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Subordination and superordination for meromorphic functions associated with the multiplier transformation

by

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Graduate School of Education

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Subordination and superordination for meromorphic functions associated with the multiplier transformation
(승수변환과 관련된 유리형함수들의 종속과 초종속)

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승수변환과 관련된 유리형함수들의 종속과 초종속

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요 약

Miller Mocanu와 Reade[10]는 미분종속 원리를 이용하여, 개단위원 상에서 정의된 해석함수들에 대한 비선형 적분연산자들에 대한 종속보존 성질들을 조사하였다. 또한, Miller Mocanu[9]는 미분 종속의 쌍대 개념으로서 미분 초종속 개념을 소개하였다.

본 논문에서는 유리형 함수들의 승수변환 $D(n,\lambda)$를 소개하고, 이 승수변환에 대한 종속보존 성질들과 초종속보존 성질들을 조사하였다. 또한, 종속 및 초종속 보존 성질들을 결합한 sandwich 형태의 새로운 결과를 연구하였다.

1. Introduction

Let $\mathcal{H} = \mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N} = \{1, 2, \cdots\}$, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}.$$

Let f and F be members of $\mathcal{H}$. The function f is said to be subordinate to F, or F is said to be superordinate to f, if there exists a function w analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = F(w(z))$. In such a case, we write $f \prec F$ or $f(z) \prec F(z)$. If the function F is univalent in $\mathbb{U}$, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$ (cf. [9,14]).

Definition 1.1 [8]. Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the differential subordination

$$\phi(p(z), zp'(z)) \prec h(z), \tag{1.1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant if $p \prec q$ for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.1) is said to be the best dominant.

Definition 1.2 [9]. Let $\varphi : \mathbb{C}^2 \to \mathbb{C}$ and let h be analytic in $\mathbb{U}$. If p and $\varphi(p(z), zp'(z))$ are univalent in $\mathbb{U}$ and satisfy the differential superordination

$$h(z) \prec \varphi(p(z), zp'(z)), \tag{1.2}$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant if $q \prec p$ for all p satisfying (1.2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (1.2) is said to be the best subordinant.

Definition 1.3 [9]. We denote by $\mathcal{Q}$ the class of functions f that are analytic and injective on $\overline{\mathbb{U}}\backslash E(f)$, where

$$E(f) = \left\{ \zeta \in \partial\mathbb{U} : \lim_{z\to\zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U}\backslash E(f)$.

Let $\mathcal{M}$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k$$

which are analytic in the punctured open unit disk $\mathbb{D} = \{z \in \mathbb{C} : 0 < |z| < 1\}$. For any $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we denote the multiplier transformations $D(n, \lambda)$ of functions $f \in \mathcal{M}$ by

$$D(n, \lambda)f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \left(\frac{k+1+\lambda}{\lambda} \right)^n a_k z^k \quad (\lambda > 0;\; z \in \mathbb{U}). \tag{1.3}$$

Obviously, we have

$$D(s, \lambda)(D(t, \lambda)f(z)) = D(s+t, \lambda)f(z)$$

for all nonnegative integers s and t. The operators $D(n, \lambda)$ and $D(n, 1)$ are the multiplier transformations introduced and studied by Sarangi and Uraligaddi [13] and Uralegaddi and Somanatha [15,16], respectively. It is easily verified from (1.3) that

$$z(D(n,\lambda)f(z))' = \lambda D(n+1,\lambda)f(z) - (\lambda+1)D(n,\lambda)f(z). \tag{1.4}$$

By using of the principle of subordination, Miller et al. [10] obtained some subordination theorems involving certain integral operators for analytic functions in $\mathbb{U}$. Also Owa and Srivastava [11] investigated the subordination properties of certain integral operators (see also [1]). Moreover, Miller and Mocanu [9] considered differential superordinations, as the dual problem of differential subordinations (see also [2]). In the present paper, we investigate the subordination and superordination preserving properties of the multiplier transformation $D(n,\lambda)$ defined by (1.3) with the sandwich-type theorems.

The following lemmas will be required in our present investigation.

Lemma 1.1 [6]. *Suppose that the function $H : \mathbb{C}^2 \to \mathbb{C}$ satisfies the condition:*

$$\mathrm{Re}\{H(is,t)\} \leq 0,$$

for all real s and $t \leq -n(1+s^2)/2$, where n is a positive integer. If the function $p(z) = 1 + p_n z^n + \cdots$ is analytic in $\mathbb{U}$ and

$$\mathrm{Re}\{H(p(z), zp'(z))\} > 0 \quad (z \in \mathbb{U}),$$

then $\mathrm{Re}\{p(z)\} > 0$ *in* $\mathbb{U}$.

Lemma 1.2 [7]. *Let $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and let $h \in \mathcal{H}(\mathbb{U})$ with $h(0) = c$. If* $\mathrm{Re}\{\beta h(z) + \gamma\} > 0$ *for $z \in \mathbb{U}$, then the solution of the differential equation:*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z) \quad (z \in \mathbb{U})$$

with $q(0) = c$ *is analytic in* $\mathbb{U}$ *and satisfies* $\mathrm{Re}\{\beta q(z) + \gamma\} > 0$ *for* $z \in \mathbb{U}$.

Lemma 1.3 [8]. *Let* $p \in \mathcal{Q}$ *with* $p(0) = a$ *and let* $q(z) = a + a_n z^n + \cdots$ *be analytic in* $\mathbb{U}$ *with* $q(z) \not\equiv a$ *and* $n \in \mathbb{N}$. *If* q *is not subordinate to* p, *then there exist points* $z_0 = r_0 e^{i\theta} \in \mathbb{U}$ *and* $\zeta_0 \in \partial\mathbb{U} \setminus E(f)$, *for which* $q(\mathbb{U}_{r_0}) \subset p(\mathbb{U})$,

$$q(z_0) = p(\zeta_0) \quad \text{and} \quad z_0 q'(z_0) = m\zeta_0 p'(\zeta_0) \quad (m \geq n).$$

A function $L(z,t)$ defined on $\mathbb{U} \times [0,\infty)$ is the subordination chain (or Löwner chain) if $L(\cdot,t)$ is analytic and univalent in $\mathbb{U}$ for all $t \in [0,\infty)$, $L(z,\cdot)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$ and $L(z,s) \prec L(z,t)$ for $0 \leq s < t$.

Lemma 1.4 [9]. *Let* $q \in \mathcal{H}[a,1]$, *let* $\varphi : \mathbb{C}^2 \to \mathbb{C}$ *and set* $\varphi(q(z), zq'(z)) \equiv h(z)$. *If* $L(z,t) = \varphi(q(z), tzq'(z))$ *is a subordination chain and* $p \in \mathcal{H}[a,1] \cap \mathcal{Q}$, *then*

$$h(z) \prec \varphi(p(z), zp'(z))$$

implies that

$$q(z) \prec p(z).$$

Furthermore, if $\varphi(q(z), zq'(z)) = h(z)$ *has a univalent solution* $q \in \mathcal{Q}$, *then* q *is the best subordinant.*

Lemma 1.5 [12]. *The function* $L(z,t) = a_1(t)z + \cdots$, *with* $a_1(t) \neq 0$ *and* $\lim_{t\to\infty} |a_1(t)| = \infty$, *is a subordination chain if and only if*

$$\mathrm{Re}\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} > 0 \quad (z \in \mathbb{U};\ 0 \leq t < \infty).$$

2. Main Results

Firstly, we begin by proving the following subordination theorem involving the multiplier transformation $D(n,\lambda)$ defined by (1.3).

Theorem 2.1. *Let $f, g \in \mathcal{M}$. Suppose that*

$$\mathrm{Re}\left\{1+\frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta \tag{2.1}$$

$\big(\phi(z) := (1-\alpha)z^2 D(n+1,\lambda)g(z) + \alpha z^2 D(n,\lambda)g(z);\ \lambda > 1-\alpha;\ 0 \leq \alpha < 1;\ z \in \mathbb{U}\big)$,

where

$$\delta = \frac{(1-\alpha)^2 + ((\lambda-1)+\alpha)^2 - |(1-\alpha)^2 - ((\lambda-1)+\alpha)^2|}{4((\lambda-1)+\alpha)(1-\alpha)}. \tag{2.2}$$

If f and g satisfy the following subordination condition :

$$(1-\alpha)z^2 D(n+1,\lambda)f(z) + \alpha z^2 D(n,\lambda)f(z) \prec \phi(z), \tag{2.3}$$

then

$$z^2 D(n,\lambda)f(z) \prec z^2 D(n,\lambda)g(z). \tag{2.4}$$

Moreover, the function $z^2 D(n,\lambda)g(z)$ is the best dominant.

Proof. Let us define the functions F and G, respectively, by

$$F(z) := z^2 D(n,\lambda)f(z) \ \text{ and } \ G(z) := z^2 D(n,\lambda)g(z), \tag{2.5}$$

We first show that, if the function q is defined by

$$q(z) := 1+\frac{zG''(z)}{G'(z)} \quad (z \in \mathbb{U}), \tag{2.6}$$

then

$$\mathrm{Re}\{q(z)\} > 0 \quad (z \in \mathbb{U}).$$

Taking the logarithmic differentiation on both sides of the second equation in (2.5) and using (1.4) for $g \in \mathcal{M}$, we obtain

$$\lambda\phi(z) = (\lambda - 1 + \alpha)G(z) + (1 - \alpha)zG'(z). \tag{2.7}$$

Now, by differentiating both sides of (2.7), we obtain the relationship:

$$\begin{aligned} 1 + \frac{z\phi''(z)}{\phi'(z)} &= 1 + \frac{zG''(z)}{G'(z)} + \frac{zq'(z)}{q(z) + (\lambda - 1 + \alpha)/(1 - \alpha)} \\ &= q(z) + \frac{zq'(z)}{q(z) + (\lambda - 1 + \alpha)/(1 - \alpha)} \equiv h(z). \end{aligned} \tag{2.8}$$

We see from (2.1) that

$$\mathrm{Re}\left\{h(z) + \frac{(\lambda - 1 + \alpha)}{1 - \alpha}\right\} > 0 \quad (z \in \mathbb{U}),$$

holds true and by using Lemma 1.2, we conclude that the differential equation (2.8) has a solution $q \in \mathcal{H}(\mathbb{U})$ with $q(0) = h(0) = 1$. Let us put

$$H(u, v) = u + \frac{v}{u + (\lambda - 1 + \alpha)/(1 - \alpha)} + \delta, \tag{2.9}$$

where δ is given by (2.2). From (2.1), (2.8) and (2.9), we obtain

$$\mathrm{Re}\{H(q(z), zq'(z))\} > 0 \quad (z \in \mathbb{U}).$$

Now we proceed to show that $\mathrm{Re}\{H(is, t)\} \leq 0$ for all real s and $t \leq -(1 + s^2)/2$. From (2.9), we have

$$
\begin{aligned}
\text{Re}\{H(is,t)\} &= \text{Re}\left\{is + \frac{t}{is + (\lambda - 1 + \alpha)/(1-\alpha)} + \delta\right\} \\
&= \frac{t(\lambda - 1 + \alpha)/(1-\alpha)}{|(\lambda - 1 + \alpha)/(1-\alpha) + is|^2} + \delta \\
&\leq -\frac{E_\delta(s)}{2|(\lambda - 1 + \alpha)/(1-\alpha) + is|^2},
\end{aligned} \tag{2.10}
$$

where

$$
E_\delta(s) := \left(\frac{\lambda - 1 + \alpha}{1-\alpha} - 2\delta\right) s^2 - \frac{\lambda - 1 + \alpha}{1-\alpha}\left(2\delta\frac{\lambda - 1 + \alpha}{1-\alpha} - 1\right). \tag{2.11}
$$

For δ given by (2.2), we can prove easily that the expression $E_\delta(s)$ given by (2.11) is positive or equal to zero. Hence from (2.9), we see that $\text{Re}\{H(is,t)\} \leq 0$ for all real s and $t \leq -(1+s^2)/2$. Thus, by using Lemma 1.1, we conclude that $\text{Re}\{q(z)\} > 0$ for all $z \in \mathbb{U}$. That is, q is convex in $\mathbb{U}$.

Next, we prove that the subordination condition (2.3) implies that

$$
F(z) \prec G(z) \tag{2.12}
$$

for the functions F and G defined by (2.5). Without loss of generality, we can assume that G is analytic and univalent on $\overline{\mathbb{U}}$ and $G'(\zeta) \neq 0$ for $|\zeta| = 1$. For this purpose, we consider the function $L(z,t)$ given by

$$
L(z,t) := \frac{\lambda - 1 + \alpha}{\lambda} G(z) + \frac{(1-\alpha)(1+t)}{\lambda} zG'(z) \quad (z \in \mathbb{U};\ 0 \leq t < \infty).
$$

We note that

$$
\left.\frac{\partial L(z,t)}{\partial z}\right|_{z=0} = G'(0)\left(\frac{\lambda + t(1-\alpha)}{\lambda}\right) \neq 0 \quad (0 \leq t < \infty;\ \lambda > 0).
$$

This shows that the function

$$L(z,t) = a_1(t)z + \cdots$$

satisfies the condition $a_1(t) \neq 0$ for all $t \in [0,\infty)$. Furthermore, we have

$$\mathrm{Re}\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} = \mathrm{Re}\left\{\frac{\lambda - 1 + \alpha}{1-\alpha} + (1+t)\left(1 + \frac{zG''(z)}{G'(z)}\right)\right\} > 0,$$

since G is convex and $(\lambda - 1 + \alpha)/(1-\alpha) > 0$. Therefore, by virtue of Lemma 1.5, $L(z,t)$ is a subordination chain. We observe from the definition of a subordination chain that

$$L(\zeta,t) \notin L(\mathbb{U},0) = \phi(\mathbb{U}) \quad (\zeta \in \partial\mathbb{U};\ 0 \leq t < \infty)$$

Now suppose that F is not subordinate to G, then by Lemma 1.3, there exists points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}$ such that

$$F(z_0) = G(\zeta_0) \quad \text{and} \quad z_0F'(z_0) = (1+t)\zeta_0 G'(\zeta_0) \quad (0 \leq t < \infty).$$

Hence we have

$$\begin{aligned} L(\zeta_0,t) &= \frac{\lambda-1+\alpha}{\lambda}G(\zeta_0) + \frac{(1-\alpha)(1+t)}{\lambda}\zeta_0 G'(\zeta_0) \\ &= \frac{\lambda-1+\alpha}{\lambda}F(z_0) + \frac{1-\alpha}{\lambda}z_0F'(z_0) \\ &= (1-\alpha)z_0^2 D(n+1,\lambda)f(z_0) + \alpha z_0^2 D(n,\lambda)f(z_0) \in \phi(\mathbb{U}), \end{aligned}$$

by virtue of the subordination condition (2.3). This contracts the above observation that $L(\zeta_0,t) \notin \phi(\mathbb{U})$. Therefore, the subordination condition (2.3) must imply the subordination given by (2.12). This evidently completes the proof of Theorem 2.1.

We next prove a dual problem of Theorem 2.1, in the sense that the subordinations are replaced by superordinations.

Theorem 2.2. *Let $f, g \in \mathcal{M}$. Suppose that*

$$\mathrm{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta$$

$$\big(\phi(z) := (1-\alpha)z^2 D(n+1,\lambda)g(z) + \alpha z^2 D(n,\lambda)g(z);\ \ \lambda > (1-\alpha);\ \ 0 \leq \alpha < 1;\ \ z \in \mathbb{U}\big),$$

where δ is given by (2.2)*. If $(1-\alpha)z^2 D(n+1,\lambda)f(z) + \alpha z^2 D(n,\lambda)f(z)$ is univalent in $\mathbb{U}$ and $z^2 D(n,\lambda)f(z) \in \mathcal{H}[0,1] \cap \mathcal{Q}$, then*

$$\phi(z) \prec (1-\alpha)z^2 D(n+1,\lambda)f(z) + \alpha z^2 D(n,\lambda)f(z) \tag{2.13}$$

implies that

$$z^2 D(n,\lambda)g(z) \prec z^2 D(n,\lambda)f(z).$$

Moreover, the function $z^2 D(n,\lambda)g(z)$ is the best subordinant.

Proof. Let us define the functions F and G, respectively, by (2.5). We first note that, if the function q is defined by (2.6), by using (2.7), then we obtain

$$\begin{aligned}\phi(z) &= \frac{\lambda - 1 + \alpha}{\lambda}G(z) + \frac{(1-\alpha)(1+t)}{\lambda}zG'(z) \\ &=: \varphi(G(z), zG'(z)).\end{aligned} \tag{2.14}$$

After a simple calculation, the equation (2.13) yields the relationship:

$$1 + \frac{z\phi''(z)}{\phi'(z)} = q(z) + \frac{zq'(z)}{q(z) + (\lambda - 1 + \alpha)/(1-\alpha)}.$$

Then by using the same method as in the proof of Theorem 2.1, we can prove that $\text{Re}\{q(z)\} > 0$ for all $z \in \mathbb{U}$. That is, G defined by (2.5) is convex(univalent) in $\mathbb{U}$.

Next, we prove that the subordination condition (2.13) implies that

$$F(z) \prec G(z) \tag{2.15}$$

for the functions F and G defined by (2.5). Now considering the function $L(z,t)$ defined by

$$L(z,t) := \frac{\lambda - 1 + \alpha}{\lambda} G(z) + \frac{(1-\alpha)t}{\lambda} zG'(z) \quad (z \in \mathbb{U};\ 0 \leq t < \infty).$$

we can prove easily that $L(z,t)$ is a subordination chain as in the proof of Theorem 2.1. Therefore according to Lemma 1.4, we conclude that the superordination condition (2.13) must imply the superordination given by (2.15). Furthermore, since the differential equation (2.14) has the univalent solution G, it is the best subordinant of the given differential superordination. Therefore we complete the proof of Theorem 2.2.

If we combine this Theorem 2.1 and Theorem 2.2, then we obtain the following sandwich-type theorem.

Theorem 2.3. *Let $f, g_k \in \mathcal{M}(k = 1, 2)$. Suppose that*

$$\text{Re}\left\{1 + \frac{z\phi_k''(z)}{\phi_k'(z)}\right\} > -\delta$$

$$\left(\phi_k(z) := (1-\alpha)z^2 D(n+1,\lambda)g_k(z) + \alpha z^2 D(n,\lambda)g_k(z);\ k = 1,2;\ \lambda > (1-\alpha);\ 0 \leq \alpha < 1;\ z \in \mathbb{U}\right), \tag{2.16}$$

where δ *is given by* (2.2)*. If* $(1-\alpha)z^2D(n+1,\lambda)f(z)+\alpha z^2D(n,\lambda)f(z)$ *is univalent in* $\mathbb{U}$ *and* $z^2D(n,\lambda)f(z)\in\mathcal{H}[0,1]\cap\mathcal{Q}$*, then*

$$\phi_1(z)\prec(1-\alpha)z^2D(n+1,\lambda)f(z)+\alpha z^2D(n,\lambda)f(z)\prec\phi_2(z)$$

implies that

$$z^2D(n,\lambda)g_1(z)\prec z^2D(n,\lambda)f(z)\prec z^2D(n,\lambda)g_2(z).$$

Moreover, the functions $z^2D(n,\lambda)g_1(z)$ *and* $z^2D(n+1,\lambda)g_2(z)$ *are the best subordinant and the best dominant, respectively.*

The assumption of Theorem 2.3, that the functions $(1-\alpha)z^2D(n+1,\lambda)f(z)+\alpha z^2D(n,\lambda)f(z)$ and $z^2D(n,\lambda)f(z)$ need to be univalent in $\mathbb{U}$, may be replaced another condition in the following result.

Corollary 2.1. *Let* $f,g_k\in\mathcal{M}(k=1,2)$*. Suppose that the condition* (2.16) *is satisfied and*

$$\operatorname{Re}\left\{1+\frac{z\psi''(z)}{\psi'(z)}\right\}>-\delta \tag{2.17}$$

$$\left(\psi(z):=(1-\alpha)z^2D(n+1,\lambda)f(z)+\alpha z^2D(n,\lambda)f(z);\ z\in\mathbb{U}\right),$$

where δ *is given by* (2.2)*. Then*

$$\phi_1(z)\prec(1-\alpha)z^2D(n+1,\lambda)f(z)+\alpha z^2D(n,\lambda)f(z)\prec\phi_2(z)$$

implies that

$$z^2D(n,\lambda)g_1(z) \prec z^2D(n,\lambda)f(z) \prec z^2D(n,\lambda)g_2(z).$$

Moreover, the functions $z^2D(n,\lambda)g_1(z)$ *and* $z^2D(n,\lambda)g_2(z)$ *are the best subordinant and the best dominant, respectively.*

Proof. In order to prove Corollary 2.1, we have to show that the condition (2.17) implies the univalence of $\psi(z)$ and $F(z) := z^2D(n,\lambda)f(z)$. Since δ given by (2.2) in Theorem 2.1 satisfies the inequality $0 < \delta \leq 1/2$, the condition (2.17) means that ψ is a close-to-convex function in $\mathbb{U}$ (see [4]) and hence ψ is univalent in $\mathbb{U}$. Furthermore, by using the same techniques as in the proof of Theorem 2.1, we can prove the convexity(univalence) of F and so the details may be omitted. Therefore, from Theorem 2.3, we obtain Corollary 2.1.

Setting $n = 0$, $\lambda = 2$ and $\alpha = 0$ in Theorem 2.3, we have the following result.

Corollary 2.2. *Let* $f, g_k \in \mathcal{M}(k = 1,2)$. *Suppose that*

$$\mathrm{Re}\left\{1 + \frac{z\phi_k''(z)}{\phi_k'(z)}\right\} > -\frac{1}{2}$$

$$\left(\phi_k(z) := \frac{z^2g_k''(z) + 3z^2g_k(z)}{2};\ k = 1,2;\ z \in \mathbb{U}\right).$$

If $(z^2f''(z) + 3z^2f(z))/2$ *is univalent in* $\mathbb{U}$ *and* $z^2f(z) \in \mathcal{H}[0,1] \cap \mathcal{Q}$, *then*

$$\phi_1(z) \prec \frac{z^2f''(z) + 3z^2f(z)}{2} \prec \phi_2(z)$$

implies that

$$z^2g_1 \prec z^2f(z) \prec z^2g_2(z).$$

Moreover, the functions $z^2 g_1(z)$ *and* $z^2 g_2(z)$ *are the best subordinant and the best dominant, respectively.*

By using the same method as in the proof of Theorem 2.3, we have the following sandwich-type theorem.

Theorem 2.4. *Let* $f, g_k \in \mathcal{M}(k = 1, 2)$. *Suppose that*

$$\mathrm{Re}\left\{1 + \frac{z\phi_k''(z)}{\phi_k'(z)}\right\} > -\delta$$

$$(\phi_k(z) := (1-\alpha)zD(n+1,\lambda)g_k(z) + \alpha zD(n,\lambda)g_k(z);\ k = 1, 2;\ \lambda > 0;\ 0 \leq \alpha < 1;\ z \in \mathbb{U}),$$

where

$$\delta = \frac{(1-\alpha)^2 + \lambda^2 - |(1-\alpha)^2 - \lambda^2|}{4\lambda(1-\alpha)}.$$

If $(1-\alpha)zD(n+1,\lambda)f(z) + \alpha zD(n,\lambda)f(z)$ *is univalent in* $\mathbb{U}$ *and* $zD(n,\lambda)f(z) \in \mathcal{H}[1,1] \cap \mathcal{Q}$, *then*

$$\phi_1(z) \prec (1-\alpha)zD(n+1,\lambda)f(z) + \alpha zD(n,\lambda)f(z) \prec \phi_2(z)$$

implies that

$$zD(n,\lambda)g_1(z) \prec zD(n,\lambda)f(z) \prec zD(n,\lambda)g_2(z).$$

Moreover, the functions $zD(n,\lambda)g_1(z)$ *and* $zD(n,\lambda)g_2(z)$ *are the best subordinant and the best dominant, respectively.*

Next, we consider the integral operator F_c $(c > 0)$ defined by (cf. [3,5,15,16])

$$F_c(f)(z) := \frac{c}{z^{c+1}} \int_0^z t^c f(t)dt \quad (f \in \mathcal{M};\ c > 0). \tag{2.18}$$

Now, we obtain the following sandwich-type result involving the integral operator defined by (2.18).

Theorem 2.5. *Let* $f, g_k \in \mathcal{M}(k = 1, 2)$. *Suppose that*

$$\mathrm{Re}\left\{1 + \frac{z\phi_k''(z)}{\phi_k'(z)}\right\} > -\delta \tag{2.19}$$

$$\left(\phi_k(z) := z^2 D(n, \lambda) g_k(z);\ k = 1, 2;\ \lambda > 0;\ c > 1;\ z \in \mathbb{U}\right),$$

where

$$\delta = \frac{1 + (c-1)^2 - |1 - (c-1)^2|}{4(c-1)} \quad (c > 1). \tag{2.20}$$

If $z^2 D(n, \lambda) f(z)$ *is univalent in* $\mathbb{U}$ *and* $z^2 D(n, \lambda) F_c(f)(z) \in \mathcal{H}[0, 1] \cap \mathcal{Q}$, *then*

$$z^2 D(n, \lambda) g_1(z) \prec z^2 D(n, \lambda) f(z) \prec z^2 D(n, \lambda) g_2(z)$$

implies that

$$z^2 D(n, \lambda) F_c(g_1)(z) \prec z^2 D(n, \lambda) F_c(f)(z) \prec z^2 D(n, \lambda) F_c(g_2)(z).$$

Moreover, the functions $z^2 D(n, \lambda) F_c(g_1)(z)$ *and* $z^2 D(n, \lambda) F_c(g_2)(z)$ *are the best subordinant and the best dominant, respectively.*

Proof. Let us define the functions F and G_k $(k = 1, 2)$ by

$$F(z) := z^2 D(n, \lambda) F_c(f)(z) \ \text{ and } \ G_k(z) := z^2 D(n, \lambda) F_c(g_k)(z),$$

respectively. From the definition of the integral operator F_c defined by (2.18), we obtain

$$z(D(n,\lambda)F_c(f)(z))' = cD(n,\lambda)f(z) - (c+1)D(n,\lambda)F_c(f)(z) \tag{2.21}$$

Then from (2.19) and (2.21), we have

$$c\phi_k(z) = (c-1)G_k(z) + zG_k'(z). \tag{2.22}$$

Setting

$$q_k(z) = 1 + \frac{zG_k''(z)}{G_k'(z)} \quad (k = 1, 2;\ z \in \mathbb{U}),$$

and differentiating both sides of (2.22), we obtain

$$1 + \frac{z\phi_k''(z)}{\phi_k'(z)} = q_k(z) + \frac{zq_k'(z)}{q_k(z) + c - 1}.$$

The remaining part of the proof is similar to that of Theorem 2.1 and so we may omit for the proof involved.

By using the same methods as in the proof of Corollary 2,1, we have the following result.

Corollary 2.3. *Let $f, g_k \in \mathcal{M}(k = 1, 2)$. Suppose that the condition (2.19) is satisfied and*

$$\operatorname{Re}\left\{1 + \frac{z\psi''(z)}{\psi'(z)}\right\} > -\delta$$

$$\left(\psi(z) := z^2 D(n,\lambda)f(z);\ \lambda > 0;\ z \in \mathbb{U}\right),$$

where δ *is given by* (2.20). *Then*

$$z^2D(n,\lambda)g_1(z) \prec z^2D(n,\lambda)f(z) \prec z^2D(n,\lambda)g_2(z)$$

implies that

$$z^2D(n,\lambda)F_c(g_1)(z) \prec z^2D(n,\lambda)F_c(f)(z) \prec z^2D(n,\lambda)F_c(g_2)(z).$$

Moreover, the functions $z^2D(n,\lambda)F_c(g_1)(z)$ *and* $z^2D(n,\lambda)F_c(g_2)(z)$ *are the best subordinant and the best dominant, respectively.*

Taking $n = 0$ in Theorem 2.5, we have the following result.

Corollary 2.4. *Let* $f, g_k \in \mathcal{M}(k = 1, 2)$. *Suppose that*

$$\mathrm{Re}\left\{1 + \frac{z\phi_k''(z)}{\phi_k'(z)}\right\} > -\delta$$
$$\left(\phi_k(z) := z^2g_k(z);\ k = 1, 2;\ \lambda > 0;\ c > 1;\ z \in \mathbb{U}\right),$$

where δ *is given by* (2.20). *If* $z^2f(z)$ *is univalent in* $\mathbb{U}$ *and* $z^2F_c(f)(z) \in \mathcal{H}[0, 1] \cap \mathcal{Q}$, *then*

$$z^2g_1(z) \prec z^2f(z) \prec z^2g_2(z)$$

implies that

$$z^2F_c(g_1)(z) \prec z^2F_c(f)(z) \prec z^2F_c(g_2)(z).$$

Moreover, the functions $z^2F_c(g_1)(z)$ *and* $z^2F_c(g_2)(z)$ *are the best subordinant and the best dominant, respectively.*

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