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Robustness of Mann's algorithm for Non-Lipschitzian Mappings

by

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Robustness of Mann's Algorithm for Non-Lipschitzian Mappings
(비-Lipschitz 사상에 대한 Mann 알고리즘의 확고성)

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비-Lipschitz 사상에 대한 Mann 알고리즘의 확고성

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요 약

본 논문에서는 먼저 Banach 공간의 공집합이 아닌 닫힌 볼록집합 C상에 정의된 전 점근적 비확대 (total asymptotically nonexpansive (in brief,TAN)) 라 불리우는 어떤 비-Lipschitz 사상을 소개한다. 더욱, 연속인 TAN사상의 반닫힘 원리를 이용하여 Mann 알고리즘의 확고성을 밝힌다.

[정리]

X는 균등볼록인 Banach공간이고 Kadec-Klee 성질 또는 Opial 성질을 만족한다고 하자. 사상 $T: X \to X$ 가 Fix(T)$\neq \phi$와 다음 두 조건 (C1)과 (C2)를 만족한다고 가정하자.

(C1) 모든 $t \geq \beta$에 대하여 $\phi(t) \leq \alpha_0 t$를 만족하는 $\alpha_0, \beta > 0$가 존재한다.

(C2) $\sum_{n=1}^{\infty} c_n < \infty$이고 $\sum_{n=1}^{\infty} d_n < \infty$이다.

초기치 $x_0 \in X$ 에서 시작하여 섭동된 Mann 알고리즘,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(T^n x_n + e_n), \quad n \geq 0$$

에 의하여 정의된 수열 $\{x_n\}$이 매개변수들의 수열 $\{\alpha_n\}$과 오차 항$\{e_n\}$의 적당한 조건하에서 사상 T의 어떤 부동점에 약수렴한다.

1 Introduction

Let X be a real Banach space with norm $\|\cdot\|$ and let X^* be the dual of X. Let $\{x_n\}$ be a sequence in X, $x \in X$. We denote by $x_n \to x$ the strong convergence of $\{x_n\}$ to x and by $x_n \rightharpoonup x$ the weak convergence of $\{x_n\}$ to x. Also, we denote by $\omega_w(x_n)$ the weak ω-limit set of $\{x_n\}$, that is,

$$\omega_w(x_n) = \{x : \exists\, x_{n_k} \rightharpoonup x\}.$$

Let C be a nonempty closed convex subset of X and let $T : C \to C$ be a mapping. Now let $Fix(T)$ be the fixed point set of T; namely,

$$Fix(T) := \{x \in C : Tx = x\}.$$

Recall that T is a *Lipschitzian* mapping if, for each $n \geq 1$, there exists a constant $k_n > 0$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \tag{1.1}$$

for all $x, y \in C$ (we may assume that all $k_n \geq 1$). A Lipschitzian mapping T is called *uniformly k-Lipschitzian* if $k_n = k$ for all $n \geq 1$, *nonexpansive* if $k_n = 1$ for all $n \geq 1$, and *asymptotically nonexpansive* if $\lim_{n\to\infty} k_n = 1$, respectively. The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [7] as a generalization of the class of nonexpansive mappings. They proved that if C is a nonempty bounded closed convex subset of a uniformly convex Banach space X, then every asymptotically nonexpanisve mapping $T : C \to C$ has a fixed point.

On the other hand, as the classes of non-Lipschitzian mappings, there appear in the literature two definitions, one is due to Kirk who says that T is a mapping of *asymptotically nonexpansive type* [14] if for each $x \in C$,

$$\limsup_{n\to\infty} \sup_{y\in C}(\|T^n x - T^n y\| - \|x-y\|) \le 0 \tag{1.2}$$

and T^N is continuous for some $N \ge 1$. The other is the stronger concept due to Bruck, Kuczumov and Reich [2]. They say that T is *asymptotically nonexpansive in the intermediate sense* if T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x-y\|) \le 0 \tag{1.3}$$

In this case, observe that if we define

$$\delta_n := \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x-y\|) \vee 0, \tag{1.4}$$

(here $a \vee b := \max\{a,b\}$), then $\delta_n \ge 0$ for all $n \ge 1$, $\delta_n \to 0$ as $n \to \infty$, and thus (1.3) immediately reduces to

$$\|T^n x - T^n y\| \le \|x-y\| + \delta_n \tag{1.5}$$

for all x, $y \in C$ and $n \ge 1$.

Recently, Alber et al. [1] introduced the wider class of total asymptotically nonexpansive mappings to unify various definitions of classes of nonlinear mappings associated with the class of asymptotically nonexpansive mappings; see also Definition 1 of [3]. They say that a mapping $T: C \to C$ is *total asymptotically nonexpansive* (TAN, in brief) [1] (or [3]) if there exist two nonnegative real sequences $\{c_n\}$ and $\{d_n\}$ with c_n, $d_n \to 0$ and $\phi \in \Gamma(\mathbb{R}^+)$ such that

$$\|T^n x - T^n y\| \le \|x-y\| + c_n \phi(\|x-y\|) + d_n, \tag{1.6}$$

for all x, $y \in K$ and $n \ge 1$, where $\mathbb{R}^+ := [0,\infty)$ and

$\phi \in \Gamma(\mathbb{R}^+) \Leftrightarrow \phi$ is strictly increasing, continuous on $\mathbb{R}^+$ and $\phi(0) = 0$.

Remark 1.1. If $\varphi(t) = t$, then (1.6) reduces to

$$\|T^n x - T^n y\| \leq \|x - y\| + c_n \|x - y\| + d_n$$

for all x, $y \in C$ and $n \geq 1$. In addition, if $d_n = 0$, $k_n = 1 + c_n$ for all $n \geq 1$, then the class of total asymptotically nonexpansive mappings coincides with the class of asymptotically nonexpansive mappings. If $c_n = 0$ and $d_n = 0$ for all $n \geq 1$, then (1.6) reduces to the class of nonexpansive mappings. Also, if we take $c_n = 0$ and $d_n = \delta_n$ as in (1.4), then (1.6) reduces to (1.5); see [3] for more details.

Let C be a nonempty closed convex subset of a real Banach space X, and let $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Recall that the following Mann [15] iterative method is extensively used for solving a fixed point equation of the form $Tx = x$:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad n \geq 0, \tag{1.7}$$

where $\{a_n\}$ is a sequence in $[0, 1]$ and $x_0 \in C$ is arbitrarily chosen. In infinite-dimensional spaces, Mann's algorithm has generally only weak convergence. In fact, it is known [18] that if the sequence $\{\alpha_n\}$ is such that $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then Mann's algorithm (1.7) converges weakly to a fixed point of T provided the underlying space is a Hilbert space or more general, a uniformly convex Banach space which has a Fréchet differentiable norm or satisfies Opial's property. Furthermore, Mann's algorithm (1.7) also converges weakly to a fixed point of T if X is a uniformly convex Banach space such that its dual X^* enjoys the *Kadec-Klee property* (*KK-property*, in brief), i.e., $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\| \;\Rightarrow\; x_n \to x$. It is well known [5] that the duals of reflexive Banach spaces with a Frechet differentiable norms have the KK-property. There exist uniformly convex spaces which have neither a Fréchet differentiable norm nor the Opial property but their duals do have the KK-property; see Example 3.1 of [6].

What is the robustness of Mann's algorithm? It means that convergence of Mann's algorithm (1.7) is still stable (or robust) for approximately small perturbations. The study of the robustness of Mann's algorithm is initiated by Combettes [4] where he considered a parallel projection method (PPM, in brief) algorithm in signal synthesis (design and recovery) problems in a real Hilbert space H as follows:

$$x_{n+1} = x_n + \lambda_n \left(\sum_{i=1}^{m} w_i (P_i x_n + c_{i,n}) - x_n \right), \tag{1.8}$$

where for each i, $P_i(x)$ is the (nearest point) projection of a signal $x \in H$ onto a closed convex subset S_i of H [4] (S_i is interpreted as the i-th constraint set of the signals), $\{\lambda_n\}_{n\geq 0}$ is a sequence of relaxation parameters in $(0,2)$, $\{w_i\}_{i=1}^m$ are strictly positive weights such that $\sum_{i=1}^m w_i = 1$, and $c_{i,n}$ stands for the error made in computing the projection onto S_i at iteration n. Then he proved the following robustness result of the PPM algorithm (1.8).

Theorem 1.2. ([4]) *Assume $G := \cap_{i=1}^m S_i \neq \emptyset$. Assume also*

(i) $\sum_{n=0}^\infty \lambda_n (2-\lambda_n) = \infty$ *and*

(ii) $\sum_{n=0}^\infty \lambda_n \|\sum_{i=1}^m w_i\, c_{i,n}\| < \infty$.

Then the sequence $\{x_n\}$ generated by the PPM algorithm (1.8) converges weakly to a point in G.

It is well known [11] that Theorem 1.2 (Combettes' robustness) can be easily reformulated as follows.

Theorem 1.3. *Let H be a real Hilbert space, $T : H \to H$ a nonexpansive mapping with $Fix(T) \neq \emptyset$, and let $\{x_n\}$ be generated by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(Tx_n + e_n), \quad n \geq 0, \tag{1.9}$$

starting from an initial guess $x_0 \in H$. Assume also that two sequences $\{\alpha_n\}$ in $(0,1)$ and $\{e_n\}$ in H satisfy the following properties:

(i)′ $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$ *and*

(ii)′ $\sum_{n=0}^{\infty} \alpha_n \|e_n\| < \infty$.

Then the sequence $\{x_n\}$ converges weakly to a fixed point of T.

In 2007, Kim and Xu [11] established that Theorem 1.3 still remains true in the framework of a special Banach space, namely, a uniformly convex Banach space X whose either its dual X^* has the KK-property or X enjoys Opial's property.

Let C be a nonempty closed convex subset of a real Banach space X. In this paper, using the demiclosedness principle of continuous TAN mappings, we also establish robustness of the following perturbed Mann's algorithm

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(T^n x_n + e_n), \quad n \geq 0 \tag{1.10}$$

for a continuous TAN self-mapping T in a uniformly convex Banach space X which either X^* has the Kadec Klee property or X satisfies Opial property, under suitable conditions of parameters $\{\alpha_n\}_{n=0}^{\infty}$ and errors $\{e_n\}_{n=0}^{\infty}$.

2 Preliminaries

Here we summarize the notations used in the sequel. The convex hull of a subset A of a real Banach space X is denoted by $co\,A$, and the closed convex hull by

$\overline{co}\, A$. We put

$$\triangle^{n-1} = \{\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) : \lambda_i \geq 0 \, (i = 1, 2, \cdots, n) \text{ and } \sum_{i=1}^{n} \lambda_i = 1\}$$

and for $r > 0$

$$B_r = \{x \in X : \|x\| \leq r\}.$$

We start with the following recent result for continuous TAN mappings, called the *demiclosedness principle*.

Proposition 2.1. ([9]) *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Let $T : C \to C$ be a continuous TAN mapping. Then $I - T$ is demiclosed at zero in the sense that whenever $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \, (\in C)$ and it satisfies*

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \|x_n - T^k x_n\| = 0.$$

Then $x \in F(T)$.

We first need an inequality characterizing the uniform convexity in real Banach spaces.

Lemma 2.2. ([19]) *Given a number $r > 0$. A real Banach space X is uniformly convex if and only if there exists a continuous strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$, $\varphi(0) = 0$, such that*

$$\|\lambda x + (1 - \lambda) y\|^2 \leq \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\varphi(\|x - y\|)$$

for all $\lambda \in [0, 1]$ and $x, y \in B_r$.

Lemma 2.3. ([21]) *Let* $\{a_n\}$, $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1+\alpha_n)a_n + \beta_n$$

for all $n \geq 1$. *Suppose that* $\sum_{n=1}^{\infty} \alpha_n < \infty$ *and* $\sum_{n=1}^{\infty} \beta_n < \infty$. *Then* $\lim_{n\to\infty} a_n$ *exists. Moreover, if in addition,* $\liminf_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Assume unless other specified that X is a real Banach space, and that $T : X \to X$ is a continuous TAN mapping as in (1.6), equipped with $F(T) \neq \emptyset$. Assume also that the following additional conditions (C1) and (C2) hold:

(C1) $\exists\, \alpha_0,\ \beta > 0$ such that $\phi(t) \leq \alpha_0 t$ for all $t \geq \beta$.

(C2) $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} d_n < \infty$.

Consider a family $\mathcal{S} = \{S_n : X \to X,\ n \geq 0\}$ defined by

$$S_n = (1-\alpha_n)I + \alpha_n T^n, \quad n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. It is not hard to see that

$$Fix(T) \subset Fix(\mathcal{S}) := \cap_{n=1}^{\infty} Fix(S_n)$$

and the converse inclusion holds for all $\alpha_n \neq 0$. We now discuss the weak convergence of the sequence $\{x_n\}$ in X defined recurrently by

$$x_{n+1} = S_n x_n + u_n, \quad n \geq 1, \tag{2.1}$$

starting from an initial guess $x_1 \in X$, where $\{u_n\}$ is a sequence in X such that

(C3) $\sum_{n=0}^{\infty} \|u_n\| < \infty$.

We first discuss some useful properties of the algorithm (2.1).

Lemma 2.4. *Let $\{x_n\}$ be generated by the algorithm (2.1) and let $p \in Fix(\mathcal{S})$. Then $\lim_{n\to\infty} \|x_n - p\|$ exists. Furthermore, $\lim_{n\to\infty} d(x_n, Fix(\mathcal{S}))$ exists, where $d(x, A)$ denotes the distance from x to the set A.*

Proof. Note firstly that, since ϕ is strictly increasing on $\mathbb{R}^+$, $\phi(t) \leq \phi(\beta)$ whenever $t \leq \beta$ and condition (C1) also gives $\phi(t) \leq \alpha_0 t$ for $t \geq \beta$. In either case, we have

$$\phi(\|x_n - p\|) \leq \phi(\beta) + \alpha_0 \|x_n - p\|.$$

This implies that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|S_n x_n + u_n - p\| \\
&\leq \|S_n x_n - p\| + \|u_n\| \\
&\leq \|x_n - p\| + c_n \phi(\|x_n - p\|) + d_n + \|u_n\| \\
&\leq (1 + \alpha_0 c_n)\|x_n - p\| + \phi(\beta) c_n + d_n + \|u_n\|.
\end{aligned}$$

On viewing the hypotheses (C2) and (C3), Lemma 2.3 is applicable with $\alpha_n = \alpha_0 c_n$ and $\beta_n = \phi(\beta) c_n + d_n + \|u_n\|$ and so $\lim_{n\to\infty} \|x_n - p\|$ exists. Obviously, $\lim_{n\to\infty} d\big(x_n, Fix(T)\big)$ exists because $p \in Fix(T)$ is arbitrarily given. $\square$

Now on mimicking Lemma 2.2 and 2.3 in [16] we have the following result. For more detailed proof, see Lemma 2.2. of [13].

Lemma 2.5. ([13]) *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Let a family $\mathcal{S} = \{S_n : C \to C\}$ be such that*

$$\|S_n x - S_n y\| \leq \|x - y\| + c_n \phi(\|x - y\|) + d_n, \quad x, y \in C.$$

and that $Fix(\mathcal{S}) \neq \emptyset$. Let K be a bounded closed convex subset of C containing x^* for some $x^* \in Fix(\mathcal{S})$. Then, for $\epsilon > 0$ there exists an integers

$N_\epsilon \geq 1$ *and* δ_ϵ *with* $0 < \delta_\epsilon \leq \epsilon$ *such that* $k \geq N_\epsilon$, $x_1, x_2, \cdots, x_n \in K$ *and if* $\|x_i - x_j\| - \|S_k x_i - S_k x_j\| \leq \delta_\epsilon$ *for* $1 \leq i, j \leq n$, *then*

$$\left\| S_k \left(\sum_{i=1}^{n} \lambda_i x_i \right) - \sum_{i=1}^{n} \lambda_i S_k x_i \right\| < \epsilon$$

for all $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \triangle^{n-1}$.

Lemma 2.6. *Let* X *be uniformly convex. Then* $\lim_{n\to\infty} \|\lambda_1 x_n + \lambda_2 p - q\|$ *exists for all* p, $q \in Fix(\mathcal{S})$ *and* $\lambda = (\lambda_1, \lambda_2) \in \triangle^1$.

Proof. For integers $n, m \geq 1$ and all $x \in X$, define the mappings U_n and $S_{n,m}$ by

$$U_n x = S_n x + u_n$$

and

$$S_{n,m} x = U_{n+m-1} U_{n+m-2} \cdots U_n x.$$

Obviously, $x_{n+m} = S_{n,m} x_n$. From Lemma 2.4, it suffices to show that the conclusion remains true for $0 < \lambda_1, \lambda_2 < 1$. By Lemma 2.4, since the sequence $\{x_n\}$ is bounded, let K be a bounded closed convex subset of X containing p, q and $\{x_n\}$ for applying for Lemma 2.5 again. Put $D := \sup_{x,y\in K} \phi(\|x - y\|) < \infty$. Then, notice that

$$\|U_n x - U_n y\| = \|S_n x - S_n y\| \leq \|x - y\| + Dc_n + d_n$$

and

$$\begin{aligned} \|S_{n,m} x - S_{n,m} y\| &\leq \|x - y\| + D \sum_{i=n}^{n+m-1} c_i + \sum_{i=n}^{n+m-1} d_i \\ &\leq \|x - y\| + \tilde{d}_n \qquad (2.2) \end{aligned}$$

for all $x, y \in K$, where $\tilde{d}_n := D\sum_{i=n}^{\infty} c_i + \sum_{i=n}^{\infty} d_i \ \to\ 0$. Similarly, we can prove that the conclusion of Lemma 2.5 still remains true with $S_{n,m}$ instead of T^n, by mimicking the processes of the proof of Lemma 2.5 with $S_{n,m}$ instead of T^n. Given $\epsilon > 0$, take $N_\epsilon \geq 1$ and $\delta_{2,\epsilon}$ with $0 < \delta_{2,\epsilon} \leq \epsilon$ as in Lemma 2.5. Set

$$a_n = \|\lambda_1 x_n + \lambda_2 p - q\|$$

and

$$b_{n,m} = \|S_{n,m}(\lambda_1 x_n + \lambda_2 p) - (\lambda_1 x_{n+m} + \lambda_2 p)\|.$$

Noticing that, for all n, $m \geq 1$,

$$\begin{aligned}
\|S_{n,m}z - z\| &= \|U_{n+m-1}U_{n+m-2}\cdots U_n z - U_{n+m-1}z\| + \|U_{n+m-1}z - z\| \\
&\leq \|U_{n+m-2}\cdots U_n z - z\| + Dc_{n+m-1} + d_{n+m-1} + \|u_{n+m-1}\| \\
&\ \vdots \\
&\leq D\sum_{i=n+1}^{n+m-1} c_i + \sum_{i=n+1}^{n+m-1} d_i + \sum_{i=n}^{n+m-1} \|u_i\| \\
&\leq D\sum_{i=n+1}^{\infty} c_i + \sum_{i=n+1}^{\infty} d_i + \sum_{i=n}^{\infty} \|u_i\| \ := g_n \qquad (2.3)
\end{aligned}$$

for either $z = p$ or $z = q$ and $g_n \to 0$ as $n \to \infty$, we deduce that

$$\begin{aligned}
&\|x_n - p\| - \|S_{n,m}x_n - S_{n,m}p\| \\
\leq\ &\|x_n - p\| - \|x_{n+m} - p\| + \|S_{n,m}p - p\| \ \to\ 0 \quad \text{as } n \to \infty
\end{aligned}$$

and we can choose $n_0 \geq N_\epsilon$ such that

$$\|x_n - p\| - \|S_{n,m}x_n - S_{n,m}p\| \leq \delta_{2,\epsilon}$$

for all $n \geq n_0$ and $m \geq 1$. As a direct consequence of Lemma 2.5, we obtain

$$\|S_{n,m}(\lambda_1 x_n + \lambda_2 p) - (\lambda_1 S_{n,m}x_n + \lambda_2 S_{n,m}p)\| < \epsilon$$

for $\lambda = (\lambda_1, \lambda_2) \in \triangle^1$, $n \geq n_0$ and $m \geq 1$. Thus we have

$$\begin{aligned} b_{n,m} &\leq \|S_{n,m}(\lambda_1 x_n + \lambda_2 p) - (\lambda_1 S_{n,m} x_n + \lambda_2 S_{n,m} p)\| \\ &\quad + \lambda_2 \|S_{n,m} p - p\| \leq \epsilon + g_n \end{aligned}$$

for all $n \geq n_0$ and $m \geq 1$. This, since $\epsilon > 0$ is arbitrarily given and $g_n \to 0$, implies that

$$\lim_{n\to\infty} \sup_{m\geq 1} b_{n,m} = 0. \tag{2.4}$$

On the other hand, by the help of (2.2) and (2.3), we have

$$\begin{aligned} a_{n+m} &= \|\lambda_1 x_{n+m} + \lambda_2 p - q\| \\ &\leq b_{n,m} + \|S_{n,m}(\lambda_1 x_n + \lambda_2 p) - S_{n,m} q\| + \|S_{n,m} q - q\| \\ &\leq b_{n,m} + a_n + \tilde{d}_n + g_n \end{aligned}$$

for all $n, m \geq 1$. Taking the $\limsup$ as $m \to \infty$ at first and next the $\liminf$ as $n \to \infty$, this together with (2.4) and $\tilde{d}_n$, $g_n \to 0$ yields that $\lim_{n\to\infty} a_n$ exists. $\square$

The following lemma is also very useful to establish the robustness of Mann's algorithm.

Lemma 2.7. (see Lemma 3.2 of [6]) *Let X be a uniformly convex Banach space such that its dual X^* has the KK-property. Suppose that $\{x_n\}$ is a bounded sequence such that $\lim_{n\to\infty} \|\alpha x_n + (1-\alpha)p - q\|$ exists for all $\alpha \in [0,1]$ and $p, q \in \omega_w(x_n)$. Then $\omega_w(x_n)$ is a singleton.*

As applying for Lemma 2.6 combined with Lemma 2.7, we immediately have the following result.

Proposition 2.8. *Let X be a uniformly convex Banach space such that its dual X^* has the KK-property. If $\omega_w(x_n) \subset Fix(\mathcal{S})$, then $\omega_w(x_n)$ is a singleton.*

3 Robustness of Mann's algorithm

In this section we now present the robustness result of Mann's algorithm for continuous TAN mappings.

Theorem 3.1. *Assume X is a uniformly convex Banach space. Assume, in addition, that either X^* has the KK-property or X satisfies Opial property. Let $T : X \to X$ be a continuous TAN mapping satisfying (C1) and (C2) together with $Fix(T) \neq \emptyset$ and $\sum_{n=1}^{\infty} \|T^{n+1} - T^n\| < \infty$. Given an initial guess $x_0 \in X$, let $\{x_n\}$ be generated by the perturbed Mann's algorithm (1.10). Assume that $\{\alpha_n\}_{n=0}^{\infty} \subset (0,1)$ and $\{e_n\}_{n=0}^{\infty} \subset X$ satisfy control conditions (i)$'$ and (ii)$'$ in Theorem 1.3. Then the sequence $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Put $S_n := (1-\alpha_n)I + \alpha_n T^n$ for each $n \geq 0$ and $u_n := \alpha_n e_n \in X$. Since $Fix(T) \subset Fix(\mathcal{S}) := \cap_{n=1}^{\infty} Fix(S_n)$, it is clear that $Fix(\mathcal{S}) \neq \emptyset$. Then the algorithm (1.10) reduces to (2.1) and (C3) is clearly fulfilled by (ii)$'$. For applying the demiclosedness at 0 of $I - T$ in the sense of Proposition 2.1, we first show the following equation holds, that is,

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \|x_n - T^k x_n\| = 0.$$

Indeed, fix a $p \in Fix(T)$ and select a number $r > 0$ big enough so that $\|x_n - p\| \leq r$ for all n. Since ϕ is a strictly increasing function on $\mathbb{R}^+$, $\phi(\|x_n - p\|) \leq \phi(r)$ for all $n \geq 1$. Since $T : X \to X$ is TAN,

$$\begin{aligned} \|T^n x_n - p\| &\leq \|x_n - p\| + c_n \phi(\|x_n - p\|) + d_n \\ &\leq \|x_n - p\| + \phi(r) c_n + d_n \end{aligned} \tag{3.1}$$

for all $n \geq 1$. Now let $M > 0$ satisfy $M > 2(r + \phi(r)c_n + d_n) + \alpha_n \|e_n\|$ for all

$n \geq 1$. By Lemma 2.2 and (3.1), we have

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1-\alpha_n)(x_n - p) + \alpha_n(T^n x_n - p) + \alpha_n e_n\|^2 \\
&\leq \|(1-\alpha_n)(x_n - p) + \alpha_n(T^n x_n - p)\|^2 \\
&\quad +2\alpha_n\|e_n\|\|(1-\alpha_n)(x_n - p) + \alpha_n(T^n x_n - p)\| + \alpha_n^2\|e_n\|^2 \\
&\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|T^n x_n - p\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\varphi(\|x_n - T^n x_n\|) + M\alpha_n\|e_n\| \\
&\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\big(\|x_n - p\| + \phi(r)c_n + d_n\big)^2 \\
&\quad -\alpha_n(1-\alpha_n)\varphi(\|x_n - T^n x_n\|) + M\alpha_n\|e_n\| \\
&\leq \|x_n - p\|^2 - \alpha_n(1-\alpha_n)\varphi(\|x_n - T^n x_n\|) + \tilde{\beta}_n,
\end{aligned}
$$

where $\tilde{\beta}_n := \big[\big(r + \phi(r)c_n + d_n\big)^2 - r^2\big] + M\alpha_n\|e_n\|$. It follows that

$$\alpha_n(1-\alpha_n)\varphi(\|x_n - T^n x_n\|) \leq \|x_{n+1} - p\|^2 - \|x_n - p\|^2 + \tilde{\beta}_n.$$

Since $\sum_{n=1}^{\infty} \tilde{\beta}_n < \infty$ by (C2) and (ii)$'$, this implies that

$$\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n)\varphi(\|x_n - T^n x_n\|) < \infty.$$

Due to condition (i)$'$, we must have that $\liminf_{n\to\infty} \varphi(\|x_n - T^n x_n\|) = 0$. Hence

$$\liminf_{n\to\infty} \|x_n - T^n x_n\| = 0. \tag{3.2}$$

However, since

$$T^n x_{n+1} - x_{n+1} = (T^n x_{n+1} - T^n x_n) + (1-\alpha_n)(T^n x_n - x_n) - \alpha_n e_n$$

and $x_{n+1} - x_n = \alpha_n(T^n x_n - x_n) + \alpha_n e_n$, we have

$$\begin{aligned} \|T^n x_{n+1} - x_{n+1}\| &\leq \|T^n x_{n+1} - T^n x_n\| + (1-\alpha_n)\|T^n x_n - x_n\| + \alpha_n\|e_n\| \\ &\leq \|x_{n+1} - x_n\| + c_n\phi(\|x_{n+1} - x_n\|) + d_n \\ &\quad +(1-\alpha_n)\|T^n x_n - x_n\| + \alpha_n\|e_n\| \\ &\leq \|T^n x_n - x_n\| + Cc_n + d_n + 2\alpha_n\|e_n\|, \end{aligned}$$

where $C := \phi(2\sup_{n\geq 0}\|x_n\|) < \infty$. Since $\{x_n\}$ is bounded, there exists $L > 0$ such that $\|x_n\| \leq L$ for all n, and so we have

$$\begin{aligned} &\|T^{n+1}x_{n+1} - x_{n+1}\| \\ \leq\ &\|T^{n+1}x_{n+1} - T^n x_{n+1}\| + \|T^n x_{n+1} - x_{n+1}\| \\ \leq\ &L\|T^{n+1} - T^n\| + \|T^n x_n - x_n\| + Cc_n + d_n + 2\alpha_n\|e_n\|. \end{aligned}$$

Since $\sum_{n=1}^{\infty}\|T^{n+1}-T^n\| < \infty$, by Lemma 2.3 with $\beta_n := L\|T^{n+1}-T^n\|+Cc_n+d_n+2\alpha_n\|e_n\|$, $\lim_{n\to\infty}\|T^n x_n - x_n\|$ exists and hence, by (3.2)

$$\lim_{n\to\infty}\|T^n x_n - x_n\| = 0. \tag{3.3}$$

By virtue of (3.3), since

$$\|x_{n+1} - x_n\| \leq \alpha_n\|T^n x_n - x_n\| + \alpha_n\|e_n\| \to 0, \tag{3.4}$$

it immediately follows that

$$\|x_{n-k} - x_n\| \to 0 \tag{3.5}$$

for any fixed positive integer k. Thus we claim that

$$\lim_{n\to\infty}\|T^{n-k}x_n - x_n\| = 0 \tag{3.6}$$

for any fixed $k \geq 1$. Indeed, since $\{x_n\}$ is bounded, using (3.5) and $c_n, d_n \to 0$, we have

$$\|T^{n-k}x_n - T^{n-k}x_{n-k}\| \leq \|x_n - x_{n-k}\| + c_{n-k}\phi(\|x_n - x_{n-k}\|) + d_{n-k} \to 0$$

as $n \to \infty$ and hence, this fact combined with (3.3) and (3.5) also gives

$$\begin{aligned} \|T^{n-k}x_n - x_n\| &\leq \|T^{n-k}x_n - T^{n-k}x_{n-k}\| + \|T^{n-k}x_{n-k} - x_{n-k}\| \\ &\quad + \|x_{n-k} - x_n\| \to 0 \end{aligned}$$

as $n \to \infty$ for any fixed $k \geq 1$. Hence (3.6) is required. Using (3.6) and properties of ϕ, it is not hard to claim that

$$\limsup_{n\to\infty} \|x_n - T^n x_n\| \leq d_k \tag{3.7}$$

for any fixed $k \geq 1$ because

$$\begin{aligned} \|x_n - T^k x_n\| &\leq \|x_n - T^n x_n\| + \|T^k T^{n-k} x_n - T^k x_n\| \\ &\leq \|x_n - T^n x_n\| + \|T^{n-k}x_n - x_n\| + c_k\phi(\|T^{n-k}x_n - x_n\|) + d_k \end{aligned}$$

for sufficiently large $n \geq k$. Thus we obtain

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \|x_n - T^k x_n\| = 0.$$

Hence, by Proposition 2.1, we have

$$\omega_w(x_n) \subset Fix(T).$$

Now to prove that $\{x_n\}$ weakly converges to a fixed point of T, it suffices to show that $\omega_w(x_n)$ is a singleton. First, if X^* has KK-property, it is clear from Proposition 2.8.

Next assume that X satisfies Opial's property. Take subsequences $\{x_{n_i}\}$ and $\{x_{m_j}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p_1$ and $x_{m_j} \rightharpoonup p_2$, respectively. If $p_1 \neq p_2$, we reach the following contradiction:

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - p_1\| &= \lim_{j\to\infty} \|x_{n_i} - p_1\| \\
&< \lim_{i\to\infty} \|x_{n_i} - p_2\| = \lim_{j\to\infty} \|x_{n_j} - p_2\| \\
&< \lim_{i\to\infty} \|x_{n_j} - p_1\| \\
&= \lim_{i\to\infty} \|x_n - p_1\|.
\end{aligned}$$

In any case, we have shown that $\omega_w(x_n)$ is a singleton set and hence the proof is complete. $\square$

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