Thesis for the Degree

Master of Education

Optimal control problems for nonlinear evolution equation of parabolic type with nonlinear perturbations

by

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Pukyong National University

August 2011

Optimal control problems for nonlinear evolution equation of parabolic type with nonlinear perturbations
(방물형의 비선형발전방정식에 대한 최적제어문제)

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A thesis submitted in partial fulfillment of the requirement
for the degree of

Master of Education

Graduate School of Education
Pukyong National University

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CONTENTS

방물형의 비선형 발전방정식에 대한 최적제어문제

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요 약

이 논문은 힐버트 공간상에서 방물형의 비선형 발전방정식의 최적제어문제를 다룬다. 먼저 주어진 방정식에 의해 정의되는 비용함수에 대한 최적제어의 존재성을 밝힌 후, 최적제어의 필요조건과 극대원리를 구하고자 하였다. 본 논문의 주요 결과는 다음과 같다.

첫째로, 와 V 를 힐버트 공간으로 하고 V가 조밀한 공간으로서 그의 공액공간을 V 로 하자. $f: V \to H$가 립쉬츠연속을 만족할 때, 다음과 같이 유계선형연산자 $A: V \subset H \to V$를 포함하는 초기치 문제:

$$x'(t) = Ax(t) + f(x(t)) + Bu(t) + k(t),\ 0 < t \leq T,$$
$$x(0) = x_0.$$

에서, 조건 $x_0, k)$ $H \times L^2(0, T; V^*)$ 이 주어지면, 위의 초기치 문제의 해는 유일하게 존재하며,아울러

$$x \in L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T];\ H).$$ 이고
$$\|x\|_{(0, T; V) \cap W^{1,2}(0, T; V^*)} \leq C_1(|x_0| + \|u\|_{L^2(0, T; Y)} + \|k\|_{L^2(0, T; V^*)}).$$

임을 증명하였다.

둘째로, 제어 u와 그 해 x를 포함하는 비용함수 J에 대하여 허용 가능한 제어집합 U_{ad} 상에서 $\inf_{u \quad U_{ad}} J(u) = J(u)$를 만족하는 최적제어 $\hat{u}$의 존재성을 증명하였다. 아울러 비선형 제어계에 대해서도 비용함수에 포함된 작용소의 조건을 제시하여 최적제어의 유일성을 유도하였고, 최적제어의 특성을 규명할 수 있는 극대원리도 증명하였다.

1 Introduction

Let H and V be two complex Hilbert spaces such that V is a dense subspace of H. Identifying the antidual of H with H we may consider $V \subset H \subset V^*$. Let Y be another Hilbert space and let $U_{ad} \subset L^2(0,T;Y)(T > 0)$ be a admissible control set.

In this paper we study the optimal control problems finding a control $\hat{u} \in U_{ad}$ for a given cost function J governed by the semilinear parabolic type equation in H such that

$$\begin{cases} \inf_{u \in U_{ad}} J(u) = J(\hat{u}) \text{ satisfing} \\ x'(t) = Ax(t) + f(x(t)) + Bu(t) + k(t), \quad 0 < t \le T, \\ x(0) = x_0. \end{cases} \tag{1.1}$$

Here, A is the operator associated with a sesquilinear form defined on $V \times V$ satisfying Gårding's inequality and B is a bounded linear operator from Y to H.

Before considering a standard optimal control problems, namely, the averaging observation cost problem for semilinear systems in Hilbert spaces, we deal with the wellposedness and regularity for the semilinear equation. This approach is close to the methods dealt with in [3, 4, 5] by considered as equation in both H and V^*. It is based on the interpolation space theory and the contraction mapping principle.

The optimal control problems of linear systems have been so extensively studied by [2, 7, 8, 10, 11] and the references cited there. In [11], Papageorgiou gives the existence of the optimal control for a broad class of nonlinear evolution control systems and in [12], the author obtained necessary conditions for optimality using the penalty method first introduced in [2] for optimal control problems governed by nonlinear evolution equations with nonmonotone nonlinearities in the state on condition of the Gateaux differentiability of the nonlinear terms. However, to obtain the necessary condition for optimal control, most studies have been devoted to the systems under the rigorous conditions for the Gateaux derivative of the nonlinear term.

In this paper, our results overcome the limitations of the above works combining techniques for the linear control problems and the properties of solutions of semilinear systems in [6, 12]. Under the bounded condition of the Frechet derivative of the nonlinear term(see [9, 13]), we can obtain the optimal conditions and maximal principles for a given equation.

The paper is organized as follows. In section 2 we study the regularity and a variational of constant formula for solutions of semilinear equations. Thereafter, we prove the existence and the uniqueness of optimal control for the problem (1.1) in Section 3. In the proofs of the main theorems, we need some compactness hypothesis. So we make the natural assumption that the embedding $D(A) \subset V$ is compact. Then by using interpolation theory, we show that the mapping which maps u to the mild solution of (1.1) is a compact operator from $L^2(0, T; Y)$ to $L^2(0, T; H)$(see [1]). Moreover in section 4, we give the maximal principle for a given cost function and present the necessary conditions of optimality which are described by the adjoint state corresponding to the given equation.

2 On solutions of semilinear systems

Let H and V be Hilbert spaces whose norms will be denoted by $|\cdot|$ and $||\cdot||$, respectively. Let A be the operator associated with a sesquilinear form $b(u,v)$ which is defined Gårding's inequality

$$\mathrm{Re}\, b(u,u) \geq c_0||u||^2 - c_1|u|^2, \quad c_0 > 0,\ c_1 \geq 0,\ \text{for } u \in V,$$

that is,

$$(v, Au) = -b(u,v), \quad u,\ v \in V,$$

where $(\cdot, \cdot)$ denotes also the duality pairing between V and V^*.

Then A is a bounded linear operator from V to V^* by the Lax-Milgram theorem and its realization in H which is the restriction of A to

$$D(A) = \{u \in V; Au \in H\}$$

is also denoted by A. Here, we note that $D(A)$ is dense in V. Hence, it is also dense in H. We endow the domain $D(A)$ of A with graph norm, that is, for $u \in D(A)$, we define $||u||_{D(A)} = |u| + |Au|$. So, for the brevity, we may regard that $|u| \leq ||u|| \leq ||u||_{D(A)}$ for all $u \in V$. It is known that A generates an analytic semigroup $S(t)(t \geq 0)$ in both H and V^*(see [14]).

From the following inequalities

$$c_0||u||^2 \leq \mathrm{Re}\, b(u,u) + c_1|u|^2 \leq C|Au|\,|u| + c_1|u|^2$$

$$\leq (C|Au| + c_1|u|)|u| \leq \max\{C, c_1\}||u||_{D(A)}|u|,$$

it follows that there exists a constant $C_0 > 0$ such that

$$||u|| \leq C_0 ||u||_{D(A)}^{1/2} |u|^{1/2}. \tag{2.1}$$

Thus we have the following sequence

$$D(A) \subset V \subset H \subset V^* \subset D(A)^* \tag{2.2}$$

where each space is dense in the next one which continuous injection.

Lemma 2.1 *With the notations (2.1), (2.2), we have*

$$(V, V^*)_{1/2,2} = H,$$

$$(D(A), H)_{1/2,2} = V.$$

If X is a Banach space and $1 < p < \infty$, $L^p(0,T;X)$ is the collection of all strongly measurable functions from $(0,T)$ into X the p-th powers whose norms are integrable and $W^{m,p}(0,T;X)$ is the set of all functions f whose derivatives $D^\alpha f$ up to degree m in the distribution sense belong to $L^p(0,T;X)$. $C^m([0,T];X)$ is the set of all m-times continuously differentiable functions from $[0,T]$ into X. Let X and Y be complex Banach spaces. Denote by $\mathcal{L}(X,Y)$ the set of all bounded linear operators from X and Y. Let $\mathcal{L}(X) = \mathcal{L}(X,X)$.

First, consider the following initial value problem for the abstract linear parabolic equation

$$\begin{cases} x'(t) = Ax(t) + k(t), \quad 0 < t \leq T, \\ x(0) = x_0. \end{cases} \tag{LE}$$

By virtue of Lemma 2.1 and Theorem 3.3 of [3](or Theorem 3.1 of [5]), we have the following result on the corresponding linear equation of (LE).

Lemma 2.2 *Suppose that the assumptions for the principal operator A stated above are satisfied. Then the following properties hold:*
1) For $x_0 \in V$ and $k \in L^2(0,T;H)$, $T > 0$, there exists a unique solution x of (LE) belonging to

$$L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset C([0,T];V)$$

and satisfying

$$||x||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} \le C_1(||x_0|| + ||k||_{L^2(0,T;H)}), \tag{2.3}$$

where C_1 is a constant depending on T.
2) Let $x_0 \in H$ and $k \in L^2(0,T;V^*)$, $T > 0$. Then there exists a unique solution x of (LE) belonging to

$$L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H)$$

and satisfying

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \le C_1(|x_0| + ||k||_{L^2(0,T;V^*)}), \tag{2.4}$$

where C_1 is a constant depending on T.

From now on , we deal with the following semilinear control equation:

$$\begin{cases} x'(t) = Ax(t) + f(x(t)) + Bu(t) + k(t), \quad 0 < t \le T, \\ x(0) = x_0. \end{cases} \tag{SE}$$

The mild solution $x(t) = x(t; f, u)$ is represented by

$$x(t) = S(t)x_0 + \int_0^t S(t-s)\{f(x(s)) + Bu(s) + k(s)\}ds, \quad t \geq 0. \tag{2.5}$$

The control space will be modeled by a Banach space Y. Let the controller B is a bounded linear operator from Y to H.

We will need the following hypotheses on the data of problem (SE).

(A) The embedding $D(A) \subset V$ is compact.

(F) $f : V \longrightarrow H$ be a nonlinear mapping such that

$$|f(x) - f(y)| \leq L||x - y||.$$

for a positive constant L.

It is clear that under the condition (F) $f : V \to H$ is C^1 with Fréchet derivative $f'(\cdot)$ such that

$$||f'(x)||_{\mathcal{L}(V,H)} \leq L, \quad x \in V.$$

For $x \in V$ we set

$$F(x) = \int_0^1 f'(rx)dr, \quad x \in V.$$

Then using the assumption (F) we see the following properties(see [9]):

$$f(x) = F(x)x + f(0),$$

$$||F(x(t))||_{\mathcal{L}(V,H)} \leq L, \quad x \in C([0,T]; V),$$

$$F(\cdot) \in C(V, \mathcal{L}(V, H)).$$

Therefore, the problem (SE) can be rewritten as

$$x(t) = S(t)x_0 + \int_0^t S(t-s)[F(x(s))x(s) + Bu(s) + f(0) + k(s)]ds,$$

or by perturbations of semigroup theory,

$$x(t) = Q(t;F)x_0 + \int_0^t Q(t-s;F)[Bu(s) + f(0) + k(s)]ds,$$

where

$$Q(t-s;F)y = S(t-s)y + \int_s^t S(t-r)F(x(r))Q(r-s;F)ydr, \quad 0 \le s \le t \le T.$$

Lemma 2.3 *Let* $f \in L^2(0,T;H)$ *and* $x(t) = \int_0^t S(t-s)f(s)ds$ *for* $0 \le t \le T$. *Then there exists a constant* C *such that for* $0 \le t \le T$

$$||x||_{L^2(0,T;D(A))} \le C||f||_{L^2(0,T;H)}, \tag{2.6}$$

$$||x||_{L^2(0,T;H)} \le CT||f||_{L^2(0,T;H)}, \tag{2.7}$$

and

$$||x||_{L^2(0,T;V)} \le C_0 C\sqrt{T}||f||_{L^2(0,T;H)}. \tag{2.8}$$

Proof. The assertion (2.6) is immediately obtained by (2.3). Since

$$||x||^2_{L^2(0,T;H)} = \int_0^T |\int_0^t S(t-s)f(s)ds|^2 dt$$

$$\leq C\int_0^T (\int_0^t |f(s)|ds)^2 dt$$

$$\leq C\int_0^T t\int_0^t |f(s)|^2 dsdt$$

$$\leq C\frac{T^2}{2}\int_0^T |f(s)|^2 ds$$

it follows that

$$||x||_{L^2(0,T;H)} \leq CT||f||_{L^2(0,T;H)}.$$

From (2.1), (2.6), and (2.7) it holds that

$$||x||_{L^2(0,T;V)} \leq C_0 C\sqrt{T}||f||_{L^2(0,T;H)}.$$

□

Theorem 2.1 *Under the assumption (F) for the nonlinear mapping f, for each $k \in L^2(0,T;V^*)$ there exists a unique solution x of (SE) such that*

$$x \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H)$$

for any $x_0 \in H$. Moreover, there exists a constant C_1 such that

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \leq C_1(|x_0| + ||u||_{L^2(0,T;Y)} + ||k||_{L^2(0,T;V^*)}). \tag{2.9}$$

Proof. Let us fix $T_0 > 0$ satisfying

$$C_0 C L \sqrt{T}_0 < 1 \tag{2.10}$$

with constants C, C_0 in Lemma 2.3. For any fixed $x \in L^2(0, T_0; V)$, let y be the solution of

$$\begin{cases} y'(t) = Ay(t) + f(x(t)) + Bu(t) + k(t), \quad 0 < t \le T, \\ y(0) = x_0. \end{cases} \tag{2.11}$$

Put

$$J(x)(t) = S(t)x_0 + \int_0^t S(t-s)\{f(x(s)) + Bu(s) + k(s)\}ds.$$

We are going to show that $J(x) = y$ is strictly contractive from $L^2(0, T_0; V)$ to itself if the condition (2.10) is satisfied. From assumption (F), (2.8) and

$$J(x_1)(t) - J(x_2)(t) = \int_0^t S(t-s)\{f(x_1(s)) - f(x_2(s))\}ds$$

we have

$$\begin{aligned} ||J(x_1) - J(x_2)||_{L^2(0,T_0;V)} &\le C_0 C \sqrt{T}_0 ||f(x_1(\cdot)) - f(x_2(\cdot))||_{L^2(0,T_0;H)} \\ &\le C_0 C L \sqrt{T}_0 ||x_1(\cdot) - x_2(\cdot)||_{L^2(0,T_0;V)}. \end{aligned}$$

So by virtue of the condition (2.10) the contraction mapping principle gives that the solution of (SE) exists uniquely in $[0, T_0]$. Let x be a solution of (SE) and $x_0 \in H$. Then there exists a constant C such that

$$||S(t)x_0||_{L^2(0,T_0;V)} \le C|x_0| \tag{2.12}$$

in view of Lemma 2.2. Let

$$y(t) = \int_0^t S(t-s)\{f(x(s)) + Bu(s) + k(s)\}ds.$$

Then from (2.8), it follows

$$||y||_{L^2(0,T_0;V)} \le C_0C\sqrt{T}_0||f(x(\cdot)) + Bu + k||_{L^2(0,T_0;H)} \tag{2.13}$$

$$\le C_0C\sqrt{T}_0(L||x||_{L^2(0,T_0;V)} + ||f(\cdot,0) + Bu + k||_{L^2(0,T_0;H)}).$$

Thus, combining (2.12) with (2.13) we have

$$||x||_{L^2(0,T_0;V)} \le (1 - C_0CL\sqrt{T}_0)^{-1}C(|x_0| \tag{2.14}$$

$$+ C_0\sqrt{T}_0||f(\cdot,0) + Bu + k||_{L^2(0,T_0;H)}).$$

Now from

$$|x(T_0)| = |S(T_0)x_0 + \int_0^{T_0} S(T_0 - s)\{f(s,x(s)) + Bu(s) + k(s)\}ds|$$

$$\le M|x_0| + ML\sqrt{T_0}||x||_{L^2(0,T_0;V)} + M\sqrt{T_0}||f(0) + Bu + k||_{L^2(0,T_0;H)}$$

we can solve the equation in $[T_0, 2T_0]$ with the initial value $x(T_0)$ and obtain an analogous estimate to (2.14). Since the condition (2.10) is independent of initial values, the solution can be extended to the interval $[0, nT_0]$ for any natural number n, and so the proof is complete. □

Let x_u be the solution of (SE) corresponding to $u \in L^2(0,T;Y)$. We define the nonlinear operator $\mathcal{F}$ from $L^2(0,T;Y)$ to $L^2(0,T;H)$ defined by

$$(\mathcal{F}u)(t) = f(x_u(t)), \quad u \in L^2(0,T;Y). \tag{2.15}$$

Theorem 2.2 *Let Assumptions (A) and (F) hold. Then the nonlinear operator $\mathcal{F}$ from $L^2(0,T;Y)$ to $L^2(0,T;H)$ defined by (2.15) is compact.*

Proof. If $u \in L^2(0,T;Y)$ we have $f(x_u) \in L^2(0,T;H)$, and so $x_u \in L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)$ by (2.6)(cf. Theorem 3.2 in [3]),

$$||x_u||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} \tag{2.16}$$

$$\le C\{||x_0|| + ||f(x_u) + Bu + k||_{L^2(0,T;H)}\}$$

$$\le C\{L||x_u||_{L^2(0,T;V)} + ||x_0|| + ||Bu + f(0) + k||_{L^2(0,T;H)}\}$$

$$\le CLC_1\{||x_0|| + ||Bu + k||_{L^2(0,T;H)}\} + C\{||x_0|| + ||Bu + f(\cdot,0) + k||_{L^2(0,T;H)}\}.$$

By virtue of Theorem 2 in [1], we know that the embedding

$$L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset L^2(0,T;V)$$

is compact since the embedding $D(A) \subset V$ is compact. If u belongs to a bounded set of $L^2(0,T;Y)$, then from (2.16) it follows that x_u is also bounded in $L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)$, and hence it is relatively compact in $L^2(0,T;V)$. Thus, the mapping $u \mapsto x_u$ is a compact operator from $L^2(0,T;Y)$ to $L^2(0,T;V)$. Noting that

$$||(\mathcal{F}u)||_{L^2(0,T;H)} \le L||x_u||_{L^2(0,T;V)} \le L||x_u||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)}$$

we have that $\mathcal{F}$ is a compact operator from $L^2(0,T;Y)$ to $L^2(0,T;H)$. □

3 Optimal control for semilinear equations

Let Z be a real Hilbert space and let $C(t)$ be bounded from H to Z for each t and be continuous in $t \in [0,T]$. Let $y \in L^2(0,T;Z)$. Suppose that there exists no admissible control which satisfies $C(t)x(t;f,u) = y(t)$ for almost all t. Choose a bounded subset U of Y and call it a control set. Let us define an admissible control U_{ad} as

$$U_{ad} = \{u \in L^2(0,T;Y) : u \text{ is strongly measurable function satisfying}$$
$$u(t) \in U \text{ for almost all } t\}$$

and let $x(t;f,u)$ be a solution of (SE) associated with the nonlinear term f and a control u at time t. The solution $x(t;f,u)$ of (SE) for each admissible control u is called a trajectory corresponding to u. Then, as in section 2, it is represented by

$$x(t;f,u) = Q(t;F)x_0 + \int_0^t Q(t-s;F)\{Bu(s) + f(0) + k(s)\}ds, \tag{3.1}$$

So we consider the following cost functional as a averaging observation control given by

$$J(u) = \frac{1}{2}\int_0^T |C(t)x(t;f,u) - y(t)|^2 dt + \int_0^T (Nu(t), u(t))dt, \tag{3.2}$$

where N is a self adjoint and positive definite:

$$N \in \mathcal{L}(Z), \quad \text{and} \quad (Nu,u) \geq c_0||u||^2, \quad c_0 > 0. \tag{3.3}$$

We study the control problems finding a control $\hat{u} \in L^1(0,T;U)$ such that

$$\begin{cases} \inf_{u \in U_{ad}} J(u) = J(\hat{u}) \ \text{ satisfing} \\ x'(t) = Ax(t) + f(x(t)) + Bu(t) + k(t), \quad 0 < t \le T, \\ x(0) = x_0. \end{cases}$$

Let $u \in L^1(0,T;Y)$. Then it is well known that

$$\lim_{h \to 0} h^{-1} \int_0^h ||u(t+s) - u(t)||_Y ds = 0 \tag{3.4}$$

for almost all point of $t \in (0,T)$.

Definition 3.1 *The point t which permits (3.4) to hold is called the Lebesgue point of u.*

Theorem 3.1 *Let U be a bounded closed convex subset of Y. Then, there exists an optimal control for the cost functional (3.1).*

Proof. Let $\{u_n\}$ be a minimizing sequence of J such that

$$\inf_{u \in U_{ad}} J(u) = \lim_{n \to \infty} J(u_n).$$

Since U is bounded and weakly closed, there exists a subsequence, which we write again by $\{u_n\}$, of $\{u_n\}$ and a $\hat{u} \in U$ such that

$$u_n \to \hat{u} \quad \text{weakly in} \ \ L^2(0,T;Y).$$

Now we show that $\hat{u}$ is admissible as follows. Since U is a closed convex set of Y, by Mazur theorem as an important consequence of the Hahn-Banach theorem, there exists an $f_0 \in Y^*$ and $c \in (-\infty, \infty)$ be such that $f_0(u) \le c$ for all $u \in U$. Let s be a Lebesgue point of $\hat{u}$ and put

$$w_{\epsilon,n} = \frac{1}{\epsilon} \int_s^{s+\epsilon} u_n(t)dt$$

for each $\epsilon > 0$ and n. Then, $f_0(w_{\epsilon,n}) \le c$ and we have

$$w_{\epsilon,n\rightarrow} w_\epsilon = \frac{1}{\epsilon} \int_s^{s+\epsilon} \hat{u}(t)dt \quad \text{weakly as } n \to \infty.$$

By letting $\epsilon \to 0$, it holds that $w_\epsilon \to \hat{u}(s)$ and $f_0(\hat{u}) \le c$, so that $\hat{u}(s) \in U$. From Theorem 2.1 it follows that $\{x(t; f, u_n)\}$ is also bounded and hence weakly sequentially compact. Hence, as seen in the prove of Theorem 2.1, we have

$$x(t; f, u_n) \to x(t; f, \hat{u}) \quad \text{weakly in} \quad H.$$

Therefore, we have

$$\inf J(u) \le J(\hat{u}) \le \liminf J(u_n) = \inf J(u).$$

Thus, this $\hat{u}$ is an optimal control. □

For the sake of simplicity we assume that $Q(t; F)$ is uniformly bounded: then

$$|Q(t; F)| \le M (t \ge 0)$$

for some $M > 0$(e.g. [11, 14]).

Lemma 3.1 *(Lemma 7.2.1 of [14]) If $x \in L^1(0,T;H)$ and*

$$\int_0^t Q(t-s;F)x(s)ds = 0, \quad 0 \le t \le T,$$

then $x(t) = 0$ for almost all $t \in [0,T]$.

The optimality condition J is often used to derive the uniqueness of optimal control. So, we give the conditions for the uniqueness of optimal control as follows.

Theorem 3.2 *Let $\mathcal{B}$ defined by $(\mathcal{B}u)(\cdot) = Bu(\cdot)$. Let $\mathcal{B}$ and $C(t)(t \ge 0)$ be one to one mappings. Then the optimal control for the cost function (3.2) is unique.*

Proof. Let $\hat{u}$ be an optimal control in terms of Theorem 3.1 and $v \in U$. Let t_0 be a Lebesgue point of $\hat{u}, v$ and $\mathcal{F}(v-\hat{u})$. For $t_0 \le t_0+\epsilon < T$, put

$$u(t) = \begin{cases} v & \text{if } t_0 \le t < t_0+\epsilon \\ \hat{u}(t) & \text{otherwise.} \end{cases} \tag{3.5}$$

Then u is an admissible control. Since $x(t;f,u) - x(t;f,\hat{u}) = 0$ for $0 \le t \le t_0$ and by (3.1),

$$x(t;f,u) - x(t;f,\hat{u}) = \begin{cases} 0, & 0 \le t \le t_0, \\ \int_{t_0}^t Q(t-s;F)B(v-\hat{u}(s))ds, & t_0 < t < t_0+\epsilon, \\ \int_{t_0}^{t_0+\epsilon} Q(t-s;F)B(v-\hat{u}(s))ds, & t_0+\epsilon \le t \le T. \end{cases} \tag{3.6}$$

Noting that $v-\hat{u}$ is admissible and t_0 is Lebesgue point of $v-\hat{u}$, there exists a constant $c>0$ such that

$$||v-\hat{u}(t)||_Y \le c, \quad \text{for } \ t_0 \le t \le t_0+\epsilon.$$

Thus, we obtain

$$|x(t;f,u)-x(t;f,\hat{u})| \le \epsilon c M||B||. \tag{3.7}$$

Since $\hat{u}$ is optimal, we have

$$\begin{aligned}
0 &\le \frac{1}{\epsilon}(J(u)-J(\hat{u})) \qquad (3.8)\\
&= \frac{1}{2\epsilon}[C(t)(x(t;f,u)-y(t), C(t)x(t;f,u)-y(t))\\
&\quad -(C(t)x(t;f,\hat{u})-y(t), C(t)x(t;f,\hat{u})-y(t))]\\
&\quad +\frac{1}{\epsilon}\int_0^T \{(Nu(t),u(t))-(N\hat{u}(t),\hat{u}(t))\}dt\\
&= \frac{1}{\epsilon}\int_0^T (C(t)(x(t;f,u)-x(t;f,\hat{u})), C(t)x(t;f,\hat{u})-y(t))dt\\
&\quad +\frac{1}{2\epsilon}\int_0^T |C(t)(x(t;f,u)-x(t;f,\hat{u}))|^2 dt\\
&\quad +\frac{1}{\epsilon}\int_0^T \{(Nu(t),u(t))-(N\hat{u}(t),\hat{u}(t))\}dt\\
&= I+II+III.
\end{aligned}$$

From (3.7) it follows that

$$\lim_{\epsilon\to 0} II = 0. \tag{3.9}$$

The first term of (3.8) can be represented as

$$I = \frac{1}{\epsilon}\int_{t_0}^{T}(C(t)(x(t;f,u) - x(t;f,\hat{u})), C(t)x(t;f,\hat{u}) - y(t))dt$$

$$= \frac{1}{\epsilon}\int_{t_0}^{t_0+\epsilon} + \frac{1}{\epsilon}\int_{t_0+\epsilon}^{T} = I_1 + I_2.$$

On account of (3.7), it holds that

$$\lim_{\epsilon\to 0} I_1 = 0. \tag{3.10}$$

Let $t > t_0$ and $\epsilon \to 0$. Then, we obtain

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}(x(t;f,u) - x(t;f,\hat{u})) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{t_0}^{t_0+\epsilon} Q(t-s;F)\mathcal{B}(v-\hat{u})(s)ds$$

$$= Q(t-t_0;F)\mathcal{B}(v-\hat{u})(t_0).$$

Hence,

$$\lim_{\epsilon\to 0} I_2 = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{t_0+\epsilon}^{T}(C(t)(x(t;f,u) - x(t;f,\hat{u})), C(t)x(t;f,\hat{u}) - y(t))dt$$

$$= \int_{t_0}^{T}(C(t)Q(t-t_0;F)\mathcal{B}(v-\hat{u})(t_0), C(t)x(t;f,\hat{u}) - y(t))dt. \tag{3.11}$$

By (3.8)-(3.11), the inequality

$$\int_0^T (C(t)Q(t-s;F)\mathcal{B}(v-\hat{u})(s), C(t)x(t;f,\hat{u}) - y(t))dt$$

$$+\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^T\{(Nu(t),u(t)) - (N\hat{u}(t),\hat{u}(t))\}dt \geq 0$$

holds for every $v \in U$. Let us denote two optimal controls by u_1 and u_2 and their corresponding by x_1 and x_2. Then, by the similar procedure mentioned above, the inequalities

$$\int_0^T (C(t)Q(t-s;F)\mathcal{B}(u_2-u_1)(s), C(t)x_1(t)-y(t))dt \geq 0$$

and

$$\int_0^T (C(t)Q(t-s;F)\mathcal{B}(u_1-u_2)(s), C(t)x_2(t)-y(t))dt \geq 0$$

hold. Adding both inequalities we have

$$\int_0^T (C(t)Q(t-s;F)\mathcal{B}(u_2-u_1)(s), C(t)(x_2(t)-x_1(t)) \leq 0.$$

Noting that

$$x_2(t)-x_1(t) = \int_0^t Q(t-s;F)\mathcal{B}(u_2-u_1)(s)ds,$$

integrating the resultant inequality from 0 to T with respect to s, it holds

$$\int_0^T |C(t)(x_2(t)-x_1(t))|^2 \leq 0.$$

Since $C(t)$ is one to one, we have that $x_2(t)-x_1(t) \equiv 0$. Hence, by the property of semigroup $Q(t;F)$ in Lemma 3.1, it holds that $\mathcal{B}(u_1-u_2)(t)=0$ almost everywhere. From that $\mathcal{B}$ is one to one, $u_1(t)=u_2(t)$ holds for almost all t. □

4 Optimal conditions

In order to derive necessary optimality conditions for the optimal control for J, we will establish the maximum principle, which is derived from the optimal condition as follows.

Theorem 4.1 *Let the admissible set U_{ad} be a closed convex subset of $L^2(0,T;Y)$. Let $\hat{u}$ be an optimal control. Then the integral inequality*

$$\int_0^T (-\Lambda_Y^{-1} B_0^* z(s) + N\hat{u}(s), v(s) - \hat{u}(s))ds \geq 0$$

holds, where z(t) is a solution of the following transposed system:

$$\begin{cases} z'(s) = & -A^* z(s) - F^*(z(s))z(s) + C^*(s)(y(s) - C(s)\hat{y}(s)), \\ z(T) = & 0 \end{cases} \tag{AS}$$

in the weak sense. Here, the operator Λ_Y (resp. Λ_Z) is the canonical isomorphism of Y (resp. Z) onto Y^* (resp. Z^*).

Proof. Let $x(t) = x(t; g, 0)$ and let $x_v(t)$ stand for solution of (SE) associated with the control $v \in L^2(0,T;Y)$. Then it holds that

$$\begin{aligned} J(v) &= \int_0^T ||C(t)x_v(t) - y(t)||^2 dt + \int_0^T (Nv(t), v(t))dt \\ &= \int_0^T ||C(t)(x_v(t) - x(t)) + C(t)x(t) - y(t)||^2 dt + \int_0^T (Nv(t), v(t))dt \\ &= \pi(v,v) - 2L(v) + \int_0^T ||y(t) - Cx(t)||^2 dt \end{aligned}$$

where

$$\pi(u,v) = \int_0^T (C(t)(x_u(t) - x(t)), C(t)(x_v(t) - x(t)))dt$$

$$+ \int_0^T (Nu(t), v(t))dt$$

$$L(v) = \int_0^T (y(t) - C(t)x(t), C(t)(x_v(t) - x(t)))dt.$$

The form $\pi(u,v)$ is a continuous bilinear form in $L^2(0,T;Y)$ and from assumption of the positive definite of the operator N we have

$$\pi(v,v) \geq c||v||^2 \quad v \in L^2(0,T;Y).$$

Therefore in virtue of Theorem 1.1 of Chapter 1 in [8](or see Remark 1.5 amd Remark 1.6 in case where the admissible set $U_{ad} = L^2(0,T;Y$ and U_{ad}=Cone with vertex at the origin, respectively) there exists a unique $u \in U_{ad}$ such that (1.1). If u is an optimal control (cf. Theorem 1.4. Chapter 1 in [8]), then

$$J'(\hat{u})(v - \hat{u}) \geq 0 \quad v \in U_{ad}, \tag{4.1}$$

where $J'(\hat{u})v$ means the Fréchet derivative of J at $\hat{u}$, applied to v. It is easily seen that

$$x'_{\hat{u}}(t)(v - \hat{u}) = (v - \hat{u}, x'_{\hat{u}}(t))$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon}\{x(t; f, \hat{u} + \epsilon(v - \hat{u})) - x(t; f, \hat{u})\}$$

$$= x_v(t) - x_{\hat{u}}(t).$$

Therefore, (3.12) is equivalent to

$$\int_0^T (C(t)x_{\hat{u}}(t) - y(t), C(t)(x_v(t) - x_{\hat{u}}(t)))dt + \int_0^T (N\hat{u}, v - \hat{u})dt =$$

$$\int_0^T (C^*(t)\Lambda_Z(C(t)x_{\hat{u}}(t) - y(t), x_v(t) - x_{\hat{u}}(t))dt + \int_0^T (N\hat{u}, v - \hat{u})dt \geq 0.$$

Note that $C^*(t) \in B(Z^*, H^*)$ and for ϕ and ψ in H we have $(C^*(t)\Lambda_Z C(t)\psi, \phi)$ $= (C(t)\psi, C(t)\phi)$ where duality pairing is also denoted by $(\cdot,\cdot)$. From Fubini's theorem and

$$x_v(t) - x_{\hat{u}}(t) = \int_0^t Q(t-s;F)B(v(s) - \hat{u}(s))ds$$

we have

$$\int_0^T \int_0^t (\Lambda_Y^{-1} B^* Q^*(t-s;F)C^*(t)\Lambda_Z(C(t)x_{\hat{u}}(t) - y(t)) + N\hat{u}(s),$$

$$v(s) - \hat{u}(s))ds\, dt$$

$$= \int_0^T (\int_s^T (\Lambda_Y^{-1} B^* Q^*(t-s;F)C^*(t)\Lambda_Z(C(t)x_{\hat{u}}(t) - y(t)))dt + N\hat{u}(s),$$

$$v(s) - \hat{u}(s))ds$$

$$= \int_0^T (-\Lambda_Y{}^{-1} B^* z(s) + N\hat{u}(s), v(s) - \hat{u}(s)))ds \geq 0$$

where $z(s)$ is given by (AS), that is, $z(s)$ is following form:

$$z(s) = -\int_s^T Q^*(t-s;F)C^*(t)\Lambda_Z(C(t)x_{\hat{u}}(t) - y(t))dt.$$

□

Remark 4.1 *Identifying the antidual Y with Y (and also in case Z) we need not use the canonical isomorphism Λ_Y. But in case where $Y \subset V^*$ this leads to difficulties since H has already been identified with its dual.*

Corollary 4.1 *(Maximal principle) Let U_{ad} be bounded and $N = 0$. If u be an optimal solution for J then*

$$\max_{v \in U_{ad}} (v, \Lambda_Y^{-1} B^* z(s)) = (u, \Lambda_Y^{-1} B^* z(s))$$

where $z(s)$ is given by in Theorem 4.1.

Proof. We note that if U_{ad} is bounded then the set of elements $u \in U_{ad}$ such that (3.1) is a nonempty, closed and convex set in U_{ad}. Let t be a Lebesque point of u, $v \in U_{ad}$ and $t < t + \epsilon < T$. Further, put

$$v_\epsilon(s) = \begin{cases} v, & \text{if} \quad t < s < t + \epsilon \\ u(s), & \text{otherwise.} \end{cases}$$

Then Substituting v_ϵ for v in (3.12) and dividing the resulting inequality by ϵ, we obtain

$$\frac{1}{\epsilon}\int_t^{t+\epsilon} (-\Lambda_Y^{-1} B^* z(s), v(s) - \hat{u}(s))ds \geq 0.$$

Thus by letting $\epsilon \to 0$, the proof is complete. □

Theorem 4.2 *(Bang-Bang Principle) Let U_{ad} be bounded and $N = 0$. Let B_0^* and $C(\cdot)$ be one to one mappings. If there is not the control u such that $C(t)x_u(t) = z_d(t)$ a.e, then the optimal control $u(t)$ is a bang-bang control, i.e, $u(t)$ satisfies $u(t) \in \partial U_{ad}$ for almost all t where ∂U_{ad} denotes the boundary of U_{ad}.*

Proof. On account of Corollary 4.1 it is enough to show that $\Lambda_U^{-1} B_0^*(t)y(t) \neq 0$ for almost all t. If $B_0^*(t)y(t) = 0$, then since

$$y(s) = -\int_s^T Q^*(t-s;F)C^*(t)\Lambda_Z(C(t)x_u(t) - y(t))dt,$$

it follows that

$$C(t)x_u(t) - y(t) = 0 \quad a.e..$$

It is a contraction. □

Remark 4.2 *From (3.8) and (3.11), it is directly obtained that*

$$\int_s^T (C(t)Q(t-s;F)B(v-\hat{u})(s), C(t)x(t;f,\hat{u}) - y(t))dt \geq 0$$

holds for every $v \in U$ and for all Lebesgue points s of $\hat{u}$. Hence, we have

$$(v - \hat{u}(s), \Lambda_Y^{-1} B^* z(s)) \leq 0$$

where

$$z(s) = -\int_s^T Q^*(t-s)C^*(t)\Lambda_Z(C(t)x(t;f,\hat{u}) - y(t))dt.$$

Here, $z(s)$ is a solution of the equation (AS) in some sense Theorem 4.1.

$\square$

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