Thesis for the Degree of Doctor of Science

# Multiobjective Variational and Control Problems with Generalized Invexity

by

You Young Seo

Department of Applied Mathematics

The Graduate School

Pukyong National University

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# (일반화된 인벡시티를 갖는 다목적 변분문제와 제어문제)

Advisor: Prof. Do Sang Kim

by

You Young Seo

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A dissertation

by

You Young Seo



Approved by:

(Chairman) Jong Soo Jung, Ph. D.

(Member) Gue Myung Lee, Ph. D.

(Member) Sung Jin Cho, Ph. D.

(Member) Jin-Mun Jeong, Ph. D.

(Member) Do Sang Kim, Ph. D.

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일반화된 인벡시티를 갖는 다목적 변분문제와 제어문제

서 유 영

부 경 대 학 교 대 학 원 응 용 수 학 과

요 약

본 논문에서는 미분불가능한 다목적 최적화 문제인 변분문제와 제어문제에 대하여 일반화된 인벡스 함수조건 아래에서 최적성과 쌍대정리를 정립하였다. V-type I 인벡스 함수를 이용하여 다목적 변분문제의 효율해와 진성 효율해에 대한 충분최적정리들을 얻었고 미분불가능한 다목적 변분대칭 쌍대문제를 정식화하였다.

먼저, 적절한 인벡시티 가정하에서 효율해에 대한 약 쌍대정리, 강 썅대정리, 역 쌍대정리를 정립하였다. 또한, 여러 쌍대문제들을 만들고 그 문제들 사이의 쌍대정리가 성립함을 보였다. 인벡스 가정하에서 변분 문제에 대하여 효율성을 이용하여 약 쌍대정리, 강 쌍대정리, 역 쌍대정리를 정립하였다.

마지막으로, 미분불가능한 다목적 제어문제에 관하여 Kuhn-Tucker 필요 최적조건을 이용하여 V-$\rho$인벡스 함수조건아래에서 Kuhn-Tucker 충분최적정리를 얻었고 효율해에 대한 쌍대문제를 만들어 두 문제사이의 쌍대 정리가 성립함을 보였다.

# Chapter 1

# Introduction and Preliminaries

## 1

Multiobjective programming problems airse when more than one objective function is to be optimized over a given feasible region. Their optimums are the concept of solution that appears to be the natural extension of the optimization for a single objectives. In economic analysis [2], game [11] and system science, optimums are effective for treating such a multiplicity of values.

Under certain convexity assumptions and suitable constraint qualifications, the primal and dual problems have equal optimal objective values and hence it is possible to solve the primal problem indirectly by solving the dual problem. In 1961, Wolfe [38] formulated a dual problem for a single objective optimization problem on the basis of the Kuhn-Tucker necessary optimality conditions, which is now called the Wolfe dual problems, and proved weak and strong duality theorems. In 1981, Mond and Weir [32] gave another type of dual problem for a single objective optimization problem on the basis of the Kuhn-Tucker necessary optimality condition, which is now called the Mond-Weir dual problem. They proved weak, strong and converse duality theorems. Until now, many authors have formulated Wolfe type dual problems and Mond-Weir type dual problems for several kinds of optimization problems and have studied these problems for duality theorems. Also several authors have been interested in optimality conditions and duality theorems for nondifferentiable multiobjective programming problems, e.g., ([21], [26], [39], [40]).

Symmetric duality in nonlinear programming was introduced by Dorn [10] by defining a symmetric dual program for quadratic programs. Subsequently Dantzig, Eisenberg and Cottle [9] first formulated a pair of symmetric dual nonlinear programs in which the dual of the dual equals the prime, and established the weak and strong duality for these problems concerning convex concave functions. Mond and Weir [32] gave a different pair of symmetric dual nonlinear programming problems in which the pseudo-convexity and pseudo-concavity assumptions reduced to the convexity and concavity ones, and obtained the weak and strong duality for these problems. Weir and Mond [37] established two distinct pairs of multiple objective symmetric dual programs. Under additional assumptions the multiobjective programs are shown to be self dual.

Several authors have been interested in duality theorems for multiobjective variational problems. Bector and Husain [3]proved duality results for a multiobjective variational problem with convexity functions. Various generalizations of convexity have been made in the literature. Bector and Husain [3] proved duality results for a multiobjective variational problem with convexity functions.

On the other hand, giving continuous analogies of the results of [8], Mond and Hanson [28] extended the symmetric duality results to variational problems. Since the invexity conditions on functions were first defined by Hanson [28] as a generalization of convexity ones, many authors [5, 7, 15, 19, 34, 35] have extended the concepts of invexity and generalized invexity to those of the continuous versions of invexity and generalized invexity functions. Smart and Mond [35] extended the symmetric duality results to variational problems by using the continuous version of invexity. Kim and Lee [17] presented a pair of symmetric dual variational problems in the spirit of Mond and Weir [33] different from the one formulated by Smart and Mond [35], using the continuous version of pseudo-invexity which is a generalization of that of invexity. Kim, Lee and Lee [18] extended Kim and Lee's symmetric dual results [17] to the multiobjective symmetric dual variational problems under pseudo-invexity assumptions. Kim and Lee [17] formulated a pair of generalized symmetric dual variational problems. Weak, strong and converse duality theorems are established under invexity assumptions for these problems. Several known results [4, 19, 31, 41] are obtained as special cases.

Let $R^n$ be the $n$-dimensional Euclidean space and let $R^n_+$ be its nonnegative orthant. We denote the interior of $R^n_+$ by $intR^n_+$.

The following convention for inequalities will be used :

$$
\begin{aligned}
x = y \quad &\text{if and only if} \quad x_i = y_i, \quad i = 1, 2, \cdots, n, \\
x \leqq y \quad &\text{if and only if} \quad x_i \leqq y_i, \quad i = 1, 2, \cdots, n, \\
x \leq y \quad &\text{if and only if} \quad x_i \leqq y_i, \quad \text{but} \quad x \neq y, \\
x < y \quad &\text{if and only if} \quad x_i < y_i, \quad i = 1, 2, \cdots, n,
\end{aligned}
$$

The following problems is called a multiobjective variational problem

**(MVP)**:

(**MVP**)

$$\begin{aligned} \text{Minimize} \quad & \int_a^b f(t, x(t), \dot{x}(t))dt \\ & = \Big(\int_a^b f^1(t, x(t), \dot{x}(t))dt, \cdots, \int_a^b f^p(t, x(t), \dot{x}(t))dt\Big) \\ \text{subject to} \quad & x(a) = t_0,\ x(b) = t_f, \\ & g(t, x(t), \dot{x}(t)) \leq 0,\ t \in I. \end{aligned}$$

Let $I = [a, b]$ be a real interval, $f : I \times R^n \times R^n \to R^p$ and $g : I \times R^n \times R^n \to R^m$ are continuously differentiable functions. We shall denote the feasible set of (**MVP**) by $X_0 := \{x \in C(I, R^n)|\ x(a) = t_0,\ x(b) = t_f,\ g(t, x(t), \dot{x}(t)) \leq 0\}$. Optimization of **(MVP)**, which lets $C(I, R^n)$ denote the space of piecewise smooth functions $x(t)$ with norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, is finding efficient solutions defined as follows.

**Definition 1.1** *A point $x^*(t) \in X_0$ is said to be an efficient solution of the problem* **(MVP)** *if there exists no other $x(t) \in X_0$ such that*

$$\int_a^b f^i(t, x(t), \dot{x}(t))dt \leq \int_a^b f^i(t, x^*(t), \dot{x}^*(t))dt,\ \forall i = 1, \cdots, p$$

*and*

$$\int_a^b f^{i_0}(t, x(t), \dot{x}(t))dt < \int_a^b f^{i_0}(t, x^*(t), \dot{x}^*(t))dt,\ \textit{for some } i_0 = 1, \cdots, p.$$

**Definition 1.2** *A point* $x^*(t) \in X_0$ *is said to be a weak efficient solution of the problem* **(MVP)** *if there does not exist* $x(t) \in X_0$ *such that*

$$\int_a^b f^i(t, x(t), \dot{x}(t))dt < \int_a^b f^i(t, x^*(t), \dot{x}^*(t))dt, \ \forall i = 1, \cdots, p.$$

**Definition 1.3** *(Geoffrion [13]) A point* $x^*(t) \in X_0$ *is said to be a properly efficient solution of* **(MVP)** *if there exist a scalar* $M > 0$ *such that*

$$\int_a^b f^i(t, x^*(t), \dot{x}^*(t))dt - \int_a^b f^i(t, x(t), \dot{x}(t))dt$$

$$\leq M\{\int_a^b (f^i(t, x(t), \dot{x}(t))dt - \int_a^b (f^i(t, x^*(t), \dot{x}^*(t))dt\}, \forall i = 1, \cdots, p$$

*and*

$$\int_a^b f^{i_0}(t, x^*(t), \dot{x}^*(t))dt - \int_a^b f^{i_0}(t, x(t), \dot{x}(t))dt$$

$$< M\{\int_a^b (f^{i_0}(t, x(t), \dot{x}(t))dt - \int_a^b (f^{i_0}(t, x^*(t), \dot{x}^*(t))dt\},$$

$$\textit{for some } i_0 = 1, \cdots, p.$$

The multiobjective dual variational problem **(MMVD)** (:Mond-Weir multiobjective dual variational problem) for **(MVP)** can be expressed as the following form:

(**MMVD**)

$$\text{Maximize} \quad \int_a^b f(t, y(t), \dot{y}(t))dt$$

$$= \Big(\int_a^b f^1(t, y(t), \dot{y}(t))dt, \cdots, \int_a^b f^p(t, y(t), \dot{y}(t))dt\Big)$$

subject to $y(a) = t_0,\ y(b) = t_f,$

$$\sum_{i=1}^{p} \tau_i \{f_y^i(t, y(t), \dot{y}(t)) - \frac{d}{dt} f_{\dot{y}}^i(t, y(t), \dot{y}(t))\}$$

$$+ \sum_{j=1}^{m} \lambda_j(t) \{g_y^j(t, y(t), \dot{y}(t)) - \frac{d}{dt} g_{\dot{y}}^j(t, y(t), \dot{y}(t))\} = 0,$$

$$\int_a^b \lambda_j(t) g^j(t, y(t), \dot{y}(t)) \geqq 0,\ \forall j = 1, \cdots m,$$

$$\tau_i \in R^p, \tau_i \geq 0,$$

$$\lambda(t) \in R^p, \lambda(t) \geq 0,\ t\ \in I,$$

where $\lambda(t)$ is a function from I into $R^m$.

Efficient solutions of **(MMVD)** can be defined analogously in definition 1.1 as follows:

**Definition 1.4** *A feasible solution* $(x^*(t), \tau^*, \lambda^*(t))$ *is said to be an efficient solution of the problem* **(MMVD)** *if there does not exist a feasible solution* $(x(t), \tau, \lambda(t))$ *of* **(MMVD)** *such that*

$$\int_a^b f^i(t, x^*(t), \dot{x}^*(t))dt \leq \int_a^b f^i(t, x(t), \dot{x}(t))dt,\ \forall i = 1, \cdots, p$$

*and*

$$\int_a^b f^{i_0}(t, x^*(t), \dot{x}^*(t))dt < \int_a^b f^{i_0}(t, x(t), \dot{x}(t))dt,\ \textit{for some } i_0 = 1, \cdots, p.$$

The control problem is to choose, under given conditions, a control vector $u(t)$, such that the state vector $x(t)$ is brought from some specified initial state $x(a) = t_0$ to some specified final state $x(b) = t_f$ in such a way as to minimize a given functional.

The following problem is called a multiobjective control problem **(MCP)**:

(**MCP**)

$$\text{Minimize} \quad \Big(\int_a^b f^1(t, x(t), u(t))dt, \cdots, \int_a^b f^p(t, x(t), u(t))dt\Big)$$

$$\text{subject to} \quad x(a) = t_0,\ x(b) = t_f,$$

$$g(t, x(t), u(t)) \leq 0,\ t \in I,$$

$$h(t, x(t), u(t)) = \dot{x}(t),\ t \in I.$$

Each $f^i : I \times R^n \times R^m \to R$ for $i = 1, \cdots, p$, $g^j : I \times R^n \times R^m \to R$ for $j = 1, \cdots, k$ and $h^r : I \times R^n \times R^m \to R$ for $r = 1, \cdots, n$.

Let $X := \{x \in C(I, R^n)|\ x(a) = t_0,\ x(b) = t_f,\ g(t, x(t), u(t)) \leq 0,$ $h(t, x(t), u(t)) = \dot{x}(t)\}$, where $C(I, R^n)$ denotes the space of piecewise smooth functions $x(t)$ with norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, be the set of feasible solutions of problem **(MCP)**. Optimization of **(MCP)** is finding efficient solutions defined as follows.

**Definition 1.5** *A feasible* $(x^*(t), u^*(t))$ *of* **(MCP)** *is said to be an efficient solution of* **(MCP)** *if there does not exist a feasible solution* $(x(t), u(t))$ *of* **(MCP)** *such that*

$$\int_a^b f^i(t, x(t), u(t))dt \leq \int_a^b f^i(t, x^*(t), u^*(t))dt,\ \forall i = 1, \cdots, p$$

*and*

$$\int_a^b f^{i_0}(t, x(t), u(t))dt < \int_a^b f^{i_0}(t, x^*(t), u^*(t))dt, \quad \text{for some} \ \ i_0 = 1, \cdots, p.$$

The multiobjective dual control problem **(MMCD)** (:Mond-Weir multiobjective control dual problem) for **(MCP)** can be expressed as the following form:

(**MMCD**)

$$\text{Maximize} \quad \Big(\int_a^b f^1(t,x(t),u(t))dt, \cdots, \int_a^b f^p(t,x(t),u(t))dt\Big)$$

$$\text{subject to} \quad x(a)=t_0,\ x(b)=t_f,$$

$$\sum_{i=1}^{p}\tau_i f_x^i(t,x(t),u(t))+\sum_{j=1}^{k}\lambda_j(t)g_x^j(t,x(t),u(t))$$

$$+\sum_{r=1}^{n}\mu_r(t)h_x^r(t,x(t),u(t))+\dot{\mu}(t)=0,\ t\in I,$$

$$\sum_{i=1}^{p}\tau_i f_u^i(t,x(t),u(t))+\sum_{j=1}^{k}\lambda_j(t)g_u^j(t,x(t),u(t))$$

$$+\sum_{r=1}^{n}\mu_r(t)h_u^r(t,x(t),u(t))=0,\ t\in I,$$

$$\int_a^b\sum_{r=1}^{k}\mu_r(t)\{h^r(t,x(t),u(t))-\dot{x}(t)\}dt \geqq 0,\ t\in I,$$

$$\int_a^b\sum_{j=1}^{k}\lambda_j(t)g^j(t,x(t),u(t))dt \geqq 0,\ t\in I,$$

$$\lambda(t)\geqq 0,\ t\in I,$$

$$\tau_i \geqq 0, \sum_{i=1}^{p}\tau_i=1.$$

where $\lambda(t)$ is a function from I into $R^m$ and $\mu(t)$ is a function from I into $R^r$. Here $\lambda(t)$ and $\mu(t)$ are required to be continuous except perhaps at points of discontinuity of $u(t)$.

We can define efficient solutions of **(MMCD)** by ways similar to the case of **(MCP)**:

**Definition 1.6** *A feasible solution* $(x^*(t), u^*(t), \tau^*, \lambda^*(t), \mu^*(t))$ *of* **(MMCD)** *is said to be an efficient solution of* **(MMCD)** *if there does not exist a feasible solution* $(x(t), u(t), \tau, \lambda(t), \mu(t))$ *of* **(MMCD)** *such that*

$$\int_a^b f^i(t, x(t), u(t))dt \geq \int_a^b f^i(t, x^*(t), u^*(t))dt, \ \forall i = 1, \cdots, p$$

*and*

$$\int_a^b f^{i_0}(t, x(t), u(t))dt > \int_a^b f^{i_0}(t, x^*(t), u^*(t))dt, \ \text{for some} \ \ i_0 = 1, \cdots, p.$$

**Definition 1.7** *[30] The support function* $s(x|D)$*, being convex and everywhere finite, has a subdifferential, that is, there exists* $z$ *such that*

$$s(y|D) \geq s(x|D) + z^T(y - x) \ \text{for all} \ y \in D.$$

*Equivalently,*

$$z^T x = s(x|D).$$

*The subdifferential of* $s(x|D)$ *is given by*

$$\partial s(x|D) := \{z \in D : z^T x = s(x|D)\}.$$

*For any set $S \subset \mathbb{R}^n$, the normal cone to $S$ at a point $x \in S$ is defined by*

$$N_S(x) := \{y \in \mathbb{R}^n : y^T(z - x) \leq 0 \text{ for all } z \in S\}.$$

*It is readily verified that for a compact convex set $D$, $y$ is in $N_D(x)$ if and only if $s(y|D) = x^T y$, or equivalently, $x$ is in the subdifferential of $S$ at $y$.*

In this thesis, we formulate the nondifferentiable multiobjective variational problem and control problem for generalized invex functions. We obtain sufficient optimality theorems and duality theorems for nondifferentiable multiobjective variational problem involving generalized type I invex functions. Also, we obtain sufficient optimality theorems for nondifferentiable multiobjective control problem involving generalized V-$\rho$-invex functions.

This thesis consists of four chapters.

In Chapter 2, we formulate nondifferentiable multiobjective variational problem with equality and inequality constraints. We introduce vector type invexity along the lines of Jeyakumar and Mond [15] extending the pseudo, quasi, quasi-pseudo, pseudo-quasi type-I invexity of Kaul et al. [16]. Some sufficiency results are established. We establish the Mond-Weir type dual and general Mond-Weir type dual problems and prove weak, strong and converse duality theorems under generalized V-type I assumptions. As a special case of our duality results, we obtain the Wolfe type duality theorems.

In Chapter 3, a pair of nondifferentiable multiobjective symmetric dual variational problem is formulated. Our duality results improve and extend ones in Smart and Mond. Under suitable invexity assumptions, we establish the weak, strong and converse duality theorems for efficient solutions.

In Chapter 4, we obtain duality results for multiobjective control problems under V-$\rho$-invexity (V-$\rho$-pseudo invexity, V-$\rho$-quasi invexity) assumptions. The results of the present section extend the work of Mishra and Mukherjee [24] to more generalized V-$\rho$-invex functions. It is also shown that for V-$\rho$-invex functions, the necessary conditions for optimality in the control problem are also sufficient. Moreover, we formulate Wolfe formulate nondifferentiable multiobjective control problems. For these problems, Wolfe and Mond Weir type duals are proposed. We establish their duality relations.

# Chapter 2

# Nondifferentiable Multiobjective Variational Problem with Generalized Type I Invexity

## 2

### 2.1 Introduction

The following problems is called a nondifferentiable multiobjective variational problem with equality or inequality constraints:

**(NMVP)**

$$\begin{aligned} \text{Minimize} \quad & \int_a^b (f(t,x(t),\dot{x}(t)) + s(x(t)|D))dt \\ & = \Big( \int_a^b (f^1(t,x(t),\dot{x}(t)) + s(x(t)|D_1))dt, \\ & \qquad \cdots, \int_a^b (f^p(t,x(t),\dot{x}(t)) + s(x(t)|D_p))dt \Big) \\ \text{subject to} \quad & x(a) = t_0, \ x(b) = t_f, \\ & g^j(t,x(t),\dot{x}(t)) \leqq 0, \ j = 1,\cdots,m, \ t \in I, \end{aligned}$$

**(NMVPE)**

$$\text{Minimize} \quad \int_a^b (f(t,x(t),\dot{x}(t)) + s(x(t)|D))dt$$

$$= \Big(\int_a^b (f^1(t,x(t),\dot{x}(t)) + s(x(t)|D_1))dt,$$

$$\cdots, \int_a^b (f^p(t,x(t),\dot{x}(t)) + s(x(t)|D_p))dt\Big)$$

$$\text{subject to} \quad x(a) = t_0,\ x(b) = t_f,$$

$$g^j(t,x(t),\dot{x}(t)) \leqq 0,\ j = 1,\cdots,m,$$

$$h(t,x(t),\dot{x}(t)) = 0,\ \forall t \in I,$$

where $f : I \times R^n \times R^n \to R^p$, $g : I \times R^n \times R^n \to R^m$ and $h : I \times R^n \times R^n \to R^q$, are assumed to continuously differentiable functions. Let $I = [a,b]$ be a real interval. In order to consider $f(t,x(t),\dot{x}(t))$, where $x(t) : I \to R^n$ with derivative $\dot{x}(t)$, denote the partial derivative of $f$ with respect to $t, x(t)$, and $\dot{x}(t)$, respectively, by $f_t, f_x$ and $f_{\dot{x}}$, such that

$$f_x = \Big(\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_n}\Big), \quad f_{\dot{x}} = \Big(\frac{\partial f}{\partial \dot{x}_1},\cdots,\frac{\partial f}{\partial \dot{x}_n}\Big)$$

The partial derivatives of other functions used will be written similarly.

Let $C(I,R^n)$ denote the space of piecewise smooth functions $x(t)$ with norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differentiation operator $D$ is given by

$$u = Dx \Longleftrightarrow x(t) = t_0 + \int_a^t u(s)ds,$$

in which $\alpha(t)$ is a given boundary value. Therefore $D = \frac{d}{dt}$ except at discontinuities.

In this chapter, we are concerned with the nondifferentiable multiobjective variational problem with equality and inequality constraints. We introduce new classes of generalized V-type I vector valued functions for variational problems and consider the nondifferentiable multiobjective variational problem **(NMVP)** and **(NMVPE)**. A number of sufficiency results are established using Lagrange multiplier conditions under various types of generalized V-type I requirements. Duality theorem are proved for Mond-Weir and general Mond-Weir type duality under the above generalized V-type I assumptions and their generalizations. As special case of our duality results, we obtain the Wolfe type duality theorems.

## 2.2 Definitions and Preliminaries

Let us now denote by $X_0$ be the set of all feasible solutions of the problem **(NMVP)** given by

$$X_0 := \{x(t) \in C(I, R^n)|\ x(a) = t_0,\ x(b) = t_f,\ g(t, x(t), \dot{x}(t)) \leq 0\},$$

and $X_1$ be the set of all feasible solutions of the problem **(NMVPE)** given by

$$\begin{aligned} X_1 := \{x(t) \in C(I, R^n)| \quad & x(a) = t_0,\ x(b) = t_f, \\ & g(t, x(t), \dot{x}(t)) \leq 0,\ h(t, x(t), \dot{x}(t)) = 0\}. \end{aligned}$$

Following Aghezzaf and Hachimi [1] we define generalized type I invex functions for variational problems as follows.

**Definition 2.1** *$(f(x), g(x))$ is said to be V-type I invex with respect to $\eta(t)$, $\alpha_i(t)$ and $\beta_j(t)$ at $x^*(t)$ if for all $i = 1, \cdots, p$ and $j = 1, \cdots, m$ there exists a differentiable vector function $\eta(t) \in R^n$, and real-valued functions $\alpha_i(t) \in R_+ \setminus \{0\}$ and $\beta_j(t) \in R_+ \setminus \{0\}$ such that*

$$\int_a^b (f^i(t, x(t), \dot{x}(t)) + s(x(t)|D_i))dt - \int_a^b (f^i(t, x^*(t), \dot{x}^*(t)) + s(x^*(t)|D_i))dt$$

$$\geq \int_a^b \alpha_i(x(t),\ x^*(t), \dot{x}(t), \dot{x}^*(t))\eta(t, x(t),\ x^*(t), \dot{x}(t), \dot{x}^*(t))\{f_x^i(t, x^*(t), \dot{x}^*(t))$$

$$+w_i(t)) - \frac{d}{dt} f_{\dot{x}}^i(t, x^*(t), \dot{x}^*(t))\}dt \tag{2.1}$$

*and*

$$-\int_a^b g^j(t, x^*(t), \dot{x}^*(t))dt$$

$$\geq \int_a^b \beta_j(x(t),\ x^*(t), \dot{x}(t), \dot{x}^*(t))\eta(t, x(t),\ x^*(t), \dot{x}(t), \dot{x}^*(t))\{g_x^j(t, x^*(t), \dot{x}^*(t))$$

$$-\frac{d}{dt} g_{\dot{x}}^j(t, x^*(t), \dot{x}^*(t))\}dt. \tag{2.2}$$

*for every $x(t)$.*

*If in the above definition, (2.2) is a strict inequality, then we say that $(f(x), g(x))$ is semistrictly V-type I invex at $x^*(t)$.*

*We now define and introduce the notions of weak strictly-pseudoquasi V-type I invexity, weak quasistrictly-pseudo V-type I invexity and weak strictly-pseudo V-type I invexity for* **(NMVP)**.

**Definition 2.2** *$(f(x), g(x))$ is said to be weak strictly-pseudoquasi V-type I invex with respect to $\eta(t)$ , $\alpha_i(t)$ and $\beta_j(t)$ at $x^*(t)$ if there exists a differentiable vector function $\eta(t) \in R^n$ , and $\alpha_i(t) \in R_+ \setminus \{0\}$ and $\beta_j(t) \in R_+ \setminus \{0\}$ , such that for some vector $\tau_i \in R^p$ ,$\tau_i \geq 0$ and piecewise smooth function $\lambda(t) : I \to R^m$, $\lambda(t) \geq 0$,*

$$\int_a^b \sum_{i=1}^p \tau_i \alpha_i(x(t), x^*(t), \dot{x}(t), \dot{x}^*(t))\{f^i(t, x(t), \dot{x}(t)) + s(x(t)|D_i)\}dt$$

$$< \int_a^b \sum_{i=1}^p \tau_i \alpha_i(x(t), x^*(t), \dot{x}(t), \dot{x}^*(t))\{f^i(t, x^*(t), \dot{x}^*(t)) + s(x^*(t)|D_i)\}dt$$

$$\Longrightarrow \int_a^b \sum_{i=1}^p \tau_i \eta(t, x(t), x^*(t), \dot{x}(t), \dot{x}^*(t))\{f_x^i(t, x^*(t), \dot{x}^*(t)) + w_i(t)$$

$$-\frac{d}{dt} f_{\dot{x}}^i(t, x^*(t), \dot{x}^*(t))\}dt < 0$$

*and*

$$-\int_a^b \sum_{j=1}^m \lambda_j(t)\beta_j(x(t), x^*(t), \dot{x}(t), \dot{x}^*(t)) g^j(t, x^*(t), \dot{x}^*(t))dt \leq 0$$

$$\Longrightarrow \int_a^b \sum_{j=1}^m \lambda_j(t)\eta(t, x(t), x^*(t), \dot{x}(t), \dot{x}^*(t))\{g_x^j(t, x^*(t), \dot{x}^*(t))$$

$$-\frac{d}{dt} g_{\dot{x}}^j(t, x^*(t), \dot{x}^*(t))\}dt \leq 0.$$

*This definition is a slight extension of that of the weak strictly-pseudoquasi V-type I functions [1].*

**Definition 2.3** $(f(x), g(x))$ *is said to be weak quasi strictly-pseudo V-type I invex with respect to* $\eta(t)$ , $\alpha_i(t)$ *and* $\beta_j(t)$ *at* $x^*(t)$ *if there exists a differentiable vector function* $\eta(t) \in R^n$ *, and* $\alpha_i(t) \in R_+ \setminus \{0\}$ *and* $\beta_j(t) \in R_+ \setminus \{0\}$ *, such that for some vector* $\tau_i \in R^p$ ,$\tau_i \geq 0$ *and piecewise smooth function* $\lambda(t) : I \to R^m,\ \lambda(t) \geq 0,$

$$\int_a^b \sum_{i=1}^p \tau_i \alpha_i(x(t), x^*(t), \dot{x}(t), \dot{x}^*(t))\{f^i(t, x(t), \dot{x}(t)) + s(x(t)|D_i)\}dt$$

$$< \int_a^b \sum_{i=1}^p \tau_i \alpha_i(x(t), x^*(t), \dot{x}(t), \dot{x}^*(t))\{f^i(t, x^*(t), \dot{x}^*(t)) + s(x^*(t)|D_i)\}dt$$

$$\Longrightarrow \int_a^b \sum_{i=1}^p \tau_i \eta(t, x(t), x^*(t), \dot{x}(t), \dot{x}^*(t))\{f_x^i(t, x^*(t), \dot{x}^*(t)) + w_i(t)$$

$$-\frac{d}{dt} f_{\dot{x}}^i(t, x^*(t), \dot{x}^*(t))\}dt \leq 0$$

*and*

$$-\int_a^b \sum_{j=1}^m \lambda_j(t)\beta_j(x(t), x^*(t), \dot{x}(t), \dot{x}^*(t)) g^j(t, x^*(t), \dot{x}^*(t))dt \leq 0$$

$$\Longrightarrow \int_a^b \sum_{j=1}^m \lambda_j(t)\eta(t, x(t), x^*(t), \dot{x}(t), \dot{x}^*(t))\{g_x^j(t, x^*(t), \dot{x}^*(t))$$

$$-\frac{d}{dt} g_{\dot{x}}^j(t, x^*(t), \dot{x}^*(t))\}dt < 0.$$

**Definition 2.4** $(f(x), g(x))$ *is said to be weak strictly-pseudo V-type I invex with respect to* $\eta(t)$ , $\alpha_i(t)$ *and* $\beta_j(t)$ *at* $x^*(t)$ *if there exists a differentiable* *vector function* $\eta(t) \in R^n$ , *and* $\alpha_i(t) \in R_+ \setminus \{0\}$ *and* $\beta_j(t) \in R_+ \setminus \{0\}$ , *such that for some vector* $\tau_i \in R^p$ ,$\tau_i \geq 0$ *and piecewise smooth function* $\lambda(t) : I \to R^m,\ \lambda(t) \geq 0,$

$$\int_a^b \sum_{i=1}^{p} \tau_i \alpha_i(x(t), x^*(t), \dot{x}(t), \dot{x}^*(t))\{f^i(t, x(t), \dot{x}(t)) + s(x(t)|D_i)\}dt$$

$$< \int_a^b \sum_{i=1}^{p} \tau_i \alpha_i(x(t), x^*(t), \dot{x}(t), \dot{x}^*(t))\{f^i(t, x^*(t), \dot{x}^*(t)) + s(x^*(t)|D_i)\}dt$$

$$\Longrightarrow \int_a^b \sum_{i=1}^{p} \tau_i \eta(t, x(t), x^*(t), \dot{x}(t), \dot{x}^*(t))\{f^i_x(t, x^*(t), \dot{x}^*(t)) + w_i(t)$$

$$-\frac{d}{dt} f^i_{\dot{x}}(t, x^*(t), \dot{x}^*(t))\}dt < 0$$

*and*

$$-\int_a^b \sum_{j=1}^{m} \lambda_j(t)\beta_j(x(t), x^*(t), \dot{x}(t), \dot{x}^*(t)) g^j(t, x^*(t), \dot{x}^*(t))dt \leq 0$$

$$\Longrightarrow \int_a^b \sum_{j=1}^{m} \lambda_j(t)\eta(t, x(t), x^*(t), \dot{x}(t), \dot{x}^*(t))\{g^j_x(t, x^*(t), \dot{x}^*(t))$$

$$-\frac{d}{dt} g^j_{\dot{x}}(t, x^*(t), \dot{x}^*(t))\}dt < 0.$$

*Following Hanson, Pini and Singh [14] we define vector type I invexity for variational problems as follows.*

**Definition 2.5** $(f(x), g(x))$ *is said to be quasi V-type I invex at* $x^*(t)$ *with respect to* $\eta(t)$ , $\alpha_i(t)$ *and* $\beta_j(t)$ *at* $x^*(t)$ *if there exists a differentiable* *vector function* $\eta(t) \in R^n$ *and* $\alpha_i(t) \in R_+ \setminus \{0\}$ *and* $\beta_j(t) \in R_+ \setminus \{0\}$ *, such that for some vector* $\tau_i \in R^p$ *,*$\tau_i \geq 0$ *and piecewise smooth function* $\lambda(t) : I \to R^m,\ \lambda(t) \geq 0,$

$$\int_a^b \sum_{i=1}^{p} \tau_i \alpha_i(x(t), x^*(t), \dot{x}(t), \dot{x}^*(t))\{f^i(t, x(t), \dot{x}(t)) + s(x(t)|D_i)\}dt$$

$$\leq \int_a^b \sum_{i=1}^{p} \tau_i \alpha_i(x(t), x^*(t), \dot{x}(t), \dot{x}^*(t))\{f^i(t, x^*(t), \dot{x}^*(t)) + s(x^*(t)|D_i)\}dt$$

$$\Longrightarrow \int_a^b \sum_{i=1}^{p} \tau_i \eta(t, x(t), x^*(t), \dot{x}(t), \dot{x}^*(t))\{f_x^i(t, x^*(t), \dot{x}^*(t)) + w_i(t)$$

$$-\frac{d}{dt} f_{\dot{x}}^i(t, x^*(t), \dot{x}^*(t)\}dt \leq 0 \tag{2.3}$$

*and*

$$\int_a^b \sum_{j=1}^{m} \lambda_j(t) \beta_j(x(t), x^*(t), \dot{x}(t), \dot{x}^*(t)) g^j(t, x^*(t), \dot{x}^*(t))dt \geq 0$$

$$\Longrightarrow \int_a^b \sum_{j=1}^{m} \lambda_j(t) \eta(t, x(t), x^*(t), \dot{x}(t), \dot{x}^*(t))\{g_x^j(t, x^*(t), \dot{x}^*(t))$$

$$-\frac{d}{dt} g_{\dot{x}}^j(t, x^*(t), \dot{x}^*(t)\}dt \leq 0. \tag{2.4}$$

*If* $(f(x), g(x))$ *is quasi V-type I invex at each* $x^*(t)$ *, we say* $(f(x), g(x))$ *is quasi V-type I invex on* $I \times R^n \times R^n$*. If the second (implied) inequality in (2.3) is strict* $(x(t) \neq x^*(t))$ $(f(x), g(x))$ *is semi strictly quasi V-type I invex at* $x^*(t)$ *or on* $I \times R^n \times R^n$ *as the case may be.*

**Definition 2.6** *$(f(x), g(x))$ is said to be pseudo V-type I invex at $x^*(t)$ with respect to $\eta(t)$ , $\alpha_i(t)$ and $\beta_j(t)$ at $x^*(t)$ if there exists a differentiable vector function $\eta(t) \in R^n$ and $\alpha_i(t) \in R_+ \setminus \{0\}$ and $\beta_j(t) \in R_+ \setminus \{0\}$ , such that for some vector $\tau_i \in R^p$ ,$\tau_i \geq 0$ and piecewise smooth function $\lambda(t) : I \to R^m,\ \lambda(t) \geq 0$, the implications*

$$\int_a^b \sum_{i=1}^p \tau_i \eta(t, x(t), x^*(t), \dot{x}(t), \dot{x}^*(t))\{f_x^i(t, x^*(t), \dot{x}^*(t)) + w_i(t)$$

$$-\frac{d}{dt} f_{\dot{x}}^i(t, x^*(t), \dot{x}^*(t))\}dt \geq 0$$

$$\Longrightarrow \int_a^b \sum_{i=1}^p \tau_i \alpha_i(x(t), x^*(t), \dot{x}(t), \dot{x}^*(t))\{f^i(t, x(t), \dot{x}(t)) + s(x(t)|D_i)\}dt$$

$$\geq \int_a^b \sum_{i=1}^p \tau_i \alpha_i(x(t), x^*(t), \dot{x}(t), \dot{x}^*(t))\{f^i(t, x^*(t), \dot{x}^*(t)) + s(x^*(t)|D_i)\}dt \tag{2.5}$$

*and*

$$\int_a^b \sum_{j=1}^m \lambda_j(t) \eta(t, x(t), x^*(t), \dot{x}(t), \dot{x}^*(t))\{g_x^j(t, x^*(t), \dot{x}^*(t))$$

$$-\frac{d}{dt} g_{\dot{x}}^j(t, x^*(t), \dot{x}^*(t))\}dt \geq 0$$

$$\Longrightarrow \int_a^b \sum_{j=1}^m \lambda_j(t) \beta_j(x(t), x^*(t), \dot{x}(t), \dot{x}^*(t)) g^j(t, x^*(t), \dot{x}^*(t))dt \leq 0. \tag{2.6}$$

*hold. If* $(f(x), g(x))$ *is pseudo V-type I invex at each* $x^*(t)$ *,we say* $(f(x), g(x))$ *pseudo V-type I invex on* $I \times R^n \times R^n$*.If the second (implied) inequality in (2.5)(Eq.(2.6)) is strict,* $(f(x), g(x))$ *is semi strictly pseudo V-type I invex in* $f(x)$ *(in* $g(x)$*) at* $x(t)$ *or on* $I \times R^n \times R^n$ *as the case may be. If the second (implied) inequality in (2.5)and (2.6) are both strict we say that* $(f(x), g(x))$ *is strictly pseudo V-type I invex at* $x^*(t)$ *or on* $I \times R^n \times R^n$ *as the case may be.*

**Definition 2.7** $(f(x), g(x))$ *is said to be quasi pseudo V-type I invex at* $x^*(t)$ *with respect to* $\eta(t)$ *,* $\alpha_i(t)$ *and* $\beta_j(t)$ *at* $x^*(t)$ *if there exists a differentiable vector function* $\eta(t) \in R^n$ *and* $\alpha_i(t) \in R_+ \setminus \{0\}$ *and* $\beta_j(t) \in R_+ \setminus \{0\}$ *, such that for some vector* $\tau_i \in R^p$ *,*$\tau_i \geq 0$ *and piecewise smooth function* $\lambda(t) : I \to R^m$*,* $\lambda(t) \geq 0$*, the implications*

$$\int_a^b \sum_{i=1}^p \tau_i \alpha_i(x(t), x^*(t), \dot{x}(t), \dot{x}^*(t))\{f^i(t, x(t), \dot{x}(t)) + s(x(t)|D_i)\}dt$$

$$\leq \int_a^b \sum_{i=1}^p \tau_i \alpha_i(x(t), x^*(t), \dot{x}(t), \dot{x}^*(t))\{f^i(t, x^*(t), \dot{x}^*(t)) + s(x^*(t)|D_i)\}dt$$

$$\Longrightarrow \int_a^b \sum_{i=1}^p \tau_i \eta(t, x(t), x^*(t), \dot{x}(t), \dot{x}^*(t))\{f_x^i(t, x^*(t), \dot{x}^*(t)) + w_i(t)$$

$$-\frac{d}{dt} f_{\dot{x}}^i(t, x^*(t), \dot{x}^*(t))\}dt \leq 0 \tag{2.7}$$

*and*

$$\int_a^b \sum_{j=1}^{m} \lambda_j(t)\eta(t,x(t),x^*(t),\dot{x}(t),\dot{x}^*(t))\{g_x^j(t,x^*(t),\dot{x}^*(t))$$

$$-\frac{d}{dt}g_{\dot{x}}^j(t,x^*(t),\dot{x}^*(t))\}dt \geq 0$$

$$\Longrightarrow \int_a^b \sum_{j=1}^{m} \lambda_j(t)\beta_j(x(t),x^*(t),\dot{x}(t),\dot{x}^*(t))g^j(t,x^*(t),\dot{x}^*(t))dt \leq 0. \tag{2.8}$$

*hold. If $(f(x), g(x))$ is quasi pseudo V-type I invex at each $x^*(t)$ ,we say $(f(x), g(x))$ quasi pseudo V-type I invex on $I \times R^n \times R^n$ .If the second (implied) inequality in (2.8) is strict,we say that $(f(x), g(x))$ is quasi strictly pseudo V-type I invex at $x^*(t)$ or on $I \times R^n \times R^n$ as the case may be.*

In order to prove the strong duality theorem we will invoke the following lemmas due to Changkong and Haimes [6].

**Lemma 2.1** *A point $x^*(t) \in X_0$ is an efficient solution for* **(NMVP)** *if and only if $x^*(t)$ solves $\forall k = 1, \cdots, p$,*

$$\textbf{NMVP}_k(x^*(t)) \qquad \textit{Minimize} \int_a^b (f^k(t,x(t),\dot{x}(t)) + s(x(t)|D_k))dt$$

$$\textit{subject to} \quad x(a) = t_0,\ x(b) = t_f,$$

$$g^j(t,x(t),\dot{x}(t)) \leqq 0,$$

$$\int_a^b (f^j(t,x(t),\dot{x}(t)) + s(x(t)|D_j))dt$$

$$\leqq \int_a^b (f^j(t,x^*(t),\dot{x}^*(t)) + s(x^*(t)|D_j))dt,$$

$$\forall j \in \{1,\cdots,p\}, j \neq k$$

**Lemma 2.2** *A point $x^*(t) \in X_1$ is an efficient solution for* **(NMVPE)** *if and only if $x^*(t)$ solves $\forall k = 1,\cdots,p$,*

$$\textbf{NMVPE}_k(x^*(t)) \qquad \textit{Minimize} \int_a^b (f^k(t,x(t),\dot{x}(t)) + s(x(t)|D_k))dt$$

$$\textit{subject to} \quad x(a) = t_0,\ x(b) = t_f,$$

$$g^j(t,x(t),\dot{x}(t)) \leqq 0, \qquad h(t,x(t),\dot{x}(t)) = 0,$$

$$\int_a^b (f^j(t,x(t),\dot{x}(t)) + s(x(t)|D_j))dt$$

$$\leqq \int_a^b (f^j(t,x^*(t),\dot{x}^*(t)) + s(x^*(t)|D_j))dt,$$

$$\forall j \in \{1,\cdots,p\}, j \neq k$$

## 2.3 Sufficient Optimality Theorems for (NMVP)

We establish some sufficient conditions for an $x^*(t) \in X_0$ to be an efficient solution of problem **(NMVP)** under various generalized type I invexity conditions specified in the definitions given above.

**Theorem 2.1 (Sufficiency)** *Suppose that*

*(i)* $x^*(t) \in X_0$*:*

*(ii) there exists* $\tau_i^* \in R^p, \tau_i^* > 0$ *and a piecewise smooth function* $\lambda^*(t) : I \to R^m, \lambda^*(t) \geq 0$ *such that*

$$(a) \sum_{i=1}^{p} \tau_i^* (f_x^i(t, x^*(t), \dot{x}^*(t)) + w_i(t) - \frac{d}{dt} f_{\dot{x}}^i(t, x^*(t), \dot{x}^*(t)))$$

$$+ \sum_{j=1}^{m} \lambda_j^*(t)(g_x^j(t, x^*(t), \dot{x}^*(t)) - \frac{d}{dt} g_{\dot{x}}^j(t, x^*(t), \dot{x}^*(t))) = 0,$$

$$(b) \int_a^b \sum_{j=1}^{m} \lambda_j^*(t) g^j(t, x^*(t), \dot{x}^*(t)) dt = 0,$$

$$(c) < w_i(t), x^*(t) >= s(x^*(t)|D_i), \ i = 1, \cdots, p.$$

*(iii)* $(f(x) + x(t)^T w(t), g(x))$ *is quasi strictly pseudo V-type I invex at* $x^*(t)$ *with respect to* $\eta(t), \tau^*, \lambda^*(t)$ *and for some positive functions* $\alpha_i(t), \beta_j(t)$ *,for* $i = 1, \cdots, p, \ j = 1, \cdots, m,$

*Then* $x^*(t)$ *is an efficient solution for* **(NMVP)**.

*Proof.* Suppose $x^*(t)$ is not an efficient solution of **(NMVP)**. Then there exists a $x(t) \in X_0$ such that

$$\int_a^b (f^i(t, x(t), \dot{x}(t)) + s(x(t)|D_i))dt$$

$$\leqq \int_a^b (f^i(t, x^*(t), \dot{x}^*(t)) + s(x^*(t)|D_i))dt, \ \forall \ i = 1, \cdots, p$$

*and*

$$\int_a^b (f^{i_0}(t, x(t), \dot{x}(t)) + s(x(t)|D_{i_0}))dt$$

$$\leq \int_a^b (f^{i_0}(t, x^*(t), \dot{x}^*(t)) + s(x^*(t)|D_{i_0}))dt, \text{ for some } i_0 = 1, \cdots, p.$$

Since $< w_i(t), x(t) > \leqq s(x(t)|D_i), \ i = 1, \cdots, p$ and $< w_i(t), x^*(t) > = s(x^*(t)|D_i), \ i = 1, \cdots, p,$

$$\int_a^b (f^i(t, x(t), \dot{x}(t)) + \ x(t)^T w_i(t))dt$$

$$\leqq \int_a^b (f^i(t, x(t), \dot{x}(t)) + s(x(t)|D_i))dt$$

$$\leqq \int_a^b (f^i(t, x^*(t), \dot{x}^*(t)) + s(x^*(t)|D_i))dt$$

$$= \int_a^b (f^i(t, x^*(t), \dot{x}^*(t)) + \ x^*(t)^T w_i(t))dt, \ \forall i = 1, \cdots, p,$$

and

$$\int_a^b (f^{i_0}(t,x(t),\dot{x}(t)) + \ x(t)^T w_{i_0}(t))dt$$

$$\leqq \int_a^b (f^{i_o}(t,x(t),\dot{x}(t)) + s(x(t)|D_{i_0}))dt$$

$$< \int_a^b (f^{i_0}(t,x^*(t),\dot{x}^*(t)) + s(x^*(t)|D_{i_0}))dt$$

$$= \int_a^b (f^{i_0}(t,x^*(t),\dot{x}^*(t)) + \ x^*(t)^T w_{i_0}(t))dt, \text{ for some } i_0 = 1,\cdots,p,$$

which implies that $\tau_i^* > 0$

$$\int_a^b \sum_{i=1}^p \tau_i^* \alpha_i(x(t),x^*(t),\dot{x}(t),\dot{x}^*(t))(f^i(t,x(t),\dot{x}(t))$$

$$+x(t)^T w_i(t))dt$$

$$< \int_a^b \sum_{i=1}^p \tau_i^* \alpha_i(x(t),x^*(t),\dot{x}(t),\dot{x}^*(t))(f^i(t,x^*(t),\dot{x}^*(t))$$

$$+ \ x^*(t)^T w_i(t))dt.$$

From the above inequality and hypothesis (iii),it follows that

$$\int_a^b \sum_{i=1}^p \tau_i^* \eta(t,x(t),x^*(t),\dot{x}(t),\dot{x}^*(t))(f_x^i(t,x^*(t),\dot{x}^*(t))$$

$$+w_i(t) - \frac{d}{dt} f_{\dot{x}}^i(t,x^*(t),\dot{x}^*(t)))dt \leq 0. \tag{2.9}$$

By the inequality (2.9)and hypothesis (ii)(a) we have

$$\int_a^b \sum_{j=1}^m \lambda_j^*(t)\eta(t,x(t),x^*(t),\dot{x}(t),\dot{x}^*(t))(g_x^j(t,x^*(t),\dot{x}^*(t))$$

$$-\frac{d}{dt}g_{\dot{x}}^j(t,x^*(t),\dot{x}^*(t)))dt \geq 0. \tag{2.10}$$

From the above inequality and hypothesis (iii) it follows that

$$\int_a^b \sum_{j=1}^m \lambda_j^*(t)\beta_j(x(t),x^*(t),\dot{x}(t),\dot{x}^*(t))g^j(t,x^*(t),\dot{x}^*(t))dt < 0.$$

Now from hypotheses (i) and (ii)(b) it follows that

$$\int_a^b \sum_{j=1}^m \lambda_j^*(t)g^j(t,x^*(t),\dot{x}^*(t))dt = 0, \text{ for every } j,$$

which further implies that

$$\int_a^b \sum_{j=1}^m \lambda_j^*(t)\beta_j(x(t),x^*(t),\dot{x}(t),\dot{x}^*(t))g^j(t,x^*(t),\dot{x}^*(t))dt = 0.$$

The last equation contradicts the inequality (2.10) and hence $x^*(t)$ is an efficient solution of **(NMVP)**. □

## 2.4 Formulations of Four Pairs of Variational Dual Problem

We formulate four pairs of the following nondifferentiable multiobjective variational dual problems.

**(NMVD)** : Maximize

$$\int_a^b (f(t, y(t), \dot{y}(t)) + y^T(t)w(t) + \sum \lambda_A(t)g_A(t, y(t), \dot{y}(t))e)dt$$

$$= \Big(\int_a^b (f^1(t, y(t), \dot{y}(t)) + y^T(t)w_1(t) + \sum \lambda_A(t)g_A(t, y(t), \dot{y}(t)))dt,$$

$$\cdots, \int_a^b (f^p(t, y(t), \dot{y}(t)) + y^T(t)w_p(t) + \sum \lambda_A(t)g_A(t, y(t), \dot{y}(t)))dt\Big)$$

subject to

$$y(a) = t_0, \quad y(b) = t_f,$$

$$\sum_{i=1}^{p} \tau_i(f_y^i(t, y(t), \dot{y}(t)) + w_i(t) - \frac{d}{dt} f_{\dot{y}}^i(t, y(t), \dot{y}(t)))$$

$$+ \sum_{j=1}^{m} \lambda_j(t)(g_y^j(t, y(t), \dot{y}(t)) - \frac{d}{dt} g_{\dot{y}}^j(t, y(t), \dot{y}(t))) = 0,$$

$$\int_a^b \lambda_B(t)g_B(t, y(t), \dot{y}(t)) \geqq 0,$$

$$\tau_i \in R^p, \tau_i \geq 0,$$

$$\lambda(t) \in R^m, \quad \lambda(t) \geq 0, \quad t \in I,$$

where $e = (1, 1, \cdots, 1)^T \in R^p$ and $A \cup B = \{1, \cdots, m\}$. When $A = \emptyset$ and $B = \{1, \cdots, m\}$, our dual problem **(NMVD)** is reduced as follows:

**(NMMVD)** : Maximize

$$\int_a^b (f^i(t, y(t), \dot{y}(t)) + y^T(t)w_i(t))dt$$

$$= \Big( \int_a^b (f^1(t, y(t), \dot{y}(t) + y^T(t)w_1(t))dt,$$

$$\cdots, \int_a^b (f^p(t, y(t), \dot{y}(t) + y^T(t)w_p(t))dt \Big)$$

subject to

$$y(a) = t_0, \quad y(b) = t_f,$$

$$\sum_{i=1}^{p} \tau_i(f_y^i(t, y(t), \dot{y}(t)) + w_i(t) - \frac{d}{dt} f_{\dot{y}}^i(t, y(t), \dot{y}(t))$$

$$+ \sum_{j=1}^{m} \lambda_j(t)(g_y^j(t, y(t), \dot{y}(t)) - \frac{d}{dt} g_{\dot{y}}^j(t, y(t), \dot{y}(t))) = 0,$$

$$\int_a^b \lambda_j(t) g^j(t, y(t), \dot{y}(t)) \geqq 0, \qquad \forall j = 1, \cdots, m,$$

$$\tau_i \in R^p, \tau_i \geq 0, \qquad \forall i = 1, \cdots, p,$$

$$\lambda(t) \in R^m, \quad \lambda(t) \geq 0, \quad t \in I.$$

We let $Y_0$ the set of feasible solutions of problem **(NMMVD)**; i.e.,

$$Y_0 = \{(y(t), \tau, \lambda(t)) |\ y(a) = t_0, \quad y(b) = t_f,$$

$$\sum_{i=1}^{p} \tau_i (f_y^i(t, y(t), \dot{y}(t)) + w_i(t) - \frac{d}{dt} f_{\dot{y}}^i(t, y(t), \dot{y}(t)))$$

$$+ \sum_{j=1}^{m} \lambda_j(t)(g_y^j(t, y(t), \dot{y}(t)) - \frac{d}{dt} g_{\dot{y}}^j(t, y(t), \dot{y}(t))) = 0,$$

$$\int_a^b \lambda_j(t) g^j(t, y(t), \dot{y}(t)) \geqq 0, \qquad \forall j = 1, \cdots, m,$$

$$\tau_i \in R^p, \quad \tau_i \geq 0, \quad \lambda(t) \in R^m, \quad \lambda(t) \geq 0, \quad t \in I\}.$$

Analogous to **(NMMVD)** and **(NMVP)**, the following problem **(NMMVDE)** is a dual to **(NMVPE)**.

**(NMMVDE)**

Maximize

$$\int_a^b (f^i(t, y(t), \dot{y}(t)) + y^T(t) w_i(t)) dt$$

$$= (\int_a^b (f^1(t, y(t), \dot{y}(t) + y^T(t) w_1(t)) dt,$$

$$\cdots, \int_a^b (f^p(t, y(t), \dot{y}(t) + y^T(t) w_p(t)) dt)$$

subject to $\quad y(a) = t_0, \quad y(b) = t_f,$

$$\sum_{i=1}^{p} \tau_i (f_y^i(t, y(t), \dot{y}(t)) + w_i(t) - \frac{d}{dt} f_{\dot{y}}^i(t, y(t), \dot{y}(t)))$$

$$+\sum_{j=1}^{m}\lambda_j(t)(g_y^j(t,y(t),\dot{y}(t))-\frac{d}{dt}g_{\dot{y}}^j(t,y(t),\dot{y}(t))$$

$$+\sum_{l=1}^{q}\mu_l(t)(h_y^l(t,y(t),\dot{y}(t))-\frac{d}{dt}h_{\dot{y}}^l(t,y(t),\dot{y}(t)))=0,$$

$$\int_a^b \lambda_j(t)g^j(t,y(t),\dot{y}(t))\geqq 0,\qquad \forall j=1,\cdots,m,$$

$$\int_a^b \mu_l(t)h^l(t,y(t),\dot{y}(t))=0,\qquad \forall l=1,\cdots,q,$$

$$\lambda(t)>0,$$

$$\tau_i\geq 0,\qquad \sum_{i=1}^{p}\tau_i=1,\qquad t\in I.$$

We let $Y_1$ the set of feasible solutions of problem **(NMMVDE)** ; i.e.,

$$Y_1=\Big\{(y(t),\tau,\lambda(t),\mu(t))|\quad y(a)=t_0,\quad y(b)=t_f,$$

$$\sum_{i=1}^{p}\tau_i\{f_y^i(t,y(t),\dot{y}(t))+w_i(t)-\frac{d}{dt}f_{\dot{y}}^i(t,y(t),\dot{y}(t))\}$$

$$+\sum_{j=1}^{m}\lambda_j(t)\{g_y^j(t,y(t),\dot{y}(t))-\frac{d}{dt}g_{\dot{y}}^j(t,y(t),\dot{y}(t))\}$$

$$+\sum_{l=1}^{q}\mu_l(t)\{h_y^l(t,y(t),\dot{y}(t))-\frac{d}{dt}h_{\dot{y}}^l(t,y(t),\dot{y}(t)\}=0,$$

$$\int_a^b \lambda_j(t)g^j(t,y(t),\dot{y}(t))\geqq 0,\qquad \forall j=1,\cdots,m,$$

$$\int_a^b \mu_l(t)h^l(t, y(t), \dot{y}(t)) = 0, \qquad \forall l = 1, \cdots, q,$$

$$\lambda(t) \geq 0,$$

$$\tau_i \geq 0, \qquad \sum_{i=1}^{p} \tau_i = 1, \qquad t \in I \Big\}.$$

We consider the following general Mond-Weir[30] type dual problem.

**(NGMMVDE)** : Maximize

$$\Big( \int_a^b (f^1(t, y(t), \dot{y}(t)) + y^T(t)w_1(t)$$

$$+ \sum_{j \in J_0} \lambda_j(t) g^j(t, y(t), \dot{y}(t)) + \sum_{l \in K_0} \mu_l(t) h^l(t, y(t), \dot{y}(t)))dt,$$

$$\cdots, \int_a^b (f^p(t, y(t), \dot{y}(t)) + y^T(t)w_p(t)$$

$$+ \sum_{j \in J_0} \lambda_j(t) g^j(t, y(t), \dot{y}(t)) + \sum_{l \in K_0} \mu_l(t) h^l(t, y(t), \dot{y}(t)))dt \Big) \tag{2.11}$$

Subject to

$$y(a) = t_0, \quad y(b) = t_f,$$

$$\sum_{i=1}^{p} \tau_i \{ f_y^i(t, y(t), \dot{y}(t)) + w_i(t) - \frac{d}{dt} f_{\dot{y}}^i(t, y(t), \dot{y}(t)) \}$$

$$+ \sum_{j=1}^{m} \lambda_j(t) \{ g_y^j(t, y(t), \dot{y}(t)) - \frac{d}{dt} g_{\dot{y}}^j(t, y(t), \dot{y}(t)) \}$$

$$+\sum_{l=1}^{q}\mu_l(t)\{h_y^l(t,y(t),\dot{y}(t))-\frac{d}{dt}h_{\dot{y}}^l(t,y(t),\dot{y}(t))\}=0,$$

$$\int_a^b(\sum_{j\in J_\alpha}\lambda_j(t)g^j(t,y(t),\dot{y}(t))+\sum_{l\in K_\alpha}\mu_l(t)h^l(t,y(t),\dot{y}(t)))dt\geqq 0,$$

$$\alpha=1,\cdots,r, \quad (2.12)$$

$$\lambda(t)\geq 0, \quad (2.13)$$

$$\tau_i\geq 0,\qquad \sum_{i=1}^{p}\tau_i=1,\qquad t\in I. \quad (2.14)$$

where $J_\alpha\subset\{1,\cdots,m\}$, $\alpha=0,1,\cdots,r$ with $J_\alpha\cap J_\beta=\emptyset$, $\alpha\neq\beta$ and $\cup_{\alpha=0}^{r}J_\alpha=\{1,\cdots,m\}$ and $K_\alpha\subset\{1,\cdots,k\},\alpha=0,1,\cdots,r$ with $K_\alpha\cap K_\beta=\emptyset$, $\alpha\neq\beta$ and $\cup_{\alpha=0}^{r}K_\alpha=\{1,\cdots,k\}$.

## 2.5 Duality Theorems

Now we establish some duality theorems between the nondifferentiable multiobjective variational problem **(NMVP)** and its dual problem **(NM-MVD)**.

**Theorem 2.2 (Weak Duality)** *Suppose that*

*(i)* $x(t)\in X_0$ :

*(ii)* $(y(t),\tau_i,\lambda(t))\in Y_0$ *and* $\tau_i>0$

*(iii)* $(f(x)+x^T(t)w(t),g(x))$ *is pseudo V-type I invex at* $y(t)$ *with respect to* $\eta(t)$, $\tau$, $\lambda(t)$ *and for some positive function* $\alpha_i(t)$, $\beta_j(t)$, *for* $i=1,\cdots,p$, $j=1,\cdots,m$.

*Then the following cannot hold:*

$$\int_a^b (f^i(t, x(t), \dot{x}(t)) + s(x(t)|D_i))dt$$

$$\leqq \int_a^b (f^i(t, y(t), \dot{y}(t)) + y^T(t)w_i(t))dt, \ \forall i = 1, \cdots, p \quad (2.15)$$

*and*

$$\int_a^b (f^{i_0}(t, x(t), \dot{x}(t)) + s(x(t)|D_{i_0}))dt$$

$$< \int_a^b (f^{i_0}(t, y(t), \dot{y}(t)) + y^T(t)w_{i_0}(t))dt, \ \textit{for some } i_0 = 1, \cdots, p. \quad (2.16)$$

*Proof.* By hypothesis (ii) we have

$$\int_a^b \lambda_j(t)g^j(t, y(t), \dot{y}(t))dt \geqq 0, \qquad \forall\ j = 1, \cdots, m.$$

which implies that

$$\int_a^b \sum_{j=1}^m \lambda_j(t)\beta_j(x(t), y(t), \dot{x}(t), \dot{y}(t))g^j(t, y(t), \dot{y}(t))dt \geqq 0,$$

$$\forall j = 1, \cdots, m. \quad (2.17)$$

By the hypothesis (iii) and (2.18) it follows that

$$\int_a^b \sum_{j=1}^m \lambda_j(t)\eta(t, x(t), y(t), \dot{x}(t), \dot{y}(t))\{g_x^j(t, y(t), \dot{y}(t))$$

$$-\frac{d}{dt}g_{\dot{x}}^j(t, y(t), \dot{y}(t))\} \leqq 0, \qquad \forall j = 1, \cdots, m. \tag{2.18}$$

Using the inequality (2.19) and hypothesis (ii) we have

$$\int_a^b \tau_i\eta(t, x(t), y(t), \dot{x}(t), \dot{y}(t)))f_x^i(t, y(t), \dot{y}(t))$$

$$-\frac{d}{dt}f_{\dot{x}}^i(t, y(t), \dot{y}(t) + w_i(t)) \geqq 0, \quad \forall i = 1, \cdots, m. \tag{2.19}$$

Hypothesis (iii) and (2.20) give

$$\int_a^b \sum_{i=1}^p \tau_i\alpha_i(x(t), y(t), \dot{x}(t), \dot{y}(t))(f^i(t, x(t), \dot{x}(t))$$

$$+x^T(t)w_i(t))dt$$

$$\geqq \int_a^b \sum_{i=1}^p \tau_i\alpha_i(x(t), y(t), \dot{x}(t), \dot{y}(t))(f^i(t, y(t), \dot{y}(t))$$

$$+y^T(t)w_i(t))dt. \tag{2.20}$$

Suppose contrary to the result that (2.16) and (2.17) hold.

Then since each $\alpha_i(t) > 0$ and $\tau_i > 0$, we have

$$\int_a^b \sum_{i=1}^p \tau_i \alpha_i(x(t), y(t), \dot{x}(t), \dot{y}(t))(f^i(t, x(t), \dot{x}(t))$$

$$+S(x(t)|D_i))dt$$

$$< \int_a^b \sum_{i=1}^p \tau_i \alpha_i(x(t), y(t), \dot{x}(t), \dot{y}(t))(f^i(t, y(t), \dot{y}(t))$$

$$+y^T(t)w_i(t))dt.$$

Since $\quad x^T(t)w_i(t) \leqq S(x(t)|D_i), \quad i = 1, \cdots, \rho,$ we have

$$\int_a^b \sum_{i=1}^p \tau_i \alpha_i(x(t), y(t), \dot{x}(t), \dot{y}(t))(f^i(t, x(t), \dot{x}(t))$$

$$+x^T(t)w_i(t))dt$$

$$< \int_a^b \sum_{i=1}^p \tau_i \alpha_i(x(t), y(t), \dot{x}(t), \dot{y}(t))(f^i(t, y(t), \dot{y}(t))$$

$$+y^T(t)w_i(t)))dt.$$

Which contradicts (2.21). Hence the conclusion follows. □

**Corollary 2.1** *Assume that weak duality theorems* (2.2) *hold between* **(NMVP)** *and* **(NMMVD)**. *If* $(y^*(t), \tau^*, \lambda^*(t))$ *is feasible for* **(NMMVD)** *such that* $y^*(t)$ *is feasible for* **(NMVP)** *and* $y^{*T}(t)w_i(t) = s(y^*(t)|D_i)(i = 1, \cdots, p)$ *then* $y^*(t)$ *is an efficient solution for* **(NMVP)** *and* $(y^*(t), \tau^*, \lambda^*(t))$ *is an efficient solution for* **(NMMVD)**.

*Proof.* Suppose that $y^*(t)$ is not efficient for **(NMVP)**.

Then there exists same feasible $x(t)$ for **(NMVP)** such that

$$\int_a^b (f^i(t, x(t), \dot{x}(t)) + s(x(t)|D_i))dt$$

$$\leqq \int_a^b (f^i(t, y^*(t), \dot{y}^*(t)) + s(y^*(t)|D_i))dt,$$

$$\forall\ i = 1, \cdots, p$$

and

$$\int_a^b (f^{i_0}(t, x(t), \dot{x}(t)) + s(x(t)|D_{i_0}))dt$$

$$< \int_a^b (f^{i_0}(t, y^*(t), \dot{y}^*(t)) + s(y^*(t)|D_{i_0}))dt,$$

$$for \quad some \quad i_0 = 1, \cdots, p.$$

since $\quad < w_i(t), y^*(t) >= s(y^*(t)|D_i), \quad i = 1, \cdots, p,$

$$\int_a^b (f^i(t, x(t), \dot{x}(t)) + s(x(t)|D_i))dt$$

$$\leqq \int_a^b (f^i(t, y^*(t), \dot{y}^*(t)) + y^{*T}(t)w_i(t))dt, \forall\ i = 1, \cdots, p$$

and

$$\int_a^b (f^{i_0}(t, x(t), \dot{x}(t)) + s(x(t)|D_{i_0}))dt,$$

$$\leqq \int_a^b (f^{i_0}(t, y^*(t), \dot{y}^*(t)) + y^{*T}(t)w_{i_0}(t))dt, \text{ for some } i_0 = 1, \cdots, p.$$

This contradicts weak duality.

Hence $y^*(t)$ is an efficient for **(NMVP)**.

Now suppose $(y^*(t), \tau^*, \lambda^*(t))$ is not an efficient for **(NMMVD)**.

Then there exist some $(x(t), \tau, \lambda(t))$ feasible for **(NMMVD)**

such that

$$\int_a^b (f^i(t, x(t), \dot{x}(t)) + x^T(t)w_i(t))dt$$

$$\geqq \int_a^b (f^i(t, y^*(t), \dot{y}^*(t)) + y^{*T}(t)w_i(t))dt, \ i = 1, \cdots, p$$

and

$$\int_a^b (f^{i_0}(t, x(t), \dot{x}(t)) + x^T(t)w_{i_0}(t))dt$$

$$> \int_a^b (f^{i_0}(t, y^*(t), \dot{y}^*(t)) + y^*(t)w_{i_0}(t))dt, \text{ for some } i_0 = 1, \cdots, p.$$

since $\quad < w_i(t), y^*(t) >= s(y^*(t)|D_i), \quad i = 1, \cdots, p,$

$$\int_a^b (f^i(t, x(t), \dot{x}(t)) + x^T(t)w_i(t))dt$$

$$\geqq \int_a^b (f^i(t, y^*(t), \dot{y}^*(t)) + s(y^*(t)|D_i))dt, \forall \ i = 1, \cdots, p$$

and

$$\int_a^b (f^{i_0}(t, x(t), \dot{x}(t)) + x^T(t)w_{i_0}(t))dt$$

$$> \int_a^b (f^{i_0}(t, y^*(t), \dot{y}^*(t) + s(y^*(t)|D_{i_0}))dt, \text{ for some } i_0 = 1, \cdots, p.$$

This contradicts weak duality. Hence $(y^*(t), \tau^*, \lambda^*(t))$ is an efficient for **(NM-MVD)**. □

**Theorem 2.3 (Strong Duality)** *Assume that*
*(i)* $x^*(t)$ *is an efficient solution for* **(NMVP)***:*
*(ii) for all* $k = 1, \cdots, p$, $x^*(t)$ *a constraint qualification for* $\mathbf{NMVP}_k(x^*(t))$ *at* $x^*(t)$ *is satisfied.*
*Then there exists* $\tau_i^* \in R^p, \tau_i^* > 0$*, and piecewise smooth function* $\lambda^*(t) : I \longrightarrow R^m, \lambda^*(t) > 0$ *such that* $(x^*(t), \tau^*, \lambda^*(t)) \in Y_0$.
*Further, if the assumption of weak duality theorems (2.2) is satisfied, then* $(x^*(t), \tau^*, \lambda^*(t))$ *is an efficient for* **(NMMVD)**.

*Proof.* Since $x^*(t)$ is an efficient solution of **(NMVP)**, then from Lemma 2.1. $x^*(t)$ solves $\mathbf{NMVP}_k(x^*(t))$ for each $k = 1, \cdots, p$. From Kuhn-Tucker necessary conditions [23] for each $k = 1, \cdots, p$, we obtain $\tau_i^k \geq 0$ for all $i \neq k$, and $\lambda(t)(> 0) \in R^m$

such that

$$\{f_x^i(t,x^*(t),\dot{x}^*(t))+w_i(t)-\frac{d}{dt}f_{\dot{x}}^i(t,x^*(t),\dot{x}^*(t))\}$$

$$+\sum_{i\neq k}\tau_i^k\{f_x^k(t,x^*(t),\dot{x}^*(t))+w_k(t)-\frac{d}{dt}f_{\dot{x}}^k(t,x^*(t),\dot{x}^*(t))\}$$

$$+\sum_{j=1}^m\lambda_j^i(t)\{g_x^j(t,x^*(t),\dot{x}^*(t))-\frac{d}{dt}g_{\dot{x}}^j(t,x^*(t),\dot{x}^*(t))\}=0. \tag{2.21}$$

$$\int_a^b\sum_{j=1}^m\lambda_j^i(t)g^j(t,x^*(t),\dot{x}^*(t)))dt=0. \tag{2.22}$$

Summing (2.23) over $i=1,\cdots,p$, we have

$$(1+\tau_2^1+\tau_3^1+\cdots+\tau_p^1)\{f_x^1(t,x^*(t),\dot{x}^*(t))+w_1(t)-\frac{d}{dt}f_{\dot{x}}^1(t,x^*(t),\dot{x}^*(t))\}$$

$$+\sum_{j=1}^m\lambda_j^1(t)\{g_x^j(t,x^*(t),\dot{x}^*(t))-\frac{d}{dt}g_{\dot{x}}^1(t,x^*(t),\dot{x}^*(t))\}$$

$$+(\tau_1^2+1+\tau_2^2+\cdots+\tau_p^2)\{f_x^2(t,x^*(t),\dot{x}^*(t))+w_2(t)-\frac{d}{dt}f_{\dot{x}}^2(t,x^*(t),\dot{x}^*(t))\}$$

$$+\sum_{j=1}^m\lambda_j^2(t)\{g_x^j(t,x^*(t),\dot{x}^*(t))-\frac{d}{dt}g_{\dot{x}}^j(t,x^*(t),\dot{x}^*(t))\}+\cdots$$

$$+(\tau_1^p+\tau_2^p+\cdots+1)\{f_x^p(t,x^*(t),\dot{x}^*(t))+w_p(t)-\frac{d}{dt}f_{\dot{x}}^p(t,x^*(t),\dot{x}^*(t))\}$$

$$+\sum_{j=1}^m\lambda_j^p(t)\{g_x^j(t,x^*(t),\dot{x}^*(t))-\frac{d}{dt}g_{\dot{x}}^j(t,x^*(t),\dot{x}^*(t))\}=0.$$

*Let* $\quad \tau_1^* = 1 + \tau_2^1 + \tau_3^1 + \cdots + \tau_p^1,$

$$\tau_2^* = \tau_1^2 + 1 + \tau_3^2 + \cdots + \tau_p^2,$$

$$\vdots$$

$$\tau_p^* = \tau_1^p + \tau_2^p + \cdots + 1,$$

$$\sum_{k=1}^{p} \lambda_j^k(t) = \lambda_j^*(t) \quad , \quad j = 1, \cdots, m, \quad \lambda^*(t) = (\lambda_1^*(t), \cdots, \lambda_m^*(t)).$$

Then we have

$$\sum_{i=1}^{p} \tau_i^* \{ f_x^i(t, x^*(t), \dot{x}^*(t)) + w_i(t) - \frac{d}{dt} f_{\dot{x}}^i(t, x^*(t), \dot{x}^*(t)) \}$$
$$+ \sum_{j=1}^{m} \lambda_j^*(t) \{ g_x^j(t, x^*(t), \dot{x}^*(t)) - \frac{d}{dt} g_{\dot{x}}^j(t, x^*(t), \dot{x}^*(t)) \} = 0.$$

Summing (2.24) for $\quad i = 1, \cdots, p \quad$, we have

$$\int_a^b \sum_{j=1}^{m} \lambda_j^*(t) g^j(t, x^*(t), \dot{x}^*(t)) dt = 0.$$

we conclude that $(x^*(t), \tau^*, \lambda^*(t))$ is feasible for **(NMMVD)**. Efficiency of $(x^*(t), \tau^*, \lambda^*(t))$ for **(NMMVD)** now follows from Corollary 2.1. □

**Theorem 2.4 (Converse Duality)** *Suppose that*

*(i)* $\quad (y^*(t), \tau^*, \lambda^*(t)) \in Y_0$ *with* $\tau^* > 0$

*(ii)* $y^*(t) \in X_0$ :

*(iii)* $(f(x) + x^T(t)w(t), g(x))$ *is V-type I invex at* $y^*(t)$ *for some positive functions* $\alpha_i(t), \beta_j(t)$ *for* $i = 1, \cdots, p,\ j = 1, \cdots, m.$

*Then* $y^*(t)$ *is an efficient solution of* **(NMVP)**.

*Proof.* If follows by the hypothesis (i) that

$$\int_a^b \lambda_j^*(t) g^j(t, y^*(t), \dot{y}^*(t))dt \geqq 0, \qquad \forall j = 1, \cdots, m \tag{2.23}$$

By hypothesis (iii), for any $x(t) \in X_0$, we have

$$\int_a^b \sum_{i=1}^p \tau_i^*(f^i(t, x(t), \dot{x}^*(t)) + x^T(t)w_i(t))dt$$

$$- \int_a^b \sum_{i=1}^p \tau_i^*(f^i(t, y^*(t), \dot{y}^*(t)) + y^T(t)w_i(t))dt$$

$$\geqq \int_a^b \sum_{i=1}^p \tau_i^* \alpha_i(x(t), y^*(t), \dot{x}(t), \dot{y}^*(t))\eta(t, x(t), y^*(t), \dot{x}(t), \dot{y}^*(t))$$

$$(f_y^i(t, y^*(t), \dot{y}^*(t)) + w_i(t) - \frac{d}{dt} f_{\dot{y}}^i(t, y^*(t), \dot{y}^*(t)))dt,$$

$$i = 1, \cdots, p,$$

$$- \int_a^b \sum_{j=1}^m \lambda_j^*(t) g^j(t, y^*(t), \dot{y}^*(t))dt$$

$$\geqq \int_a^b \sum_{j=1}^m \lambda_j^*(t)\beta_j(x(t), y^*(t), \dot{x}(t), y^*(t))\eta(t, x(t), y^*(t), \dot{x}(t), \dot{y}^*(t))$$

$$(g_y^j(t, y^*(t), \dot{y}^*(t)) - \frac{d}{dt} g_{\dot{y}}^j(t, y^*(t), \dot{y}^*(t)))dt,$$

$$j = 1, \cdots, m. \quad (2.24)$$

Now by the facts $\alpha_i(t) > 0, \beta_j(t) > 0, \ \forall i, j$ and $\tau^* > 0, \lambda^*(t) \geqq 0$, it follows by (2.25) (2.26) that

$$\int_a^b \sum_{i=1}^p \frac{\tau_i^*}{\alpha_i(x(t), y^*(t), \dot{x}(t), \dot{y}^*(t))} \{f^i(t, x(t), \dot{x}(t)) + x^T(t) w_i(t)\} dt$$

$$- \int_a^b \sum_{i=1}^p \frac{\tau_i^*}{\alpha_i(x(t), y^*(t), \dot{x}(t), \dot{y}^*(t))} \{f^i(t, y^*(t), \dot{y}^*(t)) + y^T(t) w_i(t)\} dt$$

$$\geqq \int_a^b \sum_{i=1}^p \tau_i^* \eta(t, x(t), y^*(t), \dot{x}(t), \dot{y}^*(t)) \{f_y^i(t, y^*(t), \dot{y}^*(t)) + w_i(t)$$

$$- \frac{d}{dt} f_{\dot{y}}^i(t, y^*(t), \dot{y}^*(t))\} dt$$

$$- \int_a^b \sum_{j=1}^m \frac{\lambda_j^*(t)}{\beta_j(x(t), y^*(t), \dot{x}(t), \dot{y}^*(t))} g^j(t, y^*(t), \dot{y}^*(t)) dt$$

$$\geqq \int_a^b \sum_{j=1}^m \lambda_j^*(t) \eta(t, x(t), y^*(t), \dot{x}(t), \dot{y}^*(t)) \{g_y^j(t, y^*(t), \dot{y}^*(t))$$

$$- \frac{d}{dt} g_{\dot{y}}^j(t, y^*(t), \dot{y}^*(t))\} dt$$

$$\int_a^b \sum_{i=1}^p \frac{\tau_i^*}{\alpha_i(x(t), y^*(t), \dot{x}(t), \dot{y}^*(t))} \{f^i(t, x(t), \dot{x}(t)) + x^T(t) w_i(t)\} dt$$

$$- \int_a^b \sum_{i=1}^p \frac{\tau_i^*}{\alpha_i(x(t), y^*(t), \dot{x}(t), \dot{y}^*(t))} \{f^i(t, y^*(t), \dot{y}^*(t)) + y^T(t) w_i(t)\} dt$$

$$-\int_a^b \sum_{j=1}^m \frac{\lambda_j^*(t)}{\beta_j(x(t), y^*(t), \dot{x}(t), \dot{y}^*(t))} g^j(t, y^*(t), \dot{y}^*(t))dt$$

$$\geqq \int_a^b \eta(t, x(t), y^*(t), \dot{x}(t), \dot{y}^*(t))\Big(\sum_{i=1}^p \tau_i^*\{f_y^i(t, y^*(t), \dot{y}^*(t)) + w_i(t)$$

$$-\frac{d}{dt} f_{\dot{y}}^i(t, y^*(t), \dot{y}^*(t))\} + \sum_{j=1}^m \lambda_j^*(t)\{g_y^j(t, y^*(t), \dot{y}^*(t))$$

$$-\frac{d}{dt} g_{\dot{y}}^j(t, y^*(t), \dot{y}^*(t))\}$$

$$+\sum_{l=1}^q \mu_l^*(t)\{h_y^l(t, y^*(t), \dot{y}^*(t)) - \frac{d}{dt} h_{\dot{y}}^l(t, y^*(t), \dot{y}^*(t))\}\Big)dt$$

$$= 0. \tag{2.25}$$

From (2.24) and (2.27) it follows that

$$\int_a^b \sum_{i=1}^p \frac{\tau_i^*}{\alpha_i(x(t), y^*(t), \dot{x}(t), \dot{y}^*(t))}\{f^i(t, x(t), \dot{x}(t)) + x^T(t)w_i(t)\}dt$$

$$\geqq \int_a^b \sum_{i=1}^p \frac{\tau_i^*}{\alpha_i(x(t), y^*(t), \dot{x}(t), \dot{y}^*(t))}\{f^i(t, y^*(t), \dot{y}^*(t)) + y^T(t)w_i(t)\}dt. \tag{2.26}$$

Now suppose that $y^*(t)$ is not an efficient solution of **(NMVPE)**. Then there exists an $x(t) \in X_0$ such that

$$\int_a^b \{f^i(t, x(t), \dot{x}(t)) + x^T(t)w_i(t)\}dt$$

$$\leqq \int_a^b \{f^i(t, y^*(t), \dot{y}^*(t)) + y^T(t)w_i(t)\}dt, \ \forall i = 1, \cdots, p$$

and

$$\int_a^b \{f^{i_0}(t, x(t), \dot{x}(t)) + x^T(t)w_{i_0}(t)\}dt$$

$$< \int_a^b \{f^{i_0}(t, y^*(t), \dot{y}^*(t)) + y^T(t)w_{i_0}(t)\}dt, \text{for some} i_0 = 1, \cdots, p.$$

Which implies that

$$\int_a^b \sum_{i=1}^p \frac{\tau_i^*}{\alpha_i(x(t), y^*(t), \dot{x}(t), \dot{y}^*(t))}\{f^i(t, x(t), \dot{x}(t)) + x^T(t)w_i(t)\}dt$$

$$< \int_a^b \sum_{i=1}^p \frac{\tau_i^*}{\alpha_i(x(t), y^*(t), \dot{x}(t)), \dot{y}^*(t))}\{f^i(t, y^*(t), \dot{y}^*(t) + y(t)^T w_i(t)\}dt \tag{2.27}$$

Now (2.28)and (2.29) contradict each other. Hence the conclusion follows. □

Now we establish weak, strong and converse duality theorems between the nondifferentiable multiobjective variational problem **(NMVPE)** and its dual problem **(NMMVDE)**.

**Theorem 2.5 (Weak Duality)** *Assume that for all feasible $x(t)$ for* **(NMVPE)** *and all feasible $(y(t), \tau, \lambda(t), \mu(t))$ for* **(NMMVDE)** *any of the following holds:*

*(i) $(f(x) + y^T(t)w(t), g(x) + h(x))$ is weak strictly - pseudoquasi V-type I*

*invex at $y(t)$ with respect to $\eta(t)$ and $\tau_i > 0$ and for some positive functions $\alpha_i(t),\ \beta_j(t)$ for $i = 1, \cdots, p,\ \ j = 1, \cdots, m,$:*

*(ii) $(f(x)+y^T(t)w(t), g(x)+h(x))$ is weak strictly - pseudo V-type I invex at $y(t)$ with respect to $\eta(t)$ and $\tau_i > 0$ and for some positive functions $\alpha_i(t), \beta_j(t)$ for $i = 1, \cdots, p,\ \ j = 1, \cdots, m.$*

*Then the following inequalities cannot hold:*

$$\int_a^b (f^i(t, x(t), \dot{x}(t)) + x^T(t)w_i(t))dt$$

$$\leqq \int_a^b (f^i(t, y(t), \dot{y}(t)) + y^T(t)w_i(t))dt, \forall\ i = 1, \cdots, p \tag{2.28}$$

*and*

$$\int_a^b (f^{i_0}(t, x(t), \dot{x}(t)) + x^T(t)w_{i_0}(t))dt$$

$$< \int_a^b (f^{i_0}(t, y(t), \dot{y}(t)) + y^T(t)w_{i_0}(t))dt,\ \textit{for some}\ i_0 = 1, \cdots, p \tag{2.29}$$

*Proof.* Suppose contrary to the result that (2.30) and (2.31) hold. Then since each $\alpha_i(t) > 0$ and $\tau_i > 0$, we have

$$\int_a^b \sum_{i=1}^p \tau_i\alpha_i(x(t), y(t), \dot{x}(t), \dot{y}^*(t))(f^i(t, x(t), \dot{x}(t) + x^T(t)w_i(t))dt$$

$$< \int_a^b \sum_{i=1}^p \tau_i\alpha_i(x(t), y(t), \dot{x}(t), \dot{y}(t))(f^i(t, y(t), \dot{y}(t) + y^T(t)w_i(t))dt. \tag{2.30}$$

Since $(y(t), \tau, \lambda(t), \mu(t))$ is feasible for **(NMMVDE)**, it follows that

$$\int_a^b \sum_{j=1}^m \lambda_j(t)\beta_j(x(t), y(t), \dot{x}(t)\dot{y}(t))g^j(t, y(t), \dot{y}(t)) \geqq 0$$

and

$$\int_a^b \sum_{l=1}^q \mu_l(t)\gamma_l(x(t), y(t), \dot{x}(t)\dot{y}(t))h^l(t, y(t), \dot{y}(t)) = 0.$$

Hence

$$-\int_a^b \{\sum_{j=1}^m \lambda_j(t)\beta_j(x(t), y(t), \dot{x}(t), \dot{y}(t))g^j(t, y(t), \dot{y}(t))$$

$$+\sum_{l=1}^q \mu_l(t)\gamma_l(x(t), y(t), \dot{x}(t), \dot{y}(t))h^l(t, y(t), \dot{y}(t))\} \leqq 0. \qquad (2.31)$$

By the hypothesis (i) i.e ; $(f(x) + y^T(t)w(t), g(x) + h(x))$ is weak strictly-pseudoquasi V-type I invex, (2.32),(2.33) imply,

$$\int_a^b \Big(\sum_{i=1}^p \tau_i \eta(t, x(t), y(t), \dot{x}(t), \dot{y}(t))(f_y^i(t, y(t), \dot{y}(t))$$

$$-\frac{d}{dt} f_{\dot{y}}^i(t, y(t), \dot{y}(t)))\Big)dt < 0,$$

$$\int_a^b \eta(t, x(t), y(t), \dot{x}(t), \dot{y}(t))\Big(\sum_{j=1}^m \lambda_j(t)(g_y^j(t, y(t), \dot{y}(t)) - \frac{d}{dt} g_{\dot{y}}^j(t, y(t), \dot{y}(t)))$$

$$+\sum_{l=1}^q \mu_l(t)(h_y^l(t, y(t), \dot{y}(t)) - \frac{d}{dt} h_{\dot{y}}^l(t, y(t), \dot{y}(t)))\Big)dt \leqq 0.$$

The above inequalities give

$$\int_a^b \eta(t,x(t),y(t),\dot{x}(t),\dot{y}(t))\Big(\sum_{i=1}^{p}\tau_i(f_y^i(t,y(t),\dot{y}(t))-\frac{d}{dt}f_{\dot{y}}^i(t,y(t),\dot{y}(t)))$$

$$+\sum_{j=1}^{m}\lambda_j(t)(g_y^j(t,y(t),\dot{y}(t))-\frac{d}{dt}g_{\dot{y}}^j(t,y(t),\dot{y}(t)))$$

$$+\sum_{l=1}^{q}\mu_l(t)(h_y^l(t,y(t),\dot{y}(t))-\frac{d}{dt}h_{\dot{y}}^l(t,y(t),\dot{y}(t)))\Big)dt<0. \tag{2.32}$$

Which contradicts (2.13).

By we have the hypothesis (ii) i.e : $(f(x)+y^T(x)w(t),g(x)+h(x))$ is weak strictly pseudo V-type I invex, (2.33) and (2.34) imply

$$\int_a^b \sum_{i=1}^{p}\tau_i\eta(t,x(t),y(t),\dot{x}(t),\dot{y}(t))(f_y^i(t,y(t),\dot{y}(t))$$

$$-\frac{d}{dt}f_{\dot{y}}^i(t,y(t),\dot{y}(t)))dt<0, \tag{2.33}$$

$$\int_a^b \eta(t,x(t),y(t),\dot{x}(t),\dot{y}(t))(\sum_{j=1}^{m}\lambda_j(t)(g_y^j(t,y(t),\dot{y}(t))-\frac{d}{dt}g_{\dot{y}}^j(t,y(t),\dot{y}(t))$$

$$+\sum_{l=1}^{q}\mu_l(t)(h_y^l(t,y(t),\dot{y}(t))-\frac{d}{dt}h_{\dot{y}}^l(t,y(t),\dot{y}(t))))dt<0. \tag{2.34}$$

(2.35) and (2.36) imply (2.34), again contradicting (2.12). □

**Corollary 2.2** *Assume that weak duality holds between* **(NMVPE)** *and* **(NMMVDE)**. *If* $(y^*(t),\tau^*,\lambda^*(t),\mu^*(t))$ *is feasible for* **(NMMVDE)** *such* *that $y^*(t)$ is feasible for* **(NMVPE)**, *then $y^*(t)$ is an efficient solution for* **(NMVPE)** *and $(y^*(t), \tau^*, \lambda^*(t), \mu^*(t))$ is an efficient solution for* **(NM-MVDE)**.

*Proof.* Suppose that $y^*(t)$ is not an efficient for **(NMVPE)** : then there exists a feasible $x(t)$ for **(NMVPE)** such that (2.30) and (2.31) hold. But $(y^*(t), \tau^*, \lambda^*(t), \mu^*(t))$ is feasible for **(NMMVDE)**. Hence the result of weak duality is contradicted Therefore, $y^*(t)$ must be efficient for **(NMVPE)**. Now suppose $(y^*(t), \tau^*, \lambda^*(t), \mu^*(t))$ is not an efficient for **(NMMVDE)**. Then there exist some $(x(t), \tau, \lambda(t), \mu(t))$ feasible for **(NMMVDE)** such that

$$\int_a^b (f^i(t, x(t), \dot{x}(t)) + x^T(t)w_i(t))dt$$

$$\geq \int_a^b (f^i(t, y(t), \dot{y}(t)) + y^T(t)w_i(t))dt, \forall\ i = 1, \cdots, p$$

and

$$\int_a^b (f^{i_0}(t, x(t), \dot{x}(t)) + x^T(t)w_{i_0}(t))dt$$

$$> \int_a^b (f^{i_0}(t, y(t), \dot{y}(t)) + y^T(t)w_{i_0}(t))dt,\ \text{for some } i_0 = 1, \cdots, p.$$

This contradicts weak duality. Hence $(y^*(t), \tau^*, \lambda^*(t), \mu^*(t))$ is an efficient for **(NMMVDE)**. □

**Theorem 2.6 (Strong Duality)** *Assume that*

*(i)* $x^*(t)$ *is an efficient solution for* **(MVPE)***;*

*(ii) for all* $k = 1, \cdots, p, x^*(t)$ *a constraint qualification for problem* $\mathbf{MVPE_k}(\mathbf{x}^*(\mathbf{t}))$ *is satisfied at* $x^*(t)$*.*

*Then there exist* $\tau_i^* \in R^p, \tau_i^* > 0,$ *and piecewise smooth function* $\lambda^*(t) : I \longrightarrow R^m, \lambda^*(t) \geq 0$ *and* $\mu^*(t) : I \longrightarrow R^q$ *such that* $(x^*(t), \tau^*, \lambda^*(t), \mu^*(t))$ *is feasible for* **(NMMVDE)***.*

*Further, if also weak duality holds between* **(NMVPE)** *and* **(NMMVDE)***, then* $(x^*(t), \tau^*, \lambda^*(t), \mu^*(t))$ *is an efficient solution for* **(NMMVDE)***.*

*Proof.* Since $x^*(t)$ is an efficient solution of **(NMVPE)**. Then from Lemma 2.1 $x^*(t)$ solves $\mathbf{NMVPE}_K(x^*(t))$ for each $k = 1, \cdots, p$. From Kuhn-Tucker necessary conditions [23] for each $k = 1, \cdots, p$, we obtain $\tau_i^k > 0$ for all $i \neq k$, $\lambda^i(t) \geqq 0 \in R^m$ and $\mu^i(t) \in R^q$ such that

$$\begin{aligned}
&(f_x^i(t, x^*(t), \dot{x}^*(t) + w_i(t) - \frac{d}{dt} f_{\dot{x}}^i(t, x^*(t), \dot{x}^*(t))) \\
&+ \sum_{i \neq k} \lambda_i^k(t)(f_x^k(t, x^*(t), \dot{x}^*(t)) - \frac{d}{dt} f_{\dot{x}}^k(t, x^*(t), \dot{x}^*(t))) \\
&+ \sum_{j=1}^{m} \lambda_j^i(t)(g_x^j(t, x^*(t), \dot{x}^*(t)) - \frac{d}{dt} g_{\dot{x}}^j(t, x^*(t), \dot{x}^*(t))) \\
&+ \sum_{l=1}^{q} \mu_l^i(t)(h_x^l(t, x^*(t), \dot{x}^*(t)) - \frac{d}{dt} h_{\dot{x}}^l(t, x^*(t), \dot{x}^*(t))) = 0, \qquad (2.35)
\end{aligned}$$

$$\int_a^b \sum_{j=1}^{m} \lambda_j^i(t) g^j(t, x^*(t), \dot{x}^*(t)) dt = 0, \qquad (2.36)$$

Summing (2.37) over $i =, \cdots, p$ we have

$$(1+\tau_2^1+\cdots+\tau_p^1)\{f_x^1(t,x^*(t),\dot{x}^*(t))+w_1(t)-\frac{d}{dt}f_{\dot{x}}^1(t,x^*(t),\dot{x}^*(t))\}$$

$$+\sum_{j=1}^{m}\lambda_j^1(t)\{g_x^j(t,x^*,\dot{x}^*(t))-\frac{d}{dt}g_{\dot{x}}^j(t,x^*(t),\dot{x}^*(t))$$

$$+\sum_{l=1}^{q}\mu_l^1(t)\{h_x^l(t,x^*(t),\dot{x}^*(t))-\frac{d}{dt}h_{\dot{x}}^l(t,x^*(t),\dot{x}^*(t))$$

$$+(\tau_1^2+1\cdots+\tau_p^2)\{f_x^2(t,x^*(t),\dot{x}^*(t)+w_2(t)-\frac{d}{dt}f_{\dot{x}}^2(t,x^*(t),\dot{x}^*(t))\}$$

$$+\sum_{j=1}^{m}\lambda_j^2(t)\{g_x^j(t,x^*(t),\dot{x}^*(t))-\frac{d}{dt}g_x^j(t,x^*(t),\dot{x}^*(t))\}$$

$$+\sum_{l=1}^{q}\mu_l^2(t)\{h_x^l(t,x^*(t),\dot{x}^*(t))-\frac{d}{dt}h_{\dot{x}}^l(t,x^*(t),\dot{x}^*(t))\}+\cdots$$

$$+(\tau_1^p+\tau_2^p+\cdots+1)\{f_x^p(t,x^*(t),\dot{x}^*(t))+w_p(t)-\frac{d}{dt}f_{\dot{x}}^p(t,x^*(t),\dot{x}^*(t))\}$$

$$+\sum_{j=1}^{m}\lambda_j^p(t)\{g_x^j(t,x^*,\dot{x}^*(t))-\frac{d}{dt}g_{\dot{x}}^j(t,x^*(t),\dot{x}^*(t))\}$$

$$+\sum_{l=1}^{q}\mu_l^p(t)\{h_x^l(t,x^*(t),\dot{x}^*(t))-\frac{d}{dt}h_{\dot{x}}^l(t,x^*(t),\dot{x}^*(t))\}=0.$$

Let

$$\tau_1^*=1+\tau_2^1+\cdots+\tau_p^1,\quad \tau_2^*=\tau_1^2+1+\cdots+\tau_p^2,\quad \cdots,$$

$$\tau_p^*=\tau_1^p+\tau_2^p+\cdots+1,$$

$$\sum_{j=1}^{m} \lambda_j^k(t) = \lambda_j^*(t), \qquad (j = 1, \cdots, m) \quad \lambda^*(t) = (\lambda_1^*(t), \cdots, \lambda_m^*(t)),$$

$$\sum_{l=1}^{q} \mu_l^k(t) = \mu_l^*(t), \qquad (l = 1, \cdots, q) \quad \mu^*(t) = (\mu_1^*(t), \cdots, \mu_q^*(t)).$$

Then we have

$$\sum_{i=1}^{p} \tau^*(f_x^i(t, x^*, \dot{x}^*(t)) - \frac{d}{dt} f_{\dot{x}}^i(t, x^*(t), \dot{x}^*(t)))$$

$$+ \sum_{j=1}^{m} \lambda_j^*(t)(g_x^j(t, x^*(t), \dot{x}^*(t) - \frac{d}{dt} g_{\dot{x}}^j(t, x^*(t), \dot{x}^*(t)))$$

$$+ \sum_{l=1}^{q} \mu_l^*(t)(h_x^l(t, x^*(t), \dot{x}^*(t) - \frac{d}{dt} h_{\dot{x}}^l(t, x^*(t), \dot{x}^*(t))) = 0.$$

Summing (2.38) for $i = 1, \cdots, p$, we have

$$\int_a^b \sum_{j=1}^{m} \lambda_j^1(t) g^j(t, \dot{x}(t), \dot{x}^*(t))dt + \int_a^b \sum_{j=1}^{m} \lambda_j^2(t) g^j(t, \dot{x}(t), \dot{x}^*(t))dt + \cdots$$

$$+ \int_a^b \sum_{j=1}^{m} \lambda_j^p(t) g^j(t, \dot{x}(t), \dot{x}^*(t))dt$$

$$= \int_a^b \sum_{j=1}^{m} (\lambda_j^1(t) + \lambda_j^2(t) + \cdots + \lambda_j^p) g^j(t, x^*(t), \dot{x}^*(t))dt$$

$$= \int_a^b \sum_{j=1}^{m} \lambda_j^*(t) g^j(t, \dot{x}(t), \dot{x}^*(t))dt = 0.$$

We conclude that $(x^*(t), \tau^*, \lambda^*(t), \mu^*(t))$ is feasible for **(NMMVDE)**. Efficiency of $(x^*(t), \tau^*, \lambda^*(t), \mu^*(t))$ for **(NMMVDE)** now follows from Corollary 2.2. □

**Theorem 2.7 (Converse Duality)** *Suppose that*

*(i)* $(y^*(t), \tau^*, \lambda^*(t), \mu^*(t)) \in Y_1$ :

*(ii)* $y^*(t) \in X_1$ :

*(iii)* $(f(x) + x^T(t)w(t) + \sum_{l=1}^{q} \mu_l(t)h_l(t), g(x))$ *is V-type I invex at* $y^*(t)$ *with respect to* $\eta(t)$ *and* $\tau_i^* > 0$ *and for some positive functions* $\alpha_i(t), \beta_j(t)$ *for* $i = 1, \cdots, p,\ j = 1, \cdots, m$ :

*Then* $y^*(t)$ *is an efficient solution of* **(NMVPE)**.

*Proof.* If follows by the hypothesis (i) and (ii)

$$\int_a^b \lambda_j^*(t) g^j(t, y^*(t), \dot{y}^*(t))dt = 0, \quad \forall j = 1, \cdots, m, \tag{2.37}$$

By hypothesis (iii), for any $x(t) \in X_1$, we have

$$\int_a^b (f^i(t, x(t), \dot{x}(t) + x^T(t)w_i(t))dt - \int_a^b (f^i(t, y^*(t), \dot{y}^*(t))$$

$$+y^T(t)w_i(t) + \sum_{l=1}^{q} \mu_l^*(t)h^l(t, y^*(t), \dot{y}^*(t)))dt$$

$$\geqq \int_a^b \alpha_i(x(t), \dot{x}(t), y^*(t), \dot{y}^*(t))\eta(t, x(t), y^*(t), \dot{x}(t), \dot{y}^*(t))((f_y^i(t, y^*(t), \dot{y}^*(t))$$

$$+w_i(t) - \frac{d}{dt} f^i_{\dot{y}}(t, y^*(t), \dot{y}^*(t)))$$

$$+ \sum_{l=1}^{q} \mu_l^*(t)(h^l_y(t, y^*(t), \dot{y}^*(t)) - \frac{d}{dt} h^l_{\dot{y}}(t, y^*(t), \dot{y}^*(t))))dt,$$

$$\forall i = 1, \cdots, p \tag{2.38}$$

and

$$- \int_a^b g^j(t, y^*(t), \dot{y}^*(t))dt$$

$$\geqq \int_a^b \beta_j(x(t), \dot{x}(t), y^*(t), \dot{y}^*(t)) \eta(t, x(t), y^*(t), \dot{x}(t), \dot{y}^*(t)) \{g^j_y(t, y^*(t), \dot{y}^*(t)$$

$$- \frac{d}{dt} g^j_{\dot{y}}(t, y^*(t), \dot{y}^*(t))\}dt, \ \forall j = 1, \cdots, m. \tag{2.39}$$

Now by the facts $\alpha_i(t) > 0, \beta_j(t) > 0, \forall i, j$ and $\tau^* > 0, \lambda^*(t) \geqq 0$, it follows by (2.40)and (2.41) that

$$\int_a^b \sum_{i=1}^{p} \frac{\tau_i^*}{\alpha_i(x(t), y^*(t), \dot{x}(t), \dot{y}^*(t))} (f^i(t, x(t), \dot{x}(t)) + x^T(t) w_i(t))dt$$

$$- \int_a^b \sum_{i=1}^{p} \frac{\tau_i^*}{\alpha_i(x(t), y^*(t), \dot{x}(t), \dot{y}^*(t))} (f^i(t, y^*(t), \dot{y}^*(t)) + y^T(t) w_i(t)$$

$$+ \sum_{l=1}^{q} \mu_l^*(t) h^l(t, y^*(t), \dot{y}^*(t)))dt$$

$$- \int_a^b \sum_{j=1}^{m} \frac{\lambda_j^*(t)}{\beta_j(x(t), y^*(t), \dot{x}(t), \dot{y}^*(t))} g^j_y(t, y^*(t), \dot{y}^*(t))dt$$

$$\geqq \int_a^b \eta(t,x(t),y^*(t),\dot{x}(t),\dot{y}^*(t))\big(\sum_{i=1}^p \tau_i^*(f_i^*(t,y^*(t),\dot{y}^*(t))+w_i(t)$$

$$-\frac{d}{dt}f_{\dot{y}}^i(t,y^*(t),\dot{y}^*(t)))$$

$$+\sum_{j=1}^m \lambda_j^*(t)(g_y^j(t,y^*(t),\dot{y}^*(t))-\frac{d}{dt}g_y^j(t,y^*(t),\dot{y}^*(t)))$$

$$+\sum_{l=1}^q \mu_l^*(t)(h_y^l(t,y^*(t),\dot{y}^*(t))-\frac{d}{dt}h_{\dot{y}}^l(t,y^*(t),\dot{y}^*(t)))\big)dt=0. \quad (2.40)$$

From (2.14),(2.39) and (2.42),it follows that

$$\int_a^b \sum_{i=1}^p \frac{\tau_i^*}{\alpha_i(x(t),y^*(t),\dot{x}(t),\dot{y}^*(t))}\{f^i(t,x(t),\dot{x}(t))+x^T(t)w_i(t)\}dt$$

$$\geqq \int_a^b \sum_{i=1}^p \frac{\tau_i^*}{\alpha_i(x(t),y^*(t),\dot{x}(t),\dot{y}^*(t))}\{f^i(t,y^*(t),\dot{y}^*(t))+y^{*T}(t)w_i(t)\}dt. \quad (2.41)$$

Now suppose that $y^*(t)$ is not an efficient solution of **(NMVPE)**.

Then there exists an $x(t)\in X_1$ such that

$$\int_a^b \{f^i(t,x(t),\dot{x}(t))+x^T(t)w_i(t)\}dt$$

$$\leqq \int_a^b \{f^i(t,y^*(t),\dot{y}^*(t))+y^{*T}(t)w_i(t)\}dt, \forall \quad i=1,\cdots,p$$

and

$$\int_a^b \{f^{i_0}(t,x(t),\dot{x}(t)) + x^T(t)w_{i_0}(t)\}dt$$

$$< \int_a^b \{f^{i_0}(t,y^*(t),\dot{y}^*(t)) + y^{*T}(t)w_{i_0}(t)\}dt, \text{ for some } i_0 = 1,\cdots,p.$$

which implies that

$$\int_a^b \sum_{i=1}^{p} \frac{\tau_i^*}{\alpha_i(x(t),y^*(t),\dot{x}(t),\dot{y}^*(t))}\{f^i(t,x(t),\dot{x}(t)) + x^T(t)w_i(t)\}dt$$

$$< \int_a^b \sum_{i=1}^{p} \frac{\tau_i^*}{\alpha_i(x(t),y^*(t),\dot{x}(t),\dot{y}^*(t))}\{f^i(t,y^*(t),\dot{y}^*(t)) + y^T(t)w_i(t)\}dt. \tag{2.42}$$

Now (2.42)and (2,43) contradict each other.

Hence the conclusion follows. □

Now we establish weak, strong and converse duality theorems between the multiobjective variational problem

**(NMVPE)** and its generalized Mond-Weir dual problem **(NGMMVDE)**.

**Theorem 2.8 (Weak Duality)** *Assume that for all feasible $x(t)$ for* **(NMVPE)** *and feasible $(y(t), \tau, \lambda(t), \mu(t))$ for* **(NGMMVDE)**, *any of the following holds :*

*(i)* $\tau_i > 0$, *and* $(f(x) + x^T(t)w(t) + \sum_{j\in J_0} \lambda_j(t)g^j(x) + \sum_{l\in K_0} \mu_l(t)h^l(x),$ $\sum_{j\in J_\alpha} \lambda_j(t)g^j(x) + \sum_{l\in K_\alpha} \mu_l(t)h^l(x))$ *is weak strictly-pseudoquasi V-type I invex at* $y(t)$ *with respect to* $\eta(t)$ *for any* $\alpha$, $1 \leqq \alpha \leqq r$ *and for some positive functions* $\alpha_i(t)$, *for* $i = 1, \cdots, p$ *and* $\beta(t)$*;*

*(ii)* $(f(x)+x^T(t)w(t)+\sum_{j\in J_0} \lambda_j(t)g^j(x)+\sum_{l\in K_0} \mu_l(t)h^l(x), \sum_{j\in J_\alpha} \lambda_j(t)g^j(x) +\sum_{l\in K_\alpha} \mu_l(t)h^l(x))$ *is weak strictly-pseudo V-type I invex at* $y(t)$ *with respect to* $\eta(t)$ *and for some positive functions* $\alpha_i(t)$, *for* $i = 1, \cdots, p$ *and* $\beta(t)$*;*

*(iii)* $(f(x)+x^T(t)w(t)+\sum_{j\in J_0} \lambda_j(t)g^j(x)+\sum_{l\in K_0} \mu_l(t)h^l(x), \sum_{j\in J_\alpha} \lambda_j(t)g^j(x) + \sum_{l\in K_\alpha} \mu_l(t)h^l(x))$ *is weak quasistrictly-pseudo V-type I invex at* $y(t)$ *with respect to* $\eta(t)$ *and for some positive functions* $\alpha_i(t)$, *for* $i = 1, \cdots, p$ *and* $\beta(t)$*;*

*Then the following inequalities cannot hold :*

$$\int_a^b (f^i(t, x(t), \dot{x}(t)) + s(x(t)|D_i))dt \leqq \int_a^b (f^i(t, y(t), \dot{y}(t)) + y^T(t)w_i(t) + \sum_{j\in J_0} \lambda_j(t)g^j(t, y(t), \dot{y}(t)) + \sum_{l\in K_0} \mu_l(t)h^l(t, y(t), \dot{y}(t)))dt, \quad \forall i = 1, \cdots, p \tag{2.43}$$

*and*

$$\int_a^b (f^{i_0}(t, x(t), \dot{x}(t)) + s(x(t)|D_{i_0}))dt < \int_a^b (f^{i_0}(t, y(t), \dot{y}(t)) + y^T(t)w_{i_0}(t) + \sum_{j\in J_0} \lambda_j(t)g^j(t, y(t), \dot{y}(t)) + \sum_{l\in K_0} \mu_l(t)h^l(t, y(t), \dot{y}(t)))dt, \quad for\ some\ i_0 = 1, \cdots, p. \tag{2.44}$$

*Proof.* Suppose contrary to the result that (2.45) and (2.46) hold. Since $x(t)$ is feasible for **(NMVPE)**, and $\lambda(t) \geq 0$, (2.45) and (2.46) imply

$$\int_a^b (f^i(t, x(t), \dot{x}(t)) + s(x(t)|D_i) + \sum_{j \in J_0} \lambda_j(t) g^j(t, x(t), \dot{x}(t))$$

$$+ \sum_{l \in K_0} \mu_l(t) h^l(t, x(t), \dot{x}(t)))dt$$

$$\leqq \int_a^b (f^i(t, y(t), \dot{y}(t)) + y^T(t) w_i(t) + \sum_{j \in J_0} \lambda_j(t) g^j(t, y(t), \dot{y}(t))$$

$$+ \sum_{l \in K_0} \mu_l(t) h^l(t, y(t), \dot{y}(t)))dt, \ \forall i = 1, \cdots, p$$

and

$$\int_a^b (f^{i_0}(t, x(t), \dot{x}(t)) + s(x(t)|D_{i_0}) + \sum_{j \in J_0} \lambda_j(t) g^j(t, x(t), \dot{x}(t))$$

$$+ \sum_{l \in K_0} \mu_l(t) h^l(t, x(t), \dot{x}(t)))dt$$

$$< \int_a^b (f^{i_0}(t, y(t), \dot{y}(t)) + y^T(t) w_{i_0}(t)$$

$$+ \sum_{j \in J_0} \lambda_j(t) g^j(t, y(t), \dot{y}(t)) + \sum_{l \in K_0} \mu_l(t) h^l(t, y(t), \dot{y}(t)))dt,$$

for some $i_0 = 1, \cdots, p$.

Then since $\alpha_i(t) > 0$, (2.21) we have

$$\int_a^b \sum_{i=1}^p \tau_i \alpha_i(t)(x(t), y(t), \dot{x}(t), \dot{y}(t))(f^i(t, x(t), \dot{x}(t)) + s(x(t)|D_i)$$

$$+ \sum_{j \in J_0} \lambda_j(t) g^j(t, x(t), \dot{x}(t)) + \sum_{l \in K_0} \mu_l(t) h^l(t, x(t), \dot{x}(t)))dt$$

$$< \int_a^b \sum_{i=1}^p \tau_i \alpha_i(t)(x(t), y(t), \dot{x}(t), \dot{y}(t))(f^i(t, y(t), \dot{y}(t)) + y^T(t) w_i(t)$$

$$+ \sum_{j \in J_0} \lambda_j(t) g^j(t, y(t), \dot{y}(t)) + \sum_{l \in K_0} \mu_l(t) h^l(t, y(t), \dot{y}(t)))dt,$$

$$\forall i = 1, \cdots, p, \quad (2.45)$$

Also, from (2.13) and $\beta(t) > 0$ we have

$$-\int_a^b \beta(t)(x(t), y(t), \dot{x}(t), \dot{y}(t))(\sum_{j \in J_\alpha} \lambda_j(t) g^j(t, y(t), \dot{y}(t))$$

$$+ \sum_{l \in K_\alpha} \mu_l(t) h^l(t, y(t), \dot{y}(t)))dt \leqq 0,$$

$$for\ all \quad 1 \leqq \alpha \leqq r. \quad (2.46)$$

Using hypothesis (i), we see that (2.47) and (2.48) together give

$$\int_a^b \sum_{i=1}^p \tau_i \eta(t, x(t), y(t), \dot{x}(t), \dot{y}(t)) \Big( f_y^i(t, y(t), \dot{y}(t)) + w_i(t)$$

$$+ \sum_{j \in J_0} \lambda_j(t) g_y^j(t, y(t), \dot{y}(t)) + \sum_{l \in K_0} \mu_l(t) h_y^l(t, y(t), \dot{y}(t))$$

$$-\frac{d}{dt}(f^i_{\dot{y}}(t,y(t),\dot{y}(t))+\sum_{j\in J_0}\lambda_j(t)g^j_{\dot{y}}(t,y(t),\dot{y}(t))+\sum_{l\in K_0}\mu_l(t)h^l_{\dot{y}}(t,y(t),\dot{y}(t)))\Big)dt$$

$$=\int_a^b \eta(t,x(t),y(t),\dot{x}(t),\dot{y}(t))\Big(\sum_{i=1}^p \tau_i(f^i_y(t,y(t),\dot{y}(t))+w_i(t)$$

$$-\frac{d}{dt}f^i_{\dot{y}}(t,y(t),\dot{y}(t)))+\sum_{j\in J_0}\lambda_j(t)(g^j_y(t,y(t),\dot{y}(t))-\frac{d}{dt}g^j_{\dot{y}}(t,y(t),\dot{y}(t)))$$

$$+\sum_{l\in K_0}\mu_l(t)(h^l_y(t,y(t),\dot{y}(t))-\frac{d}{dt}h^l_{\dot{y}}(t,y(t),\dot{y}(t)))\Big)dt<0,$$

$$\int_a^b \eta(t,x(t),y(t),\dot{x}(t),\dot{y}(t))\Big(\sum_{j\in J_\alpha}\lambda_j(t)g^j_y(t,y(t),\dot{y}(t))+\sum_{l\in K_\alpha}\mu_l(t)h^l_y(t,y(t),\dot{y}(t))$$

$$-\frac{d}{dt}(\sum_{j\in J_\alpha}\lambda_j(t)g^j_{\dot{y}}(t,y(t),\dot{y}(t))+\sum_{l\in K_\alpha}\mu_l(t)h^l_{\dot{y}}(t,y(t),\dot{y}(t)))\Big)dt$$

$$=\int_a^b \eta(t,x(t),y(t),\dot{x}(t),\dot{y}(t))\Big(\sum_{j\in J_\alpha}\lambda_j(t)(g^j_y(t,y(t),\dot{y}(t))-\frac{d}{dt}g^j_{\dot{y}}(t,y(t),\dot{y}(t)))$$

$$+\sum_{l\in K_\alpha}\mu_l(t)(h^l_y(t,y(t),\dot{y}(t))-\frac{d}{dt}h^l_{\dot{y}}(t,y(t),\dot{y}(t)))\Big)dt\leqq 0,\forall \quad 1\leqq\alpha\leqq r.$$

Since the above inequalities give

$$\int_a^b \eta(t,x(t),y(t),\dot{x}(t),\dot{y}(t))\Big(\sum_{i=1}^p \tau_i(f^i_y(t,y(t),\dot{y}(t))+w_i(t)$$

$$-\frac{d}{dt}f^i_{\dot{y}}(t,y(t),\dot{y}(t)))+\sum_{j=0}^r\lambda_j(t)(g^j_y(t,y(t),\dot{y}(t))-\frac{d}{dt}g^j_{\dot{y}}(t,y(t),\dot{y}(t)))$$

$$+\sum_{l=0}^{r}\mu_l(t)(h_y^l(t,y(t),\dot{y}(t))-\frac{d}{dt}h_{\dot{y}}^l(t,y(t),\dot{y}(t)))\Big)dt<0. \quad (2.47)$$

Since $J_0, J_1, \cdots, J_r$ are partitions of $\{1,\cdots,m\}$ and $k_0, k_1, \cdots, k_r$ are partitions of $\{1,\cdots,q\}$ (2.49) is equivalent to

$$\int_a^b \eta(t,x(t),y(t),\dot{x}(t),\dot{y}(t))\Big(\sum_{i=1}^{p}\tau_i(f_y^i(t,y(t),\dot{y}(t))+w_i(t)$$

$$-\frac{d}{dt}f_{\dot{y}}^i(t,y(t),\dot{y}(t)))+\sum_{j=1}^{m}\lambda_j(t)(g_y^j(t,y(t),\dot{y}(t))-\frac{d}{dt}g_{\dot{y}}^j(t,y(t),\dot{y}(t)))$$

$$+\sum_{l=1}^{q}\mu_l(t)(h_y^l(t,y(t),\dot{y}(t))-\frac{d}{dt}h_{\dot{y}}^l(t,y(t),\dot{y}(t)))\Big)dt<0. \quad (2.48)$$

Which contradicts (2.12).

Suppose now that (ii) is satisfied.

$$\int_a^b \eta(t,x(t),y(t),\dot{x}(t),\dot{y}(t))\Big(\sum_{i=1}^{p}\tau_i(f_y^i(t,y(t),\dot{y}(t))+w_i(t)$$

$$-\frac{d}{dt}f_{\dot{y}}^i(t,y(t),\dot{y}(t)))+\sum_{j\in J_0}\lambda_j(t)(g_y^j(t,y(t),\dot{y}(t))-\frac{d}{dt}g_{\dot{y}}^j(t,y(t),\dot{y}(t)))$$

$$+\sum_{l\in k_0}\mu_l(t)(h_y^l(t,y(t),\dot{y}(t))-\frac{d}{dt}h_{\dot{y}}^l(t,y(t),\dot{y}(t)))\Big)dt<0, \quad (2.49)$$

$$\int_a^b \eta(t,x(t),y(t),\dot{x}(t),\dot{y}(t))\Big(\sum_{j\in J_\alpha}\lambda_j(t)(g_y^j(t,y(t),\dot{y}(t))$$

$$-\frac{d}{dt}g_{\dot{y}}^j(t,y(t),\dot{y}(t)))+\sum_{l\in k_\alpha}\mu_l(t)(h_y^l(t,y(t),\dot{y}(t))-\frac{d}{dt}h_{\dot{y}}^l(t,y(t),\dot{y}(t)))\Big)dt$$

$$< 0, \qquad\qquad \forall 1 \leqq \alpha \leqq r. \tag{2.50}$$

Since (2.15) the above inequality give

$$\int_a^b \eta(t, x(t), y(t), \dot{x}(t), \dot{y}(t))\Big(\sum_{i=1}^{p} \tau_i(t)(f_y^i(t, y(t), \dot{y}(t)) + w_i(t)$$

$$-\frac{d}{dt} f_{\dot{y}}^i(t, y(t), \dot{y}(t))) + \sum_{j=0}^{r} \lambda_j(t)(g_y^j(t, y(t), \dot{y}(t)) - \frac{d}{dt} g_{\dot{y}}^j(t, y(t), \dot{y}(t)))$$

$$+\sum_{l=0}^{r} \mu_l(t)(h_y^l(t, y(t), \dot{y}(t)) - \frac{d}{dt} h_{\dot{y}}^l(t, y(t), \dot{y}(t)))\Big)dt < 0 \tag{2.51}$$

and then again we have (2.50). Also we obtain a contradiction.

Using hypothesis (iii), we see that (2.47) and (2.48) together give

$$\int_a^b \eta(t, x(t), y(t), \dot{x}(t), \dot{y}(t))(\sum_{i=1}^{p} \tau_i(f_y^i(t, y(t), \dot{y}(t)) + w_i(t)$$

$$-\frac{d}{dt} f_{\dot{y}}^i(t, y(t), \dot{y}(t))) + \sum_{j \in J_0} \lambda_j(t)(g_y^j(t, y(t), \dot{y}(t)) - \frac{d}{dt} g_{\dot{y}}^i(t, y(t), \dot{y}(t)))$$

$$+\sum_{l \in K_0} \mu_l(t)(h_y^l(t, y(t), \dot{y}(t)) - \frac{d}{dt} h_{\dot{y}}^l(t, y(t), \dot{y}(t))))dt \leqq 0,$$

$$\int_a^b \eta(t, x(t), y(t), \dot{x}(t), \dot{y}(t))\Big(\sum_{j \in J_\alpha} \lambda_j(t)(g_y^j(t, y(t), \dot{y}(t)) - \frac{d}{dt} g_{\dot{y}}^j(t, y(t), \dot{y}(t)))$$

$$+\sum_{l \in K_\alpha} \mu_l(t)(h_y^l(t, y(t), \dot{y}(t)) - \frac{d}{dt} h_{\dot{y}}^l(t, y(t), \dot{y}(t)))\Big)dt < 0, \ \forall 1 \leqq \alpha \leqq r$$

and then again we have (2.50). Also we obtain a contradiction. □

**Corollary 2.3** *Assume that weak duality holds between* **(NMVPE)** *and* **(NGMMVDE)**. *If* $(y^*(t), \tau^*, \lambda^*(t), \mu^*(t))$ *is feasible for* **(NGMMVDE)** *with* $\sum_{j\in J_0} \lambda_j^*(t) g^j(t, y^*(t), \dot{y}^*(t)) = 0$, $y^{*T}(t) w_i(t) = S(y^*(t)|D_i), i = 1, \cdots, p$ *and* $y^*(t)$ *is feasible for* **(NMVPE)**, *then* $y^*(t)$ *is an efficient solution for* **(NMVPE)** *and* $(y^*(t), \tau^*, \lambda^*(t), \mu^*(t))$ *is an efficient solution for* **(NGM-MVDE)**.

*Proof.* Suppose that $y^*(t)$ is not an efficient for **(NMVPE)**.
Then there exists a feasible $x(t)$ for **(NMVPE)** such that

$$\begin{aligned}&\int_a^b (f^i(t, x(t), \dot{x}(t)) + S(x(t)|D_i))dt\\ &\leqq \int_a^b (f^i(t, y^*(t), \dot{y}^*(t)) + y^{*T}(t) w_i(t))dt,\\ &\qquad \forall\ i = 1, \cdots, p\end{aligned} \tag{2.52}$$

and

$$\begin{aligned}&\int_a^b (f^{i_0}(t, x(t), \dot{x}(t)) + S(x(t)|D_{i_0})(t))dt\\ &< \int_a^b (f^{i_0}(t, y(t), \dot{y}(t)) + y^{*T}(t) w_{i_0}(t))dt,\\ &\qquad \text{for some } i_0 = 1, \cdots, p.\end{aligned} \tag{2.53}$$

By hypotheses $\sum_{j\in J_0} \lambda_j^*(t) g^j(t, y^*(t), \dot{y}^*(t)) = 0$ and $\sum_{l=1}^{q} \mu_l^*(t) h^l(t, y^*(t), \dot{y}^*(t))$=0.

So (2.54) and (2.55) can be written as

$$\int_a^b (f^i(t,x(t),\dot{x}(t)) + S(x(t)|D_i))dt \leqq \int_a^b \Big( f^i(t,y^*(t),\dot{y}^*(t)) + y^{*T}(t)w_i(t)$$

$$+ \sum_{j\in J_0} \lambda_j^*(t) g^j(t,y^*(t),\dot{y}^*(t)) + \sum_{l=1}^{q} \mu_l^*(t) h^l(t,y^*(t),\dot{y}^*(t))\Big) dt,$$

$$\forall i = 1,\cdots,p \tag{2.54}$$

and

$$\int_a^b (f^{i_0}(t,x(t),\dot{x}(t)) + S(x(t)|D_{i_0})(t))dt < \int_a^b \Big( f^{i_0}(t,y(t),\dot{y}^*(t)) + y^{*T}(t)w_{i_0}(t)$$

$$+ \sum_{j\in J_0} \lambda_j^*(t) g^j(t,y^*(t),\dot{y}^*(t)) + \sum_{l=1}^{q} \mu_l^*(t) h^l(t,y^*(t),\dot{y}^*(t))\Big) dt,$$

$$\text{for some} \quad i_0 = 1,\cdots,p. \tag{2.55}$$

Since $(y^*(t),\tau^*,\lambda^*(t),\mu^*(t))$ is feasible in **(NGMMVED)** and $x(t)$ is feasible for **(NMVPE)** these inequalities contradict weak duality. Also suppose that $(y^*(t),\tau^*,\lambda^*(t),\mu^*(t))$ is not an efficient for **(NGMMVED)**. Then there exists a feasible $(y(t),\tau,\lambda(t),\mu(t))$ for **(NGMMVED)** such that

$$\int_a^b \Big( f^i(t,y(t),\dot{y}(t)) + y^T(t)w_i(t) + \sum_{j\in J_0} \lambda_j(t) g^j(t,y(t),\dot{y}(t))$$

$$+ \sum_{l\in k_0} \mu_l(t) h^l(t,y(t),\dot{y}(t))\Big) dt$$

$$\geqq \int_a^b \Big( f^i(t,y^*(t),\dot{y}^*(t)) + y^{*T}(t)w_i(t) + \sum_{j\in J_0} \lambda_j^*(t) g^j(t,y^*(t),\dot{y}^*(t))$$

$$+\sum_{l\in k_0}\mu_l^*(t)h^l(t,y^*(t),\dot{y}^*(t))\Big)dt,$$

$$i=1,\cdots,p \tag{2.56}$$

and

$$\int_a^b \Big(f^{i_0}(t,y(t),\dot{y}(t))+y^T(t)w_{i_0}(t)+\sum_{j\in J_0}\lambda_j(t)g^j(t,y(t),\dot{y}(t))$$

$$+\sum_{l\in k_0}\mu_l(t)h^l(t,y(t),\dot{y}(t))\}dt$$

$$\geq \int_a^b \{f^{i_0}(t,y^*(t),\dot{y}^*(t))+y^{*T}(t)w_{i_0}(t)+\sum_{j\in J_0}\lambda_j^*(t)g^j(t,y^*(t),\dot{y}^*(t))$$

$$+\sum_{l\in k_0}\mu_l^*(t)h^l(t,y^*(t),\dot{y}^*(t))\Big)dt,$$

$$\text{for some } i_0=1,\cdots,p \tag{2.57}$$

and since $\sum_{j\in J_0}\lambda_j^*(t)g^j(t,y^*(t),\dot{y}^*(t))=0$, $\sum_{l\in k_0}\mu_l^*(t)h^l(t,y^*(t),\dot{y}^*(t))=0$ and $y^{*T}(t)w_i(t)=S(y^*(t)|D_i)$, $i=1,\cdots,p$, (2.58) and (2.59) reduce to

$$\int_a^b \Big(f^i(t,y(t),\dot{y}(t))+y^T(t)w_i(t)+\sum_{j\in J_0}\lambda_j(t)g^j(t,y(t),\dot{y}(t))$$

$$+\sum_{l\in k_0}\mu_l(t)h^l(t,y(t),\dot{y}(t))\Big)dt \geqq \int_a^b (f^i(t,y^*(t),\dot{y}^*(t))+S(y^*(t)|D_i))dt,$$

$$\forall i=1,\cdots,p \tag{2.58}$$

and

$$\int_a^b \Big(f^{i_0}(t, y(t), \dot{y}(t)) + y^T(t)w_{i_0}(t) + \sum_{j\in J_0} \lambda_j(t)g^j(t, y(t), \dot{y}(t))$$

$$+ \sum_{l\in k_0} \mu_l(t)h^l(t, y(t), \dot{y}(t))\Big)dt \geqq \int_a^b (f^{i_0}(t, y^*(t), \dot{y}^*(t)) + S(y^*(t)|D_{i_0}))dt,$$

for some $i_0 = 1, \cdots, p$.

Since $y^*(t)$ is feasible for **(NMVPE)**, these inequalities contradict weak duality.

Therefore $y^*(t)$ and $(y^*(t), \tau^*, \lambda^*(t), \mu^*(t))$ are an efficient for their respective problems. □

**Theorem 2.9 (Strong Duality)** *Assume that*

*(i)* $x^*(t)$ *is an efficient solution for* **(NMVPE)**

*(ii) for all* $k = 1, \cdots, p$, $x^*(t)$ *a constraint qualification for problem* **NMVPE**$_k(x^*(t))$ *is satisfied at* $x^*(t)$ :

*Then there exist* $\tau_i^* \in R^p$, $\tau_i^* > 0$, *and piecewise smooth function* $\lambda^*(t) : I \longrightarrow R^*, \lambda^*(t) \geqq 0$ *and* $\mu^*(t) : I \longrightarrow R^q$ *such that* $(x^*(t), \tau^*, \lambda^*(t), \mu^*(t))$ *is feasible for* **(NGMMVDE)** *and* $\sum_{j\in J_0} \lambda_j^*(t)g^j(t, y^*(t), \dot{y}^*(t)) = 0$, $x^{*T}(t)w_i(t) = S(x^*(t)|D_i)$, $i = 1 \cdots, p$. *Further, if also weak duality holds between* **(NMVPE)** *and* **(NGMMVDE)**, *then* $(x^*(t), \tau^*, \lambda^*(t), \mu^*(t))$ *is an efficient solution for* **(NGMMVDE)**.

*Proof.* Similar to the proof of Theorem 2.6 and Corollary 2.2 above. □

## 2.6 Special Case

As a special case of our duality results between **(NMVPE)** and **(NGMMVDE)**, we give Wolf type duality theorems.

If $J_0 = \{1, \cdots, m\}, J_\alpha = \phi, k_0 = \{1, \cdots, k\}, k_\alpha = \phi$ then **(NGMMVDE)** reduced to the Wolf type dual [2]

**(NWMVDE)**

Maximize

$$\Big(\int_a^b (f^1(t, y(t), \dot{y}(t) + y^T(t)w_1(t) + \sum_{j=1}^{m} \lambda_j(t) g^j(t, y(t), \dot{y}(t))$$

$$+ \sum_{l=1}^{q} \mu_l(t) h^l(t, y(t), \dot{y}(t)))dt,$$

$$\cdots, \int_a^b (f^p(t, y(t), \dot{y}(t) + y^T(t)w_p(t) + \sum_{j=1}^{m} \lambda_j(t) g^j(t, y(t), \dot{y}(t))$$

$$+ \sum_{l=1}^{q} \mu_l(t) h^l(t, y(t), \dot{y}(t)))dt\Big)$$

subject to $\quad y(a) = t_0, \quad y(b) = t_f,$

$$\sum_{i=1}^{p} \tau_i (f_y^i(t, y(t), \dot{y}(t)) + w_i(t) - \frac{d}{dt} f_{\dot{y}}^i(t, y(t), \dot{y}(t)))$$

$$+ \sum_{j=1}^{m} \lambda_j(t)(g_y^j(t, y(t), \dot{y}(t)) - \frac{d}{dt} g_{\dot{y}}^j(t, y(t), \dot{y}(t)))$$

$$+ \sum_{l=1}^{q} \mu_l(t)(h_y^l(t, y(t), \dot{y}^*(t)) - \frac{d}{dt} h_{\dot{y}}^l(t, y(t), \dot{y}(t))) = 0, \tag{2.59}$$

$$\lambda(t) \geq 0,$$

$$\tau_i \geq 0, \qquad \sum_{i=1}^{p} \tau_i = 1.$$

Now we establish weak, strong, converse duality theorems between the nondifferentiable multiobjective variational problem **(NMVPE)** and its Wolfe type dual problem **(NWMVDE)**.

**Theorem 2.10 (Weak Duality)** *Suppose that*
*(i)* $x(t) \in X_1$
*(ii)* $(y(t), \tau, \lambda(t)) \in Y_1$
*(iii)* $\int_a^b \Big( \sum_{i=1}^{p} \tau_i (f^i(x) + x^T(t)w(t)) + \sum_{j=1}^{m} \lambda_j(t) g^j(x) + \sum_{l=1}^{q} \mu_l(t) h^l(x) \Big) dt$ *is pseudo invex in* $\dot{y}(t)$ *and* $y(t)$ *with respect to* $\eta(t)$ *and for some positive functions* $\alpha_i$, $\beta_j$ *for* $i = 1, \cdots, p, \quad j = 1, \cdots, m.$
*Then the following inequalities cannot hold:*

$$\int_a^b (f^i(t, x(t), \dot{x}(t)) + s(x(t)|D_i))dt \leqq \int_a^b \Big( f^i(t, y(t), \dot{y}(t)) + y^T(t) w_i(t) + \sum_{j=1}^{m} \lambda_j(t) g^j(t, y(t), \dot{y}(t)) + \sum_{l=1}^{q} \mu_l(t) h^l(t, y(t), \dot{y}(t)) \Big) dt, \quad \forall i = 1, \cdots, p \tag{2.60}$$

*and*

$$\int_a^b (f^{i_0}(t, x(t), \dot{x}(t)) + s(x(t)|D_{i_0}))dt \leqq \int_a^b \Big( f^{i_0}(t, y(t), \dot{y}(t)) + y^T(t) w_{i_0}(t)$$

$$+\sum_{j=1}^{m}\lambda_j(t)g^j(t,y(t),\dot{y}(t))+\sum_{l=1}^{q}\mu_l(t)h^l(t,y(t),\dot{y}(t))\Big)dt,$$

$$\textit{for some } i_0=1,\cdots,p. \tag{2.61}$$

*Proof.* Suppose contrary to the result that (2.61) and (2.62) hold.

Since $x(t)$ is feasible for **(NMVPE)**, $\lambda(t)\geq 0$ and $g_j(t,x(t),\dot{x}(t))\leqq 0$, $j=1,\cdots,m,\ h(t,x(t),\dot{x}(t))=0$. since $< w_i(t),x(t)>\leqq S(x(t)|D_i), i=1,\cdots,p$,

$$\int_a^b\Big(f^i(t,x(t),\dot{x}(t))+s(x(t)|D_i)+\sum_{j=1}^{m}\lambda_j(t)g^j(t,y(t),\dot{y}(t))$$

$$+\sum_{l=1}^{q}\mu_l(t)h^l(t,y(t),\dot{y}(t))\Big)dt$$

$$\leqq\int_a^b\Big(f^i(t,y(t),\dot{y}(t)+y^T(t)w_i(t)+\sum_{j=1}^{m}\lambda_j(t)g^j(t,y(t),\dot{y}(t))$$

$$+\sum_{l=1}^{q}\mu_l(t)h^l(t,y(t),\dot{y}(t))\Big)dt,$$

$$\forall i=1,\cdots,p$$

and

$$\int_a^b\Big(f^{i_0}(t,x(t),\dot{x}(t))+s(x(t)|D_{i_0})+\sum_{j=1}^{m}\lambda_j(t)g^j(t,y(t),\dot{y}(t))$$

$$+\sum_{l=1}^{q}\mu_l(t)h^l(t,y(t),\dot{y}(t))\Big)dt$$

$$< \int_a^b \Big( f^{i_0}(t, y(t), \dot{y}(t)) + y^T(t) w_{i_0}(t) + \sum_{j=1}^{m} \lambda_j(t) g^j(t, y(t), \dot{y}(t))$$

$$+ \sum_{l=1}^{q} \mu_l(t) h^l(t, y(t), \dot{y}(t)) \Big) dt,$$

for some $i_0 = 1, \cdots, p$.

since $< w_i(t), x(t) >= s(x(t)|D_i)$ $i = 1, \cdots, p$,

$$\int_a^b \Big( f^{i}(t, x(t), \dot{x}(t)) + x^T(t) w_{i}(t) + \sum_{j=1}^{m} \lambda_j(t) g^j(t, y(t), \dot{y}(t))$$

$$+ \sum_{l=1}^{q} \mu_l(t) h^l(t, y(t), \dot{y}(t)) \Big) dt$$

$$\leqq \int_a^b \Big( f^{i}(t, y(t), \dot{y}(t)) + y^T(t) w_{i}(t) + \sum_{j=1}^{m} \lambda_j(t) g^j(t, y(t), \dot{y}(t))$$

$$+ \sum_{l=1}^{q} \mu_l(t) h^l(t, y(t), \dot{y}(t)) \Big) dt,$$

$\forall i = 1, \cdots, p$

and

$$\int_a^b \Big( f^{i_0}(t, x(t), \dot{x}(t)) + x^T(t) w_{i_0}(t) + \sum_{j=1}^{m} \lambda_j(t) g^j(t, y(t), \dot{y}(t))$$

$$+ \sum_{l=1}^{q} \mu_l(t) h^l(t, y(t), \dot{y}(t)) \Big) dt$$

$$< \int_a^b \Big( f^{i_0}(t, y(t), \dot{y}(t) + y^T(t) w_{i_0}(t) + \sum_{j=1}^m \lambda_j(t) g^j(t, y(t), \dot{y}(t))$$

$$+ \sum_{l=1}^q \mu_l(t) h^l(t, y(t), \dot{y}(t)) \Big) dt,$$

$$\text{for some } i_0 = 1, \cdots, p.$$

$\alpha(t) > 0$, $\tau_i > 0$

$$\int_a^b \sum_{i=1}^p \tau_i \alpha_i(x(t), y(t), \dot{x}(t), \dot{y}(t)) \Big( f^i(t, x(t), \dot{x}(t)) + x^T(t) w_i(t)$$

$$+ \sum_{j=1}^m \lambda_j(t) g^j(t, y(t), \dot{y}(t)) + \sum_{l=1}^q \mu_l(t) h^l(t, y(t), \dot{y}(t)) \Big) dt$$

$$< \int_a^b \sum_{i=1}^p \tau_i \alpha_i(x(t), y(t), \dot{x}(t), \dot{y}(t)) \Big( f^i(t, y(t), \dot{y}(t) + y^T(t) w_i(t)$$

$$+ \sum_{j=1}^m \lambda_j(t) g^j(t, y(t), \dot{y}(t)) + \sum_{l=1}^q \mu_l(t) h^l(t, y(t), \dot{y}(t)) \Big) dt,$$

$$\forall i = 1, \cdots, p.$$

$$\int_a^b \tau_i \alpha_i(x(t), y(t), \dot{x}(t), \dot{y}(t)) \eta(t, x(t), y(t), \dot{x}(t), \dot{y}(t)) \Big( \sum_{i=1}^p (f_y^i(t, y(t), \dot{y}(t))$$

$$+ w_i(t) + \sum_{j=1}^m \lambda_j(t) g_y^j(t, y(t), \dot{y}(t)) + \sum_{l=1}^q \mu_l(t) h_y^l(t, y(t), \dot{y}(t))$$

$$- \frac{d}{dt} f_{\dot{y}}^i(t, y(t), \dot{y}(t)) + \sum_{j=1}^m \lambda_j(t) g_{\dot{y}}^j(t, y(t), \dot{y}(t)) + \sum_{l=1}^q \mu_l(t) h_{\dot{y}}^l(t, y(t), \dot{y}(t)) \Big) dt$$

$$= \int_a^b \tau_i \alpha_i(x(t), y(t), \dot{x}(t), \dot{y}(t)) \eta(t, x(t), y(t), \dot{x}(t), \dot{y}(t)) \Big( \sum_{i=1}^{p} f_y^i(t, y(t), \dot{y}(t))$$

$$+ w_i(t) - \frac{d}{dt} f_{\dot{y}}^i(t, y(t), \dot{y}(t)) + \sum_{j=1}^{m} \lambda_j(t) g_y^j(t, y(t), \dot{y}(t))$$

$$- \frac{d}{dt} g_{\dot{y}}^j(t, y(t), \dot{y}(t)) + \sum_{l=1}^{q} \mu_l(t) h_y^l(t, y(t), \dot{y}(t)) - \frac{d}{dt} h_{\dot{y}}^l(t, y(t), \dot{y}(t)) \Big) dt < 0.$$

Since (2.60), the above inequalities gives

$$\int_a^b \alpha_i(x(t), y(t), \dot{x}(t), \dot{y}(t)) \eta(t, x(t), y(t), \dot{x}(t), \dot{y}(t)) \Big( \sum_{i=1}^{p} \tau_i (f_y^i(t, y(t), \dot{y}(t))$$

$$+ w_i(t) - \frac{d}{dt} f_{\dot{y}}^i(t, y(t), \dot{y}(t))) + \sum_{j=1}^{m} \lambda_j(t) (g_y^j(t, y(t), \dot{y}(t))$$

$$- \frac{d}{dt} g_{\dot{y}}^j(t, y(t), \dot{y}(t)) + \sum_{l=1}^{q} \mu_l(t) (h_y^l(t, y(t), \dot{y}(t))$$

$$- \frac{d}{dt} h_{\dot{y}}^l(t, y(t), \dot{y}(t)) \Big) dt < 0$$

Which contradicts. □

**Corollary 2.4** *Assume that weak duality holds between* **(NMVPE)** *and* **(NWMVDE)**. *If* $(y^*(t), \tau^*, \lambda^*(t), \mu^*(t))$ *is feasible for* **(NWMVDE)** *with* $\sum_{j=1}^{m} \lambda_j^*(t) g^j(t, y^*(t), \dot{y}^*(t)) = 0$ *and* $y^{*T}(t) w_i(t) = S(y^*(t)|D_i), i = 1, \cdots, p$ *feasible for* **(NMVPE)**.

*Then* $y^*(t)$ *is an efficient for* **(NMVPE)** *and* $(y^*(t), \tau^*, \lambda^*(t), \mu^*(t))$ *is an efficient for* **(NWMVDE)**.

□

**Theorem 2.11 (Strong Duality)** *Assume that*

*(i) $x^*(t)$ is an efficient solution for* **(NMVPE)**

*(ii) for all $k = 1, \cdots, p$, $x^*(t)$ a constraint qualification for problem* **NMVPE**$_k(x^*(t))$ *is satisfied at $x^*(t)$.*

*Then there exist $\tau_i^* \in R^p$, $\tau_i^* > 0$ and piecewise smooth function $\lambda^*(t) : I \longrightarrow R^*$, $\lambda^*(t) \geqq 0$ and $\mu^*(t) : I \longrightarrow R^q$ such that $(x^*(t), \tau^*, \lambda^*(t), \mu^*(t))$ is feasible for* **(NWMVDE)** *and $\sum_{j=1}^{m} \lambda_j^*(t) g^j(t, x^*(t), \dot{x}^*(t)) + \sum_{l=1}^{q} \mu_l^*(t) h^l(t, x^*(t), \dot{x}^*(t)) = 0$ and $x^{*T}(t) w_i = S(x^*|c_i)$.*

*Further, if also weak duality holds between* **(NMVPE)** *and* **(NWMVDE)***, then $(x^*(t), \tau^*, \lambda^*(t), \mu^*(t))$ is an efficient solution for* **(NWMVDE)***.*

We conclude that $(x^*(t), \tau^*, \lambda^*(t), \mu^*(t))$ is feasible for **(NMVPE)**. Efficiency of $(x^*(t), \tau^*, \lambda^*(t), \mu^*(t))$ for **(NWMVDE)** now follows from Corollary 2.4. □

# Chapter 3

# Nondifferentiable Symmetric Duality for Multiobjective Variational Problems with V-invexity

## 3

### 3.1 Introduction

The following pair of nondifferentiable multiobjective variational problems.

(**NMSP**)

$$\text{Minimize} \int_a^b \Big( (f(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) + s(x(t)|C) - y(t)^T z(t))$$

$$-(y(t)^T(\lambda(t)^T f_y(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) - z(t)$$

$$-\frac{d}{dt}\lambda(t)^T f_{\dot{y}}(t, x(t), \dot{x}(t), y(t), \dot{y}(t)))e\Big)dt$$

$$\text{subject to} \quad x(a) = x_0,\ x(b) = x_1,\ y(a) = y_0,\ y(b) = y_1,$$

$$\lambda(t)^T(f_y(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) - z(t))$$

$$-\lambda(t)^T f_{\dot{y}}(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) \leqq 0, \tag{3.1}$$

$$\lambda(t) \geq 0,\ \lambda(t)^T e = 1,$$

(**NMSD**)

$$\text{Maximize} \quad \int_a^b \Big( (f(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) - s(v(t)|D) + u(t)^T w(t))$$

$$-(u(t)^T (\lambda(t)^T f_x(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) + w(t))$$

$$-\frac{d}{dt}\lambda(t)^T f_{\dot{x}}(t, u(t), \dot{u}(t), v(t), \dot{v}(t)))e\Big) dt$$

$$\text{subject to} \quad u(a) = x_0,\ u(b) = x_1,\ v(a) = y_0,\ v(b) = y_1,$$

$$\lambda(t)^T (f_x(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) + w(t))$$

$$-\lambda(t)^T f_{\dot{y}}(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) \geqq 0, \tag{3.2}$$

$$\lambda(t) \geq 0,\ \lambda(t)^T e = 1,$$

where (3.1) and (3.2) may fail to hold at corner of $(\dot{x}(t), \dot{y}(t))$ and $(\dot{u}(t), \dot{v}(t))$, respectively, but must be satisfied for unique right- and left-hand limits, $\lambda(t) \in R^p$ , and $e = (1, \cdots, 1)^T \in R^P$.

Let[a,b] be a real interval and $f : [a, b] \times R^n \times R^n \times R^m \times R^m \rightarrow R^p$. Consider the vector valued function $f(t, x(t), \dot{x}(t), y(t), \dot{y}(t))$, where $t \in [a, b]$, $x(t)$ and $y(t)$ are function of $t$ with $x(t) \in R^n$ and $y(t) \in R^m$, and $\dot{x}(t)$ and $\dot{y}(t)$ denote the derivatives of $x(t)$ and $y(t)$, respectively. Assume that $f$ has continuous fourth-order partial derivatives with respect to $x(t)$, $\dot{x}(t)$,$y(t)$ and $\dot{y}(t)$. $f_x$ and $f_{\dot{x}}$ denote the $p \times n$ matrices of first partial derivatives with respect to $x(t)$ and $\dot{x}(t)$, i.e.,

$$f_x^i = \Big(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\Big),\ f_{\dot{x}}^i = \Big(\frac{\partial f}{\partial \dot{x}_1}, \cdots, \frac{\partial f}{\partial \dot{x}_n}\Big),\ i = 1, \ldots, p.$$

Similarly, $f_y^i$ and $f_{\dot{y}}$ denote the $p \times m$ matrices of first partial derivatives with respect to $y(t)$ and $\dot{y}(t)$. We consider the problem of finding functions $x : [a,b] \to R^n$ and $y : [a,b] \to R^m$, with $(\dot{x}(t), \dot{y}(t))$ piecewise smooth on [a,b] to solve the following pair of multiobjective variational problems.

In this thesis, we extend the results of Kim and Lee [17] to the nondifferentiable multiobjective dual problems. We formulate a pair of nondifferentiable multiobjective symmetric dual variational problems. Under invexity assumptions, we establish the weak, strong and converse duality theorems for our variational problems by using the concept of efficiency.

## 3.2 Definitions and Preliminaries

Now we define the invexity as follows:

**Definition 3.1** *The functional $\int_a^b f$ is invex in $x$ and $\dot{x}$ if for each $y : [a,b] \to R^m$, with $\dot{y}$ piecewise smooth, there exists a function $\eta : [a,b] \times R^n \times R^n \times R^n \times R^n \to R^n$ such that $\forall i = 1, \cdots, p$,*

$$\int_a^b (f^i(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) - f^i(t, u(t), \dot{u}(t), y(t), \dot{y}(t)))dt$$

$$\geqq \int_a^b \eta(t, x(t), \dot{x}(t), u(t), \dot{u}(t))^T \Big( f_x^i(t, u(t), \dot{u}(t), y(t), \dot{y}(t))$$

$$- \frac{d}{dt} f_{\dot{x}}^i(t, u(t), \dot{u}(t), y(t), \dot{y}(t)) \Big) dt$$

*for all $x(t) : [a,b] \to R^n$, $u(t) : [a,b] \to R^n$ with $(\dot{x}(t), \dot{u}(t))$ piecewise smooth on $[a,b]$.*

**Definition 3.2** *The functional* $-\int_a^b f$ *is invex in* $y(t)$ *and* $\dot{y}(t)$ *if for each* $x(t) : [a, b] \to R^n$*, with* $\dot{x}(t)$ *piecewise smooth, there exists a function* $\xi(t) : [a, b] \times R^m \times R^m \times R^m \times R^m \to R^m$ *such that* $\forall i = 1, \cdots, p$,

$$- \int_a^b (f^i(t, x(t), \dot{x}(t), v(t), \dot{v}(t)) - f^i(t, x(t), \dot{x}(t), y(t), \dot{y}(t)))dt$$

$$\geqq - \int_a^b \xi(t, v(t), \dot{v}(t), y(t), \dot{y}(t))^T \Big( f_y^i(t, x(t), \dot{x}(t), y(t), \dot{y}(t))$$

$$- \frac{d}{dt} f_{\dot{y}}^i(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) \Big) dt$$

*for all* $v(t) : [a, b] \to R^m$*,* $y(t) : [a, b] \to R^m$ *with* $(\dot{v}(t), \dot{y}(t))$ *piecewise smooth on*$[a, b]$.

## 3.3 Symmetric Duality

Now we establish the symmetric duality theorems of **(NMHNP)** and **(NMHND)**.

**Theorem 3.1 (Weak Duality)** *Let* $(x(t), y(t), \lambda(t))$ *be feasible for* **(NMSP)** *and* $(u(t), v(t), \lambda(t))$ *be feasible for* **(NMSD)**. *Assume that either for all* $t \in [a, b]$

$$(a) \quad x(t) \neq u(t), \quad \int_a^b (f + (\cdot)^T w(t)) \text{ is strictly invex in } x(t) \text{ and } \dot{x}(t),$$

$$- \int_a^b (f - (\cdot)^T z(t)) \text{ is invex in } y(t) \text{ and } \dot{y}(t), \text{ with}$$

$$\eta(x(t), u(t)) + u(t) \geqq 0 \text{ and } \xi(v(t), y(t)) + y(t) \geqq 0; \text{ or}$$

$$(b)\quad y(t)\neq v(t),\quad \int_a^b (f+(\cdot)^T w(t)) \text{ is invex in } x(t) \text{ and } \dot{x}(t), \text{ and}$$

$$-\int_a^b (f-(\cdot)^T z(t)) \text{ is strictly invex in } y(t) \text{ and } \dot{y}(t), \text{ with}$$

$$\eta(x(t),u(t))+u(t)\geqq 0 \text{ and } \xi(v(t),y(t))+y(t)\geqq 0; \text{ or}$$

$$(c)\quad \lambda(t)>0,\quad \int_a^b (f+(\cdot)^T w(t)) \text{ is invex in } x(t) \text{ and } \dot{x}(t), \text{ and}$$

$$-\int_a^b (f-(\cdot)^T z(t)) \text{ is invex in } y(t) \text{ and } \dot{y}(t), \text{ with}$$

$$\eta(x(t),u(t))+u(t)\geqq 0 \text{ and } \xi(v(t),y(t))+y(t)\geqq 0$$

*(expect perhaps at corners of* $(\dot{x}(t),\dot{y}(t))$ *or* $(\dot{u}(t),\dot{v}(t))$*).*

*Then*

$$\begin{aligned}
&\textstyle\int_a^b [f(t,x(t),\dot{x}(t),y(t),\dot{y}(t))+s(x(t)|c)-y(t)^T z(t)\\
&-[y(t)^T(\lambda(t)^T f_y(t,x(t),\dot{x}(t),y(t),\dot{y}(t))-z(t))\\
&-\tfrac{d}{dt}\lambda(t)^T f_{\dot{y}}(t,x(t),\dot{x}(t),y(t),\dot{y}(t)))e]dt\\
&\nleq \textstyle\int_a^b [f(t,u(t),\dot{u}.(t),v(t),\dot{v}(t))-s(v(t)|D)+u(t)^T w(t)\\
&-[u(t)^T(\lambda(t)^T f_x(t,u(t),\dot{u}(t),v(t),\dot{v}(t))+w(t))\\
&-\tfrac{d}{dt}\lambda(t)^T f_{\dot{x}}(t,u(t),\dot{u}(t),v(t),\dot{v}(t))]e]dt.
\end{aligned}$$

*Proof.* (a) Assume that

$$\begin{aligned}
&\textstyle\int_a^b [f(t,x(t),\dot{x}(t),y(t),\dot{y}(t))+s(x(t)|c)-y(t)^T z(t))\\
&-[y(t)^T(\lambda(t)^T f_y(t,x(t),\dot{x}(t),y(t),\dot{y}(t))-z(t))
\end{aligned}$$

$$-\tfrac{d}{dt}\lambda(t)^T f_{\dot{y}}(t, x(t), \dot{x}(t), y(t), \dot{y}(t))]e]dt$$

$$\leq \textstyle\int_a^b [f(t, u(t), \dot{u}.(t), v(t), \dot{v}(t)) - s(v(t)|D) + u(t)^T w(t)$$

$$-[u(t)^T(\lambda(t)^T f_x(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) + w(t))$$

$$-\tfrac{d}{dt}\lambda(t)^T f_{\dot{x}}(t, u(t), \dot{u}(t), v(t), \dot{v}(t))]e]dt.$$

Since $\lambda \geq 0$ and $\lambda(t)^T e = 1$, it becomes

$$\textstyle\int_a^b [\lambda(t)^T(f(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) + s(x(t)|c) - y(t)^T z(t)))$$

$$-[y(t)^T(\lambda(t)^T f_y(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) - z(t))$$

$$-\tfrac{d}{dt}\lambda(t)^T f_{\dot{y}}(t, x(t), \dot{x}(t), y(t), \dot{y}(t))]dt$$

$$\leqq \textstyle\int_a^b [\lambda(t)^T(f(t, u(t), \dot{u}.(t), v(t), \dot{v}(t)) - s(v(t)|D) + u(t)^T w(t)))$$

$$-[u(t)^T(\lambda(t)^T f_x(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) + w(t))$$

$$-\tfrac{d}{dt}\lambda(t)^T f_{\dot{x}}(t, u(t), \dot{u}(t), v(t), \dot{v}(t))]dt.$$

By the assumption of strict invexity of $\int_a^b (f + (\cdot)^T w_i(t)),\ i = 1, \ldots, p,$

$$\int_a^b [(f^i(t, x(t), \dot{x}(t), v(t), \dot{v}(t)) + x(t)^T w_i(t)$$

$$-(f^i(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) + u(t)^T w_i(t))]dt$$

$$> \int_a^b \eta(x(t), u(t))^T [(f_x^i(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) + w_i(t)$$

$$-\frac{d}{dt} f_{\dot{x}}^i(t, u(t), \dot{u}(t), v(t), \dot{v}(t))]dt.$$

Since $\lambda \geq 0,\ \lambda(t)^T e = 1$, and $\eta(x(t), u(t)) + u(t) \geqq 0$, we obtain

$$\int_a^b [\lambda(t)^T (f(t, x(t), \dot{x}(t), v(t), \dot{v}(t)) + x(t)^T w(t)$$

$$-\lambda(t)^T (f(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) + u(t)^T w(t))] dt$$

$$> \int_a^b \eta(x(t), u(t))^T [\lambda(t)^T (f_x(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) + w(t)$$

$$-\frac{d}{dt}\lambda(t)^T f_{\dot{x}}(t, u(t), \dot{u}(t), v(t), \dot{v}(t))] dt$$

$$\geqq \int_a^b -u(t)^T [\lambda(t)^T (f_x(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) + w(t)$$

$$-\frac{d}{dt}\lambda(t)^T f_{\dot{x}}(t, u(t), \dot{u}(t), v(t), \dot{v}(t))] dt. \tag{3.3}$$

Now by invexity of $-(\int_a^b (f - (.)^T z_i(t))$,

$$\int_a^b [(f^i(t, x(t), \dot{x}(t), v(t), \dot{v}(t)) - v(t)^T z_i(t)$$

$$-(f^i(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) - y(t)^T z_i(t))] dt$$

$$\leqq \int_a^b \xi(v(t), y(t))^T [f_y^i(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) - z_i(t)$$

$$-\frac{d}{dt} f_{\dot{y}}^i(t, x(t), \dot{x}(t), y(t), \dot{y}(t))] dt.$$

Since $\lambda \geq 0$, $\lambda(t)^T e = 1$ and $\xi(v(t), y(t)) + y(t) \geqq 0$,

$$\int_a^b [\lambda(t)^T (f(t, x(t), \dot{x}(t), v(t), \dot{v}(t)) - v(t)^T z(t))$$

$$-\lambda(t)^T(f(t,x(t),\dot{x}(t),y(t),\dot{y}(t))-y(t)^Tz(t))]dt$$

$$\leqq \int_a^b \xi(v(t),y(t))^T[\lambda(t)^T(f_y(t,x(t),\dot{x}(t),y(t),\dot{y}(t))-z(t))$$

$$-\frac{d}{dt}\lambda(t)^T(f_{\dot{y}}(t,x(t),\dot{x}(t),y(t),\dot{y}(t))]dt$$

$$\leqq \int_a^b -y(t)^T[\lambda(t)^T(f_y(t,x(t),\dot{x}(t),y(t),\dot{y}(t))-z(t))$$

$$-\frac{d}{dt}\lambda(t)^T(f_{\dot{y}}(t,x(t),\dot{x}(t),y(t),\dot{y}(t))]dt. \tag{3.4}$$

Subtracting (3.4) from (3.3) and rearranging gives

$$\int_a^b[\lambda(t)^T(f(t,x(t),\dot{x}(t),y(t),\dot{y}(t))+x(t)^Tw(t)-y(t)^Tz(t))$$
$$-y(t)^T[\lambda(t)^T(f_y(t,x(t),\dot{x}(t),y(t),\dot{y}(t))-z(t))$$
$$-\tfrac{d}{dt}\lambda(t)^T(f_{\dot{y}}(t,x(t),\dot{x}(t),y(t),\dot{y}(t))]]dt$$
$$>\int_a^b[\lambda(t)^T(f(t,u(t),\dot{u}(t),v(t),\dot{v}(t))-v(t)^Tz(t)+u(t)^Tw(t)$$
$$-u(t)^T[\lambda(t)^T(f_x(t,u(t),\dot{u}(t),v(t),\dot{v}(t))+w(t))$$
$$-\tfrac{d}{dt}\lambda(t)^T(f_{\dot{x}}(t,u(t),\dot{u}(t),v(t),\dot{v}(t))]]dt. \tag{3.5}$$

Using the fact that $x(t)^Tw_i \leqq s(x(t)|C_i)$ and $v(t)^Tz_i \leqq s(v(t)|D_i)$ the above inequality becomes

$$\int_a^b[\lambda(t)^T(f(t,x(t),\dot{x}(t),y(t),\dot{y}(t))+s(x(t)|C)-y(t)^Tz(t))$$
$$-y(t)^T[\lambda(t)^T(f_y(t,x(t),\dot{x}(t),y(t),\dot{y}(t))-z(t))$$
$$-\tfrac{d}{dt}\lambda(t)^T(f_{\dot{y}}(t,x(t),\dot{x}(t),y(t),\dot{y}(t))]]dt$$

$$> \int_a^b [\lambda(t)^T (f(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) - s(v(t)|D) + u(t)^T w(t))$$

$$-u(t)^T [\lambda(t)^T (f_x(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) + w(t))$$

$$-\frac{d}{dt}\lambda(t)^T (f_{\dot{x}}(t, u(t), \dot{u}(t), v(t), \dot{v}(t))]]dt \tag{3.6}$$

which contradicts (3.3).

(b) Assume that

$$\int_a^b [f(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) + s(x(t)|c) - y(t)^T z(t))$$

$$-[y(t)^T (\lambda(t)^T f_y(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) - z(t))$$

$$-\frac{d}{dt}\lambda(t)^T f_{\dot{y}}(t, x(t), \dot{x}(t), y(t), \dot{y}(t))]e]dt$$

$$\leq \int_a^b [f(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) - s(v(t)|D) + u(t)^T w(t))$$

$$-[u(t)^T (\lambda(t)^T f_x(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) + w(t))$$

$$-\frac{d}{dt}\lambda(t)^T f_{\dot{x}}(t, u(t), \dot{u}(t), v(t), \dot{v}(t))]e]dt.$$

Since $\lambda \geq 0$ and $\lambda(t)^T e = 1$, it becomes

$$\int_a^b [\lambda(t)^T (f(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) + s(x(t)|c) - y(t)^T z(t)))$$

$$-[y(t)^T (\lambda(t)^T f_y(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) - z(t))$$

$$-\frac{d}{dt}\lambda(t)^T f_{\dot{y}}(t, x(t), \dot{x}(t), y(t), \dot{y}(t))]dt$$

$$\leqq \int_a^b [\lambda(t)^T (f(t, u(t), \dot{u}.(t), v(t), \dot{v}(t)) - s(v(t)|D) + u(t)^T w(t)))$$

$$-[u(t)^T (\lambda(t)^T f_x(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) + w(t))$$

$$-\frac{d}{dt}\lambda(t)^T f_{\dot{x}}(t, u(t), \dot{u}(t), v(t), \dot{v}(t))]dt.$$

By the assumption of invexity of $\int_a^b (f + (\cdot)^T w_i(t))$, $i = 1, \ldots, p$,

$$\int_a^b [(f^i(t, x(t), \dot{x}(t), v(t), \dot{v}(t)) + x(t)^T w_i(t))$$

$$-(f^i(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) + u(t)^T w_i(t))]dt$$

$$\geqq \int_a^b \eta(x(t), u(t))^T [(f_x^i(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) + w_i(t))$$

$$-\frac{d}{dt} f_{\dot{x}}^i(t, u(t), \dot{u}(t), v(t), \dot{v}(t))]dt.$$

Since $\lambda \geq 0$, $\lambda(t)^T e = 1$ and $\eta(x(t), u(t)) + u(t) \geqq 0$, we obtain

$$\int_a^b [\lambda(t)^T (f(t, x(t), \dot{x}(t), v(t), \dot{v}(t)) + x(t)^T w(t))$$

$$-\lambda(t)^T (f(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) + u(t)^T w(t))]dt$$

$$\geqq \int_a^b \eta(x(t), u(t))^T [\lambda(t)^T (f_x(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) + w(t))$$

$$-\frac{d}{dt} \lambda(t)^T f_{\dot{x}}(t, u(t), \dot{u}(t), v(t), \dot{v}(t))]dt$$

$$\geqq \int_a^b -u(t)^T [\lambda(t)^T (f_x(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) + w(t))$$

$$-\frac{d}{dt} \lambda(t)^T f_{\dot{x}}(t, u(t), \dot{u}(t), v(t), \dot{v}(t))]dt. \qquad (3.7)$$

Now by strict invexity of $-(\int_a^b (f - (.)^T z_i(t))$, $i = 1, \ldots, p$,

$$\int_a^b [(f^i(t,x(t),\dot{x}(t),v(t),\dot{v}(t)) - v(t)^T z_i(t))$$

$$-(f^i(t,x(t),\dot{x}(t),y(t),\dot{y}(t)) - y(t)^T z_i(t))]dt$$

$$< \int_a^b \xi(v(t),y(t))^T [f_y^i(t,x(t),\dot{x}(t),y(t),\dot{y}(t)) - z_i(t))$$

$$-\frac{d}{dt} f_{\dot{y}}^i(t,x(t),\dot{x}(t),y(t),\dot{y}(t))]dt.$$

Since $\lambda \geq 0, \lambda(t)^T e = 1$ and $\xi(v(t),y(t)) + y(t) \geqq 0$,

$$\int_a^b [\lambda(t)^T (f(t,x(t),\dot{x}(t),v(t),\dot{v}(t)) - v(t)^T z(t))$$

$$-\lambda(t)^T (f(t,x(t),\dot{x}(t),y(t),\dot{y}(t)) - y(t)^T z(t))]dt$$

$$< \int_a^b \xi(v(t),y(t))^T [\lambda(t)^T (f_y(t,x(t),\dot{x}(t),y(t),\dot{y}(t)) - z(t))$$

$$-\frac{d}{dt}\lambda(t)^T (f_{\dot{y}}(t,x(t),\dot{x}(t),y(t),\dot{y}(t))]dt$$

$$\leqq \int_a^b -y(t)^T [\lambda(t)^T (f_y(t,x(t),\dot{x}(t),y(t),\dot{y}(t)) - z(t))$$

$$-\frac{d}{dt}\lambda(t)^T (f_{\dot{y}}(t,x(t),\dot{x}(t),y(t),\dot{y}(t))]dt. \qquad (3.8)$$

Subtracting (3.8) from (3.7) and rearranging gives

$$\int_a^b [\lambda(t)^T(f(t,x(t),\dot{x}(t),y(t),\dot{y}(t)) + x(t)^T w(t) - y(t)^T z(t))$$
$$-y(t)^T[\lambda(t)^T(f_y(t,x(t),\dot{x}(t),y(t),\dot{y}(t)) - z(t))$$
$$-\frac{d}{dt}\lambda(t)^T(f_{\dot{y}}(t,x(t),\dot{x}(t),y(t),\dot{y}(t))]]dt$$
$$> \int_a^b [\lambda(t)^T(f(t,u(t),\dot{u}(t),v(t),\dot{v}(t)) - v(t)^T z(t) + u(t)^T w(t)$$
$$-u(t)^T[\lambda(t)^T(f_x(t,u(t),\dot{u}(t),v(t),\dot{v}(t)) + w(t))$$
$$-\frac{d}{dt}\lambda(t)^T(f_{\dot{x}}(t,u(t),\dot{u}(t),v(t),\dot{v}(t))]]dt. \qquad (3.9)$$

Using the fact that $x(t)^T w_i \leqq s(x(t)|C_i)$ and $v(t)^T z_i \leqq s(v(t)|D_i)$ the above inequality becomes

$$\int_a^b [\lambda(t)^T(f(t,x(t),\dot{x}(t),y(t),\dot{y}(t)) + s(x(t)|C) - y(t)^T z(t))$$
$$-y(t)^T[\lambda(t)^T(f_y(t,x(t),\dot{x}(t),y(t),\dot{y}(t)) - z(t))$$
$$-\frac{d}{dt}\lambda(t)^T(f_{\dot{y}}(t,x(t),\dot{x}(t),y(t),\dot{y}(t))]]dt$$
$$> \int_a^b [\lambda(t)^T(f(t,u(t),\dot{u}(t),v(t),\dot{v}(t)) - s(v(t)|D) + u(t)^T w(t))$$
$$-u(t)^T[\lambda(t)^T(f_x(t,u(t),\dot{u}(t),v(t),\dot{v}(t)) + w(t))$$
$$-\frac{d}{dt}\lambda(t)^T(f_{\dot{x}}(t,u(t),\dot{u}(t),v(t),\dot{v}(t))]]dt \qquad (3.10)$$

which contradicts (3.3).

(c) By the assumption of invexity of $\int_a^b (f + (\cdot)^T z_i(t))$ and $-\int_a^b (f - (\cdot)^T z_i(t))$, $i = 1, \ldots, p$, we obtain (3.7) and (3.4). Subtracting (3.4) from (3.7) gives

$$\int_a^b [\lambda(t)^T (f(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) + s(x(t)|C) - y(t)^T z(t))$$
$$-y(t)^T [\lambda(t)^T (f_y(t, x(t), \dot{x}(t), y(t), \dot{y}(t)) - z(t))$$
$$-\tfrac{d}{dt}\lambda(t)^T (f_{\dot{y}}(t, x(t), \dot{x}(t), y(t), \dot{y}(t))]]dt$$
$$\geqq \int_a^b [\lambda(t)^T (f(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) - s(v(t)|D) + u(t)^T w(t))$$
$$-u(t)^T [\lambda(t)^T (f_x(t, u(t), \dot{u}(t), v(t), \dot{v}(t)) + w(t))$$
$$-\tfrac{d}{dt}\lambda(t)^T (f_{\dot{x}}(t, u(t), \dot{u}(t), v(t), \dot{v}(t))]]dt.$$

However, since $\lambda(t) > 0$, this implies (3.3). □

**Theorem 3.2 (Strong Duality)** *Let $(x^*(t), y^*(t), \lambda^*(t), z^*(t))$ be an efficient solution for* **(NMSP)**. *Suppose that the system*

$$[p(t)^T (\lambda^*(t)^T f_{yy}(t, x^*(t), \dot{x}^*(t), y^*(t), \dot{y}^*(t))$$
$$-\frac{d}{dt}\lambda^*(t)^T f_{y\dot{y}}(t, x^*(t), \dot{x}^*(t), y^*(t), \dot{y}^*(t)))$$
$$+\frac{d}{dt}(p(t)^T \frac{d}{dt}\lambda^*(t)^T f_{\dot{y}\dot{y}}(t, x^*(t), \dot{x}^*(t), y^*(t), \dot{y}^*(t)))$$
$$+\frac{d^2}{dt^2}(-p(t)^T \lambda^*(t)^T f_{\dot{y}\dot{y}}(t, x^*(t), \dot{x}^*(t), y^*(t), \dot{y}^*(t)))]p(t) = 0 \quad (3.11)$$

*only has the solution $p(t) = 0$ for all $t \in [a, b]$ and the set*

$$\{f_y^i(t, x^*(t), \dot{x}^*(t), y^*(t), \dot{y}^*(t)) - y^*(t)$$
$$-\frac{d}{dt} f_{\dot{y}}^i(t, x^*(t), \dot{x}^*(t), y^*(t), \dot{y}^*(t)) : i = 1, 2, \cdots, p\}$$

*is linearly independent.*

*Assume that* $\lambda^*(t) > 0$, $(\int_a^b(f + (\cdot)^T z_i^*(t))$, $(i = 1, 2, \cdots, p)$ *is invex in* $x(t)$ *and* $\dot{x}(t)$, *and* $-(\int_a^b(f - (\cdot)^T z_i^*(t))$, $(i = 1, 2, \cdots, p)$ *is invex in* $y(t)$ *and* $\dot{y}(t)$ *with* $\eta(x(t), u(t)) + u(t) \geqq 0$ *and* $\xi(v(t), y(t)) + y(t) \geqq 0$*(except perhaps at corners of* $(\dot{x}(t), \dot{y}(t))$ *or* $(\dot{u}(t), \dot{v}(t))$*). Then* $(x^*(t), y^*(t), \lambda^*(t), z^*(t))$ *is an efficient solution for* **(NMSD)**, *and the optimal values of* **(NMSP)** *and* **(NMSD)** *are equal.*

*Proof.* Applying the necessary conditions of Valentine [36], if $(x^*(t), y^*(t), \lambda^*(t), z^*(t))$ is an efficient solution of **(NMSP)**, then there exist $\alpha \in R^p$, $\beta : [a, b] \to R^m$ and $\gamma \in R^p$ such that

$$\begin{aligned} H^* \equiv\ & \alpha(t)^T(f(t, x^*(t), \dot{x}^*(t), y^*(t), \dot{y}^*(t)) + x^*(t)^T w^* - y^*(t)^T z^*(t) \\ & -[y^{*T}(\lambda^*(t)^T f_y(t, x^*(t), \dot{x}^*(t), y^*(t), \dot{y}^*(t)) - \lambda^*(t)^T z^*(t) \\ & -\frac{d}{dt}\lambda^*(t)^T f_{\dot{y}}(t, x^*(t), \dot{x}^*(t), y^*(t), \dot{y}^*(t))]e) \\ & -\beta(t)^T(\frac{d}{dt}\lambda^*(t)^T f_{\dot{y}}(t, x^*(t), \dot{x}^*(t), y^*(t), \dot{y}^*(t)) \\ & -\lambda^{*T}(t)(f_y(t, x^*(t), \dot{x}^*(t), y^*(t), \dot{y}^*(t)) + \lambda^*(t)^T z^*(t)) - \gamma(t)^T \lambda^*(t) \end{aligned}$$

satisfies

$$H_y^* - \frac{d}{dt}H_{\dot{y}}^* + \frac{d^2}{dt^2}H_{\ddot{y}}^* = 0, \tag{3.12}$$

$$H_x^* - \frac{d}{dt}H_{\dot{x}}^* + \frac{d^2}{dt^2}H_{\ddot{y}}^* = 0, \tag{3.13}$$

$$(\beta(t)-(\alpha(t)^Te)y^*(t))^T(f_y^i(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t))-z_i^*(t))$$

$$-\frac{d}{dt}f_{\dot{y}}^i(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t)))-\gamma(t)=0, \tag{3.14}$$

$$\beta^T(t)(\frac{d}{dt}\lambda^{*T}(t)f_{\dot{y}}^i(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t))$$

$$-\lambda^{*T}(t)(f_y^i(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t))-z_i^*(t))))=0, \tag{3.15}$$

$$\gamma(t)^T\lambda^*(t)=0, \tag{3.16}$$

$$\alpha_i(t)y^*(t)+\lambda_i^*\beta(t)-(\alpha(t)^Te)\lambda_i^*(t)y^*(t)\in N_{D_i}(z_i^*),$$

$$i=1,\cdots,p, \tag{3.17}$$

$$x^*(t)^Tw_i=s(x^*|C_i),w_i\in C_i,\ (i=1,\cdots,p), \tag{3.18}$$

$$(\alpha(t),\beta(t),\gamma(t))\geqq 0, \tag{3.19}$$

throughout $[a,b]$(except at corners of $(\dot{x}^*(t),\dot{y}^*(t))$ where (3.16) and (3.17) hold for unique right- and left-hand limits). $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ cannot be simultaneously zero at any $t\in[a,b]$ and $\beta(t)$ is continuous except perhaps at corners of $(\dot{x}^*(t),\dot{y}^*(t))$. Equation (3.16) now becomes

$$(\alpha(t)-(\alpha(t)^Te)\lambda^*(t))^T(f_y(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t))-z^*(t)$$

$$-\frac{d}{dt}f_{\dot{y}}(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t)))$$

$$+(\beta(t)-(\alpha(t)^Te)y^*(t))^T(\lambda^*(t)^Tf_{yy}(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t))$$

$$-\frac{d}{dt}\lambda^*(t)^Tf_{y\dot{y}}(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t)))$$

$$+\frac{d}{dt}((\beta(t)-(\alpha(t)^Te)y^*(t))^T\frac{d}{dt}\lambda^*(t)^Tf_{\dot{y}\dot{y}}(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t)))$$

$$+\frac{d^2}{dt^2}(-(\beta(t)-(\alpha(t)^Te)y^*(t))^T\lambda^*(t)^Tf_{\dot{y}\dot{y}}(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t)))$$

$$=0. \tag{3.20}$$

Equation (3.17) gives

$$\alpha(t)^T(f_x(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t))+w(t))$$

$$+(\beta(t)-(\alpha(t)^Te)y^*(t))^T(\lambda^*(t)^Tf_{xy}(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t))$$

$$-\frac{d}{dt}\lambda^*(t)^Tf_{\dot{y}x}(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t)))$$

$$-\frac{d}{dt}\alpha(t)^Tf_{\dot{x}}(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t))$$

$$-\frac{d}{dt}\{(\beta(t)-(\alpha(t)^Te)y^*(t))^T(\lambda^*(t)^Tf_{y\dot{x}}(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t)))$$

$$-\frac{d}{dt}\lambda^*(t)^Tf_{\dot{y}\dot{x}}(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t))$$

$$-\lambda^*(t)^Tf_{\dot{y}x}(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t))\}$$

$$+\frac{d^2}{dt^2}\{-(\beta(t)-(\alpha(t)^Te)y^*(t))^T\lambda^*(t)^Tf_{\dot{y}\dot{x}}(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t))\}$$

$$=0. \tag{3.21}$$

Multiplying (3.22) by $\beta(t)-(\alpha(t)^Te)y^*(t)$ and then using (3.18) and (3.20) gives

$$[(\beta(t)-(\alpha(t)^Te)y^*(t))^T(\lambda^*(t)^Tf_{yy}(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t))$$

$$-\tfrac{d}{dt}\lambda^*(t)^Tf_{y\dot{y}}(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t)))$$

$$+\tfrac{d}{dt}\{(\beta(t)-(\alpha(t)^Te)y^*(t))^T\tfrac{d}{dt}\lambda^*(t)^Tf_{\dot{y}\dot{y}}(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t))\}$$

$$+\tfrac{d^2}{dt^2}\{-(\beta(t)-(\alpha(t)^Te)y^*(t))^T\lambda^*(t)^Tf_{\dot{y}\dot{y}}(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t))\}]$$
$$\times(\beta(t)-(\alpha(t)^Te)y^*(t))=0.$$

Thus by the assumption (3.14),

$$\beta(t)=(\alpha(t)^Te)y^*(t)). \tag{3.22}$$

From (3.22), we have

$$(\alpha(t)-(\alpha(t)^Te)\lambda^*(t))^T(f_y(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t))-z^*(t)$$
$$-\tfrac{d}{dt}f_{\dot{y}}(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t)))=0.$$

By the assumption (3.15)

$$\alpha(t)=(\alpha(t)^Te)\lambda^*(t). \tag{3.23}$$

This gives $\alpha(t)\neq 0$, since if $\alpha(t)=0$, then by (4.24) and (4.18), $\beta(t)=\gamma(t)=0$ everywhere, contradicting the necessary condition (4.21). Equation (3.23) with (3.24) and (3.25) now becomes

$$\lambda^{*T}(t)f_x(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t))$$
$$-\frac{d}{dt}\lambda^*(t)^Tf_{\dot{x}}(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t)))=0$$

and

$$x^{*T}\{\lambda^{*T}(t)(f_x(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t))+w(t))$$
$$-\frac{d}{dt}\lambda^*(t)^Tf_{\dot{x}}(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t))\}=0.$$

Equation (3.19) with (3.24) gives

$$y^{*T}\{\lambda^{*T}(t)(f_y(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t))+z^*(t))$$
$$-\tfrac{d}{dt}\lambda^*(t)^T f_{\dot{y}}(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t))\}=0. \qquad (3.24)$$

By (3.26),$(x^*(t),y^*(t),\lambda^*(t))$ is feasible for **(NMSD)**.From (3.26) and (3.27), (NMSP) and (NMSD) have equal objective values. Therefore from (3.19),we get $y^*(t)\in N_{D_i}(z^*(t)_i),(i=1,2,\cdot,p)$, so that $y^*(t)^T z^*(t)_i = s(y^*|D_i),(i=1,2,\cdot,p)$. Moreover,By Theorem (3.1),it follows that $x^*(t),y^*(t),\lambda^*(t),z^*(t))$ is an efficient solution of **(NMSP)**. □

A converse duality theorem may be stated: the proof would be analogous to that of Theorem 3.2.

**Theorem 3.3 (Converse Duality)** *Let* $(x^*(t),y^*(t),\lambda^*(t),w^*(t))$ *be an efficient solution for* **(NMSD)**. *Suppose that the system*

$$[p(t)^T(\lambda^*(t)^T f_{xx}(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t))$$
$$-\frac{d}{dt}\lambda^*(t)^T f_{x\dot{x}}(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t)))$$
$$+\frac{d}{dt}(p(t)^T\frac{d}{dt}\lambda^*(t)^T f_{\dot{x}\dot{x}}(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t)))$$
$$+\frac{d^2}{dt^2}(-p(t)^T\lambda^*(t)^T f_{\dot{x}\dot{x}}(t,x^*(t),\dot{x}^*(t),y^*(t),\dot{y}^*(t)))]p(t)=0.$$

*only has the solution* $p(t) = 0$ *for all* $t \in [a, b]$*, and the set*

$$\{f_x^i(t, x^*(t), \dot{x}^*(t), y^*(t), \dot{y}^*(t)) + w_i^*(t)$$
$$-\frac{d}{dt} f_{\dot{x}}^i(t, x^*(t), \dot{x}^*(t), y^*(t), \dot{y}^*(t)) : i = 1, 2, \cdot, p\}.$$

*is linearly independent.*

*Assume that* $\lambda^*(t)^T > 0$*,* $(\int_a^b (f + (.)^T z_i(t))$ *is invex in* $x(t)$ *and* $\dot{x}(t)$*, and* $-(\int_a^b (f - (.)^T z_i(t))$ *is invex in* $y(t)$ *and* $\dot{y}(t)$ *with* $\eta(x(t), u(t)) + u(t) \geqq 0$ *and* $\xi(v(t), y(t)) + y(t) \geqq 0$*(except perhaps at corners of* $(\dot{x}(t), \dot{y}(t), w(t))$ *or* $(\dot{u}(t), \dot{v}(t))$*). Then* $(x^*(t), y^*(t), \lambda^*(t), w^*(t))$ *is an efficient solution for* **(NMSP)***, and the optimal values of* **(NMSP)** *and* **(NMSD)** *are equal.*

## 3.4 Special Case

As a special cases of our duality results between (**NMSP**) and (**NMSD**), we give special case of our duality.

If $D_i = \{0\},\ i = 1, \cdots, p$, then our primal and dual models become dual programs considered in Kim and Lee [20].

# Chapter 4

# Multiobjective Control Problem with Generalized V-$\rho$ Invexity

# 4

## 4.1 Introduction

The following problem is called a nondifferentiable multiobjective control problem **(NMCP)**:

$$\textbf{(NMCP)} \quad \text{Minimize} = \Big( \int_a^b \{f^1(t,x(t),u(t)) + s(x(t)|D_1)\}dt, \cdots, \int_a^b \{f^p(t,x(t),u(t)) + s(x(t)|D_p)\}dt \Big)$$

$$\text{subject to} \quad x(a) = t_0,\ x(b) = t_f, \tag{4.1}$$

$$g(t,x(t),u(t)) \leqq 0, \qquad t \in I, \tag{4.2}$$

$$h(t,x(t),u(t)) = \dot{x}(t), \qquad t \in I, \tag{4.3}$$

Here $R^n$ denotes an $n$-dimensional Euclidean space and I=[a,b] is a real interval. Each $f^i : I \times R^n \times R^m \to R$ for ( $i = 1, \cdots, p$ ), $g = (g^1, \cdots, g^k)$, $g^j : I \times R^n \times R^m \to R$ ( $j = 1, \cdots, k$ ), and $h = (h^1, \cdots, h^n)$, $h^r : I \times R^n \times R^m \to R$( $r = 1, \cdots, n$ ) is a continuously differentiable function.

Let $x(t) : I \to R^n$ be differentiable with its derivative $\dot{x}(t)$, and let $u(t)$ :

$I \to R^m$ be a differentiable function. Denote the partial derivatives of $f$ by $f_t, f_x$ and $f_u$, that is,

$$f_t = \frac{\partial f}{\partial t}, f_x = \Big(\frac{\partial f}{\partial x^1}, \cdots, \frac{\partial f}{\partial x^n}\Big), f_u = \Big(\frac{\partial f}{\partial u^1}, \cdots, \frac{\partial f}{\partial u^m}\Big),$$

where the superscripts denote the vector components.
Similarly,we have $g_t, g_x, g_u$, and $h_t, h_x, h_u$. X is the space of continuously differentiable state functions $x(t) : I \to R^n$ such that $x(a) = t_o$ and $x(b) = t_f$ and is equipped with the norm $\|x\|=\|x\|_\infty + \|Dx\|_\infty$ ; and Y is the space of piecewise continuous control functions $u(t) : I \to R^m$, and has the uniform norm $\|\cdot\|_\infty$. The differential equation (4.3) with initial conditions expressed as $x(t) = x(a) + \int_a^t h^r(s, x(s), u(s))ds, \quad t \in I$ may be written as $\dot{x}(t) = H^r(x, u)$, where $H^r : X \times Y \to C(I, R^n)$,$C(I, R^n)$ being the space of continuous functions from I to$R^n$defined as $H^r(x, u)(t)$=$h^r(t, x(t), u(t))$.

In this chapter, we will define generalized V -$\rho$-invex functions for optimal control problems and consider a nondifferentiable multiobjective control problem **(NMCP)**. The sufficient optimality conditions of the Kuhn-Tucker type for **(NMCP)** are given under generalized invexity condition. Moreover, we formulate Wolfe type dual**(NWMCD)** and Mond-Weir type dual **(NMMCD)** for **(NMCP)**, and then establish their duality relations.

## 4.2 Definitions and Preliminaries

**Definition 4.1** *Let $h^i$ be a function from $I \times R^n \times R^n \times R^m$ into $R$ and let $H^i(x, u) = \int_a^b (h^i(t, x, \dot{x}, u)dt$. Let there exist differentiable vector* *functions* $\eta(t,x,x^*,\dot{x},\dot{x}^*,u,u^*) \in R^n$ *with* $\eta = 0$ *at* $t$ *if* $x(t) = x^*(t)$, *and* $\xi(t,x,x^*,\dot{x},\dot{x}^*,u,u^*) \in R^m$, $\zeta(t,x,x^*,\dot{x},\dot{x}^*,u,u^*) \in R^n$. *Let* $\|\zeta(x,x^*,\dot{x},\dot{x}^*,u,u^*)\| = sup_{t\in I}\|\zeta(x,x^*,\dot{x},\dot{x}^*,u,u^*)\|$ *and* $\rho_i$ *real numbers.*

*(1) A vector function* $H = (H^1,\cdots,H^n)$ *is said to be V-ρ-invex in* $x^*,\dot{x}^*$, *and* $u^*$ *on* $I$ *with respect to* $\eta$, $\xi$, $\zeta$ *and* $\alpha$ *if there exist differentiable vector functions* $\eta \in R^n$, $\xi \in R^m$, $\zeta \in R^n$, $\alpha_i(x,x^*,\dot{x},\dot{x}^*,u,u^*) \in R_+\ \{0\}$,*and* $\rho_i \in R$, $i = 1,\cdots,n$ *such that, for each* $x,x^* \in X$ *and* $u,u^* \in Y$,

$$H^i(x,u) - H^i(x^*,u^*) \geq \int_a^b \{\eta^T\alpha_i h_x^i(t,x^*,\dot{x}^*,u^*) + \frac{d\eta^T}{dt}\alpha_i h_{\dot{x}}^i(t,x^*,\dot{x}^*,u^*)$$

$$+\xi^T\alpha_i h_u^i(t,x^*,\dot{x}^*,u^*)\}dt + \rho_i\|\zeta\|^2.$$

*(2) The vector function* $H = (H^1,\cdots,H^n)$ *is said to be V-ρ-pseudo-invex in* $x^*,\dot{x}^*$, *and* $u^*$ *on* $I$ *with respect to* $\eta$, $\xi$, $\zeta$ *and* $\beta$ *if there exist* $\eta$, $\xi$, $\zeta$ *as above,* $\beta_i(x,x^*,\dot{x},\dot{x}^*,u,u^*) \in R_+\{0\}$, *and* $\rho_i \in R$, $i = 1,\cdots,n$ *such that, for each* $x,x^* \in X$ *and* $u,u^* \in Y$,

$$\int_a^b \sum_{i=1}^n \{\eta^T\alpha_i h_x^i(t,x^*,\dot{x}^*,u^*) + \frac{d\eta^T}{dt} h_{\dot{x}}^i(t,x^*,\dot{x}^*,u^*)$$

$$+\xi^T h_u^i(t,x^*,\dot{x}^*,u^*)\}dt + \sum_{i=1}^n \rho_i\|\zeta\|^2 \geq 0$$

$$\Longrightarrow \int_a^b \sum_{i=1}^n \beta_i h^i(t,x,\dot{x},u)dt \geq \int_a^b \sum_{i=1}^n \beta_i h^i(t,x^*,\dot{x}^*,u^*)dt.$$

*(3) The vector function* $H = (H^1,\cdots,H^n)$ *is said to be V-ρ-quasi-invex in* $x^*,\dot{x}^*$, *and* $u^*$ *on* $I$ *with respect to* $\eta$, $\xi$, $\zeta$ *and* $\gamma$ *if there exist* $\eta$, $\xi$, $\zeta$

*as above,the vector* $\gamma_i(x, x^*, \dot{x}, \dot{x}^*, u, u^*) \in R_+ \ \{0\}$*, and* $\rho_i \in R,\ i = 1, \cdots, n$ *such that, for each* $x, x^* \in X$ *and* $u, u^* \in Y$*,*

$$\int_a^b \sum_{i=1}^n \gamma_i h^i(t, x, \dot{x}, u)dt \leq \int_a^b \sum_{i=1}^n \gamma_i h^i(t, x^*, \dot{x}^*, u^*)dt$$

$$\Longrightarrow \int_a^b \sum_{i=1}^n \{\eta^T h_x^i(t, x^*, \dot{x}^*, u^*) + \frac{d\eta^T}{dt}\ h_{\dot{x}}^i(t, x^*, \dot{x}^*, u^*)$$

$$+\xi^T h_u^i(t, x^*, \dot{x}^*, u^*)\}dt + \sum_{i=1}^n \rho_i \|\zeta\|^2 \leq 0.$$

*Lemma 1 of [32] states that* $(x^*(t), u^*(t))$ *is an efficient solution for* **(NMCP)** *if and only if* $(x^*(t), u^*(t))$ *solves*

$\mathbf{NMCP}_k(x^*(t), u^*(t))$

$$\begin{aligned} \text{Minimize} \quad & \int_a^b \{f^k(t, x(t), u(t)) + s(x(t)|D)\}dt \\ \text{subject to} \quad & x(a) = t_0,\ x(b) = t_f, \\ & g(t, x(t), u(t)) \leqq 0, \\ & h(t, x(t), u(t)) = \dot{x}(t), \\ & \int_a^b \{f^j(t, x(t), u(t)) + s(x(t)|D_j)\}dt \\ \leqq & \int_a^b \{f^j(t, x^*(t), u^*(t)) + s(x^*(t)|D_j)\}dt, \\ & \forall j \in \{1, \cdots, p\}, \quad j \neq k. \end{aligned}$$

Chandra, Craven, and Husain [5] gave the Fritz John necessary optimality conditions for the existence of an extremal solution for the single objective control problem **(NCP)**:

**(NCP)**

$$\text{Minimize} \quad \int_a^b \{f(t,x(t),u(t)) + s(x(t)|D)\}dt$$

$$\text{subject to} \quad x(a) = t_0, \; x(b) = t_f,$$

$$g(t,x(t),u(t)) \leq 0,$$

$$h(t,x(t),u(t)) = \dot{x}(t),$$

where $f, g, h$ are as defined earlier.

Mond and Hanson [28] pointed out that if the optimal solution for (CP) is normal, then Friz John conditions reduce to Kuhn-Tucker conditions.

**Lemma 4.1** *(Kuhn-Tucker Necessary Optimality Condition).*

*Let* $(x^*(t), u^*(t)) \in X \times Y$ *be an efficient for* **(NMCP)**. *If the Frechet derivatives* $[D - H_x^i(x^*(t), u^*(t))]$ *is subjective and* $(x^*(t), u^*(t))$ *is normal for* $\mathbf{MCP}_k(x^*(t), u^*(t))$ *at least one* $k \in \{1, \cdots, p\}$, *then there exist* $\tau_i^* \in R^p$, *piecewise smooth function* $\lambda^*(t) : I \to R^k$ , *and* $\mu^*(t) : I \to R^n$ *satisfying the following equalities; for all* $t \in I$ ,

$$\sum_{i=1}^{p} \tau_i^* \{f_x^i(t, x^*(t), u^*(t) + w_i(t)\} + \sum_{j=1}^{k} \lambda_j^*(t) g_x^j(t, x^*(t), u^*(t))$$

$$+ \sum_{r=1}^{n} \mu_r^*(t) h_x^r(t, x^*(t), u^*(t)) + \dot{\mu}^*(t) = 0,$$

$$\sum_{i=1}^{p} \tau_i^* f_u^i(t, x^*(t), u^*(t)) + \sum_{j=1}^{k} \lambda_j^*(t) g_u^j(t, x^*(t), u^*(t))$$

$$+ \sum_{r=1}^{n} \mu_r^*(t) h_u^r(t, x^*(t), u^*(t)) = 0,$$

$$\sum_{j=1}^{k} \lambda_j^*(t) g^j(t, x^*(t), u^*(t)) = 0,$$

$$\tau_i^* \geq 0, \qquad \sum_{i=1}^{p} \tau_i^* = 1, \qquad \lambda^*(t) \geq 0.$$

## 4.3 Formulation of Control Dual Problem

We formulate two pairs of the following nondifferentiable multiobjective dual control problems.

The Wolfe type dual [38]:

**(NWMCD)**

Maximize

$$\Big( \int_a^b \{ f^1(t, x(t), u(t)) + x^T(t) w_1(t) + \sum_{j=1}^{k} \lambda_j(t) g^j(t, x(t), u(t))$$

$$+ \sum_{r=1}^{n} \mu_r(t)(h^r(t, x(t), u(t)) - \dot{x}(t))\} dt, \cdots, \int_a^b \{ f^p(t, x(t), u(t))$$

$$+ x^T(t) w_p(t) + \sum_{j=1}^{k} \lambda_j(t) g^j(t, x(t), u(t)) + \sum_{r=1}^{n} \mu_r(t)(h^r(t, x(t), u(t))$$

$$- \dot{x}(t))\} dt \Big)$$

subject to

$$x(a) = t_0,\ x(b) = t_f, \tag{4.4}$$

$$\sum_{i=1}^{p} \tau_i(t)\{f_x^i(t, x(t), u(t) + w_i(t)\} + \sum_{j=1}^{k} \lambda_j(t) g_x^j(t, x(t), u(t))$$

$$+ \sum_{r=1}^{n} \mu_r(t) h_x^r(t, x(t), u(t)) + \dot{\mu}(t) = 0, \quad t \in I, \tag{4.5}$$

$$\sum_{i=1}^{p} \tau_i(t) f_u^i(t, x(t), u(t)) + \sum_{j=1}^{k} \lambda_j(t) g_u^j(t, x(t), u(t))$$

$$+ \sum_{r=1}^{n} \mu_r(t) h_u^r(t, x(t), u(t)) = 0, \quad t \in I, \tag{4.6}$$

$$\int_a^b \sum_{r=1}^{n} \mu_r(t)(h^r(t, x(t), u(t)) - \dot{x}(t))dt \geq 0, \quad t \in I, \tag{4.7}$$

$$\lambda(t) \geq 0, \quad t \in I, \tag{4.8}$$

$$\tau_i \geq 0, \quad \sum_{i=1}^{p} \tau_i = 1. \tag{4.9}$$

The Mond-Weir type dual [32]

**(NMMCD)**

Maximize

$$(\int_a^b \{f^1(t, x(t), u(t)) + x^T(t) w_1(t)\}dt, \cdots,$$

$$\int_a^b \{f^p(t, x(t), u(t)) + x^T(t) w_p(t)\}dt)$$

subject to

$$x(a) = t_0,\ x(b) = t_f, \tag{4.10}$$

$$\sum_{i=1}^{p} \tau_i(t)\{f_x^i(t, x(t), u(t) + w_i(t)\} + \sum_{j=1}^{k} \lambda_j(t) g_x^j(t, x(t), u(t))$$

$$+ \sum_{r=1}^{n} \mu_r(t) h_x^r(t, x(t), u(t)) + \dot{\mu}(t) = 0, \quad t \in I, \tag{4.11}$$

$$\sum_{i=1}^{p} \tau_i(t) f_u^i(t, x(t), u(t)) + \sum_{j=1}^{k} \lambda_j(t) g_u^j(t, x(t), u(t))$$

$$+ \sum_{r=1}^{n} \mu_r(t) h_u^r(t, x(t), u(t)) = 0, \quad t \in I, \tag{4.12}$$

$$\int_a^b \sum_{r=1}^{n} \mu_r(t)(h^r(t, x(t), u(t)) - \dot{x}(t))dt \geq 0, \quad t \in I, \tag{4.13}$$

$$\int_a^b \sum_{j=1}^{k} \lambda_j(t) g^j(t, x(t), u(t))dt \geq 0, \quad t \in I, \tag{4.14}$$

$$\lambda(t) \geq 0, \quad t \in I, \tag{4.15}$$

$$\tau_i \geq 0, \quad \sum_{i=1}^{p} \tau_i = 1. \tag{4.16}$$

## 4.4 Sufficient Optimality Theorem for (NMCP)

We obtain a Kuhn-Tucker type sufficient optimality theorem of **(NMCP)** as follows:

**Theorem 4.1** *Suppose that* $(x^*(t), u^*(t))$ *is feasible for* **(NMCP)** *such that there exist* $\tau_i^* > 0$, $\lambda^*(t)$ *and* $\mu^*(t)$ *such that*

$$\sum_{i=1}^{p} \tau_i^* \{f_x^i(t, x^*(t), u^*(t) + w_i(t)\} + \sum_{j=1}^{k} \lambda_j^*(t) g_x^j(t, x^*(t), u^*(t))$$

$$+ \sum_{r=1}^{n} \mu_r^*(t) h_x^r(t, x^*(t), u^*(t)) + \dot{\mu}^*(t) = 0, \tag{4.17}$$

$$< w_i(t), x^*(t) >= s(x^*(t)|D_i), i = 1, \cdots, p, \tag{4.18}$$

$$\sum_{i=1}^{p} \tau_i^* f_u^i(t, x^*(t), u^*(t)) + \sum_{j=1}^{k} \lambda_j^*(t) g_u^j(t, x^*(t), u^*(t))$$

$$+ \sum_{r=1}^{n} \mu_r^*(t) h_u^r(t, x^*(t), u^*(t)) = 0, \tag{4.19}$$

$$\sum_{j=1}^{k} \lambda_j^*(t) g^j(t, x^*(t), u^*(t)) = 0, \tag{4.20}$$

$$\sum_{r=1}^{n} \mu_r^*(t)(h^r(t, x^*(t), u^*(t)) - \dot{x}^*(t)) = 0, \tag{4.21}$$

$$\lambda^*(t) \geq 0, \tag{4.22}$$

$$\tau_i^* \geq 0, \qquad \sum_{i=1}^{p} \tau_i^* = 1. \tag{4.23}$$

*hold through* $a \leq t \leq b$ *(except that at* $t$ *corresponding to discontinuities of* $u^*(t)$*, (4.17) holds for right and left limits).*

*If* $\int_a^b \{f^i(t, x^*(t), u^*(t)) + s(x^*(t)|D_i)\}dt$, $i = 1, \cdots, p$,

$\int_a^b \lambda_j^*(t) g^j(t, x^*(t), u^*(t))dt$, $j = 1, \cdots, k$ *and* $\int_a^b \sum_{r=1}^n \mu_r^*(t)(h^r(t, x^*(t), u^*(t)) - \dot{x}^*(t))$, $r = 1, \cdots, n$ *are all V-ρ-invex with respect to* $\eta(t), \xi(t), \zeta(t), \alpha(t)$ *and* $\sum \tau_i^*(t)\rho_i + \sum \rho_j + \sum \rho_r \geq 0$ *, then* $(x^*(t), u^*(t))$ *is an efficient solution of* **(NMCP)**.

*Proof. Suppose that* $(x^*(t), u^*(t))$ *is not an efficient solution of* **(NMCP)**.

*Then there exists* $(x(t), u(t)) \neq (x^*(t), u^*(t))$ *such that* $(x(t), u(t))$ *is feasible for* **(NMCP)** *and*

$$\int_a^b \{f^i(t, x(t), u(t)) + s(x(t)|D_i)\}dt$$

$$\leqq \int_a^b \{f^i(t, x^*(t), u^*(t)) + s(x^*(t)|D_i)\}dt,$$

$$\forall i = 1, \cdots, p$$

*and*

$$\int_a^b \{f^{i_0}(t, x(t), u(t)) + s(x(t)|D_{i_0})\}dt$$

$$< \int_a^b \{f^{i_0}(t, x^*(t), u^*(t)) + s(x^*(t)|D_{i_0})\}dt,$$

$$for \quad some \quad i_0 = 1, \cdots, p.$$

*Since* $< w_i(t), x^*(t) >= s(x^*(t)|D_i), \quad i = 1, \cdots, p,$

$$\int_a^b \{f^i(t, x(t), u(t)) + x^T(t) w_i(t))\}dt$$

$$= \int_a^b \{f^i(t, x(t), u(t)) + s(x(t)|D_i)\}dt$$

$$\leqq \int_a^b \{f^i(t, x^*(t), u^*(t)) + s(x^*(t)|D_i)\}dt$$

$$= \int_a^b \{f^i(t, x^*(t), u^*(t)) + x^{*T} w_i(t)\}dt,$$

$$\forall i = 1, \cdots, p$$

*and*

$$\int_a^b \{f^{i_0}(t, x(t), u(t)) + x^T(t) w_{i_0}(t))\}dt$$

$$= \int_a^b \{f^{i_0}(t, x(t), u(t)) + s(x(t)|D_{i_0})\}dt$$

$$\leqq \int_a^b \{f^{i_0}(t, x^*(t), u^*(t)) + s(x^*(t)|D_{i_0})\}dt$$

$$= \int_a^b \{f^{i_0}(t, x^*(t), u^*(t)) + x^{*T} w_{i_0}(t)\}dt,$$

$$for \quad some \quad i_0 = 1, \cdots, p.$$

*Since* $\int_a^b \{f^i(t, x(t), u(t)) + s(x(t)|D_i)\}dt$ *is* $V$-$\rho$-*invex,*

$$\int_a^b [\eta^T(t)\alpha_i(x(t), u(t), x^*(t), u^*(t))\{f_x^i(t, x^*(t), u^*(t)) + w_i(t)\}$$

$$+\xi^T(t)\alpha_i(x(t), u(t), x^*(t), u^*(t)) f_u^i(t, x^*(t), u^*(t))]dt + \rho_i\|\zeta\|^2 \leqq 0,$$

$$\forall i = 1, \cdots, p$$

*and*

$$\int_a^b [\eta^T(t)\alpha_{i_0}(x(t), u(t), x^*(t), u^*(t))\{f_x^{i_0}(t, x^*(t), u^*(t)) + w_{i_0}(t)\}$$

$$+\xi^T(t)\alpha_{i_0}(x(t),u(t),x^*(t),u^*(t))f_u^{i_0}(t,x^*(t),u^*(t))]dt+\rho_{i_0}\|\zeta\|^2<0,$$

$$for\ some\ i_0=1,\cdots,p.$$

*Since* $\tau_i^*>0$ *for all* $i$,

$$\int_a^b\sum_{i=1}^{p}\alpha_i(x(t),u(t),x^*(t),u^*(t))[\eta^T(t)\tau_i^*(t)\{f_x^i(t,x^*(t),u^*(t))+w_i(t)\}$$

$$+\xi^T(t)\tau_i^*(t)f_u^i(t,x^*(t),u^*(t))]dt+\sum_{i=1}^{p}\tau_i^*(t)\rho_i\|\zeta\|^2<0, \tag{4.24}$$

*From the feasibility conditions,*

$$\sum_{j=1}^{k}\lambda_j^*(t)g^j(t,x(t),u(t))\leqq 0=\sum_{j=1}^{k}\lambda_j^*(t)g^j(t,x^*(t),u^*(t))$$

$$\forall j=1,\cdots,k.$$

*By the V-ρ-invexity of* $\int_a^b\lambda_j^*(t)g^j(t,x^*(t),u^*(t))dt$, *we have*

$$\int_a^b\sum_{j=1}^{k}\beta_j(x(t),u(t),x^*(t),u^*(t))[\eta^T(t)\lambda_j^*(t)g_x^j(t,x^*(t),u^*(t))$$

$$+\xi^T(t)\lambda_j^*(t)g_u^j(t,x^*(t),u^*(t))]dt+\sum_{j=1}^{k}\rho_j\|\zeta\|^2\leq 0. \tag{4.25}$$

*From the feasibility conditions,*

$$\sum_{r=1}^{n}\mu_r^*(t)(h^r(t,x(t),u(t))-\dot{x}(t))-\sum_{r=1}^{n}\mu_r^*(t)(h^r(t,x^*(t),u^*(t))-\dot{x}^*(t))=0.$$

By the V-$\rho$-invexity of $\int_a^b \mu_r^*(t)(h^r(t, x^*(t), u^*(t)) - \dot{x}^*(t))dt$, we have

$$\int_a^b \sum_{r=1}^{n} \gamma_r(x(t), u(t), x^*(t), u^*(t))[\eta^T(t)\mu_r^*(t)h_x^r(t, x^*(t), u^*(t)) - \frac{d\eta^T(t)}{dt}\mu_r^*(t)$$

$$+\xi^T(t)\mu_r^*(t)h_u^r(t, x^*(t), u^*(t))]dt + \sum_{r=1}^{n} \rho_r \|\zeta\|^2 \leq 0. \tag{4.26}$$

By integrating $\frac{d\eta^T(t)}{dt}\mu_r^*(t)$ from $a$ to $b$ and applying the boundary condition, we have

$$\int_a^b \frac{d\eta^T(t)}{dt}\mu_r^*(t)dt = -\int_a^b \eta^T(t)\dot{\mu}_r^*(t)dt \tag{4.27}$$

Using (4.27) in (4.26), we have

$$\int_a^b \sum_{r=1}^{n} \gamma_r(x(t), u(t), x^*(t), u^*(t))[\eta^T(t)\mu_r^*(t)h_x^r(t, x^*(t), u^*(t)) + \eta^T(t)\dot{\mu}_r^*(t)$$

$$+\xi^T(t)\mu_r^*(t)h_u^r(t, x^*(t), u^*(t))]dt + \sum_{r=1}^{n} \rho_r \|\zeta\|^2 \leq 0. \tag{4.28}$$

Since (4.24), (4.25) and (4.28) hold the same $\alpha(t)$, we have

$$\int_a^b [\eta^T(t)\alpha_i(x(t), u(t), x^*(t), u^*(t))\{\sum_{i=1}^{p} \tau_i^*(t)(f_x^i(t, x^*(t), u^*(t)) + w_i(t))$$

$$+\sum_{j=1}^{k} \lambda_j^*(t)g_x^j(t, x^*(t), u^*(t)) + \sum_{r=1}^{n} \mu_r^*(t)h_x^r(t, x^*(t), u^*(t)) + \dot{\mu}^*(t)\}$$

$$+\xi^T(t)\alpha_i(x(t),u(t),x^*(t),u^*(t))\{\sum_{i=1}^{p}\tau_i^*(f_u^i(t,x^*(t),u^*(t))$$

$$+\sum_{j=1}^{k}\lambda_j^*(t)g_u^j(t,x^*(t),u^*(t))+\sum_{r=1}^{n}\mu_r^*(t)h_u^r(t,x^*(t),u^*(t))\}]dt$$

$$+\sum_{i=1}^{p}\tau_i^*\rho_i\|\zeta\|^2+\sum_{j=1}^{k}\rho_j\|\zeta\|^2+\sum_{r=1}^{n}\rho_r\|\zeta\|^2<0. \tag{4.29}$$

*From (4.17), (4.19) and the fact that* $\sum\tau_i^*\rho_i+\sum\rho_j+\sum\rho_r\geq 0$,*we have*

$$\int_a^b[\eta^T(t)\alpha_i(x(t),u(t),x^*(t),u^*(t))\{\sum_{i=1}^{p}\tau_i^*(f_x^i(t,x^*(t),u^*(t))+w_i(t))$$

$$+\sum_{j=1}^{k}\lambda_j^*(t)g_x^j(t,x^*(t),u^*(t))+\sum_{r=1}^{n}\mu_r^*(t)h_x^r(t,x^*(t),u^*(t))+\dot{\mu}^*(t)\}$$

$$+\xi^T(t)\alpha_i(x(t),u(t),x^*(t),u^*(t))\{\sum_{i=1}^{p}\tau_i^*(f_u^i(t,x^*(t),u^*(t))$$

$$+\sum_{j=1}^{k}\lambda_j^*(t)g_u^j(t,x^*(t),u^*(t))+\sum_{r=1}^{n}\mu_r^*(t)h_u^r(t,x^*(t),u^*(t))\}]dt$$

$$+\sum_{i=1}^{p}\tau_i^*(t)\rho_i\|\zeta\|^2+\sum_{j=1}^{k}\rho_j\|\zeta\|^2+\sum_{r=1}^{n}\rho_r\|\zeta\|^2\geq 0.$$

*which contradicts the inequality (4.29). Hence* $(x^*(t),u^*(t))$ *is an efficient solution of* **(NMCP)**. □

## 4.5 Duality Theorems

Now we establish some duality theorems between the nondifferentiable multiobjective control problem **(NMCP)** and its Wolfe type dual problem **(WMCD)**.

**Theorem 4.2 (Weak Duality)**. *Assume that, for all feasible* $(x^*(t), u^*(t))$ *for* **(NMCP)** *and all feasible* $(x(t), u(t), \tau, \lambda(t), \mu(t))$ *for* **(NWMCD)**,

$$(i) \quad \Big(\int_a^b \{f^1(t, x(t), u(t)) + x^T(t)w_1(t)\}, \cdots, \int_a^b \{f^p(t, x(t), u(t)) + x^T(t)w_p(t)\}\Big),$$

$$(ii) \quad \Big(\int_a^b \lambda_1(t) g^1(t, x(t), u(t))dt, \cdots, \int_a^b \lambda_k(t) g^k(t, x(t), u(t))dt\Big),$$

$$(iii) \quad \int_a^b \mu_1(t)(h^1(t, x(t), u(t)) - \dot{x}(t))dt, \cdots, \int_a^b \mu_n(t)(h^n(t, x(t), u(t)) - \dot{x}(t))dt,$$

*are all V-ρ-invex with respect to the same functions* $\eta(t)$, $\xi(t)$, $\zeta(t)$ *and* $\alpha(t)$ *and*

*(iv)* $\sum_{i=1}^p \tau_i \rho_i + \sum_{j=1}^k \rho_j + \sum_{r=1}^n \rho_r \geq 0$, *then the following inequalities cannot hold:*

$$\int_a^b (f^i(t, x^*(t), u^*(t)) + s(x^*(t)|D_i))dt$$

$$\leqq \int_a^b \{(f^i(t, x(t), u(t)) + x^T(t)w_i(t) + \sum_{j=1}^k \lambda_j(t) g^j(t, x(t), u(t))$$

$$+\sum_{r=1}^{n}\mu_r(t)(h^r(t,x(t),u(t))-\dot{x}(t))\}dt,$$

$$\forall i=1\cdots,p \tag{4.30}$$

*and*

$$\int_a^b (f^{i_0}(t,x^*(t),u^*(t))+s(x^*(t)|D_{i_0}))dt$$

$$<\int_a^b\{(f^{i_0}(t,x(t),u(t))+x^T(t)w_{i_0}(t)+\sum_{j=1}^{k}\lambda_j(t)g^j(t,x(t),u(t))$$

$$+\sum_{r=1}^{n}\mu_r(t)(h^r(t,x(t),u(t))-\dot{x}(t))\}dt,$$

$$for \quad some \quad i_0=1\cdots,p. \tag{4.31}$$

*Proof. Suppose contrary to the result that (4.30) and (4.31) hold. Then, since* $\tau_i>0$ *and* $\sum_{i=1}^{p}\tau_i=1$,

$$\int_a^b\sum_{i=1}^{p}\tau_i(t)(f^i(t,x^*(t),u^*(t))+s(x^*(t)|D_i))dt$$

$$<\int_a^b\{\sum_{i=1}^{p}\tau_i(t)(f^i(t,x(t),u(t))+x^T(t)w_i(t))+\sum_{j=1}^{k}\lambda_j(t)g^j(t,x(t),u(t))$$

$$+\sum_{r=1}^{n}\mu_r(t)(h^r(t,x(t),u(t))-\dot{x}(t))\}dt. \tag{4.32}$$

*Since* $<w_i(t),x^*(t)>\leqq s(x^*(t)|D_i),\quad i=1\cdots,p,$

$$\int_a^b\sum_{i=1}^{p}\tau_i(t)(f^i(t,x^*(t),u^*(t))+x^{*T}(t)w_i(t))dt$$

$$< \int_a^b \{\sum_{i=1}^p \tau_i(t)(f^i(t,x(t),u(t)) + x^T(t)w_i(t)) + \sum_{j=1}^k \lambda_j(t)g^j(t,x(t),u(t))$$

$$+ \sum_{r=1}^n \mu_r(t)(h^r(t,x(t),u(t)) - \dot{x}(t))\}dt.$$

*By (i), we have*

$$\int_a^b (f^i(t,x^*(t),u^*(t)) + x^{*T}(t)w_i(t))dt - \int_a^b (f^i(t,x(t),u(t)) + x^T(t)w_i(t))dt$$

$$\geq \int_a^b \{\eta^T(t)\alpha_i(x(t),u(t),x^*(t),u^*(t))(f_x^i(t,x(t),u(t)) + w_i(t))$$

$$+\xi^T(t)\alpha_i(x(t),u(t),x^*(t),u^*(t))f_u^i(t,x(t),u(t))\}dt + \rho_i\|\zeta\|^2$$

*Since* $\tau_i > 0$ *and* $\sum_{i=1}^p \tau_i = 1$ *we can get*

$$\int_a^b \sum_{i=1}^p \tau_i\{f^i(t,x^*(t),u^*(t)) + x^{*T}(t)w_i(t)\}dt$$

$$- \int_a^b \sum_{i=1}^p \tau_i\{f^i(t,x(t),u(t) + x^T(t)w_i(t))\}dt$$

$$\geq \int_a^b \sum_{i=1}^p \tau_i\alpha_i(x(t),u(t),x^*(t),u^*(t))\{\eta^T(t)(f_x^i(t,x(t),u(t))$$

$$+w_i(t)) + \xi^T(t)f_u^i(t,x(t),u(t))\}dt + \sum_{i=1}^p \tau_i(t)\rho_i\|\zeta\|^2. \qquad (4.33)$$

*By (ii), we have*

$$\int_a^b \lambda_j(t)g^j(t,x^*(t),u^*(t))dt - \int_a^b \lambda_j(t)g^j(t,x(t),u(t))dt$$

$$\geq \int_a^b \{\eta^T(t)\alpha_j(x(t),u(t),x^*(t),u^*(t))\lambda_j(t)g_x^j(t,x(t),u(t))$$

$$+\xi^T(t)\alpha_j(x(t),u(t),x^*(t),u^*(t))\lambda_j(t)g_u^j(t,x(t),u(t))\}dt$$

$$+\rho_j\|\zeta\|^2. \tag{4.34}$$

*Using (4.2) and (4.8) from (4.34) we have*

$$-\int_a^b \lambda_j(t)g^j(t,x(t),u(t))dt$$

$$\geq \int_a^b \{\eta^T(t)\alpha_j(x(t),u(t),x^*(t),u^*(t))\lambda_j(t)g_x^j(t,x(t),u(t))$$

$$+\xi^T(t)\alpha_j(x(t),u(t),x^*(t),u^*(t))\lambda_j(t)g_u^j(t,x(t),u(t))\}dt$$

$$+\rho_j\|\zeta\|^2,$$

*which implies,*

$$-\int_a^b \sum_{j=1}^k \lambda_j(t)g^j(t,x(t),u(t))dt$$

$$\geq \int_a^b \sum_{j=1}^k \alpha_j(x(t),u(t),x^*(t),u^*(t))\{\eta^T(t)\lambda_j(t)g_x^j(t,x(t),u(t))$$

$$+\xi^T(t)\lambda_j(t)g_u^j(t,x(t),u(t))\}dt + \sum_{j=1}^k \rho_j\|\zeta\|^2. \tag{4.35}$$

*By (iii), we have*

$$\int_a^b \mu_r(t)(h^r(t,x^*(t),u^*(t)) - \dot{x}^*(t))dt - \int_a^b \mu_r(t)(h^r(t,x(t),u(t)) - \dot{x}(t))dt$$

$$\geq \int_a^b \{\eta^T(t)\alpha_r(x(t),u(t),x^*(t),u^*(t))\mu_r(t)(h_x^r(t,x(t),u(t))$$

$$+\frac{d\eta^T(t)}{dt}\alpha_r(x(t),u(t),x^*(t),u^*(t))(-\mu_r(t))$$

$$+\xi^T(t)\alpha_r(x(t),u(t),x^*(t),u^*(t))\mu_r(t)(h_u^r(t,x(t),u(t))\}dt+\rho_r\|\zeta\|^2. \tag{4.36}$$

*Using (4.3) we have*

$$-\int_a^b \mu_r(t)(h^r(t,x(t),u(t))-\dot{x}(t))dt$$

$$\geq \int_a^b \{\eta^T(t)\alpha_r(x(t),u(t),x^*(t),u^*(t))\mu_r(t)(h_x^r(t,x(t),u(t))$$

$$+\frac{d\eta^T(t)}{dt}\alpha_r(x(t),u(t),x^*(t),u^*(t))(-\mu_r(t))$$

$$+\xi^T(t)\alpha_r(x(t),u(t),x^*(t),u^*(t))\mu_r(t)(h_u^r(t,x(t),u(t))\}dt+\rho_r\|\zeta\|^2,$$

*which implies*

$$-\int_a^b \sum_{r=1}^n \mu_r(t)(h^r(t,x(t),u(t))-\dot{x}(t))dt$$

$$\geq \int_a^b \sum_{r=1}^n \alpha_r(x(t),u(t),x^*(t),u^*(t))\{\eta^T(t)\mu_r(t)(h_x^r(t,x(t),u(t))$$

$$-\frac{d\eta^T(t)}{dt}\mu_r(t)+\xi^T(t)\mu_r(t)(h_u^r(t,x(t),u(t))\}dt$$

$$+\sum_{r=1}^n \rho_r\|\zeta\|^2. \tag{4.37}$$

*By integration* $\frac{d\eta^T(t)}{dt}\mu_r(t)$ *from* $a$ *to* $b$ *and applying the boundary condition, we have*

$$\int_a^b \frac{d\eta^T(t)}{dt}\mu(t)dt = \eta^T(t)\mu(t)|_a^b - \int_a^b \eta^T(t)\dot{\mu}(t)dt$$

$$= -\int_a^b \eta^T(t)\dot{\mu}_r(t)dt. \tag{4.38}$$

*Using (4.38) in (4.37),we have*

$$-\int_a^b \sum_{r=1}^n \mu_r(t)(h^r(t,x(t),u(t)) - \dot{x}(t))dt$$

$$\geq \int_a^b \sum_{r=1}^n \alpha_r(x(t),u(t),x^*(t),u^*(t))\{\eta^T(t)\mu_r(t)(h_x^r(t,x(t),u(t))$$

$$+\eta^T(t)\dot{\mu}_r(t) + \xi^T(t)\mu(t)(h_u^r(t,x(t),u(t))\}dt$$

$$+\sum_{r=1}^n \rho_r\|\zeta\|^2. \tag{4.39}$$

*Since (4.33), (4.35) and (4.39) hold the same* $\alpha(t)$*, we have*

$$\int_a^b \sum_{i=1}^p \tau_i\{f^i(t,x^*(t),u^*(t)) + x^{*T}(t)w_i(t)\}dt$$

$$-\int_a^b \sum_{i=1}^p \tau_i\{f^i(t,x(t),u(t)) + x^T(t)w_i(t))\}dt$$

$$-\int_a^b \sum_{j=1}^k \lambda_j(t)g^j(t,x(t),u(t))dt - \int_a^b \sum_{r=1}^n \mu_r(t)(h^r(t,x(t),u(t))$$

$$-\dot{x}(t))dt \geq \int_a^b \eta(t)\alpha(x(t), u(t), x^*(t), u^*(t))[\sum_{i=1}^p \tau_i\{f^i(t, x^*(t), u^*(t))$$

$$+x^{*T}(t)w_i(t)\} + \int_a^b \sum_{j=1}^k \lambda_j(t)g^j(t, x(t), u(t)) + \int_a^b \sum_{r=1}^n \mu_r(t)(h^r(t, x(t), u(t))$$

$$-\dot{\mu}_r(t))dt + \xi^T(t)\alpha(x(t), u(t), x^*(t), u^*(t))\{\sum_{i=1}^p \tau_i f_u^i(t, x(t), u(t))$$

$$+\sum_{j=1}^k \lambda_j(t)g_u^j(t, x(t), u(t)) + \sum_{r=1}^n \mu_r(t)(h_u^r(t, x(t), u(t))\}dt$$

$$+\sum_{i=1}^p \tau_i\rho_i\|\zeta\|^2 + \sum_{j=1}^k \rho_j\|\zeta\|^2 + \sum_{r=1}^n \rho_r\|\zeta\|^2 \geq 0.$$

*by (4.5), (4.6) and (iv). Hence*

$$\int_a^b \sum_{i=1}^p \tau_i\{f^i(t, x^*(t), u^*(t)) + x^{*T}(t)w_i(t)\}dt$$

$$\geq \int_a^b [\sum_{i=1}^p \tau_i\{f^i(t, x(t), u(t)) + x^T(t)w_i(t))\}$$

$$+\sum_{j=1}^k \lambda_j(t)g^j(t, x(t), u(t)) + \sum_{r=1}^n \mu_r(t)(h^r(t, x(t), u(t)) - \dot{x}(t))]dt$$

*which is a contradiction to (4.32)*

□

**Corollary 4.1** *Assume that weak duality (Theorem 4.1) holds between* **(NMCP)** *and* **(NWMCD)**. *If* $(x(t), u(t))$ *is feasible for* **(NMCP)**,

*$(x(t), u(t), \tau, \lambda(t), \mu(t))$ is feasible for* **(NWMCD)** *with $\sum_{j=1}^{k} \lambda_j(t) g^j(t, x(t), u(t)) = 0$, then $(x(t), u(t))$ is an efficient for* **(NMCP)**, *$x^T(t)w_i(t) = s(x(t)|D_i)$ and $(x(t), u(t), \tau, \lambda(t), \mu(t))$ is an efficient for* **(NWMCD)**.

*Proof.* Suppose $(x(t), u(t))$ is not an efficient for **(NMCP)**. Then there exists some feasible $(x^*(t), u^*(t))$ for **(NMCP)** such that

$$\int_a^b \{f^i(t, x^*(t), u^*(t)) + s(x^*(t)|D_i)\}dt \leqq \int_a^b \{f^i(t, x(t), u(t)) + s(x(t)|D_i)\}dt, \qquad \forall i = 1, \cdots, p$$

and

$$\int_a^b \{f^{i_0}(t, x^*(t), u^*(t)) + s(x^*(t)|D_{i_0})\}dt < \int_a^b \{f^{i_0}(t, x(t), u(t)) + s(x(t)|D_{i_0})\}dt, \qquad for \quad some \quad i_0 = 1, \cdots, p.$$

Since $< w_i(t), x(t) >= s(x(t)|D_i)$ $i = 1, \cdots, p$,

$$\int_a^b \{f^i(t, x^*(t), u^*(t)) + s(x^*(t)|D_i)\}dt \leqq \int_a^b \{f^i(t, x(t), u(t)) + x^T(t)w_i(t)\}dt, \qquad \forall i = 1, \cdots, p$$

and

$$\int_a^b \{f^{i_0}(t, x^*(t), u^*(t)) + s(x^*(t)|D_{i_0})\}dt < \int_a^b \{f^{i_0}(t, x(t), u(t)) + x^T(t)w_{i_0}(t)\}dt, \qquad for \quad some \quad i_0 = 1, \cdots, p.$$

*Since* $\sum_{j=1}^{k}\lambda_j(t)g^j(t,x(t),u(t))=0$

*and* $\sum_{r=1}^{n}\mu_r(t)(h^r(t,x(t),u(t))-\dot{x}(t))=0,$ *we get*

$$\int_a^b\{f^i(t,x^*(t),u^*(t))+s(x^*(t)|D_i)\}dt\le\int_a^b\{f^i(t,x(t),u(t))$$

$$+x^T(t)w_i(t)+\sum_{j=1}^{k}\lambda_j(t)g^j(t,x(t),u(t))$$

$$+\sum_{r=1}^{n}\mu_r(t)(h^r(t,x(t),u(t))-\dot{x}(t))\}dt,$$

$$\forall i=1,\cdots,p$$

*and*

$$\int_a^b\{f^{i_0}(t,x^*(t),u^*(t))+s(x^*(t)|D_{i_0})\}dt<\int_a^b\{f^{i_0}(t,x(t),u(t))$$

$$+x^T(t)w_{i_0}(t)+\sum_{j=1}^{k}\lambda_j(t)g^j(t,x(t),u(t))$$

$$+\sum_{r=1}^{n}\mu_r(t)(h^r(t,x(t),u(t))-\dot{x}(t))\}dt,$$

$$for\quad some\quad i_0=1,\cdots,p.$$

*This contradicts the weak duality. Hence* $(x(t),u(t))$ *is an efficient for* **(NMCP)**. *Now suppose* $(x(t),u(t),\tau,\mu(t))$ *is not an efficient for* **(NWMCD)**. *Then there exists some* $(x^*(t),u^*(t),\tau^*,\mu^*(t))$ *feasible for* **(NWMCD)** *such that*

$$\int_a^b\{f^i(t,x^*(t),u^*(t))+x^{*T}(t)w_i(t)+\sum_{j=1}^{k}\lambda_j^*(t)g^j(t,x^*(t),u^*(t))$$

$$+\sum_{r=1}^{n}\mu_r^*(t)(h^r(t,x^*(t),u^*(t))-\dot{x}^*(t))\}dt \geq \int_a^b\{f^i(t,x(t),u(t))$$

$$+x^T(t)w_i(t)+\sum_{j=1}^{k}\lambda_j(t)g^j(t,x(t),u(t))+\sum_{r=1}^{n}\mu_r(t)(h^r(t,x(t),u(t))$$

$$-\dot{x}(t))\}dt,$$

$$\forall i=1,\cdots,p$$

*and*

$$\int_a^b\{f^{i_0}(t,x^*(t),u^*(t))+x^{*T}(t)w_{i_0}(t)+\sum_{j=1}^{k}\lambda_j^*(t)g^j(t,x^*(t),u^*(t))$$

$$+\sum_{r=1}^{n}\mu_r^*(t)(h^r(t,x^*(t),u^*(t))-\dot{x}^*(t))\}dt > \int_a^b\{f^{i_0}(t,x(t),u(t))$$

$$+x^T(t)w_{i_0}(t)+\sum_{j=1}^{k}\lambda_j(t)g^j(t,x(t),u(t))+\sum_{r=1}^{n}\mu_r(t)(h^r(t,x(t),u(t))$$

$$-\dot{x}(t))\}dt,$$

$$for \quad some \quad i_0=1,\cdots,p.$$

*Since* $\sum_{j=1}^{k}\lambda_j(t)g^j(t,x(t),u(t))=0$
*and* $\sum_{r=1}^{n}\mu_r(t)(h^r(t,x(t),u(t))-\dot{x}(t))=0,$

$$\int_a^b\{f^i(t,x^*(t),u^*(t))+x^{*T}(t)w_i(t)+\sum_{j=1}^{k}\lambda_j^*(t)g^j(t,x^*(t),u^*(t)$$

$$+\sum_{r=1}^{n}\mu_r^*(t)(h^r(t,x^*(t),u^*(t))-\dot{x}^*(t))\}dt$$

$$\geq \int_a^b \{f^i(t, x(t), u(t)) + x^T(t)w_i(t)\}dt$$

$$\forall i = 1, \cdots, p,$$

*and*

$$\int_a^b \{f^{i_0}(t, x^*(t), u^*(t)) + x^{*T}(t)w_{i_0}(t) + \sum_{j=1}^{k} \lambda_j^*(t) g^j(t, x^*(t), u^*(t)$$

$$+ \sum_{r=1}^{n} \mu_r^*(t)(h^r(t, x^*(t), u^*(t)) - \dot{x}^*(t))\}dt$$

$$> \int_a^b \{f^{i_0}(t, x(t), u(t)) + x^T(t)w_{i_0}(t)\}dt$$

$$for \quad some \quad i_0 = 1, \cdots, p.$$

*Since* $< w_i(t), x^T(t) >= s(x^T(t)|D_i)$

$$\int_a^b \{f^i(t, x^*(t), u^*(t)) + x^{*T}(t)w_i(t) + \sum_{j=1}^{k} \lambda_j^*(t) g^j(t, x^*(t), u^*(t))$$

$$+ \sum_{r=1}^{n} \mu_r^*(t)(h^r(t, x^*(t), u^*(t)) - \dot{x}^*(t))\}dt$$

$$\geqq \int_a^b \{f^i(t, x(t), u(t)) + s(x^T(t)|D_i)\}dt, \ \forall i = 1, \cdots, p$$

*and*

$$\int_a^b \{f^{i_0}(t, x^*(t), u^*(t)) + x^{*T}(t)w_{i_0}(t) + \sum_{j=1}^{k} \lambda_j^*(t) g^j(t, x^*(t), u^*(t)$$

$$+\sum_{r=1}^{n}\mu_r^*(t)(h^r(t,x^*(t),u^*(t))-\dot{x}^*(t))\}dt$$

$$>\int_a^b\{f^{i_0}(t,x(t),u(t))+s(x^T(t)|D_{i_0})\}dt,\ \textit{for some } i_0=1,\cdots,p.$$

*This contradicts the weak duality. Hence* $(x(t),u(t),\tau,\lambda(t),\mu(t))$ *is an efficient for* **(NWMCD)**.

□

**Theorem 4.3 (Strong Duality)** *Let* $(x^*(t),u^*(t))$ *be an efficient for* **(NMCP)** *and assume that* $(x^*(t),u^*(t))$ *satisfies the constraint qualification for* $\mathbf{NMCP}_k(x^*(t),u^*(t))$ *for at least one* $k\in\{1,\cdots,p\}$*. Then there exist* $\tau_i^*\in R^p$ *and piecewise smooth functions* $\lambda^*(t):I\longrightarrow R^k$, $\mu^*(t):I\longrightarrow R^n$ *such that* $(x^*(t),u^*(t),\tau^*,\lambda^*(t),\mu^*(t))$ *is feasible for* **(NWMCD)**.
*and* $\sum_{j=1}^k\lambda_j^*(t)g^j(t,x^*(t),\dot{u}^*(t))=0.$
*If weak duality also holds between* **(NMCP)** *and* **(NWMCD)**, *then* $(x^*(t),u^*(t),\tau^*,\lambda^*(t),\mu^*(t))$ *is an efficient for* **(NWMCD)**.

*Proof.* *If follows from Lemma 4.1 that there exist* $\tau_i^*\in R^p$, *and piecewise smooth functions* $\lambda^*(t):I\to R^k$ , *and* $\mu^*(t):I\to R^n$ *satisfying the following relations, for all* $t\in I$:

$$\sum_{i=1}^{p}\tau_i^*\{f_x^i(t,x^*(t),u^*(t))+w_i(t)\}+\sum_{j=1}^{k}\lambda_j^*(t)g_x^j(t,x^*(t),u^*(t))$$

$$+\sum_{r=1}^{n}\mu_r^*(t)(h_x^r(t,x^*(t),u^*(t))+\dot{\mu}^*(t))=0,$$

$$\sum_{i=1}^{p} \tau_i^* \{f_u^i(t, x^*(t), u^*(t)) + \sum_{j=1}^{k} \lambda_j^*(t) g_u^j(t, x^*(t), u^*(t))$$

$$+ \sum_{r=1}^{n} \mu_r^*(t)(h_u^r(t, x^*(t), u^*(t)) = 0,$$

$$\sum_{j=1}^{k} \lambda_j^*(t) g^j(t, x^*(t), u^*(t)) = 0,$$

$$\tau_i^* \geq 0, \qquad \sum_{i=1}^{p} \tau_i^* = 1, \qquad \lambda^*(t) \geq 0.$$

*As* $(x^*(t), u^*(t))$ *is feasible for* **(NMCP)**, $\dot{x}^*(t) = h^r(t, x^*(t), u^*(t))$
*and* $\int_a^b \sum_{r=1}^{n} \mu_r(t)(h^r(t, x(t), u(t)) - \dot{x}^*(t))dt \geq 0.$
*therefore* $(x^*(t), u^*(t), \tau^*, \lambda^*(t), \mu^*(t))$ *is feasible for* **(NWMCD)**.
*The result now follows from Corollary 4.1.*

□

Now we establish weak, strong duality theorems between the nondifferentiable multiobjective control problem **(NMCP)** and its Mond-Weir type dual problem **(NMMCD)**.

**Theorem 4.4 (Weak Duality)** *Assume that, for all feasible* $(x^*(t), u^*(t))$ *for* **(NMCP)** *and all feasible* $(x(t), u(t), \tau, \lambda(t), \mu(t))$ *for* **(NMMCD)**,

$$(i) \quad \Big( \int_a^b \{f^1(t,x(t),u(t)) + x^T(t)w_1(t)\}dt, \cdots, \int_a^b \{f^p(t,x(t),u(t))$$

$$+x^T(t)w_p(t)\}dt\Big)$$

*is V-ρ-pseudo-invex with respect to* $\eta(t), \xi(t), \zeta(t)$ *and* $\alpha(t)$,

$$(ii) \quad \Big( \int_a^b \lambda_1(t)g^1(t,x(t),u(t))dt, \cdots, \int_a^b \lambda_k(t)g^k(t,x(t),u(t))dt\Big)$$

*is V-ρ-quasi-invex with respect to* $\eta(t), \xi(t), \zeta(t)$ *and* $\beta(t)$,

$$(iii) \quad \Big( \int_a^b \mu_1(t)(h^1(t,x(t),u(t)) - \dot{x}(t))dt, \cdots, \int_a^b \mu_n(t)(h^n(t,x(t),u(t))$$

$$-\dot{x}(t))dt\Big)$$

*is V-ρ-quasi-invex with respect to* $\eta(t), \xi(t), \zeta(t)$ *and* $\gamma(t)$, *and*

$$(iv) \quad \sum_{i=1}^p \tau_i(t)\rho_i + \sum_{j=1}^k \rho_j + \sum_{r=1}^n \rho_r \geq 0.$$

$$\Rightarrow \sum_{i=1}^p \tau_i(t)\rho_i + \sum_{j=1}^k \lambda_j\rho_j + \sum_{r=1}^n \mu_r(t)\rho_r \geq 0.$$

*Then the following relations cannot hold:*

$$\int_a^b \{f^i(t,x^*(t),u^*(t)) + s(x^*(t)|D_i)\}dt$$

$$\leqq \int_a^b \{f^i(t, x(t), u(t)) + x^T(t)w_i(t)\}dt,$$

$$\forall i = 1, \cdots, p \tag{4.40}$$

*and*

$$\int_a^b \{f^{i_0}(t, x^*(t), u^*(t)) + s(x^*(t)|D_{i_0})\}dt$$

$$< \int_a^b \{f^{i_0}(t, x(t), u(t)) + x^T(t)w_{i_0}(t)\}dt,$$

$$\text{for some } i_0 = 1, \cdots, p. \tag{4.41}$$

*Proof. Suppose contrary to the result that (4.40) and (4.41) hold. Since* $\alpha_i(x(t), x^*(t), \dot{x}(t), \dot{x}^*(t), u(t), \dot{u}^*(t)) > 0.$

$$\int_a^b \sum_{i=1}^p \alpha_i(x(t), x^*(t), \dot{x}(t), \dot{x}^*(t), u(t), \dot{u}^*(t))\{f^i(t, x^*(t), u^*(t))$$

$$+s(x^*(t)|D_i)\}dt$$

$$< \int_a^b \sum_{i=1}^p \alpha_i(x(t), x^*(t), \dot{x}(t), \dot{x}^*(t), u(t), \dot{u}^*(t))\{f^i(t, x(t), u(t))$$

$$+x^T(t)w_i(t)\}dt.$$

*Since* $< w_i(t), x^*t) >= s(x^*(t)|D_i)$ $i = 1, \cdots, p.$

*Then (i) yields*

$$\int_a^b \sum_{i=1}^p \{\eta^T(t)(f_x^i(t, x(t), u(t)) + w_i(t)) + \xi^T(t)f_u^i(t, x(t), u(t))\}dt$$

$$+\sum_{i=1}^p \rho_i\|\zeta\|^2 < 0.$$

*Since* $\tau_i > 0$, *we have*

$$\int_a^b \sum_{i=1}^{p} \{\eta^T(t)\tau_i(t)(f_x^i(t, x(t), u(t)) + w_i(t)) + \xi^T(t)\tau_i(t)f_u^i(t, x(t), u(t))\}dt$$

$$+\sum_{i=1}^{p} \tau_i(t)\rho_i\|\zeta\|^2 < 0. \tag{4.42}$$

*Form the feasibility conditions,*

$$\int_a^b \lambda_j(t)g^j(t, x^*(t), u^*(t))dt \leq \int_a^b \lambda_j(t)g^j(t, x(t), u(t))dt,$$

*for each* $j = 1, \cdots, k$.

*Since* $\beta_j > 0$, $\forall j = 1, \cdots, k$ *we have*

$$\int_a^b \sum_{j=1}^{k} \beta_j(t)\lambda_j(t)g^j(t, x^*(t), u^*(t))dt \leq \int_a^b \sum_{j=1}^{k} \beta_j(t)\lambda_j(t)g^j(t, x(t), u(t))dt.$$

*It now follows from (ii) that*

$$\int_a^b \sum_{j=1}^{k} \{\eta^T(t)\lambda_j(t)g_x^j(t, x(t), u(t)) + \xi^T(t)\lambda_j(t)g_u^j(t, x(t), u(t))\}dt$$

$$+\sum_{j=1}^{k} \lambda_j(t)\rho_j\|\zeta\|^2 \leq 0. \tag{4.43}$$

*From (4.3) and (4.13),we have*

$$\int_a^b \sum_{r=1}^{n} \mu_r(t)(h^r(t, x^*(t), u^*(t)) - \dot{x}^*(t))dt$$

$$\leq \int_a^b \sum_{r=1}^{n} \mu_r(t)(h^r(t, x(t), u(t)) - \dot{x}(t))dt,$$

*Since* $\gamma_r(t) > 0$, $\forall r = 1, \cdots, n$, *we have*

$$\int_a^b \sum_{r=1}^{n} \gamma_r(t)\mu_r(t)(h^r(t, x^*(t), u^*(t)) - \dot{x}^*(t))dt$$

$$\leq \int_a^b \sum_{r=1}^{n} \gamma_r(t)\mu_r(t)(h^r(t, x(t), u(t)) - \dot{x}(t))dt,$$

*From (iii) it follows that*

$$\int_a^b \sum_{r=1}^{n} \{\eta^T(t)\mu_r(t)h_x^r(t, x(t), u(t))$$

$$-\frac{d\eta^T(t)}{dt}\mu_r(t) + \xi^T(t)\mu_r(t)h_u^r(t, x(t), u(t))\}dt + \sum_{r=1}^{n} \mu_r(t)\rho_r\|\zeta\|^2$$

$$\leq 0. \tag{4.44}$$

*By integrating* $\frac{d\eta^T(t)}{dt}\mu_r(t)$ *from* $a$ *to* $b$ *and applying the boundary condition (4.1) we have*

$$\int_a^b \frac{d\eta^T(t)}{dt}\mu(t)dt = -\int_a^b \eta^T(t)\dot{\mu}(t)dt \tag{4.45}$$

*Using (4.45) in (4.44),we have*

$$\int_a^b \sum_{r=1}^{n} \{\eta^T(t)\mu_r(t)h_x^r(t,x(t),u(t)) + \eta^T(t)\dot{\mu}_r(t)$$

$$+\xi^T(t)\mu_r(t)h_u^r(t,x(t),u(t))\}dt + \sum_{r=1}^{n} \mu_r(t)\rho_r\|\zeta\|^2 \leq 0. \qquad (4.46)$$

*Adding (4.42), (4.43) and (4.46), we have*

$$\int_a^b [\eta^T(t)\{\sum_{i=1}^{p} \tau_i(f_x^i(t,x(t),u(t)) + w_i(t)) + \sum_{j=1}^{k} \lambda_j(t)g_x^j(t,x(t),u(t))$$

$$+\sum_{r=1}^{n} \mu_r(t)h_x^r(t,x(t),u(t)) + \dot{\mu}(t)\} + \xi^T(t)\{\sum_{i=1}^{p} \tau_i(f_u^i(t,x(t),u(t))$$

$$+\sum_{j=1}^{k} \lambda_j(t)g_u^j(t,x(t),u(t)) + \sum_{r=1}^{n} \mu_r(t)h_u^r(t,x(t),u(t))\}]dt + \sum_{i=1}^{p} \tau_i\rho_i\|\zeta\|^2$$

$$+\sum_{j=1}^{k} \lambda_j(t)\rho_j\|\zeta\|^2 + \sum_{r=1}^{n} \mu_r(t)\rho_r\|\zeta\|^2 < 0.$$

*which is a contradiction to (4.11), (4.12) and (iv).*

□

**Corollary 4.2** *Assume that weak duality theorem (4.4) holds between* **(NMCP)** *and* **(NMMCD)**. *If* $(x(t), u(t))$ *is feasible for* **(NMCP)** *and* $(x(t), u(t), \tau(t), \lambda(t), \mu(t))$ *is feasible for* **(NMMCD)** *and* $< x^T(t), w_i(t) >=$

*$s(x(t)|D_i)$ then $(x(t), u(t))$ is an efficient for* **(NMCP)** *and* *$(x(t), u(t), \tau, \lambda(t), \mu(t))$ is an efficient for* **(NMMCD)**.

*Proof. Suppose $(x(t), u(t))$ is not an efficient for* **(NMCP)**. *Then there exists some feasible $(x^*(t), u^*(t))$ for* **(NMCP)** *such that*

$$\int_a^b \{f^i(t, x^*(t), u^*(t)) + s(x^*(t)|D_i)\}dt$$

$$\leqq \int_a^b \{f^i(t, x(t), u(t)) + x^T(t)w_i(t)\}dt, \forall i = 1, \cdots, p$$

*and*

$$\int_a^b \{f^{i_0}(t, x^*(t), u^*(t)) + s(x^*(t)|D_{i_0})\}dt$$

$$< \int_a^b \{f^{i_0}(t, x(t), u(t)) + x^T(t)w_{i_0}(t)\}dt, \text{ for some } i_0 = 1, \cdots, p.$$

*Since $< w_i(t), x^*(t) >= s(x^*(t)|D_i)$ $i = 1, \cdots, p$,*

$$\int_a^b \{f^i(t, x^*(t), u^*(t)) + x^{*T}(t)w_i(t))\}dt$$

$$\leqq \int_a^b \{f^i(t, x(t), u(t)) + x^T(t)w_i(t)\}dt, \forall i = 1, \cdots, p,$$

*and*

$$\int_a^b \{f^{i_0}(t, x^*(t), u^*(t)) + x^{*T}(t)w_{i_0}(t))\}dt$$

$$< \int_a^b \{f^{i_0}(t, x(t), u(t)) + x^T(t) w_{i_0}(t)\} dt, \ \textit{for some } i_0 = 1, \cdots, p.$$

*But* $(x(t), u(t), \tau, \lambda(t), \mu(t))$ *is feasible for* **(NMMCD)***, hence the result of weak duality theorem is contradict. Therefore* $(x(t), u(t))$ *is an efficient for* **(NMCP)***. Now suppose* $(x(t), u(t), \tau, \lambda(t), \mu(t))$ *is not an efficient for* **(NMMCD)***. Then there exist some feasible* $(x^*(t), u^*(t), \tau, \lambda(t), \mu(t))$ *for* **(NMMCD)** *such that*

$$\int_a^b \{f^i(t, x^*(t), u^*(t)) + s(x^*(t)|D_i)\} dt$$

$$\geqq \int_a^b \{f^i(t, x(t), u(t)) + x^T(t) w_i(t)\} dt, \forall i = 1, \cdots, p,$$

*and*

$$\int_a^b \{f^{i_0}(t, x^*(t), u^*(t)) + s(x^*(t)|D_{i_0})\} dt$$

$$> \int_a^b \{f^{i_0}(t, x(t), u(t)) + x^T(t) w_{i_0}(t)\} dt, \ \textit{for some } i_0 = 1, \cdots, p.$$

*Since* $< w_i(t), x^*(t) >= s(x^*(t)|D_i) \ i = 1, \cdots, p,$

$$\int_a^b \{f^i(t, x^*(t), u^*(t)) + x^{*T}(t) w_i(t))\} dt$$

$$\geqq \int_a^b \{f^i(t, x(t), u(t)) + x^T(t) w_i(t)\} dt, \forall i = 1, \cdots, p,$$

*and*

$$\int_a^b \{f^{i_0}(t, x^*(t), u^*(t)) + x^{*T}(t)w_{i_0}(t))\}dt$$

$$> \int_a^b \{f^{i_0}(t, x(t), u(t)) + x^T(t)w_{i_0}(t)\}dt, \text{ for some } \quad i_0 = 1, \cdots, p.$$

*This contradicts weak duality. Hence* $(x(t), u(t), \tau, \lambda(t), \mu(t))$ *is an efficient for* **(NMMCD)**.

□

**Theorem 4.5 (Strong Duality)** *Let* $(x^*(t), u^*(t))$ *be an efficient for* **(NMCP)** *and assume that* $(x^*(t), u^*(t))$ *satisfies the constraint qualification for* $\mathbf{NMCP}_k(x^*(t), u^*(t))$ *for at least one* $k \in \{1, \cdots, p\}$. *Then there exist* $\tau^* \in R^p$ *and piecewise smooth functions* $\lambda^*(t) : I \longrightarrow R^k$ *and* $\mu^*(t) : I \longrightarrow R^n$ *such that* $(x^*(t), u^*(t), \tau^*, \lambda^*(t), \mu^*(t))$ *is feasible for* **(NMMCD)**.
*If also weak duality holds between* **(NMCP)** *and* **(NMMCD)**, *then* $(x^*(t), u^*(t), \tau^*, \lambda^*(t), \mu^*(t))$ *is an efficient for* **(NMMCD)**.

*Proof.* *Proceeding on the same lines as in Theorem 4.3 it follows that there exist* $\tau^* \in R^p$, *and piecewise smooth functions* $\lambda^*(t) : I \longrightarrow R^k$ *and* $\mu^*(t) : I \longrightarrow R^n$, *satisfying for all* $t \in I$ *the following relations:*

$$\sum_{i=1}^{p} \tau_i^* \{f_x^i(t, x^*(t), u^*(t)) + w_i(t)\} + \sum_{j=1}^{k} \lambda_j^*(t) g_x^j(t, x^*(t), u^*(t))$$

$$+ \sum_{r=1}^{n} \mu_r^*(t)(h_x^r(t, x^*(t), u^*(t)) + \dot{\mu}^*(t)) = 0,$$

$$\sum_{i=1}^{p} \tau_i^* \{f_u^i(t, x^*(t), u^*(t)) + \sum_{j=1}^{k} \lambda_j^*(t) g_u^j(t, x^*(t), u^*(t))$$

$$+ \sum_{r=1}^{n} \mu_r^*(t) h_u^r(t, x^*(t), u^*(t)) = 0,$$

$$\sum_{j=1}^{k} \lambda_j^*(t) g^j(t, x^*(t), u^*(t)) = 0,$$

$$\tau_i^* \geq 0, \qquad \sum_{i=1}^{p} \tau_i^* = 1, \qquad \lambda^*(t) \geq 0.$$

*The relations*

$$\int_a^b \sum_{j=1}^{k} \lambda_j^*(t) g^j(t, x^*(t), u^*(t)) = 0$$

*and*

*are obvious.*
*The above relations imply that* $(x^*(t), u^*(t), \tau^*, \lambda^*(t), \mu^*(t))$ *is feasible for* **(MMCD)**. *The result now follows from Corollary 4.2.*

□



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