f(x, y) + s(x|D) - (y^T \nabla_y (\lambda^T f)(x, y))e \\
& \quad -(y^T \nabla_{yy} (\lambda^T f)(x, y)p)e - \frac{1}{2}[p^T \nabla_{yy} (\lambda^T f)(x, y)p]e \\
\text{subject to} \quad & -[\nabla_y (\lambda^T f)(x, y) - w + \nabla_{yy} (\lambda^T f)(x, y)p \in C_2^*, \qquad (4.26) \\
& x \in C_1, \quad w \in E_i, \quad \lambda \in K^*, \quad \lambda^T e = 1, \quad e \in intK,
\end{aligned}
$$

$$
\begin{aligned}
(\textbf{NWD}) \quad \text{Maximize} \quad & G(u, v, \lambda, z, r) \\
& = f(u, v) - s(v|E) - (u^T \nabla_x (\lambda^T f)(u, v))e \\
& \quad -(u^T \nabla_{xx} (\lambda^T f)(u, v)r)e - \frac{1}{2}[r^T \nabla_{xx} (\lambda^T f)(u, v)r]e \\
\text{subject to} \quad & \nabla_x (\lambda^T f)(u, v) + z + \nabla_{xx} (\lambda^T f)(u, v)r \in C_1^*, \qquad (4.27) \\
& v \in C_2, \quad z \in D_i, \quad \lambda \in K^*, \quad \lambda^T e = 1, \quad e \in intK,
\end{aligned}
$$

where

(1) $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$ is a three times differentiable function,

(2) C_1 and C_2 are closed convex cones in $\mathbb{R}^n$ and $\mathbb{R}^m$ with nonempty interiors, respectively,

(3) C_1^* and C_2^* are positive polar cones of C_1 and C_2, respectively,

(4) K is a closed convex cone in $\mathbb{R}^k$ such that $intK \neq \emptyset$ and $\mathbb{R}^k_+ \subset K$,

(5) r, z are vectors in $\mathbb{R}^n$, p, w are vectors in $\mathbb{R}^m$,

(6) $e = (1, \cdots, 1)^T$ is vector in $\mathbb{R}^k$,

(7) D_i and $E_i (i = 1, \cdots, k)$ are compact convex sets in $\mathbb{R}^n$ and R^m, respectively.

Let $\nabla_x(\lambda^T f)(x, y)$ and $\nabla_y(\lambda^T f)(x, y)$ are gradients of $(\lambda^T f)(x, y)$ with respect to x and y. Similarly, $\nabla_{xx}(\lambda^T f)(x, y)$ and $\nabla_{yy}(\lambda^T f)(x, y)$ are the Hessian matrices of $(\lambda^T f)(x, y)$ with respect to x and y, respectively.

Now we establish the symmetric duality theorems for **(NWP)** and **(NWD)**.

Theorem 4.4 (Weak Duality) *Let (x, y, λ, w, p) and (u, v, λ, z, r) be feasible solutions of* **(NWP)** *and* **(NWD)**, *respectively. Assume that,*

(i) $(\lambda^T f)(\cdot, y) + (\cdot)^T z$ is second order invex in the first variable for fixed y with

respect to η_1,

(ii) $-(\lambda^T f)(x, \cdot) + (\cdot)^T w$ is second order invex in the second variable for fixed

x with respect to η_2,

(iii) $\eta_1(x, u) + u \in C_1$ and $\eta_2(v, y) + y \in C_2$.

Then

$$G(u, v, \lambda, z, r) - K(x, y, \lambda, w, p) \notin intK.$$

Proof. Since $(\lambda^T f)(\cdot, y) + (\cdot)^T z$ is second order invex with respect to η_1 for fixed y

$$(\lambda^T f)(x, v) + x^T z - (\lambda^T f)(u, v) - u^T z$$
$$\geqq \eta_1(x, u)^T[\nabla_x(\lambda^T f)(u, v) + z + \nabla_{xx}(\lambda^T f)(u, v)r] - \frac{1}{2}r^T\nabla_{xx}(\lambda^T f)(u, v)r$$

From (4.27) and $\eta_1(x, u) + u \in C_1$,

$$[\eta_1(x, u) + u]^T[\nabla_x(\lambda^T f)(u, v) + z + \nabla_x x(\lambda^T f)(u, v)r] \geqq 0.$$

Hence

$$(\lambda^T f)(x, v) + x^T z - (\lambda^T f)(u, v) - u^T z + \frac{1}{2}r^T\nabla_{xx}(\lambda^T f)(u, v)r$$
$$\geqq -u^T[\nabla_x(\lambda^T f)(u, v) + z + \nabla_{xx}(\lambda^T f)(u, v)r]. \tag{4.28}$$

Since $-(\lambda^T f)(x, \cdot) + (\cdot)^T w$ is second order invex with respect to η_2 for fixed x,

$$-(\lambda^T f)(x, v) + v^T w + (\lambda^T f)(x, y) - y^T w$$
$$\geqq -\eta_2(v, y)^T[\nabla_y(\lambda^T f)(x, y) - w + \nabla_{yy}(\lambda^T f)(x, y)p] + \frac{1}{2}p^T\nabla_{yy}(\lambda^T f)(x, y)p$$

From (4.26) and $\eta_2(v, y) + y \in C_2$,

$$-[\eta_2 + y]^T[\nabla_y(\lambda^T f)(x, y) - w + \nabla_{yy}(\lambda^T f)(x, y)p] \geqq 0.$$

So,

$$-(\lambda^T f)(x,v) + v^T w + (\lambda^T f)(x,y) - y^T w - \frac{1}{2}p^T \nabla_{yy}(\lambda^T f)(x,y)p$$
$$\geqq y^T[\nabla_y(\lambda^T f)(x,y) - w + \nabla_{yy}(\lambda^T f)(x,y)p] \qquad (4.29)$$

Therefore, by (4.28) and (4.29),

$$(\lambda^T f)(x,y) + x^T z - y^T[\nabla_y(\lambda^T f)(x,y)$$
$$+\nabla_{yy}(\lambda^T f)(x,y)p] - \frac{1}{2}p^T \nabla_{yy}(\lambda^T f)(x,y)p$$
$$\geqq (\lambda^T f)(u,v) - v^T w - u^T[\nabla_x(\lambda^T f)(u,v)$$
$$+\nabla_{xx}(\lambda^T f)(u,v)r] - \frac{1}{2}r^T \nabla_{xx}(\lambda^T f)(u,v)r$$

Using the fact that $x^T z \leqq s(x|D_i)$ and $v^T w \leqq s(x|E_i)$ for $i = 1, \cdots, k$, we get

$$x^T z \leqq \lambda^T s(x|D) \text{and} v^T w \leqq \lambda^T s(v|E).$$

Hence,

$$(\lambda^T f)(x,y) + \lambda^T s(x|D) - y^T[\nabla_y(\lambda^T f)(x,y)$$
$$+\nabla_{yy}(\lambda^T f)(x,y)p] - \frac{1}{2}p^T \nabla_{yy}(\lambda^T f)(x,y)p$$
$$\geqq (\lambda^T f)(u,v) - \lambda^T s(v|E) - u^T[\nabla_x(\lambda^T f)(u,v)$$
$$+\nabla_{xx}(\lambda^T f)(u,v)r] - \frac{1}{2}r^T \nabla_{xx}(\lambda^T f)(u,v)r \qquad (4.30)$$

But suppose that $G(u, v, \lambda, z, r) - K(x, y, \lambda, w, p) \in intK$. Since $\lambda \in K^*$, it yields

$$\Big[(\lambda^T f)(u, v) - \lambda^T s(v|E) - u^T[\nabla_x(\lambda^T f)(u, v) + \nabla_{xx}(\lambda^T f)(u, v)r] - \frac{1}{2}r^T\nabla_{xx}(\lambda^T f)(u, v)r\Big]$$

$$-\Big[(\lambda^T f)(x, y) - \lambda^T s(x|D) - y^T[\nabla_y(\lambda^T f)(x, y) + \nabla_{yy}(\lambda^T f)(x, y)p] - \frac{1}{2}p^T\nabla_{yy}(\lambda^T f)(x, y)p\Big] > 0$$

which is a contradiction to the inequality (4.30). □

In order to prove the strong duality theorem, we now obtain necessary optimality conditions for a point to be a weak minimum of **(KP)** in Lemma 3.1.

Theorem 4.5 (Strong Duality) *Let $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p})$ be a K-weakly efficient solution for (NWP). Fix $\lambda = \overline{\lambda}$ in* **(NWD)**. *Assume that*

(i) $\nabla_{yy}(\overline{\lambda}^T f)$ is positive definite and $\overline{p}^T[\nabla_y(\overline{\lambda}^T f) - \overline{w}] \geqq 0$ or

$\nabla_{yy}\overline{\lambda}^T f$ is negative definite and $\overline{p}^T[\nabla_y(\overline{\lambda}^T f) - \overline{w}] \leqq 0$,

(ii) $\nabla_y\overline{\lambda}^T f - \overline{w} + \nabla_{yy}(\overline{\lambda}^T f)\overline{p} \neq 0$,

(iii) the set $\{\nabla_y f_1, \nabla_y f_2, \cdots, \overline{w}\}$ is linearly independent where $f = f(\overline{x}, \overline{y})$.

Then there exists $\overline{z} \in D_i (i = 1, \cdots, k)$ such that $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r} = 0)$ is a feasible solution for **(NWD)** *and objective values of* **(NWP)** *and* **(NWD)** *are equal.*

Furthermore, under the assumptions of Theorem 4.4, $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r} = 0)$ *is a K-weakly efficient solution for* **(NWD)**.

Proof. Since $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p})$ is a K-weakly efficient solution for (NWP), by Lemma 1, there exist $\alpha \in K^*$, $\beta \in C_2$, $\mu \in \beta R_+$, $\delta \in C_1^*$, and $\rho \in K$ such that

$$\alpha^T[\nabla_x f + ze] + (\beta - (\alpha^T e)\overline{y})^T \nabla_{yx}(\overline{\lambda}^T f)$$

$$+(\beta - (\alpha^T e)\overline{y} - \frac{1}{2}(\alpha^T e)\overline{p})^T \nabla_x(\nabla_{yy}(\lambda^T f))\overline{p} - \delta = 0, \tag{4.31}$$

$$(\beta - (\alpha^T e)\overline{y} - (\alpha^T e)\overline{p})^T \nabla_{yy}(\overline{\lambda}^T f)$$

$$+(\beta - (\alpha^T e)\overline{y} - (\alpha^T e)\overline{p})^T \nabla_y(\nabla_{yy}(\overline{\lambda}^T f)\overline{p}) = 0, \tag{4.32}$$

$$(\alpha^T e)\left[-\overline{y}^T \nabla_y f - \overline{y}^T(\nabla_{yy} f)\overline{p} - \frac{1}{2}\overline{p}^T \nabla_{yy} f \overline{p}\right] + \beta^T[\nabla_y f + \nabla_{yy} f \overline{p}] - \rho = 0, \tag{4.33}$$

$$\beta \in N_{E_i}(\overline{w}), \tag{4.34}$$

$$(\beta - \alpha^T e\overline{y} - (\alpha^T e)\overline{p})^T \nabla_{yy}(\overline{\lambda}^T f) = 0, \tag{4.35}$$

$$\beta^T[\nabla_y(\overline{\lambda}^T f) - \overline{w} + \nabla_{yy}(\overline{\lambda}^T f)\overline{p}] = 0, \tag{4.36}$$

$$\mu\overline{y}^T[\nabla_y(\overline{\lambda}^T f) - \overline{w} + \nabla_{yy}(\overline{\lambda}^T f)\overline{p}] = 0, \tag{4.37}$$

$$\delta^T \overline{x} = 0, \tag{4.38}$$

$$\rho^T \overline{\lambda} = 0, \tag{4.39}$$

$$z \in D_i, \ \ z^T \overline{x} = s(\overline{x}|D_i), \ \ i = 1, \cdots, k, \tag{4.40}$$

$$(\alpha, \beta, \mu, \delta, \rho) \neq 0. \tag{4.41}$$

By the assumption (i) and (4.35) yields

$$\beta = (\alpha^T e)(\overline{y} + \overline{p}). \tag{4.42}$$

If $\alpha = 0$, then (4.41),(4.31) and (4.33) give $\beta = 0$, $\delta = 0$ and $\rho = 0$. This contradicts (4.40). Therefore $\alpha > 0$. Using (4.41) in (4.32)

$$\frac{1}{2}(\alpha^T e)\overline{p}^T \nabla_y (\nabla_{yy} \overline{(\lambda^T f)})\overline{p} = 0,$$

which using the assumption (ii) implies

$$\overline{p} = 0.$$

Then (4.41) implies $\beta = (\alpha^T e)\overline{y}$. So $\overline{y} \in C_2$. Using (4.42) in (4.31)

$$\alpha^T (\nabla_x f + ze) = \delta \in C_1^*. \tag{4.43}$$

Taking $\overline{z} := z \in D_i (i = 1, \cdots, k)$, we find that $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r} = 0)$ is feasible for **(NWD)**. Multiplying (4.43) by $\overline{x}$ and using (4.37), we get

$$\overline{x}^T \left[\nabla_x (\overline{\lambda}^T f) - \overline{w} \right] = 0. \tag{4.44}$$

Consequently, using (4.44), (4.45) and (4.46),

$$\begin{aligned} K(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p} = 0) &= f(\overline{x}, \overline{y}) + s(\overline{x}|D) - (\overline{y}^T \nabla_y ((\overline{\lambda}^T f))(\overline{x}, \overline{y}))e \\ &= f(\overline{x}, \overline{y}) - (\overline{z}^T \overline{x})e - (\overline{y}^T \overline{w})e \\ &= f(\overline{x}, \overline{y}) - s(\overline{y}|E) - \overline{x}^T \nabla_x ((\overline{\lambda}^T f))(\overline{x}, \overline{y})e \\ &= G(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r} = 0). \end{aligned}$$

Thus objective values of **(NWP)** and **(NWD)** are equal. We will now show that $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r} = 0)$ is a K-weakly efficient solution for **(NWD)**, otherwise there exists a feasible solution $(u, v, \overline{\lambda}, z, r = 0)$ for **(NWD)** such that

$$G(u, v, \overline{\lambda}, z, r = 0) - G(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r} = 0) \in intK$$

Since objective values of **(NWP)** and **(NWD)** are equal.

$$G(u, v, \overline{\lambda}, z, r = 0) - K(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p} = 0) \in intK$$

which contradits weak duality theorem. Hence the result hold. □

Theorem 4.6 **(Converse Duality)** *Let $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{z}, \overline{r})$ be a K-weakly efficient solution for (NWD). Fix $\lambda = \overline{\lambda}$ in* **(NWP)**. *Assume that*

(i) $\nabla_{yy}(\overline{\lambda}^T f)$ is positive definite and $\overline{p}^T[\nabla_y(\overline{\lambda}^T f) - \overline{w}] \geqq 0$ or

$\nabla_{yy}\overline{\lambda}^T f$ is negative definite and $\overline{p}^T[\nabla_y(\overline{\lambda}^T f) - \overline{w}] \leqq 0$,

(ii) $\nabla_y \overline{\lambda}^T f - \overline{w} + \nabla_{yy}(\overline{\lambda}^T f)\overline{p} \neq 0$,

(iii) the set $\{\nabla_y f_1, \nabla_y f_2, \cdots, \overline{w}\}$ is linearly independent where $f = f(\overline{x}, \overline{y})$.

Then there exists $\overline{w} \in E_i (i = 1, \cdots, k)$ such that $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{w}, \overline{p} = 0)$ is a feasible solution for **(NWP)** *and objective values of* **(NWP)** *and* **(NWD)** *are equal. Furthermore, under the assumptions of Theorem 4.4, $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{w}, \overline{p} = 0)$ is a weakly efficient solution for* **(NWP)**.

Proof. It follows on the lines of Theorem 4.5. □

4.5 Special Cases

We give some special cases of our symmetric duality.

First of all, if $C_1 = \mathbb{R}^n_+$ and $C_2 = \mathbb{R}^m_+$, then our results reduce to the following programming problems.

(1) If $k = 1$, , then (**NWP**) and (**NWD**) become the pair of Mond-Weir symmetric dual programs considered in X.M. Yang et al. [83] for the same B and D.

(2) If $k = 1$, then **(NMP)** and **(NMD)** are redued to the second order symmetric dual programs in Hou and Yang [31].

(3) Let $D \in \mathbb{R}^n \times \mathbb{R}^n$ and $E \in \mathbb{R}^m \times \mathbb{R}^m$ are positive semidifinite symmetric matrices. If $s(x|B) = (x^T Dx)^{\frac{1}{2}}$ where $B = \{Dz|z^T Dz \leqq 1\}$ and $s(y|C) = (y^T Ey)^{\frac{1}{2}}$ where $C = \{Ew|w^T ew \leqq 1\}$, $C_1 = \mathbb{R}^n_+$ and $C_2 = \mathbb{R}^m_+$, then **(NMP)** and **(NMD)** become nondifferentiable second order symmetric duality in multiobjective programming in Ahmad and Husain [5].

(4) Let $D \in \mathbb{R}^n \times \mathbb{R}^n$ and $E \in \mathbb{R}^m \times \mathbb{R}^m$ are positive semidifinite symmetric matrices. If $s(x|B) = (x^T Dx)^{\frac{1}{2}}$ where $B = \{Dz|z^T Dz \leqq 1\}$ and $s(y|C) = (y^T Ey)^{\frac{1}{2}}$ where $C = \{Ew|w^T ew \leqq 1\}$, $C_1 = \mathbb{R}^n_+$ and $C_2 = \mathbb{R}^m_+$, then **(NWP)** and **(NWD)** is reduced to nondifferentiable second order symmetric duality in multiobjective programming. In addition, if $k = 1$, then we get second

order symmetric dual programs on nondifferentiable studied by Ahmad and Husain [4].

Next, if $D = \{0\}$ and $E = \{0\}$, then our programs become a pair or symmetric differentiable dual programs.

(1) If $D = \{0\}$, $E = \{0\}$, $C_1 = \mathbb{R}^n_+$ and $C_2 = \mathbb{R}^m_+$, then (**NWP**) and (**NWD**) become the pair of Mond-Weir symmetric dual programs considered in X.M. Yang et al. [84].

(2) If $B = \{0\}$ and $D = \{0\}$, then (**NMP**) and (**NMD**) reduced to the second order symmetric dual programs in Mishra and Lai [56].

(3) If $B = \{0\}$, $D = \{0\}$ and we remove the second order terms in (**NMP**) and (**NMD**), we get the problems **(P)** and **(D)** given by Khurana [49].

(4) $B = \{0\}$, $D = \{0\}$, $C_1 = \mathbb{R}^n_+$ and $C_2 = \mathbb{R}^m_+$ in (**NMP**) and (**NMD**), then our results reduce to the results obtained by Suneja et al. [76].

(5) If $k = 1$, $B = \{0\}$, $D = \{0\}$, $C_1 = \mathbb{R}^n_+$ and $C_2 = \mathbb{R}^m_+$, then **(NMP)** and **(NMD)** and **(NWP)** and **(NWP)** are reduced to the second order symmetric dual programs in Mishra [55].

(6) If $B = \{0\}$ and $D = \{0\}$, then **(NWP)** and **(NWD)** are reduced to the **(P)** and **(D)** in Mishra [54], and remove the second order terms, we get the first order multiobjective symmetric duality with arbitrary cones [48].

(7) If $k = 1$, $B = \{0\}$, $D = \{0\}$, $C_1 = \mathbb{R}^n_+$ and $C_2 = \mathbb{R}^m_+$ in **(NMP)** and **(NMD)**, then we get second order symmetric dual programs which studied by Bector and Chandra [10].

(8) If $B = \{0\}$, $D = \{0\}$, $C_1 = \mathbb{R}^n_+$, $C_2 = \mathbb{R}^m_+$ and $k = 1$, then we get the first order symmetric dual programs which studied by Chandra et al. [14].

(9) If $k = 1$, $C_1 = \mathbb{R}^n$ and $C_2 = \mathbb{R}^m_+$, then **(NWP)** and **(NWD)** become the pair of Wolfe type second order symmetric duality in nondifferentiable programs in Gulati and Gupta [29].

(10) If $k = 1$,$B = \{0\}$, $D = \{0\}$, $C_1 = \mathbb{R}^n_+$ and $C_2 = \mathbb{R}^m_+$, then **(NMP)** and **(NMD)** reduce to a pair of primal problem and dual problem programs studied in Yang [82].

In particular, if $p = r = 0$, then our models and results can be reduced to first order models in Gulati et al. [27], Suneja et al. [74], Khurana [40] and Mond and Schechter [60].

Chapter 5

Generalized Second Order Symmetric Duality for Multiobjective Programs

5

5.1 Introduction and Preliminaries

Symmetric duality in nonlinear programming was introduced by Dorn [20]. Subsequently, Dantzig, Eisenberg and Cottle [19] formulated a pair of the symmetric dual programs in which the dual of the dual equals the primal, and established the weak and strong duality for these problems concerning convex and concave functions. At the same time, Mond [59] presented a slightly different pair of symmetric dual nonlinear programs and obtained more generalized duality results than that of Dantzig, Eisenberg and Cottle [19].

On the other hand, Mond and Weir [63] gave a different pair of symmetric dual nonlinear programming problems in which pseudo-convexity and pseudo-concavity assumptions were reduced to the convexity and concavity ones, and obtained the weak and strong duality for these problems.

Weir and Mond [80] formulated a pair of the symmetric and self dual nonlinear programs for multiobjective nonlinear programming. Mond and Weir [61] proved symmetric duality theorems for multiobjective nonlinear programs under the assumptions of pseudo-convexity and pseudo-concavity. Very recently, the concept of symmetric duality for multiobjective variational problems has been extended to the class of multiobjective variational problems by Ahmad [2]. In 1997, Kim et al. [47] suggested another second order symmetric and self dual programs in multiobjective nonlinear programming and proved the weak, strong, and converse duality theorems under convexity and concavity conditions.

Recently, many authors [29, 3, 5, 75, 26] have studied second order symmetric duality and nondifferentiable second order symmetric duality. And Kim et al. [42], suggested multiobjective generalized nondifferentiable second order symmetric dual programs and established weak, strong and converse duality under the assumption of F-convexity.

In this chapter, we formulate a pair of generalized second order symmetric programs in multiobjective nonlinear programming. For these programs, we establish weak, strong, and converse duality theorems for efficient solutions under suitable convexity assumptions. These results are the extension of second order symmetric duality relations due to Kim et al. [47]. And we present some special cases of our duality results.

Definition 5.1 *A differentiable function $f = \big(f_1, \cdots, f_k\big) : \mathbb{R}^n \to \mathbb{R}^k$ is said to be convex(strictly convex) if for all $x, u \in \mathbb{R}^n$,*

$$f_i(x) - f_i(u) \geqq (>)(x-u)^T \nabla f_i(u), \quad \textit{for each} \quad i = 1, \cdots, k,$$

where in the case of strict convexity, $x \neq u$.

5.2 Generalized Second Order Symmetric Duality

Let f be a twice differentiable function from $\mathbb{R}^n \times \mathbb{R}^m$ into $\mathbb{R}^k$ and $N = \{1, 2, \cdots, n\}$, $M = \{1, 2, \cdots, m\}$, $A \subset N$, $I \subset M$, $N \setminus A = B$ and $M \setminus I = J$. Note that A, B, I or J can be empty. We rearrange x, y as $x = (x_A, x_B)$ and $y = (y_I, y_J)$, respectively. $\nabla_x f(x, y)$ denotes $k \times n$ matrix of first partial derivatives. If $\lambda \in \mathbb{R}^k$, then $\lambda^T f$ is a scalar valued function. Let $\nabla_x(\lambda^T f)(x, y)$ and $\nabla_y(\lambda^T f)(x, y)$ denote gradient(column) vectors with respect to x and y, respectively. Subsequently, let $\nabla_{xx}(\lambda^T f)$ and $\nabla_{yy}(\lambda^T f)$ denote respectively the $n \times n$ and $m \times m$ matrices of second partial derivatives.

We consider the following pair of generalized multiobjective symmetric dual nonlinear programs.

$$
\begin{aligned}
\textbf{(GMSP)} \quad \text{Minimize} \quad & f(x, y) - ({y_I}^T \nabla_{y_I}(\lambda^T f)(x, y))e \\
& \qquad - ({y_I}^T \nabla_{yy_I}(\lambda^T f)(x, y)p)e \\
\text{subject to} \quad & \nabla_y(\lambda^T f)(x, y) + \nabla_{yy}(\lambda^T f)(x, y)p \leqq 0, \\
& y_J^T \nabla_{y_J}(\lambda^T f)(x, y) + y_J^T \nabla_{yy_J}(\lambda^T f)(x, y)p \geqq 0, \\
& x \geqq 0, \quad \lambda > 0, \quad \lambda^T e = 1,
\end{aligned}
$$

(GMSD) Maximize $f(u,v) - ({u_A}^T \nabla_{x_A}(\lambda^T f)(u,v))e$

$$- ({u_A}^T \nabla_{xx_A}(\lambda^T f)(u,v)p)e$$

subject to $\nabla_x(\lambda^T f)(u,v) + \nabla_{xx}(\lambda^T f)(u,v)p \leqq 0,$

$$u_B^T \nabla_{x_B}(\lambda^T f)(u,v) + u_B^T \nabla_{xx_B}(\lambda^T f)(u,v)p \geqq 0,$$

$$v \geqq 0, \quad \lambda > 0, \quad \lambda^T e = 1,$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$, $\lambda \in \mathbb{R}^k$ and $e = (1, \cdots, 1)^T \in \mathbb{R}^k$. $\nabla_{x_A}(\lambda^T f)(x,y)$, $\nabla_{x_B}(\lambda^T f)(x,y)$, $\nabla_{y_I}(\lambda^T f)$ and $\nabla_{y_J}(\lambda^T f)$ are gradient vectors with respect to x_A, x_B, y_I and y_J, respectively. $\nabla_{xx} f(x,y)$ and $\nabla_{yy} f(x,y)$ are respectively the $n \times n$ and $m \times m$ symmetric Hessian matrices.

Now we establish the symmetric duality theorems for **(GMSP)** and **(GMSD)**.

Theorem 5.1 (Weak Duality) *Let* (x,y,λ,p) *be feasible for* **(GMSP)** *with*

$$\begin{pmatrix} \nabla_{xx}(\lambda^T f)(u,v) & 0 \\ 0 & -\nabla_{yy}(\lambda^T f)(x,y) \end{pmatrix} \begin{pmatrix} x-u \\ v-y \end{pmatrix} \leqq 0.$$

Assume that $f(\cdot, y)$ *is convex for fixed* y*, and* $-f(x,\cdot)$ *is convex for fixed* x*. Then*

$$f(x,y) - ({y_I}^T \nabla_{y_I}(\lambda^T f)(x,y))e - ({y_I}^T \nabla_{yy_I}(\lambda^T f)(x,y)p)e$$

$$\nleq f(u,v) - (u_A^T \nabla_{x_A}(\lambda^T f)(u,v))e - ({u_A}^T \nabla_{xx_A}(\lambda^T f)(u,v)r)e.$$

Proof. Assume to the contrary that,

$$f(x,y) - ({y_I}^T \nabla_{y_I}(\lambda^T f)(x,y))e - ({y_I}^T \nabla_{yy_I}(\lambda^T f)(x,y)p)e$$

$$\leq f(u,v) - (u_A^T \nabla_{x_A}(\lambda^T f)(u,v))e - ({u_A}^T \nabla_{xx_A}(\lambda^T f)(u,v)r)e.$$

Then, since $\lambda > 0$,

$$(\lambda^T f)(x,y) - {y_I}^T\nabla_{y_I}(\lambda^T f)(x,y) - {y_I}^T\nabla_{yy_I}(\lambda^T f)(x,y)p$$
$$< (\lambda^T f)(u,v) - u_A^T\nabla_{x_A}(\lambda^T f)(u,v) - {u_A}^T\nabla_{xx_A}(\lambda^T f)(u,v)p. \quad (5.1)$$

From the assumptions of convexity of $f(\cdot,y)$ and $-f(x,\cdot)$,

$$(\lambda^T f)(x,y) - y_I^T\nabla_{y_I}(\lambda^T f)(x,y) - y_I^T\nabla_{yy_I}(\lambda^T f)(x,y)p - (\lambda^T f)(u,v)$$
$$\geqq -({u_A}^T\nabla_{x_A}(\lambda^T f)(u,v)) - {u_A}^T\nabla_{xx_A}(\lambda^T f)(u,v)p - ({u_B}^T\nabla_{x_B}(\lambda^T f)(u,v))$$
$$-{u_B}^T\nabla_{xx_B}(\lambda^T f)(u,v)p + y_J^T\nabla_{y_J}(\lambda^T f)(x,y) + y_J^T\nabla_{yy_J}(\lambda^T f)(x,y)p$$
$$\geqq -({u_A}^T\nabla_{x_A}(\lambda^T f)(u,v)) - {u_A}^T\nabla_{xx_A}(\lambda^T f)(u,v)p$$

(by the constraints of (GMSP) and (GMSD)).

This contradicts (5.1), thus the result holds. □

In order to prove the strong duality theorem, we need the following Fritz John necessary optimality theorem.

Proposition 5.1 (Fritz John Optimality Conditions) *If* $(\bar{x},\bar{y},\bar{\lambda},\bar{p})$ *is a weakly efficient solution of (GSP), then there exists* $(\alpha,\beta,\gamma,\rho,\omega)$ *in* $\mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k$ *such that*

$$K \equiv \alpha^T\big[f - (\bar{y}_I^T\nabla_{y_I}(\bar{\lambda}^T f))e - (\bar{y}_I^T\nabla_{yy_I}(\bar{\lambda}^T f)\bar{p})e\big] + \beta^T\big[\nabla_y(\bar{\lambda}^T f) + \nabla_{yy}(\bar{\lambda}^T f)\bar{p}\big]$$
$$- \gamma\big[\bar{y}_J^T\nabla_{y_J}(\bar{\lambda}^T f) + \bar{y}_J^T\nabla_{yy_J}(\bar{\lambda}^T f)\bar{p}\big] - \rho^T x - \omega^T\bar{\lambda}$$

satisfies

$$\nabla_x K = 0,$$

$$\nabla_{y_I} K = 0,$$

$$\nabla_{y_J} K = 0,$$

$$\nabla_p K = 0,$$

$$\nabla_\lambda K = 0,$$

$$\beta^T \left[\nabla_y(\bar{\lambda}^T f) + \nabla_{yy}(\bar{\lambda}^T f)\bar{p}\right] = 0,$$

$$\gamma\left[\bar{y}_J^T \nabla_{y_J}(\bar{\lambda}^T f) + \bar{y}_J^T \nabla_{yy_J}(\bar{\lambda}^T f)\bar{p}\right] = 0,$$

$$\rho^T \bar{x} = 0,$$

$$\omega^T \bar{\lambda} = 0,$$

$$(\alpha, \beta, \gamma, \rho, \omega) \geqq 0,$$

$$(\alpha, \beta, \gamma, \rho, \omega)) \neq 0.$$

Theorem 5.2 (Strong Duality) *Let f be a three times differentiable function from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^k$. Let $\left(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p}\right)$ be an efficient solution of* **(GMSP)***: fix $\lambda = \bar{\lambda}$ and $p = \bar{p}$ in* **(GMSD)***. Assume that the assumption of the Theorem 5.1 hold. Suppose that*

(i) $\nabla_{yy}\left(\bar{\lambda}^T f\right)(\bar{x}, \bar{y})$ is non-singular,

(ii) $\nabla_{y_J}(\bar{\lambda}^T f)(\bar{x}, \bar{y}) + \nabla_{yy_J}(\bar{\lambda}^T f)(\bar{x}, \bar{y})\bar{p} \neq 0$ and

(iii) the set $\{\nabla_{y_J} f_i(\bar{x}, \bar{y})\}_{i=1,\cdots,k}$ is linearly independent.

Then $\left(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p} = 0\right)$ is an efficient solution of **(GMSD)** *and the objective values of* **(GMSP)** *and* **(GMSD)** *are equal.*

Proof. Since $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})$ is an efficient solution of **(GMSP)**, it follows from proposition 2.1 that there exist $\alpha \in \mathbb{R}^k$, $\beta \in \mathbb{R}^m$, $\gamma \in \mathbb{R}$, $\rho \in \mathbb{R}^n$ and $\omega \in \mathbb{R}^k$ such that the following Fritz John conditions are satisfied at $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})$:

$$\nabla_x\left(\alpha^T f\right) - \left(\nabla_{y_I x}\left(\bar{\lambda}^T f\right) \quad \nabla_{y_J x}\left(\bar{\lambda}^T f\right)\right)\begin{pmatrix}\left(\alpha^T e\right)\bar{y}_I - \beta_I \\ \gamma\bar{y}_J - \beta_J\end{pmatrix}$$

$$-\nabla_x\left\{\nabla_{yy}(\bar{\lambda}^T f)\bar{p}\begin{pmatrix}(\alpha^T e)\bar{y}_I - \beta_I \\ \gamma\bar{y}_J - \beta_J\end{pmatrix}\right\} - \rho = 0, \tag{5.2}$$

$$-\left(\nabla_{y_I y_I}(\bar{\lambda}^T f) \quad \nabla_{y_J y_I}(\bar{\lambda}^T f)\right)\begin{pmatrix}(\alpha^T e)\bar{y}_I - \beta_I + (\alpha^T e)\bar{p}_I \\ \gamma\bar{y}_J - \beta_J + (\alpha^T e)\bar{p}_J\end{pmatrix}$$

$$-\nabla_{y_I}\left\{\left(\nabla_{yy_I}(\bar{\lambda}^T f)\bar{p} \quad \nabla_{yy_J}(\bar{\lambda}^T f)\bar{p}\right)\begin{pmatrix}(\alpha^T e)\bar{y}_I - \beta_I \\ \gamma\bar{y}_J - \beta_J\end{pmatrix}\right\} = 0, \tag{5.3}$$

$$(\alpha - \gamma\bar{\lambda})^T \nabla_{y_J} f - \left(\nabla_{y_I y_J}(\bar{\lambda}^T f) \quad \nabla_{y_J y_J}(\bar{\lambda}^T f)\right)\begin{pmatrix}(\alpha^T e)\bar{y}_I - \beta_I + \gamma\bar{p}_I \\ \gamma\bar{y}_J - \beta_J + \gamma\bar{p}_J\end{pmatrix}$$

$$-\nabla_{y_J}\left\{\left(\nabla_{yy_I}(\bar{\lambda}^T f)\bar{p} \quad \nabla_{yy_J}(\bar{\lambda}^T f)\bar{p}\right)\begin{pmatrix}(\alpha^T e)\bar{y}_I - \beta_I \\ \gamma\bar{y}_J - \beta_J\end{pmatrix}\right\} = 0, \tag{5.4}$$

$$\nabla_{yy}(\bar{\lambda}^T f)\begin{pmatrix}\beta_I - (\alpha^T e)\bar{y}_I \\ \beta_J - \gamma\bar{y}_J\end{pmatrix} = 0, \tag{5.5}$$

$$\left(\nabla_{y_I} f \quad \nabla_{y_J} f\right)\begin{pmatrix}\beta_I - (\alpha^T e)\bar{y}_I \\ \beta_J - \gamma\bar{y}_J\end{pmatrix}$$

$$-\left(\nabla_{yy_I} f\bar{p} \quad \nabla_{yy_J} f\bar{p}\right)\begin{pmatrix}(\alpha^T e)\bar{y}_I - \beta_I \\ \gamma\bar{y}_J - \beta_J - \omega\end{pmatrix} = 0, \tag{5.6}$$

$$\beta^T(\nabla_y(\bar{\lambda}^T f) + \nabla_{yy}(\bar{\lambda}^T f)\bar{p}) = 0, \tag{5.7}$$

$$\gamma(\bar{y}_J^T \nabla_{y_J}(\bar{\lambda}^T f) + \bar{y}_J^T \nabla_{yy_J}(\bar{\lambda}^T f)\bar{p}) = 0, \tag{5.8}$$

$$\rho^T \bar{x} = 0, \tag{5.9}$$

$$\omega^T \bar{\lambda} = 0, \tag{5.10}$$

$$(\alpha, \beta, \gamma, \rho, \omega) \geqq 0, \tag{5.11}$$

$$(\alpha, \beta, \gamma, \rho, \omega) \neq 0. \tag{5.12}$$

Since $\nabla_{yy}(\bar{\lambda}^T f)$ is non-singular, (5.5) yields

$$\beta_I = (\alpha^T e)\bar{y}_I \quad \text{and} \quad \beta_J = \gamma \bar{y}_J. \tag{5.13}$$

From (5.3) and (5.13), we have

$$(\alpha^T e)\nabla_{yy}(\bar{\lambda}^T f)\bar{p}) = 0. \tag{5.14}$$

Suppose that $\alpha = 0$. From (5.4),

$$\gamma(\nabla_{y_J}(\bar{\lambda}^T f) + \nabla_{yy_J}(\bar{\lambda}^T f)\bar{p}) = 0.$$

Since $\nabla_{y_J}(\bar{\lambda}^T f) + \nabla_{yy_J}(\bar{\lambda}^T f)\bar{p} \neq 0$, $\gamma = 0$. From (5.13) and (5.6), we have $\beta = 0$ and $\omega = 0$. From (5.2), we have$\rho = 0$. This is a contradiction to (5.12). Hence $\alpha \neq 0$. Since $\nabla_{yy}(\bar{\lambda}^T f)\bar{p}$ is non-singular, we get from (5.14),

$$\bar{p} = 0. \tag{5.15}$$

Using (5.2),(5.13) and (5.15), we get

$$\nabla_x(\bar{\alpha}^T f)(\bar{x}, \bar{y}) \geqq 0. \tag{5.16}$$

From (5.4),(5.12) and (5.14), we have $(\alpha - \gamma\bar{\lambda})^T \nabla_{y_J} f = 0.$,

Since $\{\nabla_{y_J} f_i(\bar{x}, \bar{y})\}_{i=1,\cdots,k}$ is linearly independent,

$$\alpha = \gamma\bar{\lambda}. \tag{5.17}$$

If $\gamma = 0$ in (5.17), $\alpha = 0$. From (5.13), $\beta = 0$. In (5.6) and (5.2), $\omega = 0$ and $\rho = 0$. This is contradiction to (5.12). Hence $\rho > 0$ and $\alpha > 0$. Substituting (5.17) in (5.16), we have

$$\gamma \nabla_x (\bar{\lambda}^T f)(\bar{x}, \bar{y}) \geqq 0.$$

Since $\gamma > 0$,

$$\nabla_x (\bar{\lambda}^T f)(\bar{x}, \bar{y}) \geqq 0. \tag{5.18}$$

Using $\bar{p} = 0$, and (5.18),

$$\nabla_x (\bar{\lambda}^T f) + \nabla_{xx} (\bar{\lambda}^T f)\bar{p} \geqq 0 \tag{5.19}$$

and

$$\bar{x_B}^T \nabla_{x_B} (\bar{\lambda}^T f) + \bar{x_B}^T \nabla_{xx_B} (\bar{\lambda}^T f)\bar{p} = 0. \tag{5.20}$$

Now multiplying (5.6) by λ and using (5.7),(5.8) and (5.10) gives

$$\bar{y}_I^T \nabla_{y_I} (\bar{\lambda}^T f)(\bar{x}, \bar{y}) + \bar{y}_I^T \nabla_{yy_I} (\bar{\lambda}^T f)(\bar{x}, \bar{y})\bar{p} = 0. \tag{5.21}$$

Hence from (5.19) and (5.20), $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p} = 0)$ is feasible solution of **(GMSD)** and the objective values of **(GMSP)** and **(GMSD)** are equal there. □

By the similar method of Theorem 5.2, we can prove the following converse duality theorem.

Theorem 5.3 (Converse Duality) *Let f be a three times differentiable function from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^k$. Let $\left(\bar{u}, \bar{v}, \bar{\lambda}, \bar{p}\right)$ be an efficient solution of* **(GMSD)***: fix $\lambda = \bar{\lambda}$ and $p = \bar{p}$ in* **(GMSP)***. Assume that the assumptions of Theorem 5.1 hold. Suppose that*

(i) $\nabla_{xx}\left(\bar{\lambda}^T f\right)(\bar{u}, \bar{v})$ is non-singular,

(ii) $\nabla_{x_B}(\bar{\lambda}^T f)(\bar{u}, \bar{v}) + \nabla_{xx_B}(\bar{\lambda}^T f)(\bar{u}, \bar{v})\bar{r} \neq 0$ and

(iii) the set $\{\nabla_{x_B} f_i(\bar{u}, \bar{v})\}_{i=1,\cdots,k}$ is linearly independent.

Then $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{p} = 0)$ is an efficient solution of **(GMSD)** *and the objective values of* **(GMSD)** *and* **(GMSP)** *are equal.*

5.3 Special Cases

If $I = M$ and $A = N$, then our pair of programs ((**GMSP**) and (**GMSD**) are reduced to the following (**WSP**) and (**WSD**).

$$\textbf{(WSP)} \quad \text{Minimize} \quad f(x,y) - (y^T \nabla_y (\lambda^T f)(x,y))e - (y^T \nabla_{yy}(\lambda^T f)(x,y)p)e$$

$$\text{subject to} \quad \nabla_y(\lambda^T f)(x,y) + \nabla_{yy}(\lambda^T f)(x,y)p \leqq 0,$$

$$x \geqq 0, \quad \lambda > 0, \quad \lambda^T e = 1,$$

$$\textbf{(WSD)} \quad \text{Maximize} \quad f(u,v) - (u^T \nabla_x (\lambda^T f)(u,v))e - (u^T \nabla_{xx}(\lambda^T f)(u,v)p)e$$

$$\text{subject to} \quad \nabla_x(\lambda^T f)(u,v) + \nabla_{xx}(\lambda^T f)(u,v)p \leqq 0,$$

$$v \geqq 0, \quad \lambda > 0, \quad \lambda^T e = 1,$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$, $\lambda \in \mathbb{R}^k$ and $e = (1, \cdots, 1)^T \in \mathbb{R}^k$. In Kim et al. [47] proved the following duality theorems under suitable convexity assumptions.

Theorem 5.4 (Weak Duality) *Let (x, y, λ, p) be feasible for* **(WSP)**, *and (u, v, λ, p) be feasible for* **(WSD)** *with*

$$\begin{pmatrix} \nabla_{xx}(\lambda^T f)(u, v) & 0 \\ 0 & -\nabla_{yy}(\lambda^T f)(x, y) \end{pmatrix} \begin{pmatrix} x - u \\ v - y \end{pmatrix} \leqq 0.$$

Assume that $f(\cdot, y)$ is convex for fixed y, and $-f(x, \cdot)$ is convex for fixed x. Then

$$\begin{aligned} & f(x, y) - (y^T \nabla_y(\lambda^T f)(x, y))e - (y^T \nabla_{yy}(\lambda^T f)(x, y)p)e \\ & \nleq f(u, v) - (u^T \nabla_x(\lambda^T f)(u, v))e - (u^T \nabla_{xx}(\lambda^T f)(u, v)r)e. \end{aligned}$$

Theorem 5.5 (Strong Duality) *Let $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})$ be an efficient solution of (WSP): fix $\lambda = \bar{\lambda}$ and $p = \bar{p}$ in (WSD). Let $\nabla_{yy}(\bar{\lambda}^T f)(\bar{x}, \bar{y})$ be positive definite, and the set $\{\nabla_y f_i(\bar{x}, \bar{y})\}_{i=1,\cdots,k}$ be linearly independent. Assume that the assumptions of the Theorem 5.4 hold. Then the objective values of* **(WSP)** *and* **(WSD)** *are equal, and $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{p})$ is an efficient solution of* **(WSD)**.

Theorem 5.6 (Converse Duality) *Let $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{p})$ be an efficient solution of* **(WSD)**: *fix $\lambda = \bar{\lambda}$ and $p = \bar{p}$ in (WSD). Let $\nabla_{xx}(\bar{\lambda}^T f)(\bar{u}, \bar{v})$ be negative definite, and the set $\{\nabla_x f_i(\bar{u}, \bar{v})\}_{i=1,\cdots,k}$ be linearly independent. Assume that the assumptions of Theorem 5.4 hold. Then the objective valued of* **(WSP)** *and* **(WSD)** *are equal, and $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{p})$ is an efficient solution of* **(WSP)**.

If $I = \emptyset$ and $A = \emptyset$, then our pair of programs **(GMSP)** and **(GMSD)** are reduced to the following **(MSP)** and **(MSD)**.

$$
\begin{aligned}
\textbf{(MSP)} \quad & \text{Minimize} \quad f(x, y) \\
& \text{subject to} \quad \nabla_y(\lambda^T f)(x, y) + \nabla_{yy}(\lambda^T f)(x, y)p \leqq 0, \\
& \qquad y^T \nabla_y(\lambda^T f)(x, y) + y^T \nabla_{yy}(\lambda^T f)(x, y)p \geqq 0, \\
& \qquad x \geqq 0, \lambda > 0, \lambda^T e = 1,
\end{aligned}
$$

$$
\begin{aligned}
(\textbf{(MSD)} \quad & \text{Maximize} \quad f(u, v) \\
& \text{subject to} \quad \nabla_x(\lambda^T f)(u, v) + \nabla_{xx}(\lambda^T f)(u, v)p \leqq 0, \\
& \qquad u^T \nabla_x(\lambda^T f)(u, v) + u^T \nabla_{xx}(\lambda^T f)(u, v)p \geqq 0, \\
& \qquad v \geqq 0, \lambda > 0, \lambda^T e = 1,
\end{aligned}
$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$, $\lambda \in \mathbb{R}^k$ and $e = (1, \cdots, 1)^T \in \mathbb{R}^k$. We can obtain weak, strong, converse duality theorems between **(MSP)** and **(MSD)**.

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