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Strong Convergence Theorems of Total Asymptotically Nonexpansive Families

by

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Strong Convergence Theorems of Total Asymptotically Nonexpansive Families
(전점근적비확대족에 대한 강수렴 정리)

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요　약

본 논문에서는 균등 볼록 (uniformly convex)인 Banach 공간에서 어떤 조건 A를 만족하는 전점근적비확대 (total asymptotically nonexpansive(in brief, TAN)) 족에 대한 오차(error) 항을 갖는 Ishikawa (and Mann) iteration 과정에 관한 어떤 강수렴정리를 증명한다.

1 Introduction

Let C be a nonempty closed convex subset of a real Banach space X and let $T : C \to C$ be a mapping. Then T is said to be a *Lipschitzian* mapping if, for each $n \geq 1$, there exists a constant $k_n > 0$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \tag{1.1}$$

for all $x, y \in C$ (we may assume that all $k_n \geq 1$). A Lipschitzian mapping T is called *uniformly k-Lipschitzian* if $k_n = k$ for all $n \geq 1$, *nonexpansive* if $k_n = 1$ for all $n \geq 1$, and *asymptotically nonexpansive* [5] if $\lim_{n\to\infty} k_n = 1$, respectively. The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [5] as a generalization of the class of nonexpansive mappings. They proved that if C is a nonempty bounded closed convex subset of a uniformly convex Banach space X, then every asymptotically nonexpanisve mapping $T : C \to C$ has a fixed point.

On the other hand, as the classes of non-Lipschitzian mappings, there appear in the literature two definitions, one is due to Kirk who says that T is a mapping of *asymptotically nonexpansive type* [12] if for each $x \in C$,

$$\limsup_{n\to\infty} \sup_{y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0 \tag{1.2}$$

and T^N is continuous for some $N \geq 1$. The other is the stronger concept due to Bruck, Kuczumov and Reich [2]. They say that T is *asymptotically nonexpansive in the intermediate sense* if T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0 \tag{1.3}$$

Recently, Alber et al. [1] introduced the wider class of total asymptotically nonexpansive mappings to unify various definitions of classes of nonlinear mappings associated with the class of asymptotically nonexpansive mappings; see also Definition 1 of [3]. They say that a mapping $T : C \to C$ is said to be *total asymptotically nonexpansive* (in brief, TAN) [1] (or [3]) if there exists two nonnegative real sequences $\{c_n\}$ and $\{d_n\}$ with $c_n,\ d_n \to 0$, $\phi \in \Gamma(\mathbb{R}^+)$ and $n_0 \in \mathbb{N}$ such that

$$\|T^n x - T^n y\| \leq \|x - y\| + c_n\, \phi(\|x - y\|) + d_n, \tag{1.4}$$

for all $x, y \in C$ and $n \geq n_0$, where $\mathbb{R}^+ := [0, \infty)$ and $\phi \in \Gamma(\mathbb{R}^+)$ means that ϕ is strictly increasing, continuous on $\mathbb{R}^+$ and $\phi(0) = 0$. In this case, T is often said to be TAN on C with respect to $\{c_n\}$, $\{d_n\}$ and ϕ.

Recently, motivated and stimulated by (1.4), Kim and Park [11] introduced a discrete family $\Im = \{T_n : C \to C\}$ of non-Lipschitzian mappings, called TAN on C, namely, $\Im = \{T_n : C \to C\}$ is said to be TAN on C with respect to $\{c_n\}$, $\{d_n\}$ and ϕ if there exist nonnegative real sequences $\{c_n\}$ and $\{d_n\}$, $n \geq 1$ with $c_n,\ d_n \to 0$ and $\phi \in \Gamma(\mathbb{R}^+)$ such that

$$\|T_n x - T_n y\| \leq \|x - y\| + c_n\, \phi(\|x - y\|) + d_n, \tag{1.5}$$

for all $x, y \in C$ and $n \geq 1$. Furthermore, we say that $\Im$ is *continuous* on C provided each $T_n \in \Im$ is continuous on C; see [11] for examples of *continuous* TAN families. Then they established necessary and sufficient conditions for strong convergence of the sequence $\{x_n\}$ defined recursively by the following explicit algorithm

$$x_{n+1} = T_n x_n, \qquad n \geq 1, \tag{1.6}$$

starting from an initial guess $x_1 \in C$, to a common fixed point of $\Im$ in Banach spaces.

For a single mapping T of C into itself, we consider the following Ishikawa iterative scheme of the type (Kim-Kim [9], cf. Xu [19]) emphasizing the randomness of errors as follows:

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily,} \\ x_{n+1} = \alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n, \\ y_n = \alpha_n' x_n + \beta_n' T^n x_n + \gamma_n' v_n = (1-\gamma_n')\left[\frac{\alpha_n'}{1-\gamma_n'} x_n + \frac{\beta_n'}{1-\gamma_n'} T^n x_n\right] + \gamma_n' v_n, \end{cases} \tag{1.7}$$

where $\{\alpha_n\}$, $\{\beta_n\}$,$\{\gamma_n\}$, $\{\alpha_n'\}$, $\{\beta_n'\}$,$\{\gamma_n'\}$ are real sequences in [0,1] and $\{u_n\}$, $\{v_n\}$ are two bounded sequences in C such that

(i) $\alpha_n + \beta_n + \gamma_n = \alpha_n' + \beta_n' + \gamma_n' = 1$ for all $n \geq 1$,

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n' < \infty$.

If $\gamma_n = \gamma_n' = 0$ for all $n \geq 1$, then the iteration process (1.7) reduces to the modified Ishikawa iteration process Schu [16] (cf. Ishikawa [8]), while setting $\beta_n' = 0$ and $\gamma_n' = 0$ for all $n \geq 1$, (1.7) reduces to the Mann iteration process with errors which is a generalized case of the Mann iteration process [13].

Let a discrete family $\Im = \{T_n : C \to C\}$ be continuous TAN on C with respect to $\{c_n\}$, $\{d_n\}$ and ϕ and $F := \cap_{n=1}^{\infty} Fix(T_n)$. On replacing T^n in (1.7) by T_n and setting $S_n := \frac{\alpha_n'}{1-\gamma_n'} I + \frac{\beta_n'}{1-\gamma_n'} T_n$ for each $n \geq 1$, the above algorithm (1.7) can be modifies as follows:

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily,} \\ x_{n+1} = \alpha_n x_n + \beta_n T_n y_n + \gamma_n u_n, \\ y_n = (1-\gamma_n') S_n x_n + \gamma_n' v_n. \end{cases} \tag{1.8}$$

Then notice that the family $\mathcal{S} := \{S_n : C \to C\}$ is also TAN on C with respect to the same $\{c_n\}$, $\{d_n\}$ and ϕ because for each $n \geq 1$,

$$\begin{aligned} \|S_n x - S_n y\| &\leq \frac{\alpha_n'}{1-\gamma_n'}\|x-y\| + \frac{\beta_n'}{1-\gamma_n'}\|T_n x - T_n y\| \\ &\leq \|x-y\| + c_n \phi(\|x-y\|) + d_n \end{aligned} \tag{1.9}$$

for all $x, y \in C$ and $F \subset \cap_{n=1}^{\infty} Fix(S_n)$ in general even if the equality holds for all $\beta_n' > 0$.

In 1994, Rhoades [15] proved that if X is a uniformly convex Banach space, C is a nonempty bounded closed convex subset of X, and $T : C \to C$ is a completely continuous asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $k_n \geq 1$, $\sum_{n=1}^{\infty}(k_n^r - 1) < \infty$, $r \geq 2$, then for any $x_1 \in C$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1,$$

where $\{\alpha_n\}$ satisfy $a \leq \alpha_n \leq 1 - a$ for all $n \geq 1$ and some $a > 0$, converges strongly to some fixed point of T. This result extended the result of Schu [16] to uniformly convex Banach spaces. In 1999, Huang [7] generalized the results due to Rhoades [15] to a more general Ishikawa (and Mann) iteration scheme. In 2001, Kim and Kim generalized the results due to Huang [7] to a more general Ishikawa (and Mann) type scheme for non-Lipschitzian self mapping.

In this paper, we prove that the Ishikawa (and Mann) iteration process (1.8) with errors converges strongly to some common fixed point of $\Im$ under some additional conditions whenever X is a real uniformly convex Banach space and $\Im = \{T_n : C \to C\}$ is a continuous TAN family on C with respect to $\{c_n\}$, $\{d_n\}$ and ϕ and $F := \cap_{n=1}^{\infty} Fix(T_n) \neq \emptyset$.

2 Preliminaries and some lemmas

Throughout this paper we denote by X a real Banach space. Let C be a nonempty closed convex subset of X and let T be a mapping from C into itself. Then we denote by $Fix(T)$ the set of all fixed points of T, namely,

$$Fix(T) = \{x \in C : Tx = x\}.$$

A Banach space X is said to be *uniformly convex* if the modulus of convexity $\delta_X = \delta_X(\epsilon)$, $0 < \epsilon \leq 2$, of X defined by

$$\delta_X(\epsilon) = \inf\{1 - \frac{\|x+y\|}{2} : x, y \in E, \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\}$$

satisfies the inequality $\delta_X(\epsilon) > 0$ for every $\epsilon \in (0, 2]$; see [17] for more details. When $\{x_n\}$ is a sequence in X, then $x_n \to x$ will denote strong convergence of the sequence $\{x_n\}$ to x.

Let T be a single TAN mapping on C with respect to $\{c_n\}$, $\{d_n\}$ and ϕ. At first let us mention the following remarks.

Remark 2.1. Note firstly that the property (1.4) with $c_n = 0$ for all $n \geq 1$ is equivalent to (1.3). Indeed, taking $c_n \equiv 0$ in (1.4) firstly, we have

$$\sup_{x,y \in C} \{\|T^n x - T^n y\| - \|x - y\|\} \leq d_n$$

for each $n \geq 1$, and next taking the lim sup on both sides as $n \to \infty$ immediately gives the property (1.3) because $d_n \to 0$ as $n \to \infty$. Conversely, taking

$$d_n := \max\{0, \sup_{x,y \in C} \{\|T^n x - T^n y\| - \|x - y\|\}\}$$

for each $n \geq 1$, (1.3) immediately implies $d_n \to 0$ as $n \to \infty$; see also [3] for more details. Note also that a mapping of asymptotically nonexpansive in the intermediate sense is non-Lipschitzian; see [9]. Also, if we take $\phi(t) = t$ for all $t \geq 0$ and $d_n = 0$ for all $n \geq 1$ in (1.4), it can be reduced to the *asymptotically nonexpansive* mapping. Furthermore, in addition, taking $c_n = 0$ for all $n \geq 1$, it is *nonexpansive*.

Next, let $\Im = \{T_n : C \to C\}$ be *continuous* be a *continuous* TAN family on C with respect to $\{c_n\}$, $\{d_n\}$ and ϕ. Let us introduce one example given in [11].

Example 2.2. [11] *Let* $X = \mathbb{R}$, $C = [0, \infty)$ *and, for each* $n \geq 1$, *define*

$$T_n x = \Big(1 + \frac{1}{n}\Big) x + \frac{1}{n} \tan^{-1} x, \quad x \in C.$$

Then the family $\Im = \{T_n : C \to C\}$ *is continuous TAN on* C *with respect to* $c_n := \frac{1}{n}$, $d_n := \frac{\pi}{n}$ *and* $\phi(t) = t$. *In fact, use* $|\tan^{-1} x| < \frac{\pi}{2}$ *to get*

$$|T_n x - T_n y| \leq \Big(1 + \frac{1}{n}\Big)|x - y| + \frac{\pi}{n}$$

for all $x, y \in C$ *and* $n \geq 1$.

We first review the following result due to [11].

Theorem 2.3. [11] *Let* X *be a real Banach space,* C *a nonempty closed convex subset of* X. *Let a discrete family* $\Im = \{T_n : C \to C\}$ *be continuous TAN on* C *w.r.t.* $\{c_n\}$, $\{d_n\}$ *and* ϕ *with* $F := \cap_{n=1}^{\infty} Fix(T_n) \neq \emptyset$. *Assume that* $\{c_n\}$, $\{d_n\}$ *and* ϕ *satisfy the following two conditions:*

(C1) $\exists\, \alpha,\ \beta > 0$ *such that* $\phi(t) \leq \alpha t$ *for all* $t \geq \beta$;

(C2) $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$.

Then the sequence $\{x_n\}$ *defined by the explicit iteration method (1.6) converges strongly to a common fixed point of* $\Im$ *if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$, *where* $d(x_n, F) = \inf_{z \in F} \|x_n - z\|$.

It is natural to ask whether Theorem 2.3 still remains true or not for the following algorithm with errors instead of (1.6).

$$x_{n+1} = (1 - \gamma_n) T_n x_n + \gamma_n u_n, \quad n \geq 1. \tag{2.1}$$

Note that taking $\beta'_n \equiv 0$, $\gamma'_n \equiv 0$ and $\alpha'_n \equiv 0$ in (1.8) reduces quickly to (2.1).

For our argument, we need the following two subsequent lemmas.

Lemma 2.4. [14] *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of nonnegative real numbers such that $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} c_n < \infty$ and*

$$a_{n+1} \leq (1+b_n)a_n + c_n$$

for all $n \geq 1$. Then $\lim\limits_{n\to\infty} a_n$ exists.

Lemma 2.5. [6, 17] *Let X be a uniformly convex Banach space. Let $x, y \in X$. If $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x-y\| \geq \epsilon > 0$, then $\|\lambda x+(1-\lambda)y\| \leq 1-2\lambda(1-\lambda)\delta_X(\epsilon)$ for $0 \leq \lambda \leq 1$.*

Lemma 2.6. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and let a discrete family $\Im = \{T_n : C \to C\}$ be TAN on C with respect to $\{c_n\}$, $\{d_n\}$ and ϕ with $F := \cap_{n=1}^{\infty} Fix(T_n) \neq \emptyset$. Suppose also that $\{c_n\}, \{d_n\}$ and ϕ satisfy two conditions (C1) and (C2) in Theorem 2.3. Let the sequence $\{x_n\}$ be defined by (1.8). Then $\lim_{n\to\infty} \|x_n - z\|$ exists for any $z \in F$.*

Proof. Let $y_n := (1-\gamma_n')S_n x_n + \gamma_n' v_n$. Then for any $z \in F$, since $\{u_n\}$ and $\{v_n\}$ are bounded, let

$$M := 1 \vee \phi(\beta) \vee \sup_{n\geq 1} \|u_n - z\| \vee \sup_{n\geq 1} \|v_n - z\| < \infty.$$

From (I) and strict increasing of ϕ, we obtain

$$\phi(t) \leq \phi(\beta) + \alpha t, \quad t \geq 0. \tag{2.2}$$

By using (2.2) and (1.9), we obtain

$$\begin{aligned}
\|y_n - z\| &= \|(1-\gamma_n')S_n x_n + \gamma_n' v_n - z\| \\
&\le \|S_n x_n - z\| + \gamma_n'\|v_n - z\| \\
&\le \|x_n - z\| + c_n\phi(\|x_n - z\|) + d_n + \gamma_n' M \\
&\le \|x_n - z\| + c_n[\phi(\beta) + \alpha\|x_n - z\|] + d_n + \gamma_n' M \\
&\le (1+\alpha c_n)\|x_n - z\| + c_n\phi(\beta) + d_n + \gamma_n' M \\
&\le (1+\alpha c_n)\|x_n - z\| + \eta_n M,
\end{aligned}$$

where $\eta_n = c_n + d_n + \gamma_n'$ and $\sum_{n=1}^{\infty}\eta_n < \infty$. Thus

$$\begin{aligned}
\phi(\|y_n - z\|) &\le \phi(\beta) + \alpha\|y_n - z\| \\
&\le \phi(\beta) + \alpha(1+\alpha c_n)\|x_n - z\| + \alpha\eta_n M \\
&\le \alpha(1+\alpha c_n)\|x_n - z\| + (1+\alpha\eta_n)M,
\end{aligned}$$

and hence

$$\begin{aligned}
&\|T_n y_n - z\| \\
\le\ & \|y_n - z\| + c_n\phi(\|y_n - z\|) + d_n \\
\le\ & (1+\alpha c_n)\|x_n - z\| + \eta_n M + c_n[\alpha(1+\alpha c_n)\|x_n - z\| + (1+\alpha\eta_n)M] + d_n \\
\le\ & (1+2\alpha c_n + \alpha^2 c_n^2)\|x_n - z\| + (\eta_n + c_n + \alpha c_n\eta_n + d_n)M \\
\le\ & (1+\mu_n)\|x_n - z\| + \rho_n M,
\end{aligned}$$

where $\mu_n = 2\alpha c_n + \alpha^2 c_n^2$, $\rho_n = \eta_n + c_n + \alpha c_n\eta_n + d_n$, $\sum_{n=1}^{\infty}\mu_n < \infty$ and

$\sum_{n=1}^{\infty} \rho_n < \infty$. Hence

$$
\begin{aligned}
\|x_{n+1} - z\| &= \|\alpha_n x_n + \beta_n T_n y_n + \gamma_n u_n - z\| \\
&\le \alpha_n \|x_n - z\| + \beta_n \|T_n y_n - z\| + \gamma_n \|u_n - z\| \\
&\le \alpha_n \|x_n - z\| + \beta_n \{(1+\mu_n)\|x_n - z\| + \rho_n M\} + \gamma_n M \\
&= (1-\gamma_n)\|x_n - z\| + \beta_n \mu_n \|x_n - z\| + \beta_n \rho_n M + \gamma_n M \\
&\le (1+\mu_n)\|x_n - z\| + (\rho_n + \gamma_n) M.
\end{aligned}
$$

By Lemma 2.4, we see that $\lim_{n\to\infty} \|x_n - z\|$ exists. □

3 Strong convergence theorems

Now we shall present the following strong convergence for a continuous TAN family as our main result.

Theorem 3.1. *Let X be a uniformly convex Banach space and let C be a nonempty closed convex subset of X and let a discrete family $\Im = \{T_n : C \to C\}$ be continuous TAN on C with respect to $\{c_n\}$, $\{d_n\}$ and ϕ with $F \neq \emptyset$. Suppose that $\{c_n\}, \{d_n\}$ and ϕ satisfy two conditions (C1) and (C2) in Theorem 2.3, and that $0 < a \le \beta_n \le b < 1$, $\limsup\limits_{n\to\infty} \beta'_n \le b < 1$ for all $n \ge 1$ and some $a, b \in (0,1)$. For the sequence defined by (1.8), assume also that there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that*

(C3) $\limsup_{n\to\infty} f(d(x_n, F)) \le \limsup_{k\to\infty} \limsup_{n\to\infty} \|x_n - T_k x_n\|$.

Then $\{x_n\}$ converges strongly to some common fixed point of $\Im$.

Proof. For any $z \in F$, by Lemma 2.6, $\{x_n\}$ is bounded. Since $\{u_n\}$ and $\{v_n\}$

are bounded in C, we set

$$\begin{aligned} W &:= \sup_{n\geq 1}\|u_n - z\| \vee \sup_{n\geq 1}\|x_n - z\| \vee \sup_{n\geq 1}\|v_n - z\| \\ &\quad \vee \sup_{n\geq 1}\|x_n - u_n\| \vee \sup_{n\geq 1}\|x_n - v_n\| \end{aligned}$$

and $M := 1 \vee \phi(\beta) \vee W < \infty$. By Lemma 2.6, we see that $\lim_{n\to\infty}\|x_n - z\|(\equiv r)$ exists. Without loss of generality, we assume $r > 0$. As in the proof of Lemma 2.6, we obtain

$$\begin{aligned} \|T_n y_n - z\| &\leq (1+\mu_n)\|x_n - z\| + \rho_n M \\ &\leq \|x_n - z\| + \mu_n M + \rho_n M \\ &= \|x_n - z\| + \tau_n M, \end{aligned}$$

where $\tau_n := \mu_n + \rho_n$ and $\sum_{n=1}^{\infty}\tau_n < \infty$. Thus

$$\begin{aligned} \|T_n y_n - z + \gamma_n(u_n - x_n)\| &\leq \|T_n y_n - z\| + \gamma_n\|u_n - x_n\| \\ &\leq \|x_n - z\| + \tau_n M + \gamma_n M \\ &= \|x_n - z\| + (\tau_n + \gamma_n)M, \end{aligned}$$

and hence

$$\begin{aligned} \|x_n - z + \gamma_n(u_n - x_n)\| &\leq \|x_n - z\| + \gamma_n\|u_n - x_n\| \\ &\leq \|x_n - z\| + \gamma_n M \\ &\leq \|x_n - z\| + (\tau_n + \gamma_n)M. \end{aligned}$$

Since $x_{n+1} - z = \beta_n[T_n y_n - z + \gamma_n(u_n - x_n)] + (1-\beta_n)[x_n - z + \gamma_n(u_n - x_n)]$ is easily computed, by using Lemma 2.5, we obtain

$$\begin{aligned} &\|x_{n+1} - z\| \\ = \ &\|\beta_n[T_n y_n - z + \gamma_n(u_n - x_n)] + (1-\beta_n)[x_n - z + \gamma_n(u_n - x_n)]\| \\ \leq \ &\Big(\|x_n - z\| + (\tau_n + \gamma_n)M\Big)\Big[1 - 2\beta_n(1-\beta_n)\delta_X\Big(\frac{\|T_n y_n - x_n\|}{\|x_n - z\| + (\tau_n + \gamma_n)M}\Big)\Big]. \end{aligned}$$

Hence we obtain

$$\begin{aligned}&2\beta_n(1-\beta_n)\Big(\|x_n-z\|+(\tau_n+\gamma_n)M\Big)\delta_X\Big(\tfrac{\|T_ny_n-x_n\|}{\|x_n-z\|+(\tau_n+\gamma_n)M}\Big)\\ &\qquad \le \|x_n-z\|-\|x_{n+1}-z\|+(\tau_n+\gamma_n)M.\end{aligned}$$

Since

$$2a(1-b)\sum_{n=1}^{\infty}\Big(\|x_n-z\|+(\tau_n+\gamma_n)M\Big)\delta_X\Big(\frac{\|T_ny_n-x_n\|}{\|x_n-z\|+(\tau_n+\gamma_n)M}\Big)<\infty,$$

and δ_E is strictly increasing and continuous, we obtain

$$\lim_{n\to\infty}\|T_ny_n-x_n\|=0. \tag{3.1}$$

Since $S_n := \frac{\alpha'_n}{1-\gamma'_n}I+\frac{\beta'_n}{1-\gamma'_n}T_n$, we have

$$\begin{aligned}\|x_n-y_n\| &= \|x_n-[(1-\gamma'_n)S_nx_n+\gamma'_nx_n]\|\\ &\le (1-\gamma'_n)\|x_n-S_nx_n\|+\gamma'_n\|x_n-v_n\|\\ &\le \beta'_n\|x_n-T_nx_n\|+\gamma'_nM, \end{aligned}\tag{3.2}$$

we obtain

$$\begin{aligned}\phi(\|x_n-y_n\|) &\le \phi(\beta)+\alpha\|x_n-y_n\|\\ &\le M+\alpha\beta'_n\|T_nx_n-x_n\|+\alpha\gamma'_nM\\ &\le M+\alpha\{\|T_nx_n-z\|+\|z-x_n\|\}+\alpha\gamma'_nM\\ &\le M+\alpha\{\|x_n-z\|+c_n\phi(\|x_n-z\|)+d_n+\|z-x_n\|\}+\alpha\gamma'_nM\\ &\le M+\alpha\{2M+c_n[\phi(\beta)+\alpha\|x_n-z\|]+d_n\}+\alpha\gamma'_nM\\ &\le M+\alpha\{2M+c_nM+\alpha c_nM+d_n\}+\alpha\gamma'_nM\\ &\le M+\alpha(2+c_n+\alpha c_n+d_n+\gamma'_n)M. \end{aligned}\tag{3.3}$$

By using (3.2) and (3.3), we obtain

$$
\begin{aligned}
& \|T_n x_n - x_n\| \\
\leq\ & \|T_n x_n - T_n y_n\| + \|T_n y_n - x_n\| \\
\leq\ & \|x_n - y_n\| + c_n \phi(\|x_n - y_n\|) + d_n + \|T_n y_n - x_n\| \\
\leq\ & \beta'_n \|T_n x_n - x_n\| + \gamma'_n M + c_n[M + \alpha(2 + c_n + \alpha c_n + d_n + \gamma'_n)M] + d_n \\
& + \|T_n y_n - x_n\|,
\end{aligned}
$$

and thus

$$
\begin{aligned}
(1 - \beta'_n)\|T_n x_n - x_n\| \ \leq\ & \gamma'_n M + c_n[M + \alpha(2 + c_n + \alpha c_n + d_n + \gamma'_n)M] \\
& + d_n + \|T_n y_n - x_n\|. \qquad (3.4)
\end{aligned}
$$

Since $\limsup_{n\to\infty} \beta'_n \leq b < 1$, it easily follows from (3.1) and (3.4) that

$$\lim_{n\to\infty} \|T_n x_n - x_n\| = 0. \qquad (3.5)$$

Since

$$
\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n x_n + \beta_n T_n y_n + \gamma_n u_n - x_n\| \\
&\leq \beta_n \|T_n y_n - x_n\| + \gamma_n \|u_n - x_n\| \\
&\leq b\|T_n y_n - x_n\| + \gamma_n M
\end{aligned}
$$

and by (3.1), we get

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Then it is not hard to see that

$$\lim_{n\to\infty} \|x_k - x_n\| = 0 \qquad (3.6)$$

for fixed $k \geq n$. Also, since

$$\begin{aligned}
& \|T_k x_n - x_n\| \\
\leq \ & \|T_k x_n - T_k x_k\| + \|T_k x_k - x_k\| + \|x_k - x_n\| \\
\leq \ & 2\|x_k - x_n\| + c_k \phi(\|x_k - x_n\|) + d_k + \|T_k x_k - x_k\| \\
\leq \ & (2 + \alpha c_k)\|x_k - x_n\| + M c_k + d_k + \|T_k x_k - x_k\|
\end{aligned}$$

for fixed $k \geq n$. Taking the $\limsup$ as $n \to \infty$ on both sides at first, we have

$$\limsup_{n\to\infty} \|T_k x_n - x_n\| \leq M c_k + d_k + \|T_k x_k - x_k\|$$

by virtue of (3.6). Next taking the $\limsup$ as $k \to \infty$, since the right side converges to 0 by (3.5), we obtain

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \|T_k x_n - x_n\| = 0. \tag{3.7}$$

By (III), we have $\lim_{n\to\infty} f(d(x_n, F)) = 0$. On the other hand, as in the proof of Lemma 2.6, we obtain

$$\|x_{n+1} - z\| \leq (1 + \mu_n)\|x_n - z\| + (\rho_n + \gamma_n)M \tag{3.8}$$

for all $z \in F$. Thus

$$\inf_{z\in F} \|x_{n+1} - z\| \leq (1 + \mu_n) \inf_{z\in F} \|x_n - z\| + (\rho_n + \gamma_n)M.$$

By using Lemma 2.4, we see that $\lim\limits_{n\to\infty} d(x_n, F)(\equiv c)$ exists. We first claim that $\lim\limits_{n\to\infty} d(x_n, F) = 0$. In fact, assume that $c = \lim\limits_{n\to\infty} d(x_n, F) > 0$. Then we can choose $n_0 \in N$ such that $0 < \frac{c}{2} < d(x_n, F)$ for all $n \geq n_0$. Then we obtain

$$0 < f(\frac{c}{2}) \leq f(d(x_n, F)) \to 0$$

as $n \to \infty$. This is a contradiction. So, it must be $c = 0$. We can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\|x_{n_i} - w_i\| \le 2^{-i} \tag{3.9}$$

for all $i \ge 1$ and some sequence $\{w_i\}$ in F. Next, we claim that $\{w_i\}$ is a Cauchy sequence by using the idea of [18]. Substituting n_i and w_i for n and z in (3.8), respectively, we obtain

$$\|x_{n_i+1} - w_i\| \le (1 + \mu_{n_i})\|x_{n_i} - w_i\| + (\rho_{n_i} + \gamma_{n_i})M$$

On setting $n_{i+1} := n_i + l_i$ for each $i \ge 1$ and repeating this inequality continuously, we arrive at

$$\begin{aligned}
&\|x_{n_{i+1}} - w_i\| = \|x_{n_i+l_i} - w_i\| \\
\le\ & (1 + \mu_{n_i+l_i-1})\|x_{n_i+l_i-1} - w_i\| + (\rho_{n_i+l_i-1} + \gamma_{n_i+l_i-1})M \\
\le\ & \cdots\cdots \\
\le\ & \prod_{j=1}^{l_i}(1 + \mu_{n_i+l_i-j})\|x_{n_i} - w_i\| + M\sum_{k=1}^{l_i}\prod_{j=1}^{k-1}(1 + \mu_{n_i+l_i-j})(\rho_{n_i+l_i-k} + \gamma_{n_i+l_i-k})
\end{aligned}$$

Since $\sum_{n=1}^{\infty} \mu_n < \infty$, we can set $\prod_{n=1}^{\infty}(1 + \mu_n) := K < \infty$ and so this with (3.9) implies

$$\begin{aligned}
\|x_{n_{i+1}} - w_i\| &\le K\|x_{n_i} - w_i\| + MK\sum_{k=1}^{l_i}(\rho_{n_i+l_i-k} + \gamma_{n_i+l_i-k}) \\
&\le K[2^{-i} + M\sum_{k=1}^{l_i}(\rho_{n_i+l_i-k} + \gamma_{n_i+l_i-k})].
\end{aligned} \tag{3.10}$$

Then it follows from (3.9) and (3.10) that

$$\begin{aligned}
\|w_{i+1} - w_i\| &\le \|w_{i+1} - x_{n_{i+1}}\| + \|x_{n_{i+1}} - w_i\| \\
&\le 2^{-(i+1)} + K[2^{-i} + M\sum_{k=1}^{l_i}(\rho_{n_i+l_i-k} + \gamma_{n_i+l_i-k})].
\end{aligned}$$

This implies that $\{w_i\}$ is a Cauchy sequence. Since F is closed, we obtain $w_i \to w \in F$. We also obtain $x_{n_i} \to w$. By using Lemma 2.6, we obtain $\lim_{n\to\infty} \|x_n - w\| = 0$. $\square$

As a direct consequence, taking $\beta'_n = 0$ and $\gamma'_n = 0$ for all $n \geq 1$ in Theorem 3.1, we immediately have the following result.

Theorem 3.2. *Let X be a uniformly convex Banach space and let C be a nonempty closed convex subset of X and let a discrete family $\Im = \{T_n : C \to C\}$ be continuous TAN on C with respect to $\{c_n\}$, $\{d_n\}$ and ϕ with $F \neq \emptyset$. Suppose that $\{c_n\}, \{d_n\}$ and ϕ satisfy two conditions (C1) and (C2) in Theorem 2.3. For x_1 in C, the sequence $\{x_n\}$ defined by*

$$x_{n+1} = \alpha_n x_n + \beta_n T_n x_n + \gamma_n u_n,$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0,1]$ satisfying $0 < a \leq \beta_n \leq b < 1$, $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$ and some $a, b \in (0,1)$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\{u_n\}$ is bounded sequence in C. Suppose also that there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ satisfying the condition (C3) in Theorem 3.1. Then $\{x_n\}$ converges strongly to some common fixed point of $\Im$.

Finally, we shall give an example of a continuous family $\Im = \{T_n : C \to C, n \geq 1\}$ with $F \neq \emptyset$ which is not Lipschitzian but satisfies the property (C3).

Example 3.3. *Let $X := \mathbb{R}$ and $C := [0,1]$. For each $n \geq 2$, define $T_n : C \to C$ by*

$$T_n x = \begin{cases} \frac{1}{2}, & x \in [0, 1/2]; \\ -\frac{1}{\sqrt{2}}(1 - \frac{2}{n})\sqrt{x - \frac{1}{2}} + \frac{1}{2}, & x \in [1/2, 1]. \end{cases}$$

Obviously, $F(T) = \{1/2\}$. *At first we prove that the property (C3) holds for any sequence* $\{x_n\}$ *in* C. *Indeed, note at first that if* $x_n \in [0, 1/2]$, *then* $T_k x_n = \frac{1}{2}$ *for any* $k \geq 2$ *and so* $d(x_n, F(T)) = |x_n - 1/2| = |x_n - T_k x_n|$ *for* $k \geq 2$. *Next, if* $x_n \in [1/2, 1]$ *then*

$$\begin{aligned} |x_n - T_k x_n| &= x_n - T_k x_n = x_n - 1/2 + \frac{1}{\sqrt{2}}(1 - \frac{2}{k})\sqrt{x_n - \frac{1}{2}} \\ &\geq x_n - 1/2 = |x_n - 1/2| \end{aligned}$$

for $k \geq 2$. *Hence (III) is easily satisfied. Finally, it is not hard to show that* T_n *is non-Lipschitzian on* C.

References

[1] Y. I. Alber, C. E. Chidume and H. Zegeye, *Approximating fixed points of total asymptotically nonexpansive mappings*, Fixed Point Theory and Appl. **2006** (2006) article ID 10673, 20 pages.

[2] R. E. Bruck, T. Kuczumow and S. Reich, *Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property*, Colloq. Math. **65** (1993) 169–179.

[3] C. E. Chidume and E. U. Ofoedu, *Approximation of common fixed points for finite families of total asymptotically nonexpansive mappings*, J. Math. Anal. Appl. **333** (1) (2007), 128–141.

[4] C. E. Chidume and E. U. Ofoedu, *A new iteration process for approximation of common fixed points for finite families of total asymptotically nonexpansive*

mappings, Internat. J. Math. Math. Sci. **2009** (2009), Article ID 615 107, 17p.

[5] K. Goebel and W. A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. **35** (1972), 171–174.

[6] C. W. Groetsch, *A note on segmenting Mann iterates*, J. Math. Anal. Appl. **40** (1972), 369–372.

[7] Z. Huang, *Mann and Ishikawa iterations with errors for asymptotically nonexpansive mappings*, Comp. & Math. Appl. **37** (1999) 1–7.

[8] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. **44** (1974), 147–150.

[9] G. E. Kim and T. H. Kim, *Mann and Ishikawa iterations with errors for non-Lipschitzian mappings in Banach spaces*, Comp. & Math. Appl. **42** (2001) 1565–1570.

[10] G. E. Kim and T. H. Kim, *Strong convergence theorems of total asymptotically nonexpansive mappings*, Preprint.

[11] T. H. Kim and Y. K. Park, Approximation of Common Fixed Points for a Family of Non-Lipschitzian Mappings, *Kyungpook Math. J.*, **49** (2009), 701–712.

[12] W. A. Kirk, Fixed point theorems for non-Lipschitzian mappings of asymptotically nonexpansive type, *Israel J. Math.,* **17** (1974), 339–346.

[13] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.

[14] L. Qihou, *Iterative sequences for asymptotically quasi-nonexpansive mappings with error member*, J. Math. Anal. Appl. **259** (2001), 18–24.

[15] B. E. Rhoades, *Fixed point iterations for certain nonlinear mappings*, J. Math. Anal. Appl. **183** (1994), 118–120.

[16] J. Schu, *Iterative contraction of fixed points of asymptotically nonexpansive mappings*, J. Math. Anal. Appl. **158** (1991), 407–413.

[17] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama-Publishers, 2000.

[18] K. K. Tan and H. K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa Iteration process*, J. Math. Anal. Appl. **178** (1993), 301–308.

[19] Y. Xu, *Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator equations*, J. Math. Anal. Appl. **224** (1998), 91–101.