Thesis for the Degree of Doctor of Philosophy

Approaches to Multiple Attribute Group Decision Making under Linguistic Environment

by
Min Gui Gwak
Department of Applied Mathematics
The Graduate School
Pukyong National University

February 2012

Approaches to Multiple Attribute Group Decision Making under Linguistic Environment

언어적 환경에서 다속성 집단의사결정의 해결 방법

Advisor : Prof. Jin Han Park

by

Min Gui Gwak

A thesis submitted in partial fulfillment of the requirement
for the degree of

Doctor of Philosophy

in the Department of Applied Mathematics, The Graduate School,
Pukyong National University

February 2012

Approaches to Multiple Attribute Group Decision Making under Linguistic Environment

A dissertation

by

Min Gui Gwak

Approved by :

(Chairman) Young Chel Kwun, Ph. D.

(Member) Yong-Soo Pyo, Ph. D.

(Member) Sung-Jin Cho, Ph. D.

(Member) Jin-Soo Hwang, Ph. D.

(Member) Jin Han Park, Ph. D.

February 24, 2012

CONTENTS

Abstract(Korean) i

1. Introduction 1

2. Linguistic harmonic mean operators and their applications to group decision making 7

2.1. Some new aggregation operators 8

2.2. A method for group decision making with linguistic preference relations 16

2.3. Application I 17

2.4. Application II 20

2.4.1. Approach to AMT selection 20

2.4.2. Practical example 21

2.5. Conclusions 23

3. Uncertain linguistic harmonic mean operators and their applications to multiple attribute group decision making 24

3.1. Some operational laws of uncertain linguistic variables 25

3.2. Some new uncertain linguistic aggregation operators 27

3.3. An approach to multiple attribute group decision making 34

3.4. Illustrative example 36

3.5. Comparison with other methods 38

3.6. Conclusions and discussions 40

4. Generalized induced linguistic harmonic mean operators based approach to multiple attribute group decision making 43

4.1. Generalized induced linguistic aggregation operators 44

4.1.1. The GILOWHM and GIULOWHM operators 44

4.1.2. Some properties of the GILOWHM operator 50

4.2. An approach to group decision making 52

4.3. Illustrative example 54

4.4. Conclusions 60

Bibliography 61

언어적 환경에서 다속성 집단의사결정의 해결 방법

곽 민 귀

부경대학교 대학원 응용수학과

요 약

본 논문에서는 언어적 조화평균 연산자에 기초한 다속성 집단의사결정의 해결 방법을 연구 조사한 것으로 내용은 다음과 같이 요약된다.

첫째, 언어적 변수의 용어들을 가지는 선호 정보를 모으는데 이용할 수 있는 LHM 연산자, LWHM 연산자, LOWHM 연산자, 그리고 LHHM 연산자와 같은 새로운 집성연산자를 소개하고 이 연산자들의 몇 가지 성질을 조사한다. LHM 연산자와 LHHM 연산자를 기초로 언어적 선호 관계를 가지는 집단의사 결정에 대한 실제적인 방법과 예를 구하고, 또한 LWHM 연산자와 LHHM 연산자를 기초로 한 집단의사결정 문제의 해결 방법을 개선된 정보처리기술 선택 방법으로 적용하였다.

둘째, ULWHM 연산자, ULOWHM 연산자, 그리고 ULHHM 연산자와 같은 불확실한 언어적 집성연산자에 대한 여러 가지 성질을 조사하고, ULWHM 연산자와 ULHHM 연산자를 이용하여 불확실한 언어적 정보를 가지는 다속성 집단의사결정 문제로의 해결 방법과 실제적인 예제를 구하였다.

셋째, 일반화된 유도된 언어적 집성연산자인 GILOWHM 연산자와 GIULOWHM 연산자에 대한 정의를 소개하고 다양한 성질을 조사하였다. ILOWHM 연산자와 LOWHM 연산자, 그리고 IULOWHM 연산자와 ULOWHM 연산자는 각각 GILOWHM 연산자와 GIULOWHM 연산자의 특별한 경우임을 보였다. GILOWHM 연산자와 GIULOWHM 연산자를 기초로 하여, 가중치 값이 언어적 변수 또는 불확실한 언어적 변수로 표현된 다속성 집단의사결정 문제를 해결하기 위한 두 가지 방법과 그 예를 제시하였다.

Chapter 1

Introduction

Information aggregation is an essential process of gathering relevant information from multiple sources. Many techniques have been developed to aggregate data information [12, 13, 20-23, 27]. Yager and Filev [40] introduced an induced aggregation operator called the induced ordered weighted averaging (IOWA) operator, which takes as its argument pairs, called OWA pairs, in which one component is used to induce an ordering over the second components which are exact numerical values and then aggregated. Later, some new induced aggregation operators have been developed, including the induced ordered weighted geometric (IOWG) operator [26], induced fuzzy integral aggregation (IFIA) operator [37] and induced Choquet ordered averaging (ICOA) operator [38]. Xu and Da [26] introduced two more general aggregation techniques called generalized IOWA (GIOWA) and generalized IOWG (GIOWG) operators, and proved that the OWA and IOWA operators are the special cases of the GIOWA operator, and that the OWG and IOWG operators are the special cases of the GIOWG operator.

Decision making problems generally consist of finding the most desirable alternative(s) from a given alternative set. The increasing complexity of the socio-economic environment makes it less and less possible for single decision maker to consider all relevant aspects of a problem [16]. As a result, many decision making processes, in the real world, take place in group settings. Group decision making problems follow a common resolution scheme composed by the following two phases:

- Aggregation phase: It combines the individual preferences to obtain a collective preference.

- Exploitation phase: It orders the collective preference values to obtain the best alternative(s).

Recently, a number of studies have focused on the group decision making with linguistic preference relations [5, 7-15, 28, 32, 33]. Herrera et al. [8] developed a consensus model for group decision making under linguistic assessments. It is based on the use of linguistic preferences to provide individuals' opinions, and on the use of fuzzy majority of consensus, represented by means of linguistic quantifier. Herrera et al. [9, 11] combined the linguistic ordered weighted averaging (LOWA) operator with linguistic preference relations and the concept of dominance and nondominance to show its use in the field of group decision making, and presented three models of group decision making based on LOWA operator, and presented a consensus model in complete linguistic framework for group decision making. Herrera and Herrera-Viedma [15] analyzed the steps to follow in linguistic decision analysis of group decision making problem with linguistic preference relations. Herrera and Martínez [13] developed a linguistic representation model for representing the linguistic information with the 2-tuples without loss of information. Motivated by this idea, Xu [28] proposed some linguistic aggregation operators such as linguistic geometric (LG) operator, linguistic weighted geometric (LWG) operator, linguistic ordered weighted geometric (LOWG) operator and linguistic hybrid geometric (LHG) operator, and developed an approach to group decision making with linguistic relations, which is straightforward and has no loss of information. Xu [34] defined two generalized induced linguistic aggregation operators, including generalized induced linguistic ordered weighted averaging (GILOWA) operator and generalized induced linguistic ordered weighted geometric (GILOWG) operator, and proved that the induced linguistic ordered weighted averaging (ILOWA) operator and LOWA operator are the special cases of the GILOWA operator, and induced linguistic ordered weighted geometric (ILOWG) operator and LOWG operator are the special cases of the GILOWG operator.

Xu [29] proposed uncertain linguistic aggregation operators such as uncertain linguistic weighted averaging (ULWA) operator, uncertain linguistic ordered weighted averaging (ULOWA) operator and uncertain linguistic hybrid averaging (ULHA) operator, and developed an approach to multiple group decision making with uncertain linguistic information. Xu [32] proposed some uncertain linguistic aggregation operators including the uncertain linguistic geometric mean (ULGM) operator, uncertain linguistic weighted geometric mean (ULWGM) operator, and induced uncertain linguistic ordered weighted geometric (IULOWG) operator, and developed an approach to group decision making with uncertain multiplicative linguistic relation. In some situations, however, the decision makers either are willing to provide only uncertain linguistic information, or take the input arguments as the form of uncertain linguistic variables rather than numerical ones because of time pressure, lack of knowledge, or data, and their limited expertise related to the problem domain. So based on induced ordered weighted averaging (IOWA) operator proposed by Yager and Filev [40], Xu [33] introduced induced uncertain linguistic ordered weighted averaging (IULOWA) operator which take as their argument pair, called ULOWA pair, in which one component is used to induce an ordering over the second components which are given in the form of uncertain linguistic variables, and applied the IULOWA operator to group decision making with uncertain linguistic information. Xu [34] proposed two generalized induced uncertain linguistic aggregation operators, including generalized induced uncertain linguistic ordered weighted averaging (GIULOWA) operator and generalized induced uncertain linguistic ordered weighted geometric (GIULOWG) operator, and showed that the IULOWA operator and ULOWA operator are the special cases of the GIULOWA operator, and IULOWG operator and ULOWGM operator are the special cases of the GIULOWG operator. Xu [34] developed various generalized induced linguistic aggregation operators, such as the generalized induced linguistic ordered weighted averaging (GILOWA) and generalized induced linguistic ordered weighted geometric (GILOWG) operator, both of which can be used to deal with the linguistic information, and generalized induced uncertain linguistic ordered weighted averaging (GIULOWA) operator and generalized induced uncertain linguistic ordered weighted geometric (GIULOWG) operator, both of which can be used to deal with the uncertain linguistic information.

Recently, to meet the challenge of global competitiveness, manufacturing organizations are now facing the problems of selecting appropriate manufacturing strategies, product and process designs, manufacturing processes and technologies, and machinery and equipment. The selection decisions become more complex as the decision makers in manufacturing environment have to assess a wide range of alternatives based on a set of conflicting criteria. To aid these selection processes, various multiple attribute decision making methods applied in the group decision making are available. For example, Chuu [4] developed a fuzzy multiple attribute decision making applied in the group decision making to improving advanced manufacturing technology selection process. Yong [43] proposed an approach for selecting plant location under linguistic environments using the TOPSIS method taken from group decision making. On the other side, fuzzy set theory, which was introduced by Zadeh [45], has emerged as powerful mathematical tool and has been applied in many applied research fields. Since the field of interconnected systems is so broad as to cover the fundamental theory of modeling, optimization and control aspects and applications, the stability problem of interconnected system have been concerned by many researchers [3, 42]. In particular, since the factor of time-delay complicates the analysis, the stability problem of interconnected fuzzy models with time delays in subsystems is studied by Chen et al. [3]. We are going to evolve this theory in our method in order to propose a more applied decision making algorithm.

Information aggregation is essential process of gathering relevant information from multiple sources. Many techniques, such as the max and min operators, the weighted geometric mean operator, the weighted arithmetic average (WAA) operator, the weighted harmonic mean (WHM) operator, the ordered weighted averaging (OWA) operator, and so on have been developed to aggregate data information [23, 25, 26, 27, 35, 36, 41]. Harmonic mean is a conservative average to be used to provide for aggregation lying between the max and min operators.

Harmonic mean is widely used to aggregate central tendency data. In the existing literature, the harmonic mean is generally considered as a fusion technique of numerical data, in the real-life situations, the input data sometimes cannot be obtained exactly, but linguistic data can be given. Therefore, “how to aggregate linguistic data by using the harmonic mean?” is an interesting research topic and is worth paying attention too.

We briefly summarize the contents of the each chapter as follows.

In Chapter 2, we develop some linguistic harmonic mean (LHM) operators, such as linguistic weighted harmonic mean (LWHM) operator, linguistic ordered weighted harmonic mean (LOWHM) operator and linguistic hybrid harmonic mean (LHHM) operator, and then study some desirable properties of the operator, and then present an approach to group decision making based on the developed operator, illustrate the presented approach with a numerical example. Based on the LWHM and LHHM operators, develops a multiple attribute decision making applied in the group decision making to improving advanced manufacturing technology selection process and present some concluding remarks.

In Chapter 3, we develop some uncertain linguistic aggregation operators, such as uncertain linguistic weighted harmonic mean(ULWHM) operator, uncertain linguistic aggregation operators, such as uncertain linguistic ordered weighted harmonic mean(ULOWHM) operator and uncertain linguistic hybrid harmonic mean(ULHHM) operator, and then study some desirable properties of the operator. We present an approach to group decision making based on the developed operator and illustrate the presented approach with a practical example. Finally, some concluding remarks is pointed out.

In Chapter 4, we shall develop two new aggregation operators called generalized induced linguistic ordered weighted harmonic mean (GILOWHM) operator and generalized induced uncertain linguistic ordered weighted harmonic mean (GIULOWHM) operator, which can be used to deal with linguistic information or uncertain linguistic information, and study some of their desirable properties.

Each object processed by these operator consists of three components, where the first component represents the importance degree or character of the second component, and the second component is used to induce an ordering, through the first component, over the third components which are linguistic variables or uncertain linguistic variables and then aggregated. It is shown that the induced linguistic ordered weighted harmonic mean (ILOWHM)[21] operator and linguistic ordered weighted harmonic mean (LOWHM)[21] operator are the special cases of the GILOWHM operapor and that the induced uncertain linguistic ordered weighted harmonic mean (IULOWHM) operator and uncertain linguistic ordered weighted harmonic mean (ULOWHM)operator are the special cases of the GIULOWHM operapor. Two procedures based on the GILOWHM and GIULOWHM operators respectively, are developed to solve the multiple attribute decision making (MADM) problems where all decision information about attribute values take the forms of linguistic variables or uncertain linguistic variables. Finally, an illustrative example is pointed out.

Chapter 2

Linguistic harmonic mean operators and their applications to group decision making

Harmonic mean is reciprocal of arithmetic mean of reciprocal, which is a conservative average to be used to provide for aggregation lying between max and min operators. In this chapter, we develop some new aggregation operators such as linguistic harmonic mean (LHM) operator, linguistic weighted harmonic mean (LWHM) operator, linguistic ordered weighted harmonic mean (LOWHM) operator, and linguistic hybrid harmonic mean (LHHM) operator, which can be utilized to aggregate preference information taking the form of linguistic variables, and then study some desirable properties of the operators. Based on the LHM and the LHHM operators, we propose a practical method for group decision making with linguistic preference relations, and also give an illustrative example. Furthermore, based on the LWHM and LHHM operators, we develop a multiple attribute decision making applied in the group decision making to improving advanced manufacturing technology selection process.

2.1 Some new aggregation operators

Definition 2.1.1 (Harsanyi [6]) Let WAA : $R^n \to R$, if

$$\mathrm{WAA}_w(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j a_j, \tag{2.1}$$

where a_j $(j = 1, 2, \ldots, n)$ is a collection of positive real numbers, $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of a_j $(j = 1, 2, \ldots, n)$, with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$, R is the set of real numbers, then WAA is called the weighted arithmetic averaging (WAA) operator. Especially, if $w_i = 1$, $w_j = 0$, $j \neq i$, then $\mathrm{WAA}_w(a_1, a_2, \ldots, a_n) = a_i$; if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the WAA operator is reduced to the arithmetic averaging (AA) operator, i.e.,

$$\mathrm{AA}_w(a_1, a_2, \ldots, a_n) = \frac{1}{n} \sum_{j=1}^{n} a_j. \tag{2.2}$$

Definition 2.1.2 (Bullen et al. [1]) Let WHM : $(R^+)^n \to R^+$, if

$$\mathrm{WHM}_w(a_1, a_2, \ldots, a_n) = \frac{1}{\sum_{j=1}^{n} \frac{w_i}{a_i}}, \tag{2.3}$$

where a_j $(j = 1, 2, \ldots, n)$ is a collection of positive real numbers, $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of a_j $(j = 1, 2, \ldots, n)$, with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$, R^+ is the set of all positive real numbers, then WHM is called the weighted harmonic mean (WHM) operator. Especially, if $w_i = 1$, $w_j = 0$, $j \neq i$, then $\mathrm{WHM}_w(a_1, a_2, \ldots, a_n) = a_i$; if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the WHM operator is reduced to the harmonic mean (HM) operator, i.e.,

$$\mathrm{HM}_w(a_1, a_2, \ldots, a_n) = \frac{n}{\sum_{j=1}^{n} \frac{1}{a_j}}. \tag{2.4}$$

The WAA and the WHM operators first weight all the given data, and then aggregate all these weighted data into a collective one. Yager [35, 36] introduced and studied the OWA operator that weights the ordered positions of the data instead of weighting the data themselves.

Definition 2.1.3 (Yager [35]) An OWA operator of dimension n is a mapping OWA : $R^n \to R$ that has an associated vector $w = (w_1, w_2, \ldots, w_n)^T$ such that $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$. Furthermore,

$$\mathrm{OWA}_w(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j, \tag{2.5}$$

where b_j is the jth largest of a_i $(i = 1, 2, \ldots, n)$. Especially, if $w_i = 1$, $w_j = 0$, $j \neq i$, then $b_n \leq \mathrm{OWA}_w(a_1, a_2, \ldots, a_n) = b_i \leq b_1$; if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then

$$\begin{aligned} \mathrm{OWA}_w(a_1, a_2, \ldots, a_n) &= \frac{1}{n}\sum_{j=1}^{n} b_j \\ &= \frac{1}{n}\sum_{j=1}^{n} a_j \\ &= \mathrm{AA}(a_1, a_2, \ldots, a_n). \end{aligned} \tag{2.6}$$

The WAA, the WHM and the OWA operators have only been used in situation in which the input arguments are the exact values. However, judgements of people depend on personal psychological aspects such as experience, learning, situation, state of mind, and so forth. It is more suitable to provide their preferences by means of linguistic variables rather than numerical ones. In the following, based on these operators, which can be used to accommodate the situations where the input arguments are linguistic variables.

Let $S = \{s_i : i = 1, 2, \ldots, t\}$ be a finite and totally ordered discrete term set. Any label, s_i, represents a possible value for a linguistic variable, and it must have the following characteristics [8]:

(1) The set is ordered: $s_i \geq s_j$ if $i \geq j$;

(2) There is the negation operator: $\mathrm{neg}(s_i) = s_j$ such that $j = t + 1 - i$.

(3) Max operator: $\max(s_i, s_j) = s_i$ if $s_i \geq s_j$;

(4) Min operator: $\min(s_i, s_j) = s_i$ if $s_i \leq s_j$.

For example, S can be defined so as its elements are uniformly distributed on a scale on which a total order is defined:

$$\begin{aligned} S = \{ & s_1 = \text{extremely poor},\ s_2 = \text{very poor},\ s_3 = \text{poor},\ s_4 = \text{slightly poor}, \\ & s_5 = \text{fair},\ s_6 = \text{slightly good},\ s_7 = \text{good},\ s_8 = \text{very good}, \\ & s_9 = \text{extremely good}\}. \end{aligned}$$

To preserve all the given information, we extend the discrete term set S to a continuous linguistic term set $\bar{S} = \{s_\alpha : s_1 \leq s_\alpha \leq s_t, \alpha \in [1, t]\}$, where, if $s_\alpha \in S$, then we call s_α an original linguistic term, otherwise, we call s_α the virtual linguistic term [32]. The decision maker, in general, uses the original linguistic terms to evaluate alternatives, and the virtual linguistic terms can only appear in operations.

Consider any two linguistic variables s_α and s_β, then we define the operations $s_\alpha \oplus s_\beta$, λs_α and $\frac{1}{s_\alpha}$ as follows:

(1) $s_\alpha \oplus s_\beta = \min\{s_{\alpha+\beta}, s_t\}$;

(2) $\lambda s_\alpha = s_{\lambda\alpha}$, where $\lambda \in [0, 1]$;

(3) $\frac{1}{s_\alpha} = s_{\frac{1}{\alpha}}$.

Based on the operational laws, we extend the WHM operator to linguistic environment:

Definition 2.1.4 Let LWHM : $\bar{S}^n \to \bar{S}$, if

$$\begin{aligned} \mathrm{LWHM}_w(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}) &= \frac{1}{\frac{w_1}{s_{\alpha_1}} \oplus \frac{w_2}{s_{\alpha_2}} \oplus \cdots \oplus \frac{w_n}{s_{\alpha_n}}} \\ &= \frac{1}{s_{\frac{w_1}{\alpha_1}} \oplus s_{\frac{w_2}{\alpha_2}} \oplus \cdots \oplus s_{\frac{w_n}{\alpha_n}}} \\ &= \frac{1}{s_{\sum_{j=1}^{n} \frac{w_j}{\alpha_j}}}, \end{aligned} \tag{2.7}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of the s_{α_j} with $w_j \in [0, 1]$, $\sum_{j=1}^n w_j = 1$, $s_{\alpha_j} \in \bar{S}$, then LWHM is called the linguistic weighted harmonic mean (LWHM) operator.

Especially, if $w_i = 1$ and $w_j = 0$, $j \neq i$, then $\text{LWHM}(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}) = s_{\alpha_i}$; if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then LWHM operator is called the linguistic harmonic mean (LHM) operator, i.e.,

$$\text{LHM}(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}) = \frac{n}{s_{\sum_{j=1}^n \frac{1}{\alpha_j}}}. \tag{2.8}$$

Example 2.1.5 Assume $w = (0.3, 0.1, 0.4, 0.2)^T$, then

$$\begin{aligned} \text{LWHM}_w(s_4, s_7, s_3, s_1) &= \frac{1}{\frac{0.3}{s_4} \oplus \frac{0.1}{s_7} \oplus \frac{0.4}{s_3} \oplus \frac{0.2}{s_1}} \\ &= \frac{1}{s_{\frac{0.3}{4}} \oplus s_{\frac{0.1}{7}} \oplus s_{\frac{0.4}{3}} \oplus s_{\frac{0.2}{1}}} \\ &= s_{2.36}. \end{aligned}$$

Theorem 2.1.6 (Boundedness)

$$\text{Min}_j(s_{\alpha_j}) \leq \text{LWHM}_w(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}) \leq \text{Max}_j(s_{\alpha_j})$$

Proof Let $\text{Max}_j(s_{\alpha_j}) = s_\beta$ and $\text{Min}_j(s_{\alpha_j}) = s_\alpha$, then

$$\begin{aligned} \text{LWHM}_w(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}) &= \frac{1}{s_{\frac{w_1}{\alpha_1}} \oplus s_{\frac{w_2}{\alpha_2}} \oplus \cdots \oplus s_{\frac{w_n}{\alpha_n}}} \\ &\leq \frac{1}{s_{\frac{w_1}{\beta}} \oplus s_{\frac{w_2}{\beta}} \oplus \cdots \oplus s_{\frac{w_n}{\beta}}} \\ &= \frac{1}{s_{\frac{\sum_{j=1}^n w_j}{\beta}}} \\ &= s_\beta, \end{aligned}$$

$$\text{LWHM}_w(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}) = \frac{1}{s_{\frac{w_1}{\alpha_1}} \oplus s_{\frac{w_2}{\alpha_2}} \oplus \cdots \oplus s_{\frac{w_n}{\alpha_n}}}$$

$$
\begin{aligned}
&\geq \frac{1}{s_{\frac{w_1}{\alpha}} \oplus s_{\frac{w_2}{\alpha}} \oplus \cdots \oplus s_{\frac{w_n}{\alpha}}} \\
&= \frac{1}{s_{\frac{\sum_{j=1}^{n} w_j}{\alpha}}} \\
&= s_{\alpha}.
\end{aligned}
$$

Hence

$$
\mathrm{Min}_j(s_{\alpha_j}) \leq \mathrm{LWHM}_w(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}) \leq \mathrm{Max}_j(s_{\alpha_j}).
$$

Based on the OWA and the LWHM operators and the operation law, we define a LOWHM operator as below:

Definition 2.1.7 A LOWHM operator of dimension n is a mapping LOWHM : $\bar{S}^n \to \bar{S}$, which has an associated vector $w = (w_1, w_2, \ldots, w_n)^T$ with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, such that

$$
\begin{aligned}
\mathrm{LOWHM}_w(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}) &= \frac{1}{\frac{w_1}{s_{\beta_1}} \oplus \frac{w_2}{s_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{s_{\beta_n}}} \\
&= \frac{1}{s_{\frac{w_1}{\beta_1}} \oplus s_{\frac{w_2}{\beta_2}} \oplus \cdots \oplus s_{\frac{w_n}{\beta_n}}} \\
&= \frac{1}{s_{\sum_{j=1}^{n} \frac{w_j}{\beta_j}}},
\end{aligned} \tag{2.9}
$$

where s_{β_j} is the jth largest of the s_{α_i}.

Especially, if there is a tie between s_{α_i} and s_{α_j}, then we replace each of s_{α_i} and s_{α_j} by their average $(s_{\alpha_i} \oplus s_{\alpha_j})/2$ in the process of aggregation. If k items are tied, then we replace these by k replicas of their average. The weighted vector $w = (w_1, w_2, \ldots, w_n)^T$ can be determined by using some weight determining methods like the normal distribution based method.

Example 2.1.8 Assume $w = (0.3, 0.1, 0.4, 0.2)^T$, then

$$
\mathrm{LOWHM}_w(s_4, s_7, s_3, s_1) = \frac{1}{\frac{0.3}{s_7} \oplus \frac{0.1}{s_4} \oplus \frac{0.4}{s_3} \oplus \frac{0.2}{s_1}}
$$

$$
\begin{aligned}
&= \frac{1}{s_{\frac{0.3}{7}} \oplus s_{\frac{0.1}{4}} \oplus s_{\frac{0.4}{3}} \oplus s_{\frac{0.2}{1}}} \\
&= s_{2.49}.
\end{aligned}
$$

In the following, let us look at some desirable properties associated with the LOWHM operator.

Theorem 2.1.9 (Commutativity)

$$\mathrm{LOWHM}_w(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}) = \mathrm{LOWHM}_w(s'_{\alpha_1}, s'_{\alpha_2}, \ldots, s'_{\alpha_n}),$$

where $(s'_{\alpha_1}, s'_{\alpha_2}, \ldots, s'_{\alpha_n})$ is a permutation of $(s_{\alpha_1},\ s_{\alpha_2},\ \ldots, s_{\alpha_n})$.

Proof Let

$$
\begin{aligned}
\mathrm{LOWHM}_w(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}) &= \frac{1}{\frac{w_1}{s_{\beta_1}} \oplus \frac{w_2}{s_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{s_{\beta_n}}}, \\
\mathrm{LOWHM}_w(s'_{\alpha_1}, s'_{\alpha_2}, \ldots, s'_{\alpha_n}) &= \frac{1}{\frac{w_1}{s'_{\beta_1}} \oplus \frac{w_2}{s'_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{s'_{\beta_n}}}.
\end{aligned}
$$

Since $(s'_{\alpha_1}, s'_{\alpha_2}, \ldots, s'_{\alpha_n})$ is a permutation of $(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n})$, we have $s_{\beta_j} = s'_{\beta_j}$ $(j = 1, 2, \ldots, n)$, then

$$\mathrm{LOWHM}_w(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}) = \mathrm{LOWHM}_w(s'_{\alpha_1}, s'_{\alpha_2}, \ldots, s'_{\alpha_n}).$$

Theorem 2.1.10 (Idempotency) If $s_{\alpha_j} = s_\alpha$, for all j, then

$$\mathrm{LOWHM}_w(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}) = s_\alpha.$$

Proof Since $s_{\alpha_j} = s_\alpha$, for all j, it follows that

$$
\begin{aligned}
\mathrm{LOWHM}_w(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}) &= \frac{1}{\frac{w_1}{s_{\beta_1}} \oplus \frac{w_2}{s_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{s_{\beta_n}}} \\
&= \frac{1}{s_{\frac{\sum_{j=1}^n w_j}{\alpha}}} \\
&= s_\alpha.
\end{aligned}
$$

Theorem 2.1.11 (Monotonicity) If $s_{\alpha_j} \leq s^*_{\alpha_j}$, for all j, then

$$\mathrm{LOWHM}_w(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}) \leq \mathrm{LOWHM}_w(s^*_{\alpha_1}, s^*_{\alpha_2}, \ldots, s^*_{\alpha_n}).$$

Proof Let

$$\mathrm{LOWHM}_w(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}) = \frac{1}{\frac{w_1}{s_{\beta_1}} \oplus \frac{w_2}{s_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{s_{\beta_n}}},$$

$$\mathrm{LOWHM}_w(s^*_{\alpha_1}, s^*_{\alpha_2}, \ldots, s^*_{\alpha_n}) = \frac{1}{\frac{w_1}{s^*_{\beta_1}} \oplus \frac{w_2}{s^*_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{s^*_{\beta_n}}}.$$

Since $s_{\alpha_j} \leq s^*_{\alpha_j}$, for all j, it follows that $s_{\beta_j} \leq s^*_{\beta_j}$, then

$$\mathrm{LOWHM}_w(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}) \leq \mathrm{LOWHM}_w(s^*_{\alpha_1}, s^*_{\alpha_2}, \ldots, s^*_{\alpha_n}).$$

Theorem 2.1.12 (Boundedness)

$$\mathrm{Min}_j(s_{\alpha_j}) \leq \mathrm{LOWHM}_w(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}) \leq \mathrm{Max}_j(s_{\alpha_j}).$$

Proof Let $\mathrm{Max}_j(s_{\alpha_j}) = s_\beta$ and $\mathrm{Min}_j(s_{\alpha_j}) = s_\alpha$, then

$$\begin{aligned}
\mathrm{LOWHM}_w(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}) &= \frac{1}{s_{\frac{w_1}{\beta_1}} \oplus s_{\frac{w_2}{\beta_2}} \oplus \cdots \oplus s_{\frac{w_n}{\beta_n}}} \\
&\leq \frac{1}{s_{\frac{w_1}{\beta}} \oplus s_{\frac{w_2}{\beta}} \oplus \cdots \oplus s_{\frac{w_n}{\beta}}} \\
&= \frac{1}{s_{\frac{\sum_{j=1}^n w_j}{\beta}}} \\
&= s_\beta,
\end{aligned}$$

$$\begin{aligned}
\mathrm{LOWHM}_w(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}) &= \frac{1}{s_{\frac{w_1}{\beta_1}} \oplus s_{\frac{w_2}{\beta_2}} \oplus \cdots \oplus s_{\frac{w_n}{\beta_n}}} \\
&\geq \frac{1}{s_{\frac{w_1}{\alpha}} \oplus s_{\frac{w_2}{\alpha}} \oplus \cdots \oplus s_{\frac{w_n}{\alpha}}} \\
&= \frac{1}{s_{\frac{\sum_{j=1}^n w_j}{\alpha}}} \\
&= s_\alpha.
\end{aligned}$$

Hence

$$\mathrm{Min}_j(s_{\alpha_j}) \leq \mathrm{LOWHM}_w(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}) \leq \mathrm{Max}_j(s_{\alpha_j}).$$

Especially, if the associated weighting vector $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the LOWHM operator is reduced to the LHM operator, i.e.,

$$\begin{aligned}\mathrm{LOWHM}_w(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}) &= \frac{1}{\frac{w_1}{s_{\beta_1}} \oplus \frac{w_2}{s_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{s_{\beta_n}}} \\ &= \frac{n}{s_{\frac{1}{\beta_1}} \oplus s_{\frac{1}{\beta_2}} \oplus \cdots \oplus s_{\frac{1}{\beta_n}}} \\ &= \frac{n}{s_{\sum_{j=1}^{n} \frac{1}{\alpha_j}}}.\end{aligned}$$

The LWHM operator weights the linguistic argument, while the LOWHM operator weights the ordered position of the linguistic argument instead of weighting the argument itself, weights represent different aspects in both the LWHM and the LOWHM operators. However, both the operators consider only one of them. To solve this drawback, in the following we shall propose a LHHM operator.

Definition 2.1.13 A LHHM operator of dimension n is a mapping LHHM : $\bar{S}^n \to \bar{S}$, which has an associated vector $w = (w_1, w_2, \ldots, w_n)^T$ with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$, such that

$$\begin{aligned}\mathrm{LHHM}_{\omega,w}(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}) &= \frac{1}{\frac{w_1}{s_{\beta_1}} \oplus \frac{w_2}{s_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{s_{\beta_n}}} \\ &= \frac{1}{s_{\sum_{j=1}^{n} \frac{w_j}{\beta_j}}}, \qquad (2.10)\end{aligned}$$

where s_{β_j} is the jth largest of the $\bar{s}_{\alpha_i}$ ($\bar{s}_{\alpha_i} = n\omega_i s_{\alpha_i}$, $i = 1, 2, \ldots, n$), $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of s_{α_j} ($j = 1, 2, \ldots, n$) with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$, and n is the balancing coefficient.

Example 2.1.14 Assume $\omega = (0.3, 0.1, 0.4, 0.2)^T$, $w = (0.35,\ 0.15, 0.20, 0.30)^T$, and

$$s_{\alpha_1} = s_4,\ s_{\alpha_2} = s_7,\ s_{\alpha_3} = s_2,\ s_{\alpha_4} = s_1.$$

By Definition 2.1.13, we have

$$\begin{aligned} \bar{s}_{\alpha_1} &= 4 \times 0.3 \times s_4 = s_{4.8},\ \bar{s}_{\alpha_2} = 4 \times 0.1 \times s_7 = s_{2.8}, \\ \bar{s}_{\alpha_3} &= 4 \times 0.4 \times s_2 = s_{3.2},\ \bar{s}_{\alpha_4} = 4 \times 0.2 \times s_1 = s_{0.8} \end{aligned}$$

and thus

$$s_{\beta_1} = s_{4.8},\ s_{\beta_2} = s_{3.2},\ s_{\beta_3} = s_{2.8},\ s_{\beta_4} = s_{0.8}.$$

Therefore,

$$\begin{aligned} \mathrm{LHHM}_{\omega,w}(s_4, s_7, s_2, s_1) &= \frac{1}{\frac{0.35}{s_{4.8}} \oplus \frac{0.15}{s_{3.2}} \oplus \frac{0.20}{s_{2.8}} \oplus \frac{0.30}{s_{0.8}}} \\ &= s_{1.77}. \end{aligned}$$

Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then $\bar{s}_{\alpha_i} = s_{\alpha_i}$, $i = 1, 2, \ldots, n$, in this case, the LHHM operator is reduced to the LOWHM operator; if $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the LHHM operator is reduced to LWHM operator. Thus, we know that the LHMM operator generalizes both the LWHM and LOWHM operators, and reflects the importance degrees of both the given argument and its ordered position.

2.2 A method for group decision making with linguistic preference relations

Based on the LHM and the LHHM operators, we develop a practical method for group decision making with linguistic preference relations as follows:

Step 1: For a group decision making problem with linguistic preference, let $X = \{x_1, x_2, \ldots, x_n\}$ be the set of alternatives and $D = \{d_1, d_2, \ldots, d_m\}$ be the set of decision-makers, and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)^T$ be the weight vector of decision-makers, where $\lambda_k \geq 0$ and $\sum_{k=1}^{m} \lambda_k = 1$. The decision-maker $d_k \in D$ compares these alternatives with respect to a single criterion by linguistic terms in the set $S = \{s_i\}$ $(i = 1, 2, \ldots, t)$, and constructs the linguistic preference relation $R_k = (r_{ij}^{(k)})_{n\times n}$, where the diagonal elements in R_k are expressed as "$-$", which mean "undefined", and $r_{ij}^{(k)} \oplus r_{ji}^{(k)} = s_t$, $i, j = 1, 2, \ldots, n$; $i \neq j$.

Step 2: Utilize the LHM operator

$$\begin{aligned} z_i^{(k)} &= \mathrm{LHM}(r_{i1}^{(k)}, r_{i2}^{(k)}, \ldots, r_{in}^{(k)}) \\ &= \frac{n-1}{\frac{1}{r_{i1}^{(k)}} \oplus \frac{1}{r_{i2}^{(k)}} \oplus \ldots \oplus \frac{1}{r_{in}^{(k)}}}, \quad i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, m \end{aligned}$$

to aggregate the preference information $r_{ij}^{(k)}$ $(i \neq j)$ in the ith line of R_k, and then get the preference degree $s_i^{(k)}$ of the ith alternative over all the other alternatives (corresponding to $d_k \in D$).

Step 3: Utilize the LHHM operator

$$z_i = \mathrm{LHHM}_w(z_i^{(1)}, z_i^{(2)}, \ldots, z_i^{(m)})$$

to aggregate $z_I^{(k)}$ $(k = 1, 2, \ldots, m)$ corresponding to the alternatives x_i, and then get the preference degree z_i of the ith alternative over all the other alternatives, where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)^T$ is the weight vector of decision-makers, where $\lambda_k \geq 0$ and $\sum_{k=1}^{m} \lambda_k = 1$; $w = (w_1, w_2, \ldots, w_m)^T$ is the associated weight vector of the LHHM operator with $w_k \in [0, 1]$ and $\sum_{k=1}^{m} w_k = 1$.

Step 4: Rank all the alternatives and select the optimal one(s) in accordance with the values of z_i $(i = 1, 2, \ldots, n)$.

Step 5: End.

2.3 Application I

In this section, we consider that a group decision making problem involves the evaluation of five schools x_i $(i = 1, 2, 3, 4, 5)$ of a university (adapted from [28]).

One main criterion used is research. There are three decision-makers d_k ($k = 1,2,3$), whose weight vector is $\lambda = (0.3, 0.4, 0.3)^T$. The decision-makers compare these five schools with respect to the criterion research by using the linguistic terms in the set $S = \{s_1 =$ extremely poor, $s_2 =$ very poor, $s_3 =$ poor, $s_4 =$ slightly poor, $s_5 =$ fair, $s_6 =$ slightly good, $s_7 =$ good, $s_8 =$ very good, $s_9 =$ extremely good$\}$, and construct, respectively, the linguistic preference relations R_k ($k = 1,2,3$) as listed in Tables 2.1-2.3.

Table 2.1: Linguistic preference relation R_1

	x_1	x_2	x_3	x_4	x_5
x_1	$-$	s_2	s_4	s_3	s_7
x_2	s_8	$-$	s_5	s_4	s_6
x_3	s_6	s_5	$-$	s_2	s_4
x_4	s_7	s_6	s_8	$-$	s_3
x_5	s_3	s_4	s_6	s_7	$-$

To get the best school(s), the following steps are involved:

Step 1: Utilize the LHM operator to aggregate the preference information in the ith line of the R_k ($k = 1,2,3$), and then get the preference degree $z_i^{(k)}$ of the

Table 2.2: Linguistic preference relation R_2

	x_1	x_2	x_3	x_4	x_5
x_1	$-$	s_3	s_4	s_6	s_5
x_2	s_7	$-$	s_7	s_4	s_5
x_3	s_6	s_3	$-$	s_4	s_6
x_4	s_4	s_6	s_6	$-$	s_4
x_5	s_5	s_5	s_4	s_6	$-$

Table 2.3: Linguistic preference relation R_3

	x_1	x_2	x_3	x_4	x_5
x_1	$-$	s_2	s_6	s_4	s_7
x_2	s_8	$-$	s_4	s_3	s_4
x_3	s_4	s_6	$-$	s_5	s_7
x_4	s_6	s_7	s_5	$-$	s_3
x_5	s_3	s_6	s_3	s_7	$-$

ith school over all the other schools:

$$z_1^{(1)} = s_{3.26},\ z_1^{(2)} = s_{4.21},\ z_1^{(3)} = s_{3.77},$$
$$z_2^{(1)} = s_{5.40},\ z_2^{(2)} = s_{5.43},\ z_2^{(3)} = s_{4.18},$$
$$z_3^{(1)} = s_{3.58},\ z_3^{(2)} = s_{4.36},\ z_3^{(3)} = s_{5.27},$$
$$z_4^{(1)} = s_{5.21},\ z_4^{(2)} = s_{4.80},\ z_4^{(3)} = s_{4.74},$$
$$z_5^{(1)} = s_{4.48},\ z_5^{(2)} = s_{4.90},\ z_5^{(3)} = s_{4.10}.$$

Step 2: Utilize the LHHM operator (whose weight vector $w = (0.3, 0.4, 0.3)^T$) to aggregate $z_i^{(k)}$ $(k = 1, 2, 3)$ corresponding to the school x_i, and then get the preference degree z_i of the ith school over all the other schools:

$$z_1 = s_{3.57},\ z_2 = s_{4.76},\ z_3 = s_{4.35},\ z_4 = s_{5.00},$$
$$z_5 = s_{4.55}.$$

Step 3: Utilize the values of z_i $(i = 1, 2, 3, 4, 5)$ to rank the schools:

$$x_4 \succ x_2 \succ x_5 \succ x_3 \succ x_1$$

and thus the best school is x_4.

2.4 Application II

In this section, a new advanced manufacturing technology (AMT) selection method using linguistic multiple attributes analysis as well as group decision making is proposed.

2.4.1 Approach to AMT selection

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of n feasible alternatives (courses of AMT), and $G = \{G_1, G_2, \ldots, G_m\}$ be a set of m attributes, whose weight vector is $w = (w_1, w_2, \ldots, w_m)^T$, where $w_i \geq 0$ and $\sum_{i=1}^{m} w_i = 1$, and let $D = \{d_1, d_2, \ldots, d_l\}$ be the set of decision-makers, whose weight vector is $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)^T$, where $\lambda_k \geq 0$ and $\sum_{k=1}^{l} \lambda_k = 1$. The decision maker $d_k \in D$ may provide the linguistic decision matrix $R_k = (r_{ij}^{(k)})_{m\times n}$, where $r_{ij}^{(k)}$ is a performance rating (attribute value), which takes the form of linguistic variable, of the alternative $x_j \in X$ with respect to the attribute $G_i \in G$ for all $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$; $k = 1, 2, \ldots, l$.

In the following, based on the LWHM and LHHM operators, we shall develop a multiple attribute decision making applied in the group decision making to improving advanced manufacturing technology selection process.

Step 1: Utilize the LWHM operator:

$$\begin{aligned} r_j^{(k)} &= \mathrm{LWHM}_w(r_{1j}^{(k)}, r_{2j}^{(k)}, \ldots, r_{mj}^{(k)}) \\ &= \frac{1}{\frac{w_1}{r_{1j}^{(k)}} \oplus \frac{w_2}{r_{2j}^{(k)}} \oplus \cdots \oplus \frac{w_m}{r_{mj}^{(k)}}} \end{aligned}$$

to aggregate all the elements in the jth column of R_k and get the overall attribute value $r_j^{(k)}$ of the alternative x_j corresponding to the decision maker d_k.

Step 2: Utilize the LHHM operator:

$$\begin{aligned} r_j &= \mathrm{LHHM}_\omega(r_j^{(1)}, r_j^{(2)}, \ldots, r_j^{(l)}) \\ &= \frac{1}{\frac{\omega_1}{\dot{r}_j^{\sigma(1)}} \oplus \frac{\omega_2}{\dot{r}_j^{\sigma(2)}} \oplus \cdots \oplus \frac{\omega_l}{\dot{r}_j^{\sigma(l)}}} \end{aligned}$$

to aggregate the overall attribute values $r_j^{(k)}$ $(k = 1, 2, \ldots, l)$ corresponding to the decision-maker d_k $(k = 1, 2, \ldots, l)$ and get the collective overall attribute value r_j, where $\dot{r}_j^{\sigma(k)}$ is the kth largest of the weighted data $\dot{r}_j^{(k)}$ $(\dot{r}_j^{(k)} = l\lambda_k r_j^{(k)},\ k = 1, 2, \ldots, l)$, $\omega = (\omega_1, \omega_2, \ldots, \omega_l)^T$ is the weighting vector of the LHHM operator with $\omega_k \geq 0$ and $\sum_{k=1}^{l} \omega_k = 1$.

Step 3: Rank all the alternatives x_j $(j = 1, 2, \ldots, n)$, and then select the most desirable one in accordance with the collective overall preference values r_j $(j = 1, 2, , \ldots, n)$.

Step 4: End.

2.4.2 Practical example

The following practical case was adapted from [4]. Due to increasing customization, a leading Taiwan firm in the bicycle industry needs a flexible manufacturing system (FMS) to produce a customized bike, which is designing for customer's requirements. After performing task analysis, it has been identified that this system should be produce mountain bikes and road racing bikes for a customized order. After preliminary screening, three competing alternatives, x_1, x_2 and x_3 are identified that are capable of performing this production task. A committee of three decision-makers, d_1, d_2 and d_3 has been formed to conduct further evaluation and to select the most suitable FMS. The attributes which are considered here in assessment of x_j $(j = 1, 2, 3)$ are: (1) G_1 is process flexibility; (2) G_2 is product quality; (3) G_3 is learning; (4) G_4 is exposure to labor unrest. The decision-maker d_k $(k = 1, 2, 3)$ evaluates the performance of FMS x_j $(j = 1, 2, 3)$ according to the attributes G_i $(j = 1, 2, 3, 4)$ by using the linguistic terms in the set

$$\begin{aligned} S \;=\; & \{s_1 = \text{extremely low}, s_2 = \text{very low}, s_3 = \text{low}, \\ & s_4 = \text{slightly low}, s_5 = \text{middle}, s_6 = \text{slightly high}, \\ & s_7 = \text{high}, s_8 = \text{very high}, s_9 = \text{extremely high}\}. \end{aligned}$$

and constructs, respectively, the linguistic decision matrix R_k ($k = 1, 2, 3$) as listed in Tables 2.4-2.6. Let $w = (0.35, 0.15, 0.20, 0.30)^T$ be the weight vector of the attributes G_i ($i = 1, 2, 3, 4$), and $\lambda = (0.3, 0.4, 0.3)^T$ be the weight vector of the decision-makers d_k ($k = 1, 2, 3$).

Table 2.4: Linguistic decision matrix R_1

	x_1	x_2	x_3
G_1	s_6	s_3	s_7
G_2	s_5	s_6	s_4
G_3	s_7	s_6	s_5
G_4	s_4	s_6	s_5

Table 2.5: Linguistic decision matrix R_2

	x_1	x_2	x_3
G_1	s_5	s_5	s_7
G_2	s_6	s_3	s_8
G_3	s_2	s_7	s_7
G_4	s_5	s_6	s_5

Table 2.6: Linguistic decision matrix R_3

	x_1	x_2	x_3
G_1	s_7	s_5	s_6
G_2	s_4	s_7	s_7
G_3	s_3	s_8	s_7
G_4	s_3	s_6	s_6

To get the best alternative(s), the following steps are involved:

Step 1: Utilize the LWHM operator to aggregate all the elements in the jth column of R_k and get the overall attribute value $r_j^{(k)}$:

$$r_1^{(1)} = s_{5.21},\ r_2^{(1)} = s_{4.44},\ r_3^{(1)} = s_{5.33},$$
$$r_1^{(2)} = s_{3.92},\ r_2^{(2)} = s_{5.04},\ r_3^{(2)} = s_{6.36},$$
$$r_1^{(3)} = s_{3.93},\ r_2^{(3)} = s_{6.01},\ r_3^{(3)} = s_{6.32}.$$

Step 2: Utilize the LHHM operator (suppose that its weight vector is $\omega = (0.2, 0.5, 0.3)^T$) to aggregate the overall attribute values $r_j^{(k)}$ $(k = 1, 2, 3)$ corresponding to the decision maker d_k $(k = 1, 2, 3)$, and get the collective overall attribute value r_j $(j = 1, 2, 3)$:

$$r_1 = s_{4.28},\ r_2 = s_{4.99},\ r_3 = s_{5.96}.$$

Step 3: Utilize the values of r_j $(j = 1, 2, 3)$ to rank the alternatives:

$$x_3 \succ x_2 \succ x_1$$

and thus the best alternative is x_3.

2.5 Conclusions

In this chapter, we have defined operational law of linguistic variables and developed some new aggregation operators including the LHM, the LWHM, the LOWHM and LHHM operators, which can be utilized to aggregate preference information taking the form of linguistic variables. Based on the LHM and the LHHM operators, we have proposed a practical method for group decision making with linguistic preference relations. Theoretical analysis and the numerical results show that the method is straightforward and has no loss of information. Moreover, a new AMT selection method using linguistic multiple attributes analysis as well as group decision making is proposed. In the future, we shall continue working in the application and extension of the LWHM operator in other domain.

Chapter 3

Uncertain linguistic harmonic mean operators and their applications to multiple attribute group decision making

In this chapter, some uncertain linguistic aggregation operators called uncertain linguistic weighted harmonic mean (ULWHM) operator, uncertain linguistic ordered weighted harmonic mean (ULOWHM) operator and uncertain linguistic hybrid harmonic mean (ULHHM) operator are proposed. An approach to multiple attribute group decision making with uncertain linguistic information is developed based on the ULWHM and the ULHHM operators. Finally, a practical application of the developed approach to multiple attribute group decision making problem with uncertain linguistic information is given.

3.1 Some operational laws of uncertain linguistic variables

Let $\tilde{s} = [s_\alpha, s_\beta]$, where $s_\alpha, s_\beta \in \bar{S}$, s_α and s_β are the lower and upper limits, respectively. We call $\tilde{s}$ the uncertain linguistic variables. Let $\tilde{S}$ be the set of all the uncertain linguistic variables.

Consider any three uncertain linguistic variables $\tilde{s} = [s_\alpha, s_\beta]$, $\tilde{s}_1 = [s_{\alpha_1}, s_{\beta_1}]$ and $\tilde{s}_2 = [s_{\alpha_2}, s_{\beta_2}]$, and let $\lambda \in [0, 1]$, then we define their operations as follows:

(1) $\tilde{s}_1 \oplus \tilde{s}_2 = [s_{\alpha_1}, s_{\beta_1}] \oplus [s_{\alpha_2}, s_{\beta_2}] = [s_{\alpha_1} \oplus s_{\alpha_2}, s_{\beta_1} \oplus s_{\beta_2}] = [s_{\alpha_1+\alpha_2}, s_{\beta_1+\beta_2}]$;

(2) $\lambda\tilde{s} = \lambda[s_\alpha, s_\beta] = [\lambda s_\alpha, \lambda s_\beta] = [s_{\lambda\alpha}, s_{\lambda\beta}]$;

(3) $\frac{1}{\tilde{s}} = \frac{1}{[s_\alpha, s_\beta]} = [\frac{1}{s_\beta}, \frac{1}{s_\alpha}] = [s_{\frac{1}{\beta}}, s_{\frac{1}{\alpha}}]$.

In order to compare uncertain linguistic variables, Xu [32] provided the following definition:

Definition 3.1.1 Let $\tilde{s}_1 = [s_{\alpha_1}, s_{\beta_1}]$ and $\tilde{s}_2 = [s_{\alpha_2}, s_{\beta_2}]$ be two uncertain linguistic variables, and let $\text{len}(\tilde{s}_1) = \beta_1 - \alpha_1$ and $\text{len}(\tilde{s}_2) = \beta_2 - \alpha_2$, then the degree of possibility of $\tilde{s}_1 \geq \tilde{s}_2$ is defined as

$$p(\tilde{s}_1 \geq \tilde{s}_2) = \frac{\max\{0, \text{len}(\tilde{s}_1) + \text{len}(\tilde{s}_2) - \max(\beta_2 - \alpha_1, 0)\}}{\text{len}(\tilde{s}_1) + \text{len}(\tilde{s}_2)} \tag{3.1}$$

From Definition 3.1.1, we can easily get the following results:

(1) $0 \leq p(\tilde{s}_1 \geq \tilde{s}_2) \leq 1$, $0 \leq p(\tilde{s}_2 \geq \tilde{s}_1) \leq 1$;

(2) $p(\tilde{s}_1 \geq \tilde{s}_2) + p(\tilde{s}_2 \geq \tilde{s}_1) = 1$. Especially, $p(\tilde{s}_1 \geq \tilde{s}_1) = p(\tilde{s}_2 \geq \tilde{s}_2) = \frac{1}{2}$.

Wei and Yi [24] introduced the concept of fuzzy triangular linguistic variable as follow:

Definition 3.1.2 Let $\hat{s} = (s_\alpha, s_\beta, s_\gamma)$, where $s_\alpha, s_\beta, s_\gamma \in \bar{S}$, s_α,s_β, and s_γ are the lower, modal and upper values of $\hat{s}$, respectively, then we called $\hat{s}$ a triangular fuzzy linguistic variable, which characterized by the following membership function:

$$\mu_{\hat{s}}(s_\theta) = \begin{cases} 0, & s_1 \leq s_\theta \leq s_\alpha, \\ \frac{d(s_\theta, s_\alpha)}{d(s_\beta, s_\alpha)}, & s_\alpha \leq s_\theta \leq s_\beta, \\ \frac{d(s_\theta, s_\gamma)}{d(s_\beta, s_\gamma)}, & s_\beta \leq s_\theta \leq s_\gamma, \\ 0, & s_\gamma \leq s_\theta \leq s_t, \end{cases} \tag{3.2}$$

where $d(s_\alpha, s_\beta) = |\beta - \alpha|$ is the distance between s_α and s_β.

Clearly, s_β gives the maximal grade of $\mu_{\hat{s}}(s_\theta)$ ($\mu_{\hat{s}}(s_\beta) = 1$), s_α and s_γ are the lower and upper bounds with limit in the field of possible evaluation. If $s_\alpha = s_\beta = s_\gamma$, then $\hat{s}$ is reduced to a linguistic variable. If $s_\alpha = s_\beta$ or $s_\beta = s_\gamma$, then $\hat{s}$ is reduced to an uncertain linguistic variable.

In the following, Wei and Yi [24] introduced a formula for comparing triangular fuzzy linguistic variables.

Definition 3.1.3 Let $\hat{s}_1 = (s_{\alpha_1}, s_{\beta_1}, s_{\gamma_1})$ and $\hat{s}_2 = (s_{\alpha_2}, s_{\beta_2}, s_{\gamma_2})$ be any two triangular fuzzy linguistic variables, then the degree of possibility of $\hat{s}_1 \geq \hat{s}_2$ is defined as

$$\begin{aligned} p(\hat{s}_1 \geq \hat{s}_2) = & \lambda \max\left\{1 - \max\left[\frac{d(s_{\beta_2}, s_{\alpha_1})}{d(s_{\beta_1}, s_{\alpha_1}) + d(s_{\beta_2}, s_{\alpha_2})}, 0\right], 0\right\} \\ & +(1-\lambda)\max\left\{1 - \max\left[\frac{d(s_{\gamma_2}, s_{\beta_1})}{d(s_{\gamma_1}, s_{\beta_1}) + d(s_{\gamma_2}, s_{\beta_2})}, 0\right], 0\right\} \end{aligned} \tag{3.3}$$

Definition 3.1.4 The α-cut of a triangular fuzzy linguistic variable is a subset of $\bar{S}$ and is denoted by

$$[\hat{s}]_\alpha = \{s_\theta \in \bar{S} : \mu_{\hat{s}}(s_\theta) \geq \alpha\}, \tag{3.4}$$

where $\mu_{\hat{s}}(s_\theta)$ is the membership function of $\hat{s}$ and $\alpha \in [0, 1]$.

The lower and upper points of any α-cut, $[\hat{s}]_\alpha$, are represented by $[\hat{s}]_\alpha^L$ and $[\hat{s}]_\alpha^U$, respectively, and suppose that both are finite.

Remark 3.1.5 If $\hat{s} = [[\hat{s}]_\alpha^L, [\hat{s}]_\alpha^U]$, then by choosing $\alpha = 1$ we can identify the modal value of $\hat{s}$, and by $\alpha = 0$ we can identify the lower and upper values of $\hat{s}$.

3.2 Some new uncertain linguistic aggregation operators

Definition 3.2.1 Let ULHM : $\tilde{S}^n \to \tilde{S}$, if

$$\mathrm{ULHM}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \frac{n}{\frac{1}{\tilde{s}_1} \oplus \frac{1}{\tilde{s}_2} \oplus \cdots \oplus \frac{1}{\tilde{s}_n}} \tag{3.5}$$

where $\tilde{s} \in \tilde{S}$, $i = 1, 2, \ldots, n$, then ULHM is called the uncertain linguistic harmonic mean (ULHM) operator.

Example 3.2.2 Given the collection of uncertain linguistic variables: $\tilde{s}_1 = [s_2, s_3]$, $\tilde{s}_2 = [s_1, s_2]$, $\tilde{s}_3 = [s_3, s_4]$, $\tilde{s}_4 = [s_4, s_5]$, then by (3.5) and the operational laws of uncertain linguistic variables, we have

$$\begin{aligned} \mathrm{ULHM}(\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4) &= \frac{4}{\frac{1}{\tilde{s}_1} \oplus \frac{1}{\tilde{s}_2} \oplus \frac{1}{\tilde{s}_3} \oplus \frac{1}{\tilde{s}_4}} \\ &= \frac{4}{\frac{1}{[s_2,s_3]} \oplus \frac{1}{[s_1,s_2]} \oplus \frac{1}{[s_3,s_4]} \oplus \frac{1}{[s_4,s_5]}} \\ &= [s_{1.92}, s_{3.13}]. \end{aligned}$$

Definition 3.2.3 Let ULWHM : $\tilde{S}^n \to \tilde{S}$, if

$$\mathrm{ULWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \frac{1}{\frac{w_1}{\tilde{s}_1} \oplus \frac{w_2}{\tilde{s}_2} \oplus \cdots \oplus \frac{w_n}{\tilde{s}_n}}, \tag{3.6}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the weighting vector of $\tilde{s}_j$ $(j = 1, 2, \ldots, n)$, with $w_j \geq 0$ and $\sum_{j=1}^n w_j = 1$, then ULWHM is called the uncertain linguistic weighted harmonic mean (ULWHM) operator.

Especially, if $w_i = 1$, $w_j = 0$, $j \neq i$, then $\mathrm{ULWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \tilde{s}_i$; if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the ULWHM operator is reduced to the ULHM operator. Furthermore, the ULWHM operator has the following property similar to that of the LWHM operator:

$$\min_j(\tilde{s}_j) \leq \mathrm{ULWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \leq \max_j(\tilde{s}_j).$$

Example 3.2.4 Given the collection of uncertain linguistic variables: $\tilde{s}_1 = [s_2, s_3]$, $\tilde{s}_2 = [s_1, s_2]$, $\tilde{s}_3 = [s_3, s_4]$, $\tilde{s}_4 = [s_4, s_5]$, and let $w = (0.3, 0.2, 0.3, 0.2)^T$ be the weight vector of $\tilde{s}_j$ $(j = 1, 2, 3, 4)$, then by (3.6), we have

$$\begin{aligned}\mathrm{ULWHM}_w(\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4) &= \frac{1}{\frac{0.3}{\tilde{s}_1} \oplus \frac{0.2}{\tilde{s}_2} \oplus \frac{0.3}{\tilde{s}_3} \oplus \frac{0.2}{\tilde{s}_4}} \\ &= \frac{1}{\frac{0.3}{[s_2,s_3]} \oplus \frac{0.2}{[s_1,s_2]} \oplus \frac{0.3}{[s_3,s_4]} \oplus \frac{0.2}{[s_4,s_5]}} \\ &= [s_{2.00}, s_{3.17}].\end{aligned}$$

Definition 3.2.5 An uncertain linguistic ordered weighted harmonic mean (ULOWHM) operator of dimension n is a mapping $\mathrm{ULOWHM} : \tilde{S}^n \to \tilde{S}$, which has associated weighting vector $w = (w_1, w_2, \ldots, w_n)^T$ such that $w_j \in [0, 1]$, $j = 1, 2, \ldots, n$, and $\sum_{j=1}^n w_j = 1$. Furthermore:

$$\mathrm{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \frac{1}{\frac{w_1}{\tilde{s}_{\beta_1}} \oplus \frac{w_2}{\tilde{s}_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{\tilde{s}_{\beta_n}}}, \tag{3.7}$$

where $\tilde{s}_{\beta_j}$ is the jth largest of the $\tilde{s}_i \in \tilde{S}$.

Especially, if $w_i = 1$, $w_j = 0$, $j \neq i$, then $\mathrm{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \tilde{s}_i$; if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the ULOWHM operator is reduced to the ULHM operator. The weighting vector $w = (w_1, w_2, \ldots, w_n)^T$ can be determined by using some weight determining methods like the normal distribution based method (see, Refs.33, 34, 40 for more details).

To rank these arguments $\tilde{s}_i$ $(i = 1, 2, \ldots, n)$, we first compared each argument $\tilde{s}_i$ with all arguments $\tilde{s}_j$ $(j = 1, 2, \ldots, n)$ by using (3.1), and let $p_{ij} = p(\tilde{s}_i \geq \tilde{s}_j)$.

Then we construct a complementary matrix $\mathbf{P} = (p_{ij})_{n\times n}$ where:

$$p_{ij} \geq 0, p_{ij} + p_{ji} = 1, p_{ii} = \frac{1}{2},\ i, j = 1, 2, \ldots, n.$$

Summing all elements in each line of matrix $\mathbf{P}$, we have $p_i = \sum_{j=1}^{n} p_{ij},\ i = 1, 2, \ldots, n$. Then, in accordance with the values of p_i $(i = 1, 2, \ldots, n)$, we rank the arguments $\tilde{s}_i$ $(i = 1, 2, \ldots, n)$ in descending order.

Example 3.2.6 Given the collection of uncertain linguistic variables: $\tilde{s}_1 = [s_2, s_3]$, $\tilde{s}_2 = [s_1, s_3]$, $\tilde{s}_3 = [s_2, s_4]$, $\tilde{s}_4 = [s_3, s_4]$. To rank these arguments, we first compare each argument $\tilde{s}_i$ with all arguments $\tilde{s}_j$ $(j = 1, 2, \ldots, n)$ by using (3.1), let $p_{ij} = p(\tilde{s}_i \geq \tilde{s}_j)$ $(j = 1, 2, 3, 4)$, then we utilize these possibility degrees to construct the following matrix $\mathbf{P} = (p_{ij})_{4\times 4}$:

$$\mathbf{P} = \begin{pmatrix} 0.500 & 0.667 & 0.333 & 0.000 \\ 0.333 & 0.500 & 0.250 & 0.000 \\ 0.667 & 0.750 & 0.500 & 0.333 \\ 1.000 & 1.000 & 0.667 & 0.500 \end{pmatrix}.$$

Summing all elements in each line of matrix $\mathbf{P}$, we have

$$p_1 = 1.500,\ p_2 = 1.083,\ p_3 = 2.250,\ p_4 = 3.167.$$

Then we rank the arguments $\tilde{s}_i$ $(i = 1, 2, 3, 4)$ in descending order in accordance with the values of p_i $(i = 1, 2, 3, 4)$:

$$\tilde{s}_{\beta_1} = \tilde{s}_4 = [s_3, s_4],\ \tilde{s}_{\beta_2} = \tilde{s}_3 = [s_2, s_4],\ \tilde{s}_{\beta_3} = \tilde{s}_1 = [s_2, s_3],\ \tilde{s}_{\beta_4} = \tilde{s}_2 = [s_1, s_3].$$

Suppose that the weighting vector $w = (w_1, w_2, w_3, w_4)^T$ of the ULOWHM operator is $w = (0.3, 0.2, 0.3, 0.2)^T$, then by (3.7), we get

$$\begin{aligned} \mathrm{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4) &= \frac{1}{\frac{0.3}{\tilde{s}_4} \oplus \frac{0.2}{\tilde{s}_3} \oplus \frac{0.3}{\tilde{s}_1} \oplus \frac{0.2}{\tilde{s}_2}} \\ &= \frac{1}{\frac{0.3}{[s_3,s_4]} \oplus \frac{0.2}{[s_2,s_4]} \oplus \frac{0.3}{[s_2,s_3]} \oplus \frac{0.2}{[s_1,s_3]}} \\ &= [s_{1.82}, s_{3.42}]. \end{aligned}$$

Based on Definition 2.1.7, we have the following properties of the ULOWHM operator:

Theorem 3.2.7 Let $\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n$ be a collection of uncertain linguistic variables and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of the ULOWHM operator with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$; then we have the following.

(1) (Idempotency): If all $\tilde{s}_j$ $(j = 1, 2, \ldots, n)$ are equal, i.e., $\tilde{s}_j = \tilde{s}$ for all j, then

$$\mathrm{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \tilde{s}.$$

(2) (Boundedness):

$$\min_j(\tilde{s}_j) \leq \mathrm{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \leq \max_j(\tilde{s}_j).$$

(3) (Monotonicity): If $\tilde{s}_j \leq \tilde{s}_j^*$, for all j, then

$$\mathrm{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \leq \mathrm{ULOWHM}_w(\tilde{s}_1^*, \tilde{s}_2^*, \ldots, \tilde{s}_n^*).$$

(4) (Commutativity): If $(\tilde{s}'_1, \tilde{s}'_2, \ldots, \tilde{s}'_n)$ is a permutation of $(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n)$, then

$$\mathrm{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \mathrm{ULOWHM}_w(\tilde{s}'_1, \tilde{s}'_2, \ldots, \tilde{s}'_n).$$

Proof (1) Since $\tilde{s}_j = \tilde{s}$, for all i, we have

$$\begin{aligned}\mathrm{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) &= \frac{1}{\frac{w_1}{\tilde{s}_{\beta_1}} \oplus \frac{w_2}{\tilde{s}_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{\tilde{s}_{\beta_n}}} \\ &= \frac{1}{\frac{w_1}{\tilde{s}} \oplus \frac{w_2}{\tilde{s}} \oplus \cdots \oplus \frac{w_n}{\tilde{s}}} \\ &= \tilde{s}.\end{aligned}$$

(2) Let $\max_j(\tilde{s}_j) = \tilde{s}_k$ and $\min_j(\tilde{s}_j) = \tilde{s}_l$, then

$$\begin{aligned}\mathrm{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) &= \frac{1}{\frac{w_1}{\tilde{s}_{\beta_1}} \oplus \frac{w_2}{\tilde{s}_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{\tilde{s}_{\beta_n}}} \\ &\leq \frac{1}{\frac{w_1}{\tilde{s}_k} \oplus \frac{w_2}{\tilde{s}_k} \oplus \cdots \oplus \frac{w_n}{\tilde{s}_k}} \\ &= \tilde{s}_k,\end{aligned}$$

$$
\begin{aligned}
\text{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) &= \frac{1}{\frac{w_1}{\tilde{s}_{\beta_1}} \oplus \frac{w_2}{\tilde{s}_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{\tilde{s}_{\beta_n}}} \\
&\geq \frac{1}{\frac{w_1}{\tilde{s}_l} \oplus \frac{w_2}{\tilde{s}_l} \oplus \cdots \oplus \frac{w_n}{\tilde{s}_l}} \\
&= \tilde{s}_l.
\end{aligned}
$$

Hence

$$
\min_j(\tilde{s}_j) \leq \text{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \leq \max_j(\tilde{s}_j).
$$

(3) Since $\tilde{s}_j \leq \tilde{s}_j^*$, for all j, it follows that $\tilde{s}_{\beta_j} \leq \tilde{s}^*_{\beta_j}$, then

$$
\begin{aligned}
\text{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) &= \frac{1}{\frac{w_1}{\tilde{s}_{\beta_1}} \oplus \frac{w_2}{\tilde{s}_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{\tilde{s}_{\beta_n}}} \\
&\leq \frac{1}{\frac{w_1}{\tilde{s}^*_{\beta_1}} \oplus \frac{w_2}{\tilde{2}^*_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{\tilde{s}^*_{\beta_n}}} \\
&= \text{ULOWHM}_w(\tilde{s}_1^*, \tilde{s}_2^*, \ldots, \tilde{s}_n^*).
\end{aligned}
$$

(4) Since $(\tilde{s}'_1, \tilde{s}'_2, \ldots, \tilde{s}'_n)$ is a permutation of $(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n)$, we have $\tilde{s}_{\beta_j} = \tilde{s}'_{\beta_j}$, for all j, then

$$
\begin{aligned}
\text{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) &= \frac{1}{\frac{w_1}{\tilde{s}_{\beta_1}} \oplus \frac{w_2}{\tilde{s}_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{\tilde{s}_{\beta_n}}} \\
&= \frac{1}{\frac{w_1}{\tilde{s}'_{\beta_1}} \oplus \frac{w_2}{\tilde{s}'_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{\tilde{s}'_{\beta_n}}} \\
&= \text{ULOWHM}_w(\tilde{s}'_1, \tilde{s}'_2, \ldots, \tilde{s}'_n).
\end{aligned}
$$

Besides the above properties, the ULOWHM operator has the following desirable results.

Theorem 3.2.8 Let $\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n$ be a collection of uncertain linguistic variables and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of the ULOWHM operator with $w_j \geq 0$ and $\sum_{j=1}^n w_j = 1$; then we have the following.

(1) If $w = (1, 0, \dots, 0)^T$, then

$$\text{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_n) = \max_j(\tilde{s}_j).$$

(2) If $w = (0, 0, \dots, 1)^T$, then

$$\text{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_n) = \min_j(\tilde{s}_j).$$

(3) If $w_j = 1$ and $w_i = 0$ $(i \neq j)$, then

$$\text{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_n) = \tilde{s}_{\beta_j},$$

where $\tilde{s}_{\beta_j}$ is the jth largest of $\tilde{s}_j$ $(j = 1, 2, \dots, n)$.

Proof (1) Since $w = (1, 0, \dots, 0)^T$, we have

$$\begin{aligned}\text{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_n) &= \frac{1}{\frac{1}{\tilde{s}_{\beta_1}} \oplus \frac{0}{\tilde{s}_{\beta_2}} \oplus \cdots \oplus \frac{0}{\tilde{s}_{\beta_n}}}\\ &= \tilde{s}_{\beta_1}\\ &= \max_j(\tilde{s}_j).\end{aligned}$$

(2) Since $w = (0, 0, \dots, 1)^T$, we have

$$\begin{aligned}\text{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_n) &= \frac{1}{\frac{0}{\tilde{s}_{\beta_1}} \oplus \frac{0}{\tilde{s}_{\beta_2}} \oplus \cdots \oplus \frac{1}{\tilde{s}_{\beta_n}}}\\ &= \tilde{s}_{\beta_n}\\ &= \min_j(\tilde{s}_j).\end{aligned}$$

(3) Since $w_j = 1$ and $w_i = 0$ $(i \neq j)$, we have

$$\begin{aligned}\text{ULOWHM}_w(\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_n) &= \frac{1}{\frac{0}{\tilde{s}_{\beta_1}} \oplus \frac{0}{\tilde{s}_{\beta_2}} \oplus \cdots \oplus \frac{1}{\tilde{s}_{\beta_j}} \oplus \cdots \oplus \frac{0}{\tilde{s}_{\beta_n}}}\\ &= \tilde{s}_{\beta_j}.\end{aligned}$$

Clearly, the fundamental characteristic of the ULWHM operator is that it considers the importance of each given uncertain linguistic variable, whereas the fundamental characteristic of the ULOWHM operator is the reordering step, and it weights all the ordered positions of uncertain linguistic variables instead of weighting the given uncertain linguistic variables themselves. In the following, by combining the advantages of the ULWHM and ULOWHM operators, we develop a ULHHM operator that weights both the given uncertain linguistic variables and their ordered positions.

Definition 3.2.9 An uncertain linguistic hybrid harmonic mean (ULHHM) operator of dimension n is a mapping ULHHM : $\tilde{S}^n \to \tilde{S}$, which has associated weighting vector $w = (w_1, w_2, \ldots, w_n)^T$ such that $w_j \in [0, 1]$, $j = 1, 2, \ldots, n$, and $\sum_{j=1}^{n} w_j = 1$, such that

$$\mathrm{ULHHM}_{\omega,w}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = \frac{1}{\frac{w_1}{\tilde{s}_{\beta_1}} \oplus \frac{w_2}{\tilde{s}_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{\tilde{s}_{\beta_n}}}, \tag{3.8}$$

where $\tilde{s}_{\beta_j}$ is the jth largest of the weighted uncertain linguistic variables $\dot{\tilde{s}}_i$ ($\dot{\tilde{s}}_i = n\omega_i \tilde{s}_i$, $i = 1, 2, \ldots, n$), $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $\tilde{s}_i$ ($i = 1, 2, \ldots, n$) with $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n} \omega_i = 1$, and n is the balancing coefficient.

Especially, if $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then $\dot{\tilde{s}}_i = \tilde{s}_i$, $i = 1, 2, \ldots, n$, in this case, the ULHHM operator is reduced to ULOWHM operator; if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the ULHHM operator is reduced to ULWHM operator.

Example 3.2.10 Given the collection of uncertain linguistic variables: $\tilde{s}_1 = [s_2, s_3]$, $\tilde{s}_2 = [s_1, s_3]$, $\tilde{s}_3 = [s_2, s_4]$, $\tilde{s}_4 = [s_3, s_4]$, and let $w = (0.3, 0.2, 0.3, 0.2)^T$ be the weight vector of $\tilde{s}_j$ ($j = 1, 2, 3, 4$). Then we get the weighted uncertain linguistic variables:

$$\begin{aligned}
\dot{\tilde{s}}_1 &= 4 \times 0.3 \times [s_2, s_3] = [s_{2.4}, s_{3.6}], \\
\dot{\tilde{s}}_2 &= 4 \times 0.2 \times [s_1, s_3] = [s_{0.8}, s_{2.4}], \\
\dot{\tilde{s}}_3 &= 4 \times 0.3 \times [s_2, s_4] = [s_{2.4}, s_{4.8}], \\
\dot{\tilde{s}}_4 &= 4 \times 0.2 \times [s_3, s_4] = [s_{2.4}, s_{3.2}].
\end{aligned}$$

By using (3.1), we construct the following matrix:

$$\mathbf{P} = \begin{pmatrix} 0.500 & 1.000 & 0.333 & 0.600 \\ 0.000 & 0.500 & 0.000 & 0.000 \\ 0.667 & 1.000 & 0.500 & 0.750 \\ 0.400 & 1.000 & 0.250 & 0.500 \end{pmatrix}.$$

Summing all elements in each line of matrix $\mathbf{P}$, we have

$$p_1 = 2.433,\ p_2 = 0.500,\ p_3 = 2.917,\ p_4 = 2.150.$$

Then we rank the arguments $\tilde{s}_i$ $(i = 1, 2, 3, 4)$ in descending order in accordance with the values of p_i $(i = 1, 2, 3, 4)$:

$$\tilde{s}_{\beta_1} = \tilde{s}_3 = [s_2, s_4],\ \tilde{s}_{\beta_2} = \tilde{s}_1 = [s_2, s_3],\ \tilde{s}_{\beta_3} = \tilde{s}_4 = [s_3, s_4],\ \tilde{s}_{\beta_4} = \tilde{s}_2 = [s_1, s_3].$$

Suppose that the weighting vector $w = (w_1, w_2, w_3, w_4)^T$ of the ULHHM operator is $w = (0.3, 0.2, 0.3, 0.2)^T$, then by (3.8), we get

$$\begin{aligned} \mathrm{ULHHM}_w(\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4) &= \frac{1}{\frac{0.3}{\tilde{s}_3} \oplus \frac{0.2}{\tilde{s}_1} \oplus \frac{0.3}{\tilde{s}_4} \oplus \frac{0.2}{\tilde{s}_2}} \\ &= \frac{1}{\frac{0.3}{[s_2,s_4]} \oplus \frac{0.2}{[s_2,s_3]} \oplus \frac{0.3}{[s_3,s_4]} \oplus \frac{0.2}{[s_1,s_3]}} \\ &= [s_{1.82}, s_{3.53}]. \end{aligned}$$

3.3 An approach to multiple attribute group decision making

In this section, we consider a MAGDM problem, let $X = \{x_1, x_2, \ldots, x_n\}$ be a discrete set of n feasible alternatives and $G = \{G_1, G_2, \ldots, G_m\}$ be a set of m attributes, whose weight vector is $w = (w_1, w_2, \ldots, w_m)^T$, where $w_i \geq 0$ and $\sum_{i=1}^m w_i = 1$, and let $D = \{d_1, d_2, \ldots, d_l\}$ be the set of decision makers, whose weight vector is $v = (v_1, v_2, \ldots, v_l)^T$, where $v_k \geq 0$ and $\sum_{k=1}^l v_k = 1$. The decision maker $d_k \in D$ may provide the uncertain linguistic decision matrix

$R_k = (\tilde{r}_{ij}^{(k)})_{m\times n}$, where $\tilde{r}_{ij}^{(k)}$ is an attribute value, which takes the form of uncertain linguistic variable, of the alternative $x_j \in X$ with respect to the attribute $G_i \in G$ for all $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$; $k = 1, 2, \ldots, l$.

In the following, we shall develop an approach based on the ULWHM and ULHHM operators to MAGDM with uncertain linguistic preference relations.

Step 1. Utilize the ULWHM operator:

$$\begin{aligned}\tilde{r}_j^{(k)} &= \mathrm{ULWHM}_w(\tilde{r}_{1j}^{(k)}, \tilde{r}_{2j}^{(k)}, \ldots, \tilde{r}_{mj}^{(k)}) \\ &= \frac{1}{\frac{w_1}{\tilde{r}_{1j}^{(k)}} \oplus \frac{w_2}{\tilde{r}_{2j}^{(k)}} \oplus \cdots \oplus \frac{w_n}{\tilde{r}_{mj}^{(k)}}}\end{aligned} \tag{3.9}$$

to aggregate all the elements in the jth column of R_k and get the overall attribute value $\tilde{r}_j^{(k)}$ of the alternative x_j corresponding to the decision maker d_k.

Step 2. Utilize the ULHHM operator:

$$\begin{aligned}\tilde{r}_j &= \mathrm{ULHHM}_\omega(\tilde{r}_j^{(1)}, \tilde{r}_j^{(2)}, \ldots, \tilde{r}_j^{(l)}) \\ &= \frac{1}{\frac{\omega_1}{\dot{\tilde{r}}_j^{\sigma(1)}} \oplus \frac{\omega_2}{\dot{\tilde{r}}_j^{\sigma(2)}} \oplus \cdots \oplus \frac{\omega_l}{\dot{\tilde{r}}_j^{\sigma(l)}}}\end{aligned} \tag{3.10}$$

to aggregate the overall attribute values $\tilde{r}_j^{(k)}$ $(k = 1, 2, \ldots l)$ corresponding to the decision maker d_k $(k = 1, 2, \ldots, l)$ and get the collective overall attribute value $\tilde{r}_j$, where $\dot{\tilde{r}}_j^{\sigma(k)}$ is the kth largest of the weighted data $\dot{\tilde{r}}_j^{(k)}$ $(\dot{\tilde{r}}_j^{(k)} = lv_k\tilde{r}_j^{(k)}$, $k = 1, 2, \ldots, l)$, $\omega = (\omega_1, \omega_2, \ldots, \omega_l)^T$ is the weighting vector of the ULHHM operator with $\omega_k \geq 0$ and $\sum_{k=1}^{l} \omega_k = 1$.

Step 3. Compare each $\tilde{r}_j$ with all $\tilde{r}_i$ $(i = 1, 2, \ldots, n)$ by using (3.1), and let $p_{ij} = p(\tilde{r}_i \geq \tilde{r}_j)$, and then construct a possibility matrix $\mathbf{P} = (p_{ij})_{n\times n}$, where $p_{ij} \geq 0$, $p_{ij} + p_{ji} = 1$, $p_{ii} = 0.5$, $i, j = 1, 2, \ldots, n$. Summing all elements in each line of matrix $\mathbf{P}$, we have $p_i = \sum_{j=1}^{n} p_{ij}$, $i = 1, 2, \ldots, n$, and then reorder $\tilde{r}_j$ $(j = 1, 2, \ldots, n)$ in descending order in accordance with the values of p_j $(j = 1, 2, \ldots, n)$.

Step 4. Rank all the alternatives x_j $(j = 1, 2, \ldots n)$ by the ranking of $\tilde{r}_j$ $(j = 1, 2, \ldots, n)$, and then select the most desirable one.

Step 5. End.

3.4 Illustrative example

In this section, we use a MAGDM problem of determining what kind of air-conditioning systems should be installed in a library (adapted from [44]) to illustrate the proposed approach.

A city is planning to build a municipal library. One of the problems facing the city development commissioner is to determine what kind of air-conditioning systems should be installed in the library. The contractor offers five feasible alternatives, which might be adapted to the physical structure of the library. The offered air-conditioning system must take a decision according to the following four attributes:

(1) G_1 is performance.

(2) G_2 is maintainability.

(3) G_3 is flexibility.

(4) G_4 is safety.

The five possible alternatives x_j $(j = 1, 2, 3, 4, 5)$ are to be evaluated using the uncertain linguistic variables by three decision makers (whose weight vector is $v = (0.4, 0.3, 0.3)^T$) under the above four attributes (whose weight vector $w = (0.2, 0.1, 0.3, 0.4)^T$), and construct, respectively, the decision matrices $R_k = (\tilde{r}_{ij}^{(k)})_{5\times 4}$ $(k = 1, 2, 3)$ as listed in Tables 3.1-3.3:

Table 3.1: Decision matrix R_1

	x_1	x_2	x_3	x_4	x_5
G_1	$[s_5, s_7]$	$[s_3, s_4]$	$[s_2, s_4]$	$[s_4, s_5]$	$[s_2, s_3]$
G_2	$[s_4, s_5]$	$[s_1, s_3]$	$[s_3, s_4]$	$[s_3, s_5]$	$[s_4, s_6]$
G_3	$[s_2, s_4]$	$[s_5, s_6]$	$[s_1, s_3]$	$[s_6, s_7]$	$[s_4, s_5]$
G_4	$[s_3, s_4]$	$[s_2, s_3]$	$[s_3, s_5]$	$[s_2, s_3]$	$[s_3, s_4]$

To get the best alternative(s), the following steps are involved:

Step 1. Utilize the ULWHM operator to aggregate all the elements in the

Table 3.2: Decision matrix R_2

	x_1	x_2	x_3	x_4	x_5
G_1	$[s_3, s_5]$	$[s_4, s_5]$	$[s_1, s_2]$	$[s_3, s_5]$	$[s_1, s_3]$
G_2	$[s_2, s_4]$	$[s_2, s_3]$	$[s_3, s_5]$	$[s_2, s_4]$	$[s_4, s_5]$
G_3	$[s_1, s_2]$	$[s_2, s_3]$	$[s_1, s_2]$	$[s_2, s_4]$	$[s_5, s_6]$
G_4	$[s_3, s_5]$	$[s_4, s_6]$	$[s_2, s_3]$	$[s_1, s_3]$	$[s_4, s_6]$

Table 3.3: Decision matrix R_3

	x_1	x_2	x_3	x_4	x_5
G_1	$[s_2, s_3]$	$[s_3, s_5]$	$[s_1, s_3]$	$[s_2, s_3]$	$[s_4, s_5]$
G_2	$[s_3, s_4]$	$[s_1, s_3]$	$[s_4, s_5]$	$[s_3, s_4]$	$[s_3, s_4]$
G_3	$[s_1, s_3]$	$[s_3, s_5]$	$[s_2, s_3]$	$[s_4, s_5]$	$[s_3, s_4]$
G_4	$[s_2, s_3]$	$[s_2, s_4]$	$[s_4, s_5]$	$[s_1, s_2]$	$[s_2, s_4]$

jth column of R_k and get the overall attribute value $\tilde{r}_j^{(k)}$:

$$\tilde{r}_1^{(1)} = [s_{2.86}, s_{4.55}],\ \tilde{r}_2^{(1)} = [s_{2.33}, s_{3.70}],\ \tilde{r}_3^{(1)} = [s_{1.75}, s_{3.85}],$$
$$\tilde{r}_4^{(1)} = [s_{3.03}, s_{4.17}],\ \tilde{r}_5^{(1)} = [s_{3.03}, s_{4.17}],$$
$$\tilde{r}_1^{(2)} = [s_{1.82}, s_{3.33}],\ \tilde{r}_2^{(2)} = [s_{2.86}, s_{4.17}],\ \tilde{r}_3^{(2)} = [s_{1.59}, s_{2.50}],$$
$$\tilde{r}_4^{(2)} = [s_{1.49}, s_{3.70}],\ \tilde{r}_5^{(2)} = [s_{2.27}, s_{5.00}],$$
$$\tilde{r}_1^{(3)} = [s_{1.59}, s_{3.03}],\ \tilde{r}_2^{(3)} = [s_{2.13}, s_{4.35}],\ \tilde{r}_3^{(3)} = [s_{2.08}, s_{3.70}],$$
$$\tilde{r}_4^{(3)} = [s_{1.64}, s_{2.86}],\ \tilde{r}_5^{(3)} = [s_{2.63}, s_{4.17}].$$

Step 2. Utilize the ULHHM operator (suppose that its weight vector is $\omega = (0.2, 0.5, 0.3)^T$) to aggregate the overall attribute values $\tilde{r}_j^{(k)}$ $(k = 1, 2, 3)$ corresponding to the decision maker d_k $(k = 1, 2, 3)$, and get the collective overall attribute value $\tilde{r}_j$:

$$\tilde{r}_1 = [s_{1.75}, s_{3.23}],\ \tilde{r}_2 = [s_{2.38}, s_{4.00}],\ \tilde{r}_3 = [s_{1.75}, s_{3.03}],$$

$$\tilde{r}_4 = [s_{1.59}, s_{3.23}],\ \tilde{r}_5 = [s_{2.33}, s_{4.35}].$$

Step 3. Compare each $\tilde{r}_j$ with $\tilde{r}_i$ $(i = 1, 2, 3, 4, 5)$ by using (3.1), and let $p_{ij} = p(\tilde{r}_i \geq \tilde{r}_j)$, and then construct a possibility matrix:

$$\mathbf{P} = \begin{pmatrix} 0.500 & 0.274 & 0.563 & 0.526 & 0.257 \\ 0.726 & 0.500 & 0.776 & 0.739 & 0.459 \\ 0.464 & 0.224 & 0.500 & 0.493 & 0.212 \\ 0.474 & 0.261 & 0.507 & 0.500 & 0.246 \\ 0.743 & 0.541 & 0.788 & 0.754 & 0.500 \end{pmatrix}.$$

Summing all elements in each line of matrix $\mathbf{P}$, we have

$$p_1 = 2.12,\ p_2 = 3.200,\ p_3 = 1.893,\ p_4 = 1.988,\ p_5 = 3.326$$

and then we rank $\tilde{r}_j$ $(j = 1, 2, 3, 4, 5)$ in descending order in accordance with the values of p_j $(j = 1, 2, 3, 4, 5)$:

$$\tilde{r}_5 > \tilde{r}_2 > \tilde{r}_1 > \tilde{r}_4 > \tilde{r}_3.$$

Step 4. Rank all alternatives x_j $(j = 1, 2, 3, 4, 5)$ by the ranking $\tilde{r}_j$ $(j = 1, 2, 3, 4, 5)$:

$$x_5 \succ x_2 \succ x_1 \succ x_4 \succ x_3$$

and thus the most desirable alternative is x_5.

3.5 Comparison with other methods

In this section, we compare the proposed method with other methods. The methods to be compared here are the methods proposed by Xu [29, 32], respectively.

Each of methods has its advantages and disadvantages and none of them can always perform better than the others in any situations. It perfectly depends on how we look at things, and not on how they are themselves.

The method proposed by Xu [32] is suitable for solving group decision making with uncertain multiplicative linguistic preference relations because the ULOWG operator combines the uncertain multiplicative linguistic variables giving weights to the values in relation to their ordering position, diminishing the importance of extreme values by increasing the importance of central ones; whereas the proposed method in this chapter and the method proposed by Xu [29] are suitable for solving MAGDM with uncertain linguistic information because the ULHA operator and ULHHM operator reflect the importance degrees of both the given uncertain linguistic variables and their ordered positions. Others of relative comparison with the methods respectively proposed by Xu [29, 32] are shown in Table 3.4.

Table 3.4: Comparison with other methods

	Xu [29]	Xu [32]	Proposed method
Problem type	MAGDM	GDM	MAGDM
Application area	Evaluating university faculty	Investment of money	Air-conditioning system selection
Decision information	Uncertain linguistic decision matrix	Uncertain multiplicative decision matrix	Uncertain linguistic decision matrix
Solution method			
Aggregation stage	ULWA operator	IULOWG operator	ULWHM operator
Exploitation stage	ULHA operator	ULOWG operator	ULHHM operator
Ranking stage	Complementary matrix	Complementary matrix	Possibility matrix
Final decision	Ranking of a number of alternatives	Ranking of a number of alternatives	Ranking of a number of alternatives

3.6 Conclusions and discussions

In this chapter, we have developed some new aggregation operators including the uncertain linguistic weighted harmonic mean (ULWHM) operator, the uncertain linguistic ordered weighted harmonic mean (ULOWHM) operator and the uncertain linguistic hybrid harmonic mean (ULHHM) operator. The ULOWHM operator, which is an extension of Chen et al.'s OWHM operator, can be used in situations where the input arguments are uncertain linguistic variables. The ULHHM operator generalizes both ULWHM operator and the ULOWHM operator, and reflects the importance degrees of both the given arguments and their ordered positions. Based on the ULWHM and ULHHM operators, we have proposed an approach to multiple attribute group decision making with uncertain linguistic information. We have also applied the proposed approach to the problem of determining what kind of air-conditioning systems should be installed in the library. Furthermore, Wei and Yi [24] proposed some harmonic aggregation operators for aggregating triangular fuzzy linguistic information, such as the fuzzy linguistic weighted harmonic mean (FLWHM) operator, fuzzy linguistic ordered weighted harmonic mean (FLOWHM) operator and fuzzy linguistic hybrid harmonic mean (FLHHM) operator, and developed an approach to multiple attribute group decision making with triangular fuzzy linguistic variables. From Definition 3.1.4, since an uncertain linguistic variable can be thought as an α-cut of triangular fuzzy linguistic variable, each triangular fuzzy linguistic variable is transform to an uncertain linguistic variable. Therefore, by Definition 3.1.4 and Remark 3.1.5, we can use the ULHM, ULWHM, ULOWHM, and ULHHM operators for aggregating triangular fuzzy linguistic information, and thus we can use the our approach for solving MAGDM problems with triangular fuzzy linguistic environment.

Appendix A. Fuzzy linguistic harmonic mean aggregation operators [24]

A fuzzy linguistic hybrid harmonic mean (FLHHM) operator is defined as follows:

$$\mathrm{FLHHM}_{\omega,w}(\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_n) = \frac{1}{\frac{w_1}{\hat{r}_1} \oplus \frac{w_2}{\hat{r}_2} \oplus \cdots \oplus \frac{w_n}{\hat{r}_n}},$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the associated weighting vector such that $w_j \in [0, 1]$, $j = 1, 2, \ldots, n$, and $\sum_{j=1}^{n} w_j = 1$, and $\hat{r}_j$ is the jth largest element of the weighted triangular fuzzy linguistic variables $\dot{\hat{s}}_i$ ($\dot{\hat{s}}_i = \frac{\hat{s}_i}{n\omega_i}$, $i = 1, 2, \ldots, n$), $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $\tilde{s}_i$ ($i = 1, 2, \ldots, n$) with $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n} \omega_i = 1$, and n is the balancing coefficient, then the function FLHHM is called the fuzzy linguistic hybrid harmonic mean (FLHHM) operator of dimension n.

Especially, if $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then $\dot{\hat{s}}_i = \hat{s}_i$, $i = 1, 2, \ldots, n$, in this case, the FLHHM operator is reduced to the fuzzy linguistic ordered weighted harmonic mean (FLOWHM) operator; if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the FLHHM operator is reduced to the fuzzy linguistic weighted harmonic mean (FLWHM) operator.

Appendix B. An approach to multiple attribute group decision making under triangular fuzzy linguistic environment [24]

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a discrete set of n alternatives and $G = \{G_1, G_2, \ldots, G_m\}$ be a set of m attributes, whose weight vector is $w = (w_1, w_2, \ldots, w_m)^T$, where $w_i \geq 0$ and $\sum_{i=1}^{m} w_i = 1$, and let $D = \{d_1, d_2, \ldots, d_l\}$ be the set of decision makers, whose weight vector is $v = (v_1, v_2, \ldots, v_l)^T$, where $v_k \geq 0$ and $\sum_{k=1}^{l} v_k = 1$. The decision maker $d_k \in D$ may provide the uncertain linguistic decision matrix $R_k = (\hat{r}_{ij}^{(k)})_{m\times n}$, where $\hat{r}_{ij}^{(k)}$ is an attribute value, which takes the form of triangular fuzzy linguistic variable, of the alternative $x_j \in X$ with respect to the attribute $G_i \in G$ for all $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$; $k = 1, 2, \ldots, l$.

Step 1. Utilize the FLWHM operator:

$$\hat{r}_j^{(k)} = \mathrm{FLWHM}_w(\hat{r}_{1j}^{(k)}, \hat{r}_{2j}^{(k)}, \ldots, \hat{r}_{mj}^{(k)})$$

to aggregate all the elements in the jth column of R_k and get the overall attribute value $\hat{r}_j^{(k)}$ of the alternative x_j corresponding to the decision maker d_k.

Step 2. Utilize the FLHHM operator:

$$\begin{aligned}\hat{r}_j &= \mathrm{FLHHM}_{v,\omega}(\hat{r}_j^{(1)}, \hat{r}_j^{(2)}, \ldots, \hat{r}_j^{(l)}) \\ &= \frac{1}{\frac{\omega_1}{\dot{\hat{r}}_j^{\sigma(1)}} \oplus \frac{\omega_2}{\dot{\hat{r}}_j^{\sigma(2)}} \oplus \cdots \oplus \frac{\omega_l}{\dot{\hat{r}}_j^{\sigma(l)}}}\end{aligned}$$

to aggregate the overall attribute values $\hat{r}_j^{(k)}$ $(k = 1, 2, \ldots l)$ corresponding to the decision maker d_k $(k = 1, 2, \ldots, l)$ and get the collective overall attribute value $\hat{r}_j$, where $\dot{\hat{r}}_j^{\sigma(k)}$ is the kth largest of the weighted data $\dot{\hat{r}}_j^{(k)}$ $(\dot{\hat{r}}_j^{(k)} = \frac{\hat{r}_j^{(k)}}{lv_k}, k = 1, 2, \ldots, l)$, $\omega = (\omega_1, \omega_2, \ldots, \omega_l)^T$ is the weighting vector of the FLHHM operator with $\omega_k \geq 0$ and $\sum_{k=1}^{l} \omega_k = 1$.

Step 3. Compare each $\hat{r}_j$ with all $\hat{r}_i$ $(i = 1, 2, \ldots, n)$ by using (3.3), and let $q_{ij} = p(\hat{r}_i \geq \hat{r}_j)$, and then construct a possibility matrix $\mathbf{Q} = (q_{ij})_{n \times n}$, where $q_{ij} \geq 0$, $q_{ij} + q_{ji} = 1$, $q_{ii} = 0.5$, $i, j = 1, 2, \ldots, n$. Summing al elements in each line of matrix $\mathbf{Q}$, we have $q_i = \sum_{j=1}^{n} q_{ij}$, $i = 1, 2, \ldots, n$, and then reorder $\hat{r}_j$ $(j = 1, 2, \ldots, n)$ in descending order in accordance with the values of q_j $(j = 1, 2, \ldots, n)$.

Step 4. Rank all the alternatives x_j $(j = 1, 2, \ldots n)$ by the ranking of $\hat{r}_j$ $(j = 1, 2, \ldots n)$, and then select the most desirable one.

Step 5. End.

Chapter 4

Generalized induced linguistic harmonic mean operators based approach to multiple attribute group decision making

Two generalized induced linguistic aggregation operator called the generalized induced linguistic ordered weighted harmonic mean (GILOWHM) operator and generalized induced uncertain linguistic ordered weighted harmonic mean (GIULOWHM) operator is defined. Each object processed by these operators consists of three components, where the first component represents the importance degree or character of the second component, and the second component is used to induce an ordering, through the first component, over the third components which are linguistic variables (or uncertain linguistic variables) and then aggregated. Based on the GILOWHM and GIULOWHM operators respectively, we develop two procedures to solve the multiple attribute group decision making problems where all attribute values are expressed in linguistic variables or uncertain linguistic variables. Finally, an example is used to illustrate the developed procedures.

4.1 Generalized induced linguistic aggregation operators

4.1.1 The GILOWHM and GIULOWHM operators

Definition 4.1.1 [21] An induced LOWHM (ILOWHM) operator is defined as follows:

$$\mathrm{ILOWHM}_w(\langle u_1, s_{\alpha_1}\rangle, \langle u_2, s_{\alpha_2}\rangle, \ldots, \langle u_n, s_{\alpha_n}\rangle) = \frac{1}{\frac{w_1}{s_{\beta_1}} \oplus \frac{w_2}{s_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{s_{\beta_n}}} \tag{4.1}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is a weighting vector, such that $w_i \in [0,1]$, $i = 1, 2, \ldots, n$, $\sum_{i=1}^{n} w_i = 1$, s_{β_i} is the s_{α_i} value of the LOWHM pair $\langle u_i, s_{\alpha_i}\rangle$ having the ith largest u_i, and u_i in $\langle u_i, s_{\alpha_i}\rangle$ is referred to as the order inducing variable and s_{α_i} as the linguistic argument variable. Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then ILOWHM is reduced to the LHM operator; if $u_i = s_{\alpha_i}$, for all i, then ILOWHM is reduced to the LOWHM operator; if $u_i =$No. i, for all i, where No. i is the ordered position of the a_i, then ILOWHM is the LHM operator.

However, if there is a tie between $\langle u_i, s_{\alpha_i}\rangle$, $\langle u_j, s_{\alpha_j}\rangle$ with respect to order-inducing variables, in this case, we can follow the policy presented by Yager and Filov [40] - to replace the arguments of the tied objects by the mean of the arguments of the tied objects (i.e., we replace the argument component of each of $\langle u_i, s_{\alpha_i}\rangle$ and $\langle u_j, s_{\alpha_j}\rangle$ by their average $(s_{\alpha_i} \oplus s_{\alpha_j})/2$). If k items are tied, we replace these by k replicas of their average.

In the following, we shall give example to specify the special cases with respect to the inducing variables.

Example 4.1.2 Consider the following collection of LOWHM pairs:

$$\langle s_4, s_3\rangle, \langle s_6, s_7\rangle, \langle s_3, s_1\rangle, \langle s_5, s_4\rangle.$$

Performing the ordering the LOWHM pairs with respect to the first component, we have

$$\langle s_6, s_7\rangle, \langle s_5, s_4\rangle, \langle s_4, s_3\rangle, \langle s_3, s_1\rangle.$$

This ordering induces the ordered linguistic arguments

$$s_{\beta_1} = s_7, s_{\beta_2} = s_4, s_{\beta_3} = s_3, s_{\beta_4} = s_1.$$

If the weighting vector $w = (0.3, 0.1, 0.4, 0.2)^T$, then we get an aggregated value:

$$\begin{aligned}&\text{ILOWHM}_w(\langle s_4, s_3\rangle, \langle s_5, s_7\rangle, \langle s_3, s_1\rangle, \langle s_6, s_4\rangle)\\ &= \frac{1}{\frac{0.3}{s_7} \oplus \frac{0.1}{s_4} \oplus \frac{0.4}{s_3} \oplus \frac{0.2}{s_1}} = s_{2.49}.\end{aligned}$$

Definition 4.1.3 An induced uncertain LOWHM (IULOWHM) operator is defined as follows:

$$\text{IULOWHM}_w(\langle u_1, \tilde{s}_1\rangle, \langle u_2, \tilde{s}_2\rangle, \ldots, \langle u_n, \tilde{s}_n\rangle) = \frac{1}{\frac{w_1}{\tilde{s}_{\beta_1}} \oplus \frac{w_2}{\tilde{s}_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{\tilde{s}_{\beta_n}}} \qquad (4.2)$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is a weighting vector, such that $w_i \in [0, 1]$, $i = 1, 2, \ldots, n$, $\sum_{i=1}^{n} w_i = 1$, $\tilde{s}_{\beta_i}$ is the $\tilde{s}_i$ value of the ULOWHM pair $\langle u_i, \tilde{s}_i\rangle$ having the ith largest u_i, and u_i in $\langle u_i, \tilde{s}_i\rangle$ is referred to as the order inducing variable and $\tilde{s}_i$ as the uncertain linguistic argument variable. Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then IULOWHM is reduced to the ULHM operator; if $u_i = \tilde{s}_i$, for all i, then IULOWHM is reduced to the ULOWHM operator; if $u_i =$No. i, for all i, where No. i is the ordered position of the a_i, then IULOWHM is the ULHM operator.

However, if there is a tie between $\langle u_i, \tilde{s}_i\rangle$, $\langle u_j, \tilde{s}_j\rangle$ with respect to order-inducing variables. In this case, we can replace the argument component of each of $\langle u_i, \tilde{s}_i\rangle$ and $\langle u_j, \tilde{s}_j\rangle$ by their average $(\tilde{s}_i \oplus \tilde{s}_j)/2)$. If k items are tied, we replace these by k replicas of their average.

Example 4.1.4 Consider the following collection of ULOWHM pairs:

$$\langle 0.5, [s_3, s_4]\rangle, \langle 0.3, [s_6, s_7]\rangle, \langle 0.7, [s_2, s_3]\rangle, \langle 0.4, [s_2, s_4]\rangle.$$

Performing the ordering the ULOWHM pairs with respect to the first component, we have

$$\langle 0.7, [s_2, s_3]\rangle, \langle 0.5, [s_3, s_4]\rangle, \langle 0.4, [s_2, s_4]\rangle, \langle 0.3, [s_6, s_7]\rangle.$$

This ordering induces the ordered linguistic arguments

$$\tilde{s}_{\beta_1} = [s_2, s_3], \tilde{s}_{\beta_2} = [s_3, s_4], \tilde{s}_{\beta_3} = [s_2, s_4], \tilde{s}_{\beta_4} = [s_6, s_7].$$

If the weighting vector $w = (0.3, 0.1, 0.4, 0.2)^T$, then we get an aggregated value:

$$\begin{aligned}&\mathrm{ILOWHM}_w(\langle 0.5, [s_3, s_4]\rangle, \langle 0.3, [s_6, s_7]\rangle, \langle 0.7, [s_2, s_3]\rangle, \langle 0.4, [s_2, s_4]\rangle)\\ &\quad = [s_{2.40}, s_{3.94}].\end{aligned}$$

An important feature of the ILOWHM operator is that the argument ordering process is guided by a variable called the order inducing value. This operator essentially aggregate objects, which are pairs, and provide a very general family of aggregations operators. In some situations, however, when we need to provide more information about the objects, i.e. each object may consist of three components, a direct locator, an indirect locator and a prescribed value, it is unsuitable to use this induced aggregation operator as an aggregation tool. In following we shall present some more general linguistic aggregation technique.

Definition 4.1.5 A generalized induced LOWHM (GILOWHM) operator is given by

$$\begin{aligned}&\mathrm{GILOWHM}_w(\langle v_1, u_1, s_{\alpha_1}\rangle, \langle v_2, u_2, s_{\alpha_2}\rangle, \ldots, \langle v_n, u_n, s_{\alpha_n}\rangle)\\ &\quad = \frac{1}{\frac{w_1}{s_{\beta_1}} \oplus \frac{w_2}{s_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{s_{\beta_n}}}\end{aligned} \tag{4.3}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the associated weighting vector with $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, the object $\langle v_i, u_i, s_{\alpha_i}\rangle$ consists of three components, where the first component v_i represents the importance degree or character of second component u_i, and the second component u_i is used to induce an ordering, through the first component v_i, over the third component s_{α_i} which are then aggregated. Here, s_{β_j} is the s_{α_i} value of the object having the jth largest v_i. In discussing the object $\langle v_i, u_i, s_{\alpha_i}\rangle$, because of its role we shall refer to the v_i as the direct order inducing variable, the u_i as the indirect inducing variable, and s_{α_i} as the linguistic argument variable.

Especially, if $v_i = u_i$, for all i, then the GILOWHM operator is reduced to the ILOWHM operator; if $v_i = s_{\alpha_i}$, for all i, then the GILOWHM operator is reduced to the LOWHM operator; if $v_i = \text{No. } i$, for all i, where No. i is the ordered position of the s_{α_i}, then the GILOWHM operator is reduced to the LWHM operator; if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the GILOWHM operator is reduced to the LHM operator.

Example 4.1.6 Consider the collection of the objects

$$\langle \text{No. } 3, \text{Kim}, s_1\rangle, \langle \text{No. } 1, \text{Park}, s_7\rangle, \langle \text{No. } 2, \text{Lee}, s_2\rangle, \langle \text{No. } 4, \text{Jung}, s_5\rangle.$$

By the first component, we get the ordered objects

$$\langle \text{No. } 1, \text{Park}, s_7\rangle, \langle \text{No. } 2, \text{Lee}, s_2\rangle, \langle \text{No. } 3, \text{Kim}, s_1\rangle, \langle \text{No. } 4, \text{Jung}, s_5\rangle.$$

The ordering induces the ordered arguments $s_{\beta_1} = s_7$, $s_{\beta_2} = s_2$, $s_{\beta_3} = s_1$, $s_{\beta_4} = s_5$. If the weighting vector for this aggregation is $w = (0.3, 0.1, 0.2, 0.4)^T$, then we get

$$\begin{aligned}&\text{GILOWHM}_w(\langle \text{No.3}, \text{Kim}, s_1\rangle, \langle \text{No.1}, \text{Park}, s_7\rangle, \langle \text{No.2}, \text{Lee}, s_2\rangle, \langle \text{No.4}, \text{Jung}, s_5\rangle)\\ &\qquad = \frac{1}{\frac{0.3}{s_7} \oplus \frac{0.1}{s_2} \oplus \frac{0.2}{s_1} \oplus \frac{0.4}{s_5}} = s_{2.70}.\end{aligned}$$

However, if we replace the objects in Example 4.1.6 with

$$\langle \text{No. } 3, \text{Kim}, s_1\rangle, \langle \text{No. } 1, \text{Park}, s_7\rangle, \langle \text{No. } 2, \text{Lee}, s_2\rangle, \langle \text{No. } 3, \text{Jung}, s_5\rangle,$$

then there is a tie between $\langle \text{No. } 3, \text{Kim}, s_1\rangle$ and $\langle \text{No. } 3, \text{Jung}, s_5\rangle$ with respect to order direct inducing variable, in this case, we can follow the policy: we replace the linguistic argument component of each of $\langle \text{No. } 3, \text{Kim}, s_1\rangle$ and $\langle \text{No. } 3, \text{Jung}, s_5\rangle$ by their average $(s_1 \oplus s_5)/2 = s_3$. This substitution gives us ordered arguments $s_{\beta_1} = s_7$, $s_{\beta_2} = s_2$, $s_{\beta_3} = s_3$, $s_{\beta_4} = s_3$. Thus

$$\begin{aligned}&\text{GILOWHM}_w(\langle \text{No.3}, \text{Kim}, s_1\rangle, \langle \text{No.1}, \text{Park}, s_7\rangle, \langle \text{No.2}, \text{Lee}, s_2\rangle, \langle \text{No.3}, \text{Jung}, s_5\rangle)\\ &\qquad = \frac{1}{\frac{0.3}{s_7} \oplus \frac{0.1}{s_2} \oplus \frac{0.2}{s_3} \oplus \frac{0.4}{s_3}} = s_{3.44}.\end{aligned}$$

If we replace (4.3) with

$$\begin{aligned}&\text{GIULOWHM}_w(\langle v_1, u_1, \tilde{s}_1\rangle, \langle v_2, u_2, \tilde{s}_2\rangle, \ldots, \langle v_n, u_n, \tilde{s}_n\rangle)\\ &= \frac{1}{\frac{w_1}{\tilde{s}_{\beta_1}} \oplus \frac{w_2}{\tilde{s}_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{\tilde{s}_{\beta n}}}\end{aligned} \tag{4.4}$$

then by Definition 4.1.5, we get a GIULOWHM operator. Especially, if $v_i = u_i$, for all i, then the GIULOWHM operator is reduced to the IULOWGM operator; if $v_i = \tilde{s}_i$, for all i, then the GIULOWHM operator is reduced to the ULOWHM operator; if $v_i =$ No. i, for all i, where No. i is ordered position of the $\tilde{s}_i$, then the GIULOWHM operator is reduced to the ULWHM operator; if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the GIULOWHM operator is reduced to the ULHM operator.

Example 4.1.7 Consider a collection of the objects

$$\langle 0.3, \text{Kim}, [s_1, s_3]\rangle, \langle 0.1, \text{Park}, [s_7, s_8]\rangle, \langle 0.2, \text{Lee}, [s_2, s_3]\rangle.$$

Performing the ordering of the objects with respect to the first component, we get the ordered objects

$$\langle 0.3, \text{Kim}, [s_1, s_3]\rangle, \langle 0.2, \text{Lee}, [s_2, s_3]\rangle, \langle 0.1, \text{Park}, [s_7, s_8]\rangle.$$

The ordering induces the ordered uncertain linguistic arguments $\tilde{s}_{\beta_1} = [s_1, s_3]$, $\tilde{s}_{\beta_2} = [s_2, s_3]$, $s_{\beta_3} = [s_7, s_8]$. If the weighting vector for this aggregation is $w = (0, 2, 0.6, 0.2)^T$, then we have

$$\begin{aligned}&\text{GIULOWHM}_w(\langle 0.3, \text{Kim}, [s_1, s_3]\rangle, \langle 0.1, \text{Park}, [s_7, s_8]\rangle, \langle 0.2, \text{Lee}, [s_2, s_3]\rangle)\\ &= [s_{2.33}, s_{3.42}].\end{aligned}$$

If the direct order inducing variables v_i $(i = 1, 2, \ldots, n)$ take the form of uncertain linguistic variables $\tilde{s}'_i$ $(i = 1, 2, \ldots, n)$, then we shall use, to rank these uncertain linguistic variables, the procedure for ranking uncertain linguistic arguments when using the ULOWHM operator.

Example 4.1.8 Consider a collection of the objects

$$\langle [s_1, s_2], \text{Kim}, [s_2, s_4] \rangle, \langle [s_4, s_5], \text{Park}, [s_7, s_8] \rangle, \langle [s_3, s_5], \text{Lee}, [s_2, s_3] \rangle.$$

To rank the first components v_i $(i = 1, 2, 3)$ of the objects, we first compare each v_i with all these first components v_i $(i = 1, 2, 3)$ by using (3.1), and then construct a complementary matrix

$$\mathbf{P} = \begin{pmatrix} 0.500 & 0.000 & 0.000 \\ 1.000 & 0.500 & 0.667 \\ 1.000 & 0.333 & 0.500 \end{pmatrix}.$$

Summing all elements in each line of matrix $\mathbf{P}$, we have

$$p_1 = 0.500, \ p_2 = 2.167, \ p_3 = 1.833.$$

Then we rank all the variables v_i $(i = 1, 2, 3)$ in descending order in accordance with the values of p_i $(i = 1, 2, 3)$

$$v_2 = [s_4, s_5], \ v_3 = [s_3, s_5], \ v_1 = [s_1, s_2].$$

Performing the ordering of the objects with respect to the first component, we get the ordered objects

$$\langle [s_4, s_5], \text{Park}, [s_7, s_8] \rangle, \langle [s_3, s_5], \text{Lee}, [s_2, s_3] \rangle, \langle [s_1, s_2], \text{Kim}, [s_2, s_4] \rangle.$$

The ordering induces the ordered uncertain linguistic arguments $\tilde{s}_{\beta_1} = [s_7, s_8]$, $\tilde{s}_{\beta_2} = [s_2, s_3]$, $s_{\beta_3} = [s_2, s_4]$. If the weighting vector for this aggregation is $w = (0, 2, 0.6, 0.2)^T$, then we have

$$\begin{aligned} \text{GIULOWHM}_w\Big(&\langle [s_1, s_2], \text{Kim}, [s_2, s_4] \rangle, \langle [s_4, s_5], \text{Park}, [s_7, s_8] \rangle, \\ &\langle [s_3, s_5], \text{Lee}, [s_2, s_3] \rangle \Big) = [s_{2.33}, s_{3.64}]. \end{aligned}$$

4.1.2 Some properties of the GILOWHM operator

In the following we shall make an investigation on some desirable properties of the GILOWHM operator.

Theorem 4.1.9 (Commutativity) If $(\langle v'_1, u'_1, s'_{\alpha_1}\rangle, \langle v'_2, u'_2, s'_{\alpha_2}\rangle, \ldots, \langle v'_n, u'_n, s'_{\alpha_n}\rangle)$ is any permutation of $(\langle v_1, u_1, s_{\alpha_1}\rangle, \langle v_2, u_2, s_{\alpha_2}\rangle, \ldots, \langle v_n, u_n, s_{\alpha_n}\rangle)$, then

$$\begin{aligned}&\mathrm{GILOWHM}_w(\langle v_1, u_1, s_{\alpha_1}\rangle, \langle v_2, u_2, s_{\alpha_2}\rangle, \ldots, \langle v_n, u_n, s_{\alpha_n}\rangle)\\ &\qquad = \mathrm{GILOWHM}_w(\langle v'_1, u'_1, s'_{\alpha_1}\rangle, \langle v'_2, u'_2, s'_{\alpha_2}\rangle, \ldots, \langle v'_n, u'_n, s'_{\alpha_n}\rangle).\end{aligned}$$

Proof Let

$$\mathrm{GILOWHM}_w(\langle v_1, u_1, s_{\alpha_1}\rangle, \langle v_2, u_2, s_{\alpha_2}\rangle, \ldots, \langle v_n, u_n, s_{\alpha_n}\rangle) = \frac{1}{\frac{w_1}{s_{\beta_1}} \oplus \frac{w_2}{s_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{s_{\beta_n}}}$$

$$\mathrm{GILOWHM}_w(\langle v'_1, u'_1, s'_{\alpha_1}\rangle, \langle v'_2, u'_2, s'_{\alpha_2}\rangle, \ldots, \langle v'_n, u'_n, s'_{\alpha_n}\rangle) = \frac{1}{\frac{w_1}{s'_{\beta_1}} \oplus \frac{w_2}{s'_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{s'_{\beta_n}}}.$$

Since $(\langle v'_1, u'_1, s'_{\alpha_1}\rangle, \langle v'_2, u'_2, s'_{\alpha_2}\rangle, \ldots, \langle v'_n, u'_n, s'_{\alpha_n}\rangle)$ is a permutation of $(\langle v_1, u_1, s_{\alpha_1}\rangle, \langle v_2, u_2, s_{\alpha_2}\rangle, \ldots, \langle v_n, u_n, s_{\alpha_n}\rangle)$, we have $s_{\beta_j} = s'_{\beta_j}$ $(j = 1, 2, \ldots, n)$, then

$$\begin{aligned}&\mathrm{GILOWHM}_w(\langle v_1, u_1, s_{\alpha_1}\rangle, \langle v_2, u_2, s_{\alpha_2}\rangle, \ldots, \langle v_n, u_n, s_{\alpha_n}\rangle)\\ &\qquad = \mathrm{GILOWHM}_w(\langle v'_1, u'_1, s'_{\alpha_1}\rangle, \langle v'_2, u'_2, s'_{\alpha_2}\rangle, \ldots, \langle v'_n, u'_n, s'_{\alpha_n}\rangle).\end{aligned}$$

Theorem 4.1.10 (Idempotency) If $s_{\alpha_i} = s_\alpha$, for all i, then

$$\mathrm{GILOWHM}_w(\langle v_1, u_1, s_{\alpha_1}\rangle, \langle v_2, u_2, s_{\alpha_2}\rangle, \ldots, \langle v_n, u_n, s_{\alpha_n}\rangle) = s_\alpha.$$

Proof Since $s_{\alpha_i} = s_\alpha$, for all i, we have

$$\begin{aligned}&\mathrm{GILOWHM}_w(\langle v_1, u_1, s_{\alpha_1}\rangle, \langle v_2, u_2, s_{\alpha_2}\rangle, \ldots, \langle v_n, u_n, s_{\alpha_n}\rangle)\\ &= \frac{1}{\frac{w_1}{s_{\beta_1}} \oplus \frac{w_2}{s_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{s_{\beta_n}}} = \frac{1}{\frac{w_1}{s_\alpha} \oplus \frac{w_2}{s_\alpha} \oplus \cdots \oplus \frac{w_n}{s_\alpha}}\\ &= s_\alpha.\end{aligned}$$

Theorem 4.1.11 (Monotonicity) If $s_{\alpha_i} \leq s'_{\alpha_i}$, for all i, then

$$\begin{aligned}&\text{GILOWHM}_w(\langle v_1, u_1, s_{\alpha_1}\rangle, \langle v_2, u_2, s_{\alpha_2}\rangle, \ldots, \langle v_n, u_n, s_{\alpha_n}\rangle)\\&\qquad\leq \text{GILOWHM}_w(\langle v_1, u_1, s'_{\alpha_1}\rangle, \langle v_2, u_2, s'_{\alpha_2}\rangle, \ldots, \langle v_n, u_n, s'_{\alpha_n}\rangle).\end{aligned}$$

Proof Let

$$\text{GILOWHM}_w(\langle v_1, u_1, s_{\alpha_1}\rangle, \langle v_2, u_2, s_{\alpha_2}\rangle, \ldots, \langle v_n, u_n, s_{\alpha_n}\rangle) = \frac{1}{\frac{w_1}{s_{\beta_1}} \oplus \frac{w_2}{s_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{s_{\beta_n}}}$$

$$\text{GILOWHM}_w(\langle v_1, u_1, s'_{\alpha_1}\rangle, \langle v_2, u_2, s'_{\alpha_2}\rangle, \ldots, \langle v_n, u_n, s'_{\alpha_n}\rangle) = \frac{1}{\frac{w_1}{s'_{\beta_1}} \oplus \frac{w_2}{s'_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{s'_{\beta_n}}}.$$

Since $s_{\alpha_i} \leq s'_{\alpha_i}$, for all i, it follows that $s_{\beta_i} \leq s'_{\beta_i}$, then

$$\begin{aligned}&\text{GILOWHM}_w(\langle v_1, u_1, s_{\alpha_1}\rangle, \langle v_2, u_2, s_{\alpha_2}\rangle, \ldots, \langle v_n, u_n, s_{\alpha_n}\rangle)\\&\qquad\leq \text{GILOWHM}_w(\langle v_1, u_1, s'_{\alpha_1}\rangle, \langle v_2, u_2, s'_{\alpha_2}\rangle, \ldots, \langle v_n, u_n, s'_{\alpha_n}\rangle).\end{aligned}$$

Theorem 4.1.12 (Boundedness)

$$\min_i(s_{\alpha_i}) \leq \text{GILOWHM}_w(\langle v_1, u_1, s_{\alpha_1}\rangle, \langle v_2, u_2, s_{\alpha_2}\rangle, \ldots, \langle v_n, u_n, s_{\alpha_n}\rangle) \leq \max_i(s_{\alpha_i}).$$

Proof Let $\max_i(s_{\alpha_i}) = s_\beta$ and $\min_i(s_{\alpha_i}) = s_\alpha$, then

$$\begin{aligned}&\text{GILOWHM}_w(\langle v_1, u_1, s_{\alpha_1}\rangle, \langle v_2, u_2, s_{\alpha_2}\rangle, \ldots, \langle v_n, u_n, s_{\alpha_n}\rangle)\\&= \frac{1}{\frac{w_1}{s_{\beta_1}} \oplus \frac{w_2}{s_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{s_{\beta_n}}} \leq \frac{1}{\frac{w_1}{s_{\beta}} \oplus \frac{w_2}{s_{\beta}} \oplus \cdots \oplus \frac{w_n}{s_{\beta}}} = s_\beta,\end{aligned}$$

$$\begin{aligned}&\text{GILOWHM}_w(\langle v_1, u_1, s_{\alpha_1}\rangle, \langle v_2, u_2, s_{\alpha_2}\rangle, \ldots, \langle v_n, u_n, s_{\alpha_n}\rangle)\\&= \frac{1}{\frac{w_1}{s_{\beta_1}} \oplus \frac{w_2}{s_{\beta_2}} \oplus \cdots \oplus \frac{w_n}{s_{\beta_n}}} \geq \frac{1}{\frac{w_1}{s_{\alpha}} \oplus \frac{w_2}{s_{\alpha}} \oplus \cdots \oplus \frac{w_n}{s_{\alpha}}} = s_\alpha.\end{aligned}$$

Hence we have

$$\min_i(s_{\alpha_i}) \leq \text{GILOWHM}_w(\langle v_1, u_1, s_{\alpha_1}\rangle, \langle v_2, u_2, s_{\alpha_2}\rangle, \ldots, \langle v_n, u_n, s_{\alpha_n}\rangle) \leq \max_i(s_{\alpha_i}).$$

Similarly, we can prove that GIULOWHM operator also has the desirable properties above.

4.2 An approach to group decision making

For a group decision making with linguistic information, let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of alternatives, and $G = \{G_1, G_2, \ldots, G_m\}$ be the set of attributes, and $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$ be the weight vector of attributes, where $\omega_i \geq 0$, $i = 1, 2, \ldots, m$, $\sum_{i=1}^{m} \omega_i = 1$. Let $U = \{u_1, u_2, \ldots, u_l\}$ be a set of decision makers, and $V = \{v_1, v_2, \ldots, v_l\}$ be the set of importance degrees or characters of decision makers u_k $(k = 1, 2 \ldots, l)$. Suppose that $A^{(k)} = (a_{ij}^{(k)})_{m \times n}$ is the linguistic decision matrix, where $a_{ij}^{(k)} \in \bar{S}$ is preference value, which takes the form of linguistic variables, given by the decision maker $u_k \in U$, for the alternative $x_j \in X$ with respect to the attribute $G_i \in G$, for all $k = 1, 2, \ldots, l$; $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$.

In the following, we apply the GILOWHM operator (whose exponential weighting vector $w = (w_1, w_2, \ldots, w_l)^T$, $w_k \geq 0$, $k = 1, 2, \ldots l$, $\sum_{k=1}^{l} w_k = 1$) and the LWHM operator to group decision making with linguistic information:

Procedure I.

Step 1: Utilize the GILOWHM operator

$$\begin{aligned} a_{ij} &= \mathrm{GILOWHM}_w(\langle v_1, u_1, a_{ij}^{(1)} \rangle, \langle v_2, u_2, a_{ij}^{(2)} \rangle, \ldots, \langle v_l, u_l, a_{ij}^{(l)} \rangle), \\ &= \frac{1}{\frac{w_1}{a_{ij}^{(1)}} \oplus \frac{w_2}{a_{ij}^{(2)}} \oplus \cdots \oplus \frac{w_l}{a_{ij}^{(l)}}}, \ i = 1, 2, \ldots, m; \ j = 1, 2, \ldots, n \end{aligned}$$

to aggregate all the decision matrices $A^{(k)}$ $(k = 1, 2, \ldots, l)$ into a collective decision matrix $A = (a_{ij})_{m \times n}$, where v_k $(k = 1, 2, \ldots l)$ are direct order inducing variables and u_k $(k = 1, 2, \ldots l)$ are indirect order inducing variables.

Step 2: Utilize the decision information given in matrix A, and the LWHM operator

$$\begin{aligned} a_j &= \mathrm{LWHM}_\omega(a_{1j}, a_{2j}, \ldots, a_{mj}) \\ &= \frac{1}{\frac{\omega_1}{a_{1j}} \oplus \frac{\omega_2}{a_{2j}} \oplus \cdots \oplus \frac{\omega_m}{a_{mj}}}, \ j = 1, 2, \ldots, n \end{aligned}$$

to derive the collective overall preference values a_j of the alternative x_j, where $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$ be the weight vector of attributes.

Step 3: Rank all the alternatives x_j $(j = 1, 2, \ldots, n)$ and select the best one(s) in accordance with the collective overall preference values a_j $(j = 1, 2, , \ldots, n)$.

Step 4: End.

Now we consider the group decision making problems under interval uncertainty where all the attribute values are expressed in uncertain linguistic variables. The following notations are used to depict the considered problems:

Let X, G, ω, U and V be presented as above-mentioned, and let $\tilde{A}^{(k)} = (\tilde{a}_{ij}^{(k)})_{n\times m}$ be an uncertain linguistic decision matrix, where $\tilde{a}_{ij}^{(k)} \in \tilde{S}$ is preference value, which takes the form of uncertain linguistic variables, given by the decision maker $u_k \in U$, for the alternative $x_j \in X$ with respect to the attribute $G_i \in G$, for all $k = 1, 2, \ldots, l$; $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$.

Similar to the Procedure I, a procedure for solving the above problems can be described as follows:

Procedure II.

Step 1: Utilize the GIULOWHM operator

$$\begin{aligned} a_{ij} &= \mathrm{GIULOWHM}_w(\langle v_1, u_1, \tilde{a}_{ij}^{(1)}\rangle, \langle v_2, u_2, \tilde{a}_{ij}^{(2)}\rangle, \ldots, \langle v_l, u_l, \tilde{a}_{ij}^{(l)}\rangle), \\ &= \frac{1}{\frac{w_1}{\tilde{a}_{ij}^{(1)}} \oplus \frac{w_2}{\tilde{a}_{ij}^{(2)}} \oplus \cdots \oplus \frac{w_l}{\tilde{a}_{ij}^{(l)}}},\ i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n \end{aligned}$$

to aggregate all the decision matrices $\tilde{A}^{(k)}$ $(k = 1, 2, \ldots, l)$ into a collective decision matrix $\tilde{A} = (\tilde{a}_{ij})_{m\times n}$, where v_k $(k = 1, 2, \ldots l)$ are direct order inducing variables and u_k $(k = 1, 2, \ldots l)$ are indirect order inducing variables.

Step 2: Utilize the decision information given in matrix $\tilde{A}$, and the ULWHM operator

$$\begin{aligned} \tilde{a}_j &= \mathrm{ULWHM}_\omega(\tilde{a}_{1j}, \tilde{a}_{2j}, \ldots, \tilde{a}_{mj}) \\ &= \frac{1}{\frac{\omega_1}{\tilde{a}_{1j}} \oplus \frac{\omega_2}{\tilde{a}_{2j}} \oplus \cdots \oplus \frac{\omega_m}{\tilde{a}_{mj}}},\ j = 1, 2, \ldots, n \end{aligned}$$

to derive the collective overall preference values $\tilde{a}_j$ of the alternative x_j, where $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$ be the weight vector of attributes.

Step 3: To rank these collective attribute values $\tilde{a}_i$ $(i = 1, 2, \ldots, n)$, we first compare each $\tilde{a}_i$ with all $\tilde{a}_j$ $(j = 1, 2, \ldots, n)$ by using (3.1). For simplicity, we let $p_{ij} = p(\tilde{a}_i \geq \tilde{a}_j)$, then we develop a complementary matrix as $\mathbf{P} = (p_{ij})_{n\times n}$, where:

$$p_{ij} \geq 0,\ p_{ij} + p_{ji} = 1,\ p_{ii} = 0.5,\ i, j = 1, 2, \ldots, n.$$

Summing all elements in each line of matrix $\mathbf{P}$, we have

$$p_i = \sum_{j=1}^{n} p_{ij}, \quad i = 1, 2, \ldots, n.$$

Then we rank the $\tilde{a}_i$ $(i = 1, 2, \ldots, n)$ in descending order in accordance with the values of p_i $(i = 1, 2, \ldots, n)$.

Step 4: Rank all the alternatives x_i $(i = 1, 2, \ldots, n)$ and select the best one(s) in accordance with the $\tilde{a}_i$ $(i = 1, 2, \ldots n)$.

Step 5: End.

4.3 Illustrative example

Let us suppose an investment company, which wants to invest a sum of money in the best option (adapted by Herrera et al. [12]). There is a panel with five possible alternatives in which to invest the money: (1) x_1 is a car industry; (2) x_2 is a food company; (3) x_3 is a computer company; (4) x_4 is an arms company; (5) x_5 is a TV company.

The investment company must make a decision according to the following four attributes (suppose that the weight vector of four attributes is $\omega = (0.3, 0.4, 0.2, 0.1)^T$): (1) G_1 is the risk analysis; (2) G_2 is the growth analysis; (3) G_3 is the social-political impact analysis; (4) G_4 is the environmental impact analysis.

There is three decision makers u_k $(k = 1, 2, 3)$ to evaluate five alternatives as follows: u_1 is Anderson; u_2 is Smith; and u_3 is Brown, where v_1 = No. 3, v_2 = No. 2 and v_3 = No. 1 are order positions of relative importance of decision makers u_k $(k = 1, 2, 3)$, respectively. The five possible alternatives x_j $(j = 1, 2, 3, 4, 5)$ are evaluated using the linguistic scale:

$$\begin{aligned} S = \{ & s_1 = \text{extremely poor}, s_2 = \text{very poor}, s_3 = \text{poor}, \\ & s_4 = \text{slightly poor}, s_5 = \text{fair}, s_6 = \text{slightly good}, \\ & s_7 = \text{good}, s_8 = \text{very good}, s_9 = \text{extremely good}\}. \end{aligned}$$

by three decision makers under the above four attributes G_i $(i = 1, 2, 3, 4)$, and construct, respectively, the decision matrices $A^{(k)} = (a_{ij}^{(k)})_{4\times5}$ $(k = 1, 2, 3)$ as listed in Tables 4.1-4.3.

Table 4.1: Linguistic decision matrix $A^{(1)}$

	x_1	x_2	x_3	x_4	x_5
G_1	s_6	s_9	s_4	s_3	s_6
G_2	s_3	s_7	s_8	s_8	s_4
G_3	s_7	s_4	s_6	s_8	s_7
G_4	s_2	s_4	s_6	s_7	s_8

Now we utilize the proposed procedure I to prioritize these alternatives:

Step 1: Utilize the GILOWHM operator (whose weight vector is $w = (0.3, 0.4, 0.3)^T$)

$$\begin{aligned} a_{ij} = & \text{GILOWHM}_w(\langle v_1, u_1, a_{ij}^{(1)}\rangle, \langle v_2, u_2, a_{ij}^{(2)}\rangle, \langle v_3, u_3, a_{ij}^{(3)}\rangle), \\ & i = 1, 2, 3, 4; j = 1, 2, 3, 4, 5 \end{aligned}$$

to aggregate all the decision matrices $A^{(k)}$ $(k = 1, 2, 3)$ into a collective decision matrix $A = (a_{ij})_{4\times5}$ (Table 4.4).

Table 4.2: Linguistic decision matrix $A^{(2)}$

	x_1	x_2	x_3	x_4	x_5
G_1	s_6	s_8	s_4	s_7	s_3
G_2	s_3	s_6	s_8	s_8	s_4
G_3	s_7	s_4	s_6	s_7	s_9
G_4	s_2	s_3	s_4	s_6	s_8

Table 4.3: Linguistic decision matrix $A^{(3)}$

	x_1	x_2	x_3	x_4	x_5
G_1	s_6	s_8	s_4	s_7	s_2
G_2	s_4	s_6	s_8	s_7	s_4
G_3	s_7	s_3	s_7	s_9	s_8
G_4	s_3	s_4	s_4	s_7	s_7

Step 2: Utilize the decision information given in matrix A, and the LWHM operator

$$\begin{aligned} a_j &= \mathrm{LWHM}_\omega(a_{1j}, a_{2j}, a_{3j}, a_{4j}) \\ &= \frac{1}{\frac{\omega_1}{a_{1j}} \oplus \frac{\omega_2}{a_{2j}} \oplus \frac{\omega_3}{a_{3j}} \oplus \frac{\omega_4}{a_{4j}}}, \ j = 1, 2, 3, 4, 5 \end{aligned}$$

to derive the collective overall preference values a_j of the alternative x_j:

$$a_1 = s_{4.02}, a_2 = s_{5.44}, a_3 = s_{5.57}, a_4 = s_{6.55}, a_5 = s_{4.20}.$$

Step 3: Rank all the alternatives x_j ($j = 1, 2, 3, 4, 5$) and select the best one(s) in accordance with the collective overall preference values a_j ($j = 1, 2, 3, 4, 5$):

$$x_4 \succ x_3 \succ x_2 \succ x_5 \succ x_1$$

thus the best alternative is x_4.

Table 4.4: Collective linguistic decision matrix A

	x_1	x_2	x_3	x_4	x_5
G_1	$s_{6.0}$	$s_{8.3}$	$s_{4.0}$	$s_{5.0}$	$s_{3.0}$
G_2	$s_{3.2}$	$s_{6.3}$	$s_{8.0}$	$s_{7.7}$	$s_{4.0}$
G_3	$s_{7.0}$	$s_{3.6}$	$s_{6.3}$	$s_{7.8}$	$s_{8.0}$
G_4	$s_{2.2}$	$s_{3.5}$	$s_{4.4}$	$s_{6.6}$	$s_{7.7}$

If three decision makers evaluate the performance of five companies x_j $(j = 1, 2, 3, 4,\ 5)$ according to attributes G_i $(i = 1, 2, 3, 4)$ by using the uncertain linguistic terms in the set $\tilde{S}$ and constructs, respectively, the uncertain linguistic decision matrices $\tilde{A}^{(k)}$ $(k = 1, 2, 3)$ as listed in Tables 4.5-4.7.

Table 4.5: Uncertain linguistic decision matrix $\tilde{A}^{(1)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$[s_5, s_7]$	$[s_7, s_9]$	$[s_2, s_4]$	$[s_3, s_5]$	$[s_4, s_6]$
G_2	$[s_2, s_3]$	$[s_6, s_7]$	$[s_7, s_9]$	$[s_3, s_5]$	$[s_4, s_6]$
G_3	$[s_2, s_4]$	$[s_5, s_6]$	$[s_1, s_3]$	$[s_6, s_7]$	$[s_4, s_5]$
G_4	$[s_3, s_4]$	$[s_2, s_3]$	$[s_3, s_5]$	$[s_2, s_3]$	$[s_3, s_4]$

In such case, we can utilize the proposed procedure II to prioritize these alternatives as follows.

Step 1: Utilize the GIULOWHM operator (whose weight vector $w = (0.3, 0.4, 0.3)^T$)

$$\begin{aligned}\tilde{a}_{ij} &= \text{GIULOWHM}_w(\langle v_1, u_1, \tilde{a}_{ij}^{(1)}\rangle, \langle v_2, u_2, \tilde{a}_{ij}^{(2)}\rangle, \langle v_3, u_3, \tilde{a}_{ij}^{(3)}\rangle), \\ &\quad i = 1, 2, 3, 4; j = 1, 2, 3, 4, 5\end{aligned}$$

to aggregate all the uncertain linguistic decision matrices $\tilde{A}^{(k)}$ $(k = 1, 2, 3)$ into a collective uncertain linguistic decision matrix $\tilde{A} = (a_{ij})_{4\times 5}$ (Table 4.8).

Table 4.6: Uncertain linguistic decision matrix $\tilde{A}^{(2)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$[s_6, s_7]$	$[s_8, s_9]$	$[s_1, s_2]$	$[s_3, s_5]$	$[s_1, s_3]$
G_2	$[s_2, s_4]$	$[s_2, s_3]$	$[s_3, s_5]$	$[s_2, s_4]$	$[s_4, s_5]$
G_3	$[s_1, s_2]$	$[s_2, s_3]$	$[s_1, s_2]$	$[s_2, s_4]$	$[s_5, s_6]$
G_4	$[s_3, s_5]$	$[s_4, s_6]$	$[s_2, s_3]$	$[s_1, s_3]$	$[s_4, s_6]$

Table 4.7: Uncertain linguistic decision matrix $\tilde{A}^{(3)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$[s_6, s_8]$	$[s_6, s_8]$	$[s_1, s_3]$	$[s_2, s_3]$	$[s_4, s_5]$
G_2	$[s_3, s_4]$	$[s_1, s_3]$	$[s_4, s_5]$	$[s_3, s_4]$	$[s_3, s_4]$
G_3	$[s_1, s_3]$	$[s_3, s_5]$	$[s_2, s_3]$	$[s_4, s_5]$	$[s_3, s_4]$
G_4	$[s_2, s_3]$	$[s_2, s_4]$	$[s_4, s_5]$	$[s_1, s_2]$	$[s_2, s_4]$

Step 2: Utilize the decision information given in matrix $\tilde{A}$, and the ULWHM operator

$$\begin{aligned}\tilde{a}_j &= \mathrm{ULWHM}_\omega(\tilde{a}_{1j}, \tilde{a}_{2j}, \tilde{a}_{3j}, \tilde{a}_{4j})\\ &= \frac{1}{\frac{\omega_1}{\tilde{a}_{1j}} \oplus \frac{\omega_2}{\tilde{a}_{2j}} \oplus \frac{\omega_3}{\tilde{a}_{3j}} \oplus \frac{\omega_4}{\tilde{a}_{4j}}},\ j = 1, 2, 3, 4, 5\end{aligned}$$

to derive the collective overall preference values $\tilde{a}_j$ of the alternative x_j:

$$\begin{aligned}&\tilde{a}_1 = [s_{2.49}, s_{3.98}], \tilde{a}_2 = [s_{2.67}, s_{4.59}], \tilde{a}_3 = [s_{1.78}, s_{3.48}],\\ &\tilde{a}_4 = [s_{2.36}, s_{4.09}], \tilde{a}_5 = [s_{2.77}, s_{4.61}].\end{aligned}$$

Step 3: To rank these collective overall preference values $\tilde{a}_j$ $(j = 1, 2, 3, 4, 5)$, we first compare each $\tilde{a}_j$ with all $\tilde{a}_i$ $(i = 1, 2, 3, 4, 5)$ by using (3.1), and develop

Table 4.8: Collective uncertain linguistic decision matrix $\tilde{A}$

	x_1	x_2	x_3	x_4	x_5
G_1	$[s_{5.66}, s_{7.27}]$	$[s_{7.00}, s_{8.67}]$	$[s_{1.18}, s_{2.67}]$	$[s_{2.61}, s_{4.17}]$	$[s_{1.82}, s_{4.11}]$
G_2	$[s_{2.22}, s_{3.64}]$	$[s_{1.82}, s_{3.62}]$	$[s_{3.98}, s_{5.77}]$	$[s_{2.50}, s_{4.26}]$	$[s_{3.64}, s_{4.88}]$
G_3	$[s_{1.54}, s_{2.67}]$	$[s_{2.78}, s_{4.11}]$	$[s_{1.18}, s_{2.50}]$	$[s_{3.08}, s_{4.93}]$	$[s_{3.92}, s_{4.96}]$
G_4	$[s_{2.61}, s_{3.92}]$	$[s_{2.50}, s_{4.14}]$	$[s_{2.67}, s_{3.95}]$	$[s_{1.18}, s_{2.61}]$	$[s_{2.86}, s_{4.62}]$

a complementary matrix:

$$\mathbf{P} = \begin{pmatrix} 0.500 & 0.384 & 0.690 & 0.503 & 0.363 \\ 0.616 & 0.500 & 0.776 & 0.611 & 0.484 \\ 0.310 & 0.224 & 0.500 & 0.327 & 0.201 \\ 0.497 & 0.389 & 0.673 & 0.500 & 0.370 \\ 0.637 & 0.516 & 0.799 & 0.630 & 0.500 \end{pmatrix}.$$

Summing all elements in each line of the matrix $\mathbf{P}$, we have

$$p_1 = 2.440,\ p_2 = 2.987,\ p_3 = 1.562,\ p_4 = 2.429,\ p_5 = 3.082$$

and then we rank $\tilde{a}_j$ $(j = 1, 2, 3, 4, 5)$ in descending order in accordance with the values of p_j $(j = 1, 2, 3, 4, 5)$:

$$\tilde{a}_5 > \tilde{a}_2 > \tilde{a}_1 > \tilde{a}_4 > \tilde{a}_3.$$

Step 4. Rank all alternatives x_j $(j = 1, 2, 3, 4, 5)$ by the ranking $\tilde{a}_j$ $(j = 1, 2, 3, 4, 5)$:

$$x_5 \succ x_2 \succ x_1 \succ x_4 \succ x_3$$

and thus the most desirable alternative is x_5.

4.4 Conclusions

In this chapter, we have defined the GILOWHM and GIULOWHM operators, by which each object processed consists of three components, where the first component represents the importance degree or character of the second component, and the second component is used to induce an ordering, through the first component, over the third components which are linguistic variables (or uncertain linguistic variables) and then aggregated. We have also shown that the ILOWHM operator and LOWHM operator are the special cases of the GILOWHM operator, and that the IULOWHM operator and the ULOWHM operator are the special cases of the GIULOWHM operator. In the process of aggregating information, these operators can avoid losing the original linguistic or uncertain linguistic information and thus ensure exactness and rationality of the aggregated results. Moreover, based on the GILOWHM and GIULOWHM operators respectively, we have developed two procedures for solving the MADM problems where all decision information about attribute values take the forms of linguistic variables or uncertain linguistic variables. To verify the effectiveness and practicality of the developed procedures, we have given an illustrative example.

Bibliography

[1] P.S. Bullen, D.S. Mitrinovi, P.M.Vasi, Means and their inequalities, The Netherlands, Reidel (1988).

[2] H.Y. Chen, C.L. Liu, Z.H. Shen, Induced ordered weighted harmonic averaging (IOWHA) operator and its application to combination forecasting method, Chinese Journal of Management Sciences 12 (2004), 35–40.

[3] C.Y. Chen, J.R.C. Hsu, C.W. Chen, Fuzzy logic derivation of neural network models with time delays in subsystems, Int. J. Artif. Intell. T. 14 (2005), 967–974.

[4] S.J. Chuu, Selecting the advanced manufacturing technology using fuzzy multiple attributes group decision making with multiple fuzzy information, Comput. Ind. Eng. 57 (2009), 1033–1042.

[5] M. Delgado, J.L. Verdegay, M.A. Vila, Linguistic decision making models, Int. J. Intell. Syst. 8 (1993), 351–370.

[6] J.C. Harsanyi, Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility, J. Polit. Economy 63 (1955), 309–321.

[7] F. Herrera, J.L. Verdegay, Linguistic assessments in group decision, Proceedings of 11th European Congress of Fuzzy Intelligent Technology (1993), 941–948.

[8] F. Herrera, E. Herrera-Viedma, J.L. Verdegay, A model of consensus in group decision making under linguistic assessments, Fuzzy Sets Syst. 78 (1996), 73–87.

[9] F. Herrera, E. Herrera-Viedma, J.L. Verdegay, Direct approach process in group decision making using linguistic OWA operators, Fuzzy Sets Syst. 79 (1996), 175–190.

[10] F. Herrera, E. Herrera-Viedma, Aggregation operators for linguistic weighted information, IEEE Trans. Syst. Man. Cybern. 27 (1997), 646–656.

[11] F. Herrera, E. Herrera-Viedma, J.L. Verdegay, A rational consensus model in group decision making linguistic assessments, Fuzzy Sets Syst. 88 (1997), 31–49.

[12] F. Herrera, E. Herrera-Viedma, L. Martinez, A fusion approach for managing multi-granularity linguistic term sets in decision making, Fuzzy Sets Syst. 114 (2000), 43–58 .

[13] F. Herrera, L. Martinez, A 2-tuple fuzzy linguistic representation model for computing with words, IEEE Trans. Fuzzy Syst. 8 (2000), 746–752.

[14] F. Herrera, E. Herrera-Viedma, Choice functions and mechanisms for linguistic preference relations, Eur. J. Oper. Res. 120 (2000), 144–161.

[15] F. Herrera, E. Herrera-Viedma, Linguistic decision analysis: steps for solving decision problems under linguistic information, Fuzzy Sets Syst. 115 (2000), 67–82.

[16] S.H. Kim, S.H. Choi, J.K. Kim, An intractive procedure for multiple attribute group decision making with incomplete information: range-based approach. Eur. J. Oper. Res. 118 (1999), 139–152.

[17] X.W. Liu, S.L. Han, Orness and parameterized RIM quantifier aggregation with OWA operators: A summary, International Journal of Approximate Reasoning 48 (2008), 77–97.

[18] J.H. Park, M.G. Gwak, Y.C. Kwun, Linguistic harmonic mean operators and their applications to group decision making, Int. J. Adv. Manuf. Technol. 57 (2010), 411–419.

[19] J.H. Park, M.G. Gwak, Y.C. Kwun, Uncertain linguistic harmonic mean operators and their applications to multiple attribute group decision making, Computing 93 (2011), 47–64.

[20] J.H. Park, M.G. Gwak, Y.C. Kwun, Generalized induced linguistic harmonic mean operators based approach to multiple attribute group decision making, Journal of Computational Analysis and Applications (2011), in press.

[21] J.H. Park, B.Y. Lee, M.J. Son, An approach based on the LOWHM and induced LOWHM operators to group decision making under linguistic information, Int. J. Fuzzy. Logic Intell. Syst. 20 (2010), 285–291.

[22] L.I. Tong, C.C. Chen, C.H. Wang, Optimization of multi-response processes using the VIKOR method, Int. J. Adv. Manuf. Technol. 31 (2007), 1049–1057.

[23] V. Torra, Y. Narukawa, Information fusion and aggregating information, Berlin, Springer (2007).

[24] G.W. Wei, W.D. Yi, Fuzzy linguistic hybrid harmonic mean operator and its application to software selection, Journal of Software 4 (2009), 1037–1042.

[25] Z.S. Xu, Q.L. Da, The uncertain OWA operators, Int. J. Intell. Syst. 17 (2002), 569–575.

[26] Z.S. Xu, Q.L. Da, An overview of operators for aggregationg information, Int. J. Intell. Syst. 18 (2003), 953–969.

[27] Z.S. Xu, Uncertain Multiple Attribute Decision Making: Methods and Applications, Tsinghua University Press, Beijing (2004).

[28] Z.S. Xu, A method based on linguistic aggregation operators for group decision making with linguistic preference relations, Inf. Sci. 166 (2004), 19–30.

[29] Z.S. Xu, Uncertian linguistic aggregation operators based approach to multiple attribute group decision making under uncertain linguistic environment, Inf. Sci. 168 (2004), 171–184.

[30] Z.S. Xu, An overview of methods for determining OWA weights, International Journal of Intelligent Systems 20 (2005), 843–865.

[31] Z.S. Xu, Deviation measures of linguistic preference relations in group decision making, Omega 33 (2005), 249–254.

[32] Z.S. Xu, An approach based on the uncertain LOWG and induced uncertain LOWG operators to group decision making with uncertain multiplicative linguistic preference relations, Decis. Support Syst. 41 (2006), 488–499.

[33] Z.S. Xu, Induced uncertain linguistic OWA operators applied to group decision making, Inf. Fusion 7 (2006), 231–238.

[34] Z.S. Xu, On generalized induced linguistic aggregation operators, Int. J. Intell. Syst. 35 (2006), 17–28.

[35] R.R. Yager, On ordered weighted averaging aggregation operators in multicriteria decision making, IEEE Trans. Syst. Man Cybern. 18 (1988), 183–190.

[36] R.R. Yager, Families and extension of OWA aggregation, Fuzzy Sets Syst. 59 (1993), 125–148.

[37] R.R. Yager, The induced fuzzy integral aggregation operator, Int. J. Intell. Syst. 17 (2002), 1049–1065.

[38] R.R. Yager, Induced aggregation operators, Fuzzy Sets Syst. 137 (2003), 59–69.

[39] R.R. Yager, Centered OWA operators, Soft Computing 11 (2007), 631–639.

[40] R.R. Yager, D.F. Filev, Induced ordered weighted averaging operators, IEEE Trans. Syst. Man. Cybern. 29 (1999), 141–150.

[41] R.R. Yager, J. Kacprzyk, The ordered weighted averaging operator: Theory and application, Boston, Kluwer (1997).

[42] K. Yeh, C.W. Chen, C.Y. Chen, Robustness design of time-delay fuzzy systems using fuzzy Lyapunov method, Appl. Math. Comput. 205 (2008), 568–577.

[43] D. Yong, Plant location selection based on fuzzy TOPSIS, Int. J. Adv. Manuf. Technol. 28 (2006), 839–844.

[44] K. Yoon, The propagation of errors in multiple-attribute decision making analysis: A practical approach, Journal of Operational Research Soceity 40 (1989), 681–686.

[45] L.A. Zadeh, Fuzzy sets, Information and Control 8 (1965), 338–353.

[46] L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-I, Inform. Sci. 8 (1975), 199–249.

[47] L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-II, Inform. Sci. 8 (1975), 301–357.

[48] L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-III, Inform. Sci. 9 (1976), 43–80.

감사의 글

본 논문이 완성되기까지 항상 학문의 연구를 독려하시고 밝은 가르침과 날카로운 깨우침으로 이끌어주신 박진한 지도교수님께 먼저 머리 숙여 깊은 감사드립니다. 저희 졸고에 대해 아낌없는 질책과 편달로 더 아름다운 논문을 요구하시며 마지막까지 정성을 쏟을 수 있게 지도해주신 권영철 교수님, 항상 격려와 지도 조언을 해주신 황진수 교수님, 많은 지도와 격려를 해주신 표용수 교수님, 조성진 교수님께 진심으로 감사드립니다.

아울러 음으로 양으로 도움을 주신 부경대학교 응용수학과의 여러 교수님들, 깊은 관심으로 지켜봐주신 여러 선배님들과 연구실의 많은 후배들에게도 깊은 감사의 뜻을 전하고 싶습니다.

오늘이 있기 까지 항상 묵묵히 지켜봐주시고 격려해 주신 부모님과 어머님, 오빠 그리고 가족들의 은혜에 작은 보답이 되었으면 합니다. 무엇보다도 본인이 학위과정을 마칠 수 있도록 많은 어려움을 인내하며 따뜻한 사랑으로 외조해 준 남편과 이 기쁨을 함께하고 싶으며, 더욱더 열심히 정진하여 진정한 이학박사가 되도록 노력할 것을 다짐합니다.