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Regularity for the Semilinear Evolution Equations of Second Order and Perturbation Theory

by

Han-Geul Kim

Department of Applied Mathematics

The Graduate School

Pukyong National University

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(2계 준선형 발전 방정식에 대한 정규성과 섭동이론)

Advisor : Jin-Mun Jeong

by

Han-Geul Kim

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A dissertation

by

Han-Geul Kim

Approved by:

(Chairman) Gue Myung Lee

(Member) Do Sang Kim

(Member) Sung Jin Cho

(Member) Young chel Kwun

(Member) Jin-Mun Jeong

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2계 준선형 발전 방정식에 대한 정규성과 섭동이론

김 한 글

부경대학교 대학원 응용수학과

요 약

본 논문에서는 Hilbert 공간상에서 다음과 같은 2계 준선형 발전 방정식에 대한 해의 존재성, 정규성 그리고 섭동문제를 다룬다.

(SE) $$\begin{cases} u''(t) + A(t)u(t) = f(t, u(t)) + h(t) \\ u(0) = u_{0,} \quad u'(0) = u_1 \end{cases}$$

여기서, $A(t)$는 주 작용소로서 Garding 조건을 만족하는 미분연산자로부터 정의되었고 비선형항 $f(\cdot, x)$은 일반화된 Lipschitz 연속성을 만족한다.

먼저, 2장에서는 주 작용소가 시간 t에 대하여 독립인 경우, 즉, $A(t) = A$일 때, 비선형항을 포함하지 않는 선형방정식의 결과들에 대하여 이러한 비선형 방정식에도 성립 가능함을 증명한다.

3장에서는 시간 t에 대하여 변하는 비유계 작용소 $A(t)$에 대한 (SE)의 정규성을 함수적 성질을 이용하여 고찰한다.

마지막으로, 4장에서는 다음과 같이 섭동된 비동차 2계 함수미분방정식:

(PE) $$\begin{cases} u''(t) + (A(t) + B(t))u(t) = f(t) \\ u(0) = u_{0,} \quad u'(0) = u_1 \end{cases}$$

의 정규성과 기본성질을 증명하기 위해 작용소 $B(t)$의 충분조건을 유도한다.

Chapter 1
Introduction and Preliminaries

By the 1930s, the basic theory of dynamical systems was well in place, and the basic studies, which at a later time would lead to a theory of flows and semiflows for the infinite dimensional evolutionary equations arising in partial differential equations, had begun. During the period 1930 - 1970 there were many major developments in the study of the longtime dynamics of systems of ordinary differential equations, including perturbation theory for invariant manifolds, bifurcation theory, exponential dichotomies and hyperbolic structures, the Pliss reduction principle (center manifold), the Kolmogorov-Arnold-Moser theory, skew products flows for nonautonomous problems, Morse-Smale dynamical systems, the structural stability program, the role of symmetries, and index theory.

By the 1970s, the dynamical theories for dissipative partial differential equations, such as reaction diffusion equations, the Navier-Stokes equations, and the Cahn-Hilliard equation, were coming to fruition. In this area and during the subsequent 30 years, one finds the development of existence theories and dimension theories for global attractors and inertial manifolds, the use of smooth and discrete-valued Lyapunov functions to find Morse-Smale structures and Poincare-Bendixon theories, and the use of exponential trichotomies and hyperbolic structures for the perturbation theory of invariant manifolds, for example(see [20, 12, 13]).

The year 1970 is an approximate date of the merger of finite dimensional and infinite dimensional dynamical systems. Since that time, this has become a united subject, the Dynamics of Evolutionary Equations. Other major include the Melinikov method, singular perturbations, random dynamical systems, almost periodic and almost automorphic dynamics, and approximation dynamics. The subject of the Dynamics of Evolutionary Equation is only at its beginning. While it is not possible to predict the future, we sincerely hope that this paper will be helpful for scholars working in these areas and in some of the newer areas of dynamics, such as global climate modeling, numerical simulation of longtime dynamics, and control theory in time-varying media (see [16, 26, 19]).

In this paper, we consider the existence results, the regularity of the solutions, and the hyperbolic structures for the perturbation theory for the following semilinear wave equation:

$$\begin{cases} u''(t) + A(t)u(t) = f(t, u(t)) + h(t) \\ u(0) = u_0, \quad u'(0) = u_1 \end{cases} \tag{SE}$$

in a Hilbert space H.

This dissertation is organized as follows;

In Chapter 2, we obtain the regularity for (SE) in the case that $A(t) = A$ is the operator associated with a sesquilinear form defined on $V \times V$ and satisfying Gårding's inequality, where V is another Hilbert space such that $V \subset H \subset V^*$(the dual space of V). The nonlinear term $f(\cdot, x)$, which is a Lipschitz continuous operator with respect to x from V to H, is a semilinear version of the quasilinear one considered in [6, 15, 25].

As a consequence, our models for the equation (SE) are Volterra integrodifferential equations of the hyperbolic type. These equations arise naturally in the study of Viscoelasticity in Edelstein and Gurtin [5]. Our formulation of the equation (SE) is a direct attempt to generalize some results of Webb [25] and Heard [8], who studied problems similar to the equation (SE) in the case when $A(t) = A$ does not depend on t. By using the useful integral inequalities, we will show that there exists a solution for the class of nonlinear second order evolution equations by a similar method to that for the linear heat equations of [11].

In Chapter 3, we adhere to the construction of an evolution system for the equation (SE) with unbounded operator $A(t)$ constructed by Kato [12, 13]. For each $t \geq 0$, $A(t)$ is the infinitesimal generator of an analytic semigroup together with some continuity conditions on the family of bounded operators $A(t)A(s)^{-1}$. Section 3.3 is devoted to the regularity for solutions of the linear wave equations in Gelfand triple spaces. Subsequently, our construction of a local solution of the nonlinear equation (SE) is essentially based on [10]. We will show the energy inequalities for the equation (SE) with the aid of estimate of L^2-type of the solutions, which is an important role in the proof of the global solutions and in that of the regularity of solutions. Finally, a possible extension of the given equation (SE) is discussed.

In Chapter 4, we consider the following perturbed inhomogeneous second order hyperbolic equation:

$$\begin{cases} u''(t) + (A(t) + B(t))u(t) = f(t) \\ u(0) = u_0, \quad u'(0) = u_1 \end{cases} \tag{PE}$$

where $A(t)$ satisfies the conditions in Chapter 3. Let $B(t)$ be defined on $[0, T]$ as a strongly continuously differentiable operator satisfying

$$B(t)u \in C^1((0,T);H), \quad |B(t)u| \leq B|u| \quad \text{for} \quad \text{all} \quad u \in H$$

for some constant $B > 0$. In order to give the construction of an evolution system of $A(t) + B(t)$, we will assume general conditions that $A(t)$, for each $t \in [0, T]$, is self adjoint and bounded and $A(t)v$ for each $v \in V$ is strongly continuously differentiable on $[0, T]$. Our problem can be applied to second order time dependent equations by writing them as first order systems. Consequently, we deal with constructing of the fundamental solution of the linear equation explained the arguments in given in [14, 2]. In addition to assumptions of $A(t)$, Tanabe [23] dealt with a singular perturbation of evolution systems in a Banach space X with conditions that $B(t)$ is strongly continuous and there exists a real number λ_0 satisfying $\lambda_0 \in \rho(A(t))$ for all $t \in [0, T]$, such that

$$A(t)B(t)(A(t) - \lambda_0)^{-1} \in \mathcal{L}(X),$$

where $\mathcal{L}(X)$ denotes the set of all bounded linear operators from X into itself. But we will give a perturbation approach under the more general conditions that X is a Hilbert space and $B(t)v$ for each $v \in V$ is strongly continuously differentiable on $[0, T]$ instead of the above condition even in special cases of second order equations. In the last section we give an example of a partial functional equation as an application of the preceding result in a mixed problem for hyperbolic case that

$$A(t) = -\sum_{i,j=1}^{n} \frac{\partial u}{\partial x_j}(a_{ij}(t,x)\frac{\partial u}{\partial x_i}), \quad B(t) = \sum_{i=1}^{n} b_i(t,x)\frac{\partial u}{\partial x_i} + c(t,x)u,$$

where the matrix $(a_{ij}(t,x))$ is uniformly positive definite.

Chapter 2
Regularity for solutions of nonlinear second order evolution equations

2.1. Introduction

In this chapter, we consider the existence and regularity of the solutions for the following semilinear wave equation:

$$\begin{cases} u''(t) + Au(t) = f(t, u(t)) + h(t) \\ u(0) = u_0, \quad u'(0) = u_1 \end{cases} \tag{2.1.1}$$

in a Hilbert space H. Here A is the operator associated with a sesquilinear form defined on $V \times V$ and satisfying Gårding's inequality, where V is another Hilbert space such that $V \subset H \subset V^*$(the dual space of V). The nonlinear term $f(\cdot, x)$, which is a Lipschitz continuous operator with respect to x from V to H, is a semilinear version of the quasilinear one considered in [6, 15, 25]. Precise assumptions are given in the next section.

In the papers [6, 15], they investigated some results of existence and uniqueness of solutions for some problems that are related to functional differential inclusions of second order in time, containing some hereditary characteristics. The existence and regularity for the linear heat equations, which was first investigated by Brézis [3], has been developed as seen in section 4.3.1 of Barbu [1], and [11].

As a consequence, our models for (2.1.1) are Volterra integrodifferential equations of the hyperbolic type. These equations arise naturally in the study of Viscoelasticity in Edelstein and Gurtin [5]. Our formulation of (2.1.1) is a direct attempt to generalize some results of Webb [25] and Heard [8], who studied problems similar to (2.1.1) in the case when $A(t) = A$ does not depend on t. By using the useful integral inequalities, we will show that there exists a solution for the class of nonlinear second order evolution equations by a similar method to that for the linear heat equations of [11].

Section 2.2 gives some basic results on existence, uniqueness, and a representation formula of solutions for the given equation (2.1.1). In section 2.3, we will obtain the regularity for solutions of (2.1.1) by converting the problem into the contraction mapping principle when the nonlinear mapping f is Lipschitz continuous from $\mathbb{R} \times V$ into H, and obtain the norm estimate of a solution of the above nonlinear equation on $L^2(0,T;V) \cap W^{1,2}(0,T;H) \cap W^{2,2}(0,T;V^*)$ by using the results of its corresponding the linear part as seen in [20]. Finally a simple example to which our main result can be applied is given in section 2.4.

2.2. Semilinear equations

Let H be a complex Hilbert space with inner product $(\ ,\ )$ and norm $|\cdot|$. Let V be embedded in H as a dense subspace with inner product and norm by $((\ ,\ ))$ and $||\cdot||$, respectively. By considering $H = H^*$, we may write $V \subset H \subset V^*$ where H^* and V^* denote the dual spaces of H and V, respectively . For $l \in V^*$ we denoted (l, v) by the value $l(v)$ of l at $v \in V$. The norm of l as an element of V^* is given by

$$||l||_* = \sup_{v \in V} \frac{|(l,v)|}{||v||}.$$

Therefore, we assume that V has a stronger topology than H and, for the brevity, we may regard that

$$||u||_* \le |u| \le ||u||, \quad \forall u \in V.$$

Definition 2.2.1. Let X and Y be complex Banach spaces. An operator S from X to Y is called antilinear if $S(u+v) = S(u)+S(v)$ and $S(\lambda u) = \bar{\lambda} S(u)$ for $u, v \in X$ and for $\lambda \in \mathbb{C}$.

Let $a(u, v)$ be a quadratic form defined on $V \times V$ which is linear in u and antilinear in v.

We make the following assumptions:

i) $a(u, v)$ is bounded, i.e., there exists $c_0 > 0$ such that

$|a(u,v)| \le c_0 ||u|| \cdot ||v||$;

ii) $a(u, v)$ is symmetric, i.e.,

$a(u, v) = \overline{a(v,u)}$;

iii) $a(u, v)$ satisfies the Gårding's inequality, i.e.,

$\mathrm{Re}\, a(u,u) \ge \delta ||u||^2, \quad \delta > 0.$

Let A be the operator such that $(Au, v) = a(u, v) \quad u, v \in V$. Then, as seen in Theorem 2.2.3 of [23], the operator A is positive definite and self-adjoint, $D(A^{1/2}) = V$, and

$$a(u, v) = (A^{1/2}u, A^{1/2}v), \quad u, \ v \in V.$$

It is also known that the operator A is a bounded linear operator from V to V^*. The realization of A in H which is the restriction of A to $D(A) = \{v \in V : Av \in H\}$ is also denoted by A, which is structured as a Hilbert space with the norm $||v||_{D(A)} = |Av|$. Then the operator A generates an analytic semigroup in both of H and V^*. Thus we have the following sequence

$$D(A) \subset V \subset H \subset V^* \subset D(A)^*$$

where each space is dense in the next one, which is continuous injection.

If X is a Banach space and $1 < p < \infty$, $L^p(0, T; X)$ is the collection of all strongly measurable functions from $(0, T)$ into X the p-th powers whose norms are integrable and $W^{m,p}(0, T; X)$ is the set of all functions f whose derivatives $D^\alpha f$ up to degree m in the distribution sense belong to $L^p(0, T; X)$. $C^m([0, T]; X)$ is the set of all m-times continuously differentiable functions from $[0, T]$ into X.

We consider the initial value problem of the following semilinear equation

$$\begin{cases} u''(t) + Au(t) = f(t, u(t)) \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \tag{2.2.1}$$

Definition 2.2.1. *A function* $u : [0, T] \to H$ *is called a solution of equation* (2.2.1) *on* $[0, T]$ *if*

i) $u \in C([0, T]; V) \cap C^1((0, T); H) \cap C^2((0, T); V^*)$,

ii) u *satisfies* (2.2.1) *on* $[0, T]$.

Assumption (F). Let $f : [0, T] \times V \to H$ $(T > 0)$ be a nonlinear mapping such that $t \mapsto f(t, \cdot)$ is continuous on $[0, T]$ and $u \mapsto f(\cdot, u)$ is locally Lipschitz continuous on V: for any $C > 0$, there exists a constant $L_C > 0$ such that

$$|f(\cdot, u)| \le L_C, \quad |f(\cdot, u) - f(\cdot, v)| \le L_C ||u - v||$$

holds for $||u|| < C$ and $||v|| < C$.

Let us introduce a new norm in V^* as follows. For $g, k \in V^*$, putting

$$(g, k)_{-1} = a(A^{-1}g, A^{-1}k) = (AA^{-1}g, A^{-1}k) = (g, A^{-1}k),$$

in virtue of the condition of a $(g, k)_{-1}$, it satisfies the inner product properties and its norm is given by

$$||g||_{-1} = a(A^{-1}g, A^{-1}g)^{1/2}.$$

Lemma 2.2.1. The norm $||g||_{-1}$ is equivalent to $|| \cdot ||_*$, i.e., we have

$$\frac{\delta}{\sqrt{c_0}}||g||_{-1} \le ||g||_* \le \frac{c_0}{\sqrt{\delta}}||g||_{-1}.$$

Proof. From the condition iii) and i) of $a(\cdot, \cdot)$ it follows

$$\delta||A^{-1}g||^2 \le a(A^{-1}g, A^{-1}g) \le c_0||A^{-1}g||^2$$

and hence,

$$\sqrt{\delta}||A^{-1}g|| \le ||g||_{-1} \le \sqrt{c_0}||A^{-1}g||. \tag{2.2.2}$$

Since

$$||Ag||_* = \sup_{u\in V}\frac{|(Ag,u)|}{||u||} = \sup_{u\in V}\frac{|a(g,u)|}{||u||} \le c_0||g||$$

and

$$||Ag||_* \ge \frac{|(Ag,g)|}{||g||} = \frac{|a(g,g)|}{||g||} \ge \delta||g||,$$

we obtain that

$$\delta||A^{-1}g|| \le ||g||_* \le c_0||A^{-1}g||. \tag{2.2.3}$$

Combining (2.2.2) with (2.2.3), we obtain the inequality and hence $||\cdot||_*$ and $||\cdot||_{-1}$ are equivalent norms. □

If we set $X = (V \times H)^T$ with inner product and norm given by

$$\left(\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}\right) = ((u_0, v_0)) + (u_1, v_1)$$

and

$$\left\|\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}\right\|_X = \{||u_0||^2 + |u_1|^2\}^{1/2},$$

respectively. Noting that $a(u,v)$ is an inner product in V and $a(u,u)^{1/2}$ is equivalent to the norm $||u||$, we can also rewrite an inner product and a norm as

$$\left(\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}\right) = a(u_0, v_0) + (u_1, v_1)$$

and

$$\left\| \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right\|_X = \{a(u_0, u_0) + |u_1|^2\}^{1/2},$$

respectively.

Putting $\widetilde{X} = (H \times V^*)^T$, for every $\begin{pmatrix} g_0 \\ g_1 \end{pmatrix}$, $\begin{pmatrix} k_0 \\ k_1 \end{pmatrix} \in \widetilde{X}$, we define an inner product and a norm by

$$\left(\begin{pmatrix} g_0 \\ g_1 \end{pmatrix}, \begin{pmatrix} k_0 \\ k_1 \end{pmatrix} \right)_{\widetilde{X}} = (g_0, k_0) + (g_1, k_1)_{-1}$$

and

$$\left\| \begin{pmatrix} g_0 \\ g_1 \end{pmatrix} \right\|_{\widetilde{X}} = \left(|g_0|^2 + |g_1|^2_{-1} \right)^{1/2},$$

respectively. Let $\mathcal{A}$ be an operator defined by

$$D(\mathcal{A}) = (D(A) \times V)^T,$$

$$\mathcal{A} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} u_1 \\ -Au_0 \end{pmatrix} \in (V \times H)^T = X.$$

In virtue of Lax-Milgram theorem, we can also consider as

$$D(\mathcal{A}) = (V \times H)^T = X,$$

$$\mathcal{A} \begin{pmatrix} g_0 \\ g_1 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \end{pmatrix} = \begin{pmatrix} g_1 \\ -Ag_0 \end{pmatrix} \in (H \times V^*)^T = \widetilde{X}.$$

Theorem 2.2.1. The linear operator $\mathcal{A}$ as mentioned above is the infinitesimal generator of a C_0-group of unitary operators in both X and $\widetilde{X}$.

Proof. For $\binom{u_0}{u_1}, \binom{v_0}{v_1} \in X$,

$$\left(\mathcal{A}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}\right)_X = \left(\begin{pmatrix} u_1 \\ -Au_0 \end{pmatrix}, \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}\right)_X = a(u_1, v_0) + (-Au_0, v_1)$$

$$= a(u_1, v_0) - a(u_0, v_1),$$

and

$$\left(\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \mathcal{A}\begin{pmatrix} v_0 \\ v_1 \end{pmatrix}\right) = \left(\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \begin{pmatrix} v_1 \\ \mathcal{A}v_0 \end{pmatrix}\right) = a(u_0, v_1) + (u_1, -Av_0)$$

$$= a(u_0, v_1) - a(u_1, v_0),$$

Noting that A is symmetric, we have that

$$\left(\mathcal{A}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}\right)_X = -\left(\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \mathcal{A}\begin{pmatrix} v_0 \\ v_1 \end{pmatrix}\right),$$

which implies that $\mathcal{A}^* = -\mathcal{A}$ and therefore $i\mathcal{A} = (i\mathcal{A})^*$ and $i\mathcal{A}$ is self adjoint(skew self adjoint). Hence, from Stone's theorem, it follows that $\mathcal{A}$ is the infinitesimal generator of a C_0-group of unitary operators on X if and only if $i\mathcal{A}$ is self adjoint.

If $\binom{u_0}{u_1}$, $\binom{v_0}{v_1} \in \widetilde{X}$, then

$$\left(\mathcal{A}\begin{pmatrix}u_0\\u_1\end{pmatrix},\begin{pmatrix}v_0\\v_1\end{pmatrix}\right)_{\widetilde{X}} = \left(\begin{pmatrix}u_1\\-Au_0\end{pmatrix},\begin{pmatrix}v_0\\v_1\end{pmatrix}\right)_{\widetilde{X}} = (u_1, v_0) + (-Au_0, v_1)_{-1}$$

$$= (u_1, v_0) - a(u_0, A^{-1}v_1)$$

$$= (u_1, v_0) - \overline{a(A^{-1}v_1, u_0)} = (u_1, v_0) - \overline{(v_1, u_0)}$$

$$= (u_1, v_0) - (u_0, v_1)$$

and

$$\left(\begin{pmatrix}u_0\\u_1\end{pmatrix},\mathcal{A}\begin{pmatrix}v_0\\v_1\end{pmatrix}\right)_{\widetilde{X}} = \left(\begin{pmatrix}u_0\\u_1\end{pmatrix},\begin{pmatrix}v_1\\\mathcal{A}v_0\end{pmatrix}\right)_{\widetilde{X}} = (u_0, v_1) + (u_1, -Av_0)_{-1}$$

$$= (u_0, v_1) - a(A^{-1}u_1, v_0) = (u_0, v_1) - (u_1, v_0).$$

Hence, we have that

$$\left(\mathcal{A}\begin{pmatrix}u_0\\u_1\end{pmatrix},\begin{pmatrix}v_0\\v_1\end{pmatrix}\right)_{\widetilde{X}} = -\left(\begin{pmatrix}u_0\\u_1\end{pmatrix},\mathcal{A}\begin{pmatrix}v_0\\v_1\end{pmatrix}\right)_{\widetilde{X}},$$

that is, $\mathcal{A}$ is also skew self adjoint on $\widetilde{X}$. □

Let $\mathbf{x}(t) = \begin{pmatrix}u_0(t)\\u_1(t)\end{pmatrix}$ and let $F(\mathbf{x}) = \begin{pmatrix}0\\f(\cdot,u_0(\cdot))\end{pmatrix}$. Then problem (2.2.1) are equivalent to

$$\begin{cases} \mathbf{x}'(t) = \mathcal{A}\mathbf{x}(t) + F(\mathbf{x}(t)) \\ \mathbf{x}(0) = \begin{pmatrix}u_0\\u_1\end{pmatrix}. \end{cases} \tag{2.2.4}$$

Let $\mathcal{U}(t)$ be a C_0-group generated by $\mathcal{A}$. For a solution of (2.2.4) in the wide sense, we are to find a solution of the integral equation

$$\mathbf{x}(t) = \mathcal{U}(t)\mathbf{x}(0) + \int_0^t \mathcal{U}(t-s)F(\mathbf{x}(s))ds. \tag{2.2.5}$$

Now, we consider the global existence of a solution of (2.2.5).

Theorem 2.2.2. Let us assume the Assumption (F). Then for every $u_0 \in V, u_1 \in H$, the equation (2.2.4) has a unique solution on $[0, T]$ for given $T > 0$.

Proof. From Theorems 6.1.1 and 6.1.5 in [23], the equation (2.2.4) has a unique local solution on interval $[0, T_0]$ for $0 < T_0 \leq T$.

Now, we give a norm estimation of the solution of (2.2.4) and establish the global existence of solutions with the aid of norm estimations. So, it is enough to show that if u is a solution in $0 \leq t \leq T_0$, then $u(t)$ is bounded in $0 \leq t \leq T_0$, i.e., there exists a constant $C > 0$ such that

$$||u(t)|| \leq C, \quad 0 \leq t \leq T_0.$$

From

$$\left\|\mathcal{A}\begin{pmatrix} u_0(t) \\ u_1(t) \end{pmatrix}\right\|_{\tilde{X}} = \left\|\begin{pmatrix} u_1(t) \\ -Au_0(t) \end{pmatrix}\right\|_{\tilde{X}} \geq (\delta^2||u_0(t)||^2 + |u_1(t)|^2)^{\frac{1}{2}}$$

$$\geq \min\{\delta, 1\}(||u_0(t)||^2 + |u_1(t)|^2)^{\frac{1}{2}}$$

and

$$\left\|\mathcal{A}\begin{pmatrix} u_0(t) \\ u_1(t) \end{pmatrix}\right\|_{\tilde{X}} = \left\|\begin{pmatrix} u_1(t) \\ -Au_0(t) \end{pmatrix}\right\|_{\tilde{X}} = (|u_1(t)|^2 + ||Au_0(t)||_*^2)^{\frac{1}{2}}$$

$$\leq \max\{c_0, 1\}(||u_0(t)||^2 + |u_1(t)|^2)^{\frac{1}{2}},$$

it follows that

$$\min\{\delta,1\}(||u_0(t)||^2+|u_1(t)|^2)^{\frac{1}{2}} \le \left\| \mathcal{A} \begin{pmatrix} u_0(t) \\ u_1(t) \end{pmatrix} \right\|_{\widetilde{X}} \tag{2.2.6}$$

$$\le \max\{c_0,1\}(||u_0(t)||^2+|u_1(t)|^2)^{\frac{1}{2}}.$$

Therefore, from (2.2.5) and (2.2.6) we obtain that

$$\min\{\delta,1\}(||u_0(t)||^2+|u_1(t)|^2)^{\frac{1}{2}} \le \left\| \mathcal{A} \begin{pmatrix} u_0(t) \\ u_1(t) \end{pmatrix} \right\|_{\widetilde{X}}$$

$$\le \left\| \mathcal{A}\mathcal{U}(t) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right\|_{\widetilde{X}} + \left\| \mathcal{A} \int_0^t \mathcal{U}(t-s) \begin{pmatrix} 0 \\ f(s,u(s)) \end{pmatrix} ds \right\|_{\widetilde{X}}.$$

Here, we can calculate from (2.2.6) that

$$\left\| \mathcal{A}\mathcal{U}(t) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right\|_{\widetilde{X}} = \left\| \mathcal{A}\mathcal{U}(t)\mathcal{A}^{-1}\mathcal{A} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right\|_{\widetilde{X}}$$

$$\le c_1 \left\| \mathcal{A} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right\|_{\widetilde{X}}$$

$$\le c_1 \max\{c_0,1\}(||u_0||^2+|u_1|^2)^{1/2}$$

$$\le c_1 \max\{c_0,1\}(||u_0||+|u_1|),$$

where $c_1 = ||\mathcal{A}\mathcal{U}(t)\mathcal{A}^{-1}||_{B(\tilde{X})}$ and

$$\left\| \mathcal{A} \int_0^t \mathcal{U}(t-s) \begin{pmatrix} 0 \\ f(s,u(s)) \end{pmatrix} ds \right\|_{\widetilde{X}}$$

$$\le \left\| \int_0^t \mathcal{U}(t-s)\mathcal{A} \left(\begin{pmatrix} 0 \\ f(s,u(s)) \end{pmatrix} - \begin{pmatrix} 0 \\ f(s,0) \end{pmatrix} \right) ds \right\|_{\widetilde{X}}$$

$$+ \left\| \int_0^t \mathcal{U}(t-s)\mathcal{A} \begin{pmatrix} 0 \\ f(s,0) \end{pmatrix})ds \right\|_{\widetilde{X}}$$

$$\leq c_0 L_C M t + c_0 L_C M \int_0^t ||u(s)|| ds$$

$$\leq c_0 L_C M t + c_0 L_C M \int_0^t (||u(s)||^2 + |u(s)|^2)^{1/2} ds$$

where $M = \sup_{0 \leq t \leq T} ||\mathcal{U}(t)||$. Combining two inequalities above and (2.2.6), it follows from Gronwall's inequality that there exists a constant c_1 such that

$$(||u_0(t)||^2 + |u_1(t)|^2)^{1/2} \leq c_1(1 + ||u_0|| + |u_1|). \tag{2.2.7}$$

By the calculation similar to those in the proof of mentioned above, a solution $\mathbf{y}$ of

$$\begin{pmatrix} v_0(t) \\ v_1(t) \end{pmatrix} = \mathcal{U}(t - T_0) \begin{pmatrix} u_0(T_0) \\ u_1(T_0) \end{pmatrix} + \int_{T_0}^t \mathcal{U}(t-s) \begin{pmatrix} 0 \\ f(s, v_0(s)) \end{pmatrix} ds$$

exists in some interval $[T_0, T_1)$. By letting $\hat{\mathbf{x}}(t) = \mathbf{x}(t)$ for $0 \leq t \leq T_0$ and $\hat{\mathbf{x}}(t) = \mathbf{y}(t)$ for $T_0 \leq t < T_1$, it is easy to see that $\hat{\mathbf{x}}$ is a solution in $0 \leq t \leq T_1$. Therefore, $\mathbf{x}$ can be extended to the interval $[0, T_1]$ as a solution of (2.2.5). Let $\mathbf{x}$ be a bounded solution of (2.2.1): $||\mathbf{x}(t)||_X < C'$. Then, since $|| \begin{pmatrix} 0 \\ f(t, u_0(t)) \end{pmatrix} ||_X \leq L_{C'}$ for $0 \leq t < T_0$ by Assumption (F), if we put

$$\mathbf{x}(T_0) = \mathcal{U}(T_0)\mathbf{x}(0) + \int_0^{T_0} \mathcal{U}(T_0 - s) F(\mathbf{x}(s)) ds,$$

$\mathbf{x}$ is continuous in $0 \leq t \leq T_1$ and, moreover, satisfies (2.2.5). Hence, $\mathbf{x}$ can be extended to the interval $[0, T_1]$ as a solution and u_0 is the desired solution. So the proof is complete. □

2.3. L^2-regularity for solutions

Let V and H be complex Hilbert spaces forming Gelfand triple $V \subset H \subset V^*$ with pivot space H as mentioned in section 2.2.

Let $T > 0$. Define

$$W_T = \{u : u \in L^2(0,T;V),\ \dot{u} \in L^2(0,T;H),\ \ddot{u} \in L^2(0,T;V^*)\},$$

$$||u||_{W_T} = ||u||_{L^2(0,T;V)} + ||\dot{u}||_{L^2(0,T;H)} + ||\ddot{u}||_{L^2(0,T;V^*)},$$

where $\dot{u}$ denote the derivative of u in the generalized sense.

First, consider the following L^2-regularity for the abstract linear evolution equation:

$$\begin{cases} u''(t) + Au(t) = h(t), & 0 < t \le T, \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \tag{2.3.1}$$

Let $a(u,v)$ be a bounded sesquilinear form defined on $V \times V$ and satisfying Gårding's inequality:

$$\operatorname{Re} a(u,u) \ge \delta ||u||^2 - \kappa |u|^2, \quad \delta > 0, \quad \kappa \ge 0. \tag{2.3.2}$$

Let A be the operator associated with the sesquilinear form $a(u,v)$:

$$(Au, v) = a(u,v) \quad u, v \in V.$$

We begin with the following existence result(see Chapter 4 of [7]).

Proposition 2.3.1. Let $(u_0, u_1) \in V \times H$ and $h \in L^2(0,T;H)$. Then the evolution equation (2.3.1) has a unique solution $u \in W_T$. Moreover, we have

$$||u||_{W_T} \leq C_1(1 + ||u_0|| + |u_1| + ||h||_{L^2(0,T;H)}),$$

where C_1 is a constant depending on T.

Remark 2.3.1. From (2.3.1) it follows that

$$u''(t) = h(t) - Au(t) \in L^2(0,T;V^*),$$

hence it follows $u' \in C([0,T];V^*)$ and $u \in C([0,T];H)$(cf. Theorem 1.1 of Chapter 3 in [7]). Hence $(u_0, u_1) \in V \times H$ makes sense.

This section is to investigate the regularity of solutions for the following abstract semilinear second order initial value problem:

$$\begin{cases} u''(t) + Au(t) = f(t, u(t)) + h(t) \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \tag{2.3.3}$$

We assume the following hypotheses on the nonlinear term.

Assumption (F1). Let $f : [0,T] \times V \to H$ be a nonlinear mapping such that $t \mapsto f(t, \cdot)$ is measurable on $[0,T]$ and $u \mapsto f(\cdot, u)$ is Lipschitz continuous on V: there exists a constant $L > 0$ such that

$$|f(\cdot, u) - f(\cdot, v)| \leq L||u - v||, \quad u, v \in V.$$

The following lemma is from H. Brézis ([3]; Lemma A.5)

Lemma 2.3.1. Let $m \in L^1(0, T; \mathbb{R})$ satisfying $m(t) \geq 0$ for all $t \in (0, T)$ and $a \geq 0$ be a constant. Let b be a continuous function on $[0, T] \subset \mathbb{R}$ satisfying the following inequality:

$$\frac{1}{2}b^2(t) \leq \frac{1}{2}a^2 + \int_0^t m(s)b(s)ds, \quad t \in [0, T].$$

Then,

$$|b(t)| \leq a + \int_0^t m(s)ds, \quad t \in [0, T].$$

The following lemma is one of the useful integral inequalities.

Lemma 2.3.2. Let $b, a, m \in C(\mathbb{R}^+, \mathbb{R}^+)$ and suppose that the following inequality:

$$b(t) \leq a(t) + \int_{t_0}^t m(s)b(s)ds, \quad t \geq t_0.$$

Then,

$$b(t) \leq a(t) + \int_{t_0}^t [a(s)m(s)]exp\{\int_s^t m(\tau)d\tau\}ds, \quad t \geq t_0.$$

We establish the following results on the local solvability of the equation (2.3.3)

Theorem 2.3.1. Let the Assumption (F1) be satisfied. Assume that $h \in L^2(0, T; H)$ and $(u_0, u_1) \in V \times H$. Then, there exists a time $T_0 > 0$ such that the equation (2.3.3) admits a unique solution

$$u \in W_{T_0} \cap C([0, T_0]; V) \cap C^1((0, T_0); H), \quad 0 < T_0 \leq T.$$

Proof. Let us fix $T_0 > 0$ such that

$$\{(1 + 2T_0e^{2T_0})L\}^2(e^{2\kappa T_0} - 1)T_0/(2\kappa\delta) < 1. \tag{2.3.4}$$

The operator F is defined on $L^2(0, T_0; V)$ by letting $Fu = w$ be a solution of the following Cauchy problem:

$$\begin{cases} w''(t) + Aw(t) = f(t, u(t)) + h(t), \quad 0 < t \leq T_0, \\ w(0) = u_0, \quad w'(0) = u_1. \end{cases} \tag{2.3.5}$$

Invoking Proposition 2.3.1, we obtain that the problem (2.3.5) has a unique solution $w \in W_{T_0} \cap C([0, T_0]; V) \cap C^1((0, T_0); H)$. We will show that the operator F is strictly contractive from $L^2(0, T_0; V)$ to itself if the condition (2.3.4) is satisfied. □

To prove this theorem, we use the following lemma.

Lemma 2.3.3. Let w_1, w_2 be the solutions of (2.3.5) with u replaced by $u_1, u_2 \in L^2(0, T_0; V)$, respectively. Then the following inequality holds:

$$|w_1(t) - w_2(t)| \leq \alpha(t)L \int_0^t e^{\kappa(t-s)}||u_1(t) - u_2(t)||ds, \tag{2.3.6}$$

where $\alpha(t) = 1 + 2te^{2t}$.

Proof. For $i = 1,\ 2$, we consider the following equation:

$$\begin{cases} w_i''(t) + Aw_i(t) = f(t, u_i(t)) + h(t), \quad 0 < t \leq T, \\ w(0) = u_0, \quad w'(0) = u_1. \end{cases} \tag{2.3.7}$$

Then, we have that

$$(w_1(t) - w_2(t))'' + A(w_1(t) - w_2(t)) = f(t, u_1(t)) - f(t, u_2(t)) \qquad (2.3.8)$$

for $t > 0$. Acting on the both sides (2.3.8) by $w_1'(t) - w_2'(t)$, we have

$$\frac{1}{2}\frac{d}{dt}|w_1'(t) - w_2'(t)|^2 + (A(w_1(t) - w_2(t)), w_1'(t) - w_2'(t))$$

$$= (f(t, u_1(t)) - f(t, u_2(t)), w_1'(t) - w_2'(t)), \qquad (2.3.9)$$

noting that

$$2\int_0^t (Aw(s), w'(s))ds = \mathrm{Re}(Aw(t), w(t)) - \mathrm{Re}(Aw(0), w(0))$$

and integrating (2.3.9) over $(0, t)$, which implies that

$$|w_1'(t) - w_2'(t)|^2 + \mathrm{Re}(A(w_1(t) - w_2(t)), w_1(t) - w_2(t))$$

$$\leq 2L\int_0^t ||u_1(s) - u_2(s)|| \cdot |w_1'(s) - w_2'(s)|ds.$$

Putting

$$G(t) = 2L||u_1(t) - u_2(t)|| \cdot |w_1'(t) - w_2'(t)|,$$

which yields that

$$|w_1'(t) - w_2'(t)|^2 + \delta||w_1(t) - w_2(t)||^2 \qquad (2.3.10)$$

$$\leq \kappa|w_1(t) - w_2(t)|^2 + \int_0^t G(s)ds.$$

From (2.3.10), it follows that

$$
\begin{aligned}
&\frac{d}{dt}\{e^{-2\kappa t}|w_1(t)-w_2(t)|^2\} \qquad (2.3.11)\\
&= 2e^{-2\kappa t}\{\frac{1}{2}\frac{d}{dt}|w_1(t)-w_2(t)|^2-\kappa|w_1(t)-w_2(t)|^2\}\\
&= 2e^{-2\kappa t}\{\mathrm{Re}(w_1^{'}(t)-w_2^{'}(t), w_1(t)-w_2(t))-\kappa|w_1(t)-w_2(t)|^2\}\\
&\le 2e^{-2\kappa t}(|w_1^{'}(t)-w_2^{'}(t)|^2+|w_1(t)-w_2(t)|^2-\kappa|w_1(t)-w_2(t)|^2)\\
&\le 2e^{-2\kappa t}\{|w_1(t)-w_2(t)|^2+\int_0^t G(s)ds\}.
\end{aligned}
$$

Integrating (2.3.11) over $(0,t)$, we have

$$
\begin{aligned}
&e^{-2\kappa t}|w_1(t)-w_2(t)|^2\\
&\le 2\int_0^t e^{-2\kappa s}|w_1(s)-w_2(s)|^2 ds + 2\int_0^t e^{-2\kappa\tau}\int_0^\tau G(s)dsd\tau\\
&= 2\int_0^t e^{-2\kappa s}|w_1(s)-w_2(s)|^2 ds + \frac{1}{\kappa}\int_0^t (e^{-2\kappa s}-e^{-2\kappa t})G(s)ds.
\end{aligned}
$$

By Gronwall's inequality of Lemma 2.3.2, we get

$$e^{-2\kappa t}|w_1(t)-w_2(t)|^2 \le \frac{\alpha(t)}{\kappa}\int_0^t (e^{-2\kappa s}-e^{-2\kappa t})G(s)ds,$$

where $\alpha(t) = 1 + 2te^{2t}$, that is,

$$\kappa|w_1(t) - w_2(t)|^2 \leq \alpha(t) \int_0^t (e^{2\kappa(t-s)} - 1)G(s)ds. \tag{2.3.12}$$

From (2.3.10) and (2.3.12) it follows that

$$|w_1'(t) - w_2'(t)|^2 + \delta||w_1(t) - w_2(t)||^2 \leq \alpha(t) \int_0^t e^{2\kappa(t-s)}G(s)ds, \tag{2.3.13}$$

which implies

$$\begin{aligned}&\frac{1}{2}(e^{-\kappa t}|w_1'(t) - w_2'(t)|)^2 + \frac{1}{2}\delta e^{-2\kappa t}||w_1(t) - w_2(t)||^2 \\ &\leq \alpha(t) \int_0^t e^{-\kappa s} L||u_1(s) - u_2(s)|| \cdot e^{-\kappa s}|w_1'(s) - w_2'(s)|ds.\end{aligned}$$

By using Lemma 2.3.1, we obtain that

$$e^{-\kappa t}|w_1'(t) - w_2'(t)| \leq \alpha(t)L \int_0^t e^{-\kappa s}||u_1(s) - u_2(s)||ds. \tag{2.3.14}$$

□

Now we are to begin proving this theorem. From (2.3.13) and (2.3.14) it follows that

$$\begin{aligned}&|w_1'(t) - w_2'(t)|^2 + \delta||w_1(t) - w_2(t)||^2 \qquad (2.3.15)\\ &\leq 2(\alpha(t)L)^2 \int_0^t e^{2\kappa(t-s)}||u_1(s) - u_2(s)|| \int_0^s e^{\kappa(s-\tau)}||u_1(\tau) - u_2(\tau)||d\tau ds \\ &= 2(\alpha(t)L)^2 e^{2\kappa t} \int_0^t e^{-\kappa s}||u_1(s) - u_2(s)|| \int_0^s e^{-\kappa\tau}||u_1(\tau) - u_2(\tau)||d\tau ds\end{aligned}$$

$$= 2(\alpha(t)L)^2 e^{2\kappa t}\int_0^t \frac{1}{2}\frac{d}{ds}\{\int_0^s e^{-\kappa\tau}||u_1(\tau)-u_2(\tau)||d\tau\}^2 ds$$

$$= (\alpha(t)L)^2 e^{2\kappa t}\{\int_0^t e^{-\kappa\tau}||u_1(\tau)-u_2(\tau)||d\tau\}^2$$

$$\leq (\alpha(t)L)^2 e^{2\kappa t}\int_0^t e^{-2\kappa\tau}d\tau\int_0^t ||u_1(\tau)-u_2(\tau)||^2 d\tau$$

$$= (\alpha(t)L)^2 e^{2\kappa t}\frac{1-e^{-2\kappa t}}{2\kappa}\int_0^t ||u_1(\tau)-u_2(\tau)||^2 d\tau$$

$$= \frac{(\alpha(t)L)^2}{2\kappa}(e^{2\kappa t}-1)\int_0^t ||u_1(s)-u_2(s)||^2 ds.$$

Starting from initial value $u_0(t) = u_0$, consider a sequence $\{u_n(\cdot)\}$ satisfying

$$\begin{cases} u''_{n+1}(t) + Au_{n+1}(t) = f(t, u_n(t)) + h(t), & 0 < t \leq T, \\ u_{n+1}(0) = u_0, \quad u'_{n+1}(0) = u_1. \end{cases}$$

Then from (2.3.15) it follows that

$$|u'_{n+1}(t) - u'_n(t)|^2 + \delta||u_{n+1}(t) - u_n(t)||^2 \tag{2.3.16}$$
$$\leq \frac{(\alpha(t)L)^2}{2\kappa}(e^{2\kappa t}-1)\int_0^t ||u_n(s)-u_{n-1}(s)||^2 ds.$$

Hence, we obtain that

$$||u_{n+1}-u_n||^2_{L^2(0,T_0;V)} \leq \frac{(\alpha(T_0)L)^2}{2\kappa\delta}(e^{2\kappa T_0}-1)T_0||u_n-u_{n-1}||^2_{L^2(0,T_0;V)}. \tag{2.3.17}$$

So by virtue of the condition (2.3.4) the contraction principle gives that there exists $u(\cdot) \in L^2(0, T_0; V)$ such that

$$u_n(\cdot) \to u(\cdot) \quad \text{in} \quad L^2(0, T_0; V),$$

and hence, from (2.3.16) there exists $u(\cdot) \in C([0, T_0]; V) \cap C^1((0, T_0); H)$ such that

$$u_n(\cdot) \to u(\cdot) \quad \text{in} \quad C([0, T_0]; V) \cap C^1((0, T_0); H).$$

This completes the proof of Theorem. □

Now, we give a norm estimation of the solution (2.3.3) and establish the global existence of solutions with the aid of norm estimations.

Theorem 2.3.2. Let the Assumption (F1) be satisfied. Assume that $h \in L^2(0, T; H)(T > 0)$ and $(u_0, u_1) \in V \times H$. Then, the solution u of (2.3.3) exists and is unique in

$$u \in W_T \cap C([0, T]; V) \cap C^1((0, T); H), \quad T > 0.$$

Furthermore, there exists a constant C_2 depending on T such that

$$||u||_{W_T} \leq C_2(1 + ||u_0|| + |u_1| + ||h||_{L^2(0,T;H)}). \tag{2.3.18}$$

Proof. We establish the estimates of solution. Let w be the solution of

$$\begin{cases} w''(t) + Aw(t) = h(t), \quad 0 < t \leq T_0, \\ w(0) = u_0, \quad w^{'}(0) = u_1. \end{cases}$$

Then, since

$$(u(t) - w(t))'' + A(u(t) - w(t)) = f(t, u(t)),$$

by multiplying $u(t) - w(t)$ and using the monotonicity of A, we obtain

$$|u'(t) - w'(t)|^2 + \delta||u(t) - w(t)||^2 \tag{2.3.19}$$

$$\leq \kappa|u(t) - w(t)|^2 + 2L\int_0^t ||u(t)|| \cdot |u(t) - w(t)|dt.$$

By the procedure similar to (2.3.17) we have

$$||u - w||^2_{L^2(0,T_0;V)} \leq \frac{(\alpha(T_0)L)^2}{2\kappa\delta}(e^{2\kappa T_0} - 1)T_0||u||^2_{L^2(0,T_0;V)}.$$

Put

$$N^2 = \frac{(\alpha(T_0)L)^2}{2\kappa\delta}(e^{2\kappa T_0} - 1)T_0.$$

Then from Proposition 2.3.1, we have that

$$||u||_{L^2(0,T_0;V)} \tag{2.3.20}$$

$$\leq \frac{1}{1-N}||w||_{L^2(0,T_0;V)}$$

$$\leq \frac{C_1}{1-N}(1 + ||u_0|| + |u_1| + ||h||_{L^2(0,T_0;H)})$$

$$\leq C_2(1 + ||u_0|| + |u_1| + ||h||_{L^2(0,T_0;H)})$$

for some positive constant C_2. Noting that by Assumption (F1),

$$||f(\cdot, u(\cdot))||_{L^2(0,T_0;H)} \leq ||f(\cdot, u(\cdot)) - f(\cdot, 0)||_{L^2(0,T_0;H)} + ||f(\cdot, 0)||_{L^2(0,T_0;H)}$$

$$\leq \text{const.}(1 + ||u||_{L^2(0,T_0;V)})$$

and by Proposition 2.3.1,

$$||u||_{W^{2,2}(0,T_0;V^*)} \le \{1 + ||u_0|| + |u_1| + ||f(\cdot, u(\cdot)) + h||_{L^2(0,T_0;H)}\}.$$

It is easy to obtain the norm estimate of u in $W^{2,2}(0, T_0; V^*)$ satisfying (2.3.18).

Now from (2.3.15), (2.3.20) it follows that

$$|u(T_0)| \le ||u||_{C([0,T_0],H)} \tag{2.3.21}$$
$$\le C_2(1 + ||u_0|| + |u_1| + ||h||_{L^2(0,T_0;H)}).$$

So, we can solve the equation in $[T_0, 2T_0]$ and obtain an analogous estimate to (2.3.19). Since the condition (2.3.4) is independent of initial values, the solution of (2.3.3) can be extended to the internal $[0, nT_0]$ for natural number n, i.e., for the initial $u(nT_0)$ in the interval $[nT_0, (n+1)T_0]$, as analogous estimate (2.3.20) holds for the solution in $[0, (n+1)T_0]$. Furthermore, the estimate (2.3.18) is easily obtained from (2.3.19) and (2.3.21). □

2.4. Applications for nonlinear evolution equations

This section deals with the existence and uniqueness of the solutions to the nonlinear Volterra integrodifferential equations of the form

$$
\begin{cases}
\frac{\partial^2}{\partial t^2}u(t,x) - \Delta u(t,x) = \int_0^t \sum_{i=1}^n \frac{\partial}{\partial x_i}\sigma_i(s, \nabla u(s,x))ds + f(t), & \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le t < \infty, \quad x \in \Omega, & \\
u(t,x) = 0, \qquad\qquad\qquad\qquad\qquad 0 \le t < \infty, \quad x \in \partial\Omega, & \\
u(0,x) = u_0(x), \quad \frac{\partial}{\partial t}u(0,x) = u_1(x), \qquad\qquad x \in \Omega, &
\end{cases}
\tag{2.4.1}
$$

where $\sigma_i(s,\xi)$ are real-valued continuous functions defined in

$$\{(s,\xi) : 0 \le s < \infty, \ \xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n\}.$$

Here, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. These types of equations arise in the theory of viscoelasticity, and in the study of electromagnetism in rigid nonconducting material dielectrics(see [8, 4]). We study the initial-boundary value problem (2.4.1) in $L^2(\Omega)$.

We define the following spaces:

$$H^1(\Omega) = \left\{ u : u, \ \frac{\partial u}{\partial x_i} \in L^2(\Omega), \ \ i = 1, 2, \cdots, n \right\},$$

$$H^2(\Omega) = \left\{ u : u, \ \frac{\partial u}{\partial x_i}, \ \frac{\partial^2 u}{\partial x_i \partial x_j} \in L^2(\Omega), \ \ i, j = 1, 2, \cdots, n \right\},$$

where $\frac{\partial u}{\partial x_i}$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}$ are the derivative of u in the distribution sense.

The norm of $H^1(\Omega)$ is defined by

$$||u||_1 = \left\{ \int_\Omega (u(x)^2 + \sum_{i=1}^n (\frac{\partial u(x)}{\partial x_i})^2) dx \right\}^{\frac{1}{2}}.$$

Hence $H^1(\Omega)$ is a Hilbert space.

$$H_0^1(\Omega) = \{u : u \in H^1(\Omega),\ \ u|_{\partial\Omega} = 0\}$$

$$= \text{the closure of } \ C_0^\infty(\Omega) \ \text{ in } \ H^1(\Omega).$$

The norm and inner product of $H_0^1(\Omega)$ are defined by

$$||u|| = \left\{ \int_\Omega \sum_{i=1}^n (\frac{\partial u(x)}{\partial x_i})^2 dx \right\}^{\frac{1}{2}} = ||u||_1,$$

$$((u,v)) = \int_\Omega \sum_{i=1}^n \frac{\partial u(x)}{\partial x_i} \cdot \frac{\partial v(x)}{\partial x_i} dx$$

for any $u, v \in H_0^1(\Omega)$.

We put $\nabla = (\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})$. Define the operator A by

$$D(A) = \text{domain of A}$$

$$= \{u : u \in H^2(\Omega) \cap H_0^1(\Omega)\}$$

$$= \{u : u \in H^2(\Omega),\ \ u|_{\partial\Omega} = 0\},$$

$$Au = -\triangle u \quad \text{for all} \quad u \in D(A).$$

The operator A in $L^2(\Omega)$ define the following that for any $v \in H_0^1(\Omega)$ there exists $f \in L^2(\Omega)$ such that

$$((u,v)) = (f,v)$$

then, for any $u \in D(A)$, $Au = f$ and A is a positive definite self-adjoint operator.

Let $H^{-1}(\Omega)$=$H_0^1(\Omega)^*$ be a dual space of $H_0^1(\Omega)$. For any $l \in H^{-1}(\Omega)$ and $v \in H_0^1(\Omega)$, the notation (l, v) denotes the value l at v. The norm of $H^{-1}(\Omega)$ is defined by

$$||l||_{-1} = \sup_{v \in H_0^1(\Omega)} \frac{|(l, v)|}{||v||}.$$

Let u be fixed if we consider the functional $H_0^1(\Omega) \ni v \mapsto ((u, v))$, this function is continuous linear. For any $l \in H^{-1}(\Omega)$, it follow that $(l, v) = ((u, v))$. We denote that for any $u, v \in H_0^1(\Omega)$

$$((u, v)) = (\widetilde{A}u, v),$$

that is, $\widetilde{A}u = l$. The operator $\widetilde{A}$ is one to one mapping from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$. The relation of operators A and $\widetilde{A}$ satisfy the following that

$$D(A) = \{u \in H_0^1(\Omega), \ \ \widetilde{A}u \in L^2(\Omega)\}$$

$$Au = \widetilde{A}u \ \text{ for any } \ u \in D(A).$$

From now on, both A and $\widetilde{A}$ are denoted simply by A. For any $u \in D(A)$, we define the following that

$$G(t, u(t, x)) = \int_0^t \sum_{i=1}^n \frac{\partial}{\partial x_i} \sigma_i(s, \nabla u(s, x)) ds.$$

Then we treat it as the initial value problem for the abstract second order equations

$$\begin{cases} u''(t) + Au(t) = f(t, u(t)) + h(t) \\ u(0) = u_0, \qquad u'(0) = u_1. \end{cases} \tag{2.4.2}$$

In (2.4.2), A is the positive definite self-adjoint operator in $L^2(\Omega)$. We consider the equation (2.4.1) in Hilbert spaces forming a Galfand triple $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$. We have thus proved.

Theorem 2.4.1. We assume the following:

A) $\sigma_i(s, \xi)$ satisfies an uniform Lipschitz condition with respect to ξ, that is, there exists a constant $L > 0$ such that

$$|\sigma_i(s, \xi) - \sigma_i(s, \widehat{\xi})| \leq L|\xi - \widehat{\xi}|$$

where $|\cdot|$ denotes the norm of $L^2(\Omega)$. Without loss of the generality, it follows that $\sigma_i(s, 0) = 0$. Hence, there satisfies the following that

$$|\sigma_i(s, \xi)| \leq L|\xi|.$$

Assume that $h \in L^2(0, T; L^2(\Omega))$ and $(x_0, x_1) \in H_0^1(\Omega) \times L^2(\Omega)$. Then, the solution x of (2.4.1) exists and is unique in

$$L^2(0, T; H_0^1(\Omega)) \cap W^{2,2}(0, T; H^{-1}(\Omega)) \cap C([0, T]; H_0^1(\Omega)) \cap C^1((0, T); L^2(\Omega)).$$

Furthermore, there exists a constant C_2 depending on T such that

$$||x||_{L^2(0,T;H_0^1(\Omega)) \cap W^{2,2}(0,T;H^{-1}(\Omega))} \leq C_2(1 + ||x_0|| + |x_1| + ||h||_{L^2(0,T;L^2(\Omega))}).$$

Proof. Put

$$g(s,u)=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\sigma_i(s,\nabla u).$$

Then we have $g(s,u)\in H^{-1}(\Omega)$. For each $w\in H_0^1(\Omega)$, we satisfy the following that

$$(g(s,u),w)=-\sum_{i=1}^{n}(\sigma_i(s,\nabla u),\frac{\partial}{\partial x_i}w).$$

The nonlinear term is given by

$$f(t,u)=\int_0^t g(s,u)ds.$$

For any $w\in H_0^1(\Omega)$, if u and $\hat{u}$ belong to $H_0^1(\Omega)$, by the condition A) we obtain

$$|(f(t,u)-f(t,\hat{u})),w|\leq LT||u-\hat{u}||\,||w||.$$

Thus, we can apply the results of Theorem 2.3.2. □

Remark 2.4.1. The condition A) in Theorem 2.4.1 guarantees that the nonlinear term f given by

$$f(t,u)=\int_0^t g(s,u)ds$$

is Lipschitz continuous from $L^2(\Omega)$ into $H_0^1(\Omega)$, which is essential to obtain a strong solution of the case of a nonlinear partial differential equations. In this chapter, we no longer require the uniform boundedness and the uniform Lipschitz condition for $\sigma_i(s,\xi)$ and $\partial/\partial\, s\sigma_i(s,\xi)$ with respect to s in the study of [8], but instead we need L^2-regularity properties and a variation of solutions of semilinear retarded functional differential equations. So this sufficient condition in Theorem 2.4.1 is more general than the previous ones.

Chapter 3
Regular problems for semilinear hyperbolic type equations

3.1. Introduction

This chapter is concerned with the regularity of solutions for an abstract semilinear wave equation:

$$\begin{cases} u''(t) + A(t)u(t) = G(t, u(t)) + f(t) \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \tag{3.1.1}$$

The problem (3.1.1) is formulated as the following

$$\begin{cases} \frac{\partial^2}{\partial t^2}u(t,x) - \sum_{i,j=1}^n \frac{\partial}{\partial x_j}(a_{ij}(t,x)\frac{\partial u}{\partial x_i}) + c(t,x)u \\ = \int_0^t \sum_{i=1}^n \frac{\partial}{\partial x_i}\sigma_i(s, \nabla u(s,x))ds + h(t), \quad 0 \le t, \quad x \in \Omega, \\ u(t,x) = 0, \qquad 0 \le t, \quad x \in \partial\Omega, \\ u(0,x) = u_0(x), \quad \frac{\partial}{\partial t}u(0,x) = u_1(x), \qquad x \in \Omega. \end{cases} \tag{3.1.2}$$

Typical models can be found in the works of materials with biology, engineering, population models, etc.(see, for instance, [27, 5] and the bibliography therein). From the pioneering results as our linear case, the regularity for solutions of Cauchy problems for linear hyperbolic equations of second order with boundary conditions has been studied by Ikawa [9]. As the second order nonlinear functional evolutions, Kalsatos and Markov in [15] have analyzed some questions on existence of solutions for functional differential inclusions of second order in time and in [6] proved them in the case where a damping term is added. Kato [12, 13] was the first to make a successful attack on the hyperbolic type problem. In recent papers, which generalize Kato's linear theory, Tanaka [24] has proved wellposedness of the first order nonautonomous abstract Cauchy problems for strongly measurable families under a new type of quasi-stability condition from the viewpoint of the theory of finite difference approximations and Kobayashi [16] under strong continuity of A.

An example of parabolic type problems in which the nonlinear term is Lipschitz continuous but the mild solution of the equation is not a strong solution can be found in Webb [26]. We note that Lipschitz continuity of nonlinear term can be replaced by accretiveness and one still obtains, under suitable conditions, global solutions of the parabolic type equation, see Chapter 8 of Martin [19]. Recently, Kobayashi et al. [17] introduced the notion of semigroups of locally Lipschitz operators which provide us with mild solutions to the Cauchy problem for semilinear evolution equations. The regularity for the semilinear heat equations has been developed as seen in section 4.3.1 of Barbu [1] and [11].

In this chapter, we propose a different approach of the earlier works (briefly introduced in [9, 26, 8]) about the mild, strong, and classical solutions of Cauchy problems because we allow implicit arguments to occur in terms which deal with the L^2-regularity for solutions of semilinear hyperbolic equations under more general hypotheses of nonlinear term G. We are going to study that results of the linear cases to those of [9] on the L^2-regularity remain valid under the above formulation of the equation (3.1.1).

In section 3.2, we treat some basic results with the main tools of our scheme. We adhere to the construction of an evolution system for the equation (3.1.1) with unbounded operator $A(t)$ constructed by Kato [12, 13]. For each $t \geq 0$, $A(t)$ is the infinitesimal generator of an analytic semigroup together with some continuity conditions on the family of bounded operators $A(t)A(s)^{-1}$. Section 3.3 is devoted to the regularity for solutions of the linear wave equations in Gelfand triple spaces. Subsequently, our construction of a local solution of the nonlinear equation (3.1.1) is essentially based on [10]. We will show the energy inequalities for our problem (3.1.1) with the aid of estimate of L^2-type of the solutions, which is an important role in the proof of the global solutions and in that of the regularity of solutions. Finally, in section 3.4, a possible extension of the given equation (3.1.1) is discussed.

3.2. Preliminaries

Let V and H be complex Hilbert spaces forming Gelfand triple $V \subset H \subset V^*$ with pivot space H as mentioned in Chapter 2.

Let $a(t; u, v)$ be quadratic form defined on $V \times V$ and let us also make the following assumptions:

i) $a(t; u, v)$ is bounded and uniformly Lipschitz continuous and $d/dt\ a(t; u, v)$ is strong continuous with respect to t, i.e., there are some positive constants c_0, c_1 such that

$$|a(t;u,v)| \leq c_0||u||\,||v||,$$

$$|a(t;u,v) - a(s;u,v)| \leq c_1|t-s|\,||u||\,||v||,$$

$$|d/dt\ a(t;u,v)| = |\dot{a}(t;u,v)| \leq c_1||u||\,||v||;$$

ii) $a(t;u,v)$ is symmetric, i.e.,

$$a(t;u,v) = \overline{a(t;v,u)};$$

iii) $a(t;u,v)$ satisfies the Gårding's inequality, i.e.,

$$Re\ \ a(t;u,u) \geq \delta||u||^2, \quad \delta > 0.$$

Lemma 3.2.1. Let us define $A(t)$ as the operator determined by $a(t;u,v)$, i.e., we set

$$a(t;u,v) = (A(t)u,v), \quad u,v \in V.$$

Then $A(t)$ is an isomorphism V onto V^* and for $u \in V$, we have

$$\delta||u|| \leq ||A(t)u||_* \leq c_0||u||. \tag{3.2.1}$$

Proof. From assumtions i), ii) it follows that

$$||A(t)u||_* = \sup_{v\in V}\frac{|(A(t)u,v)|}{||v||} = \sup_{v\in V}\frac{|a(t;u,v)|}{||v||} \leq c_0||u||,$$

and

$$||A(t)u||_* \geq \frac{|(A(t)u,u)|}{||u||} = \frac{|a(t;u,u)|}{||u||} \geq \delta||u||,$$

which proves (3.2.1). □

The restriction of $A(t)$ to

$$D(A_H(t)) = \{u \in V;\ A(t)u \in H\}$$

is denoted by $A_H(t)$. Then it is well known that $D(A_H(t))$ is dense in H by Lax-Milgram theorem and it is easy to see that

$$\delta||u|| \leq |A_H(t)u| \leq c_0||u||_{D(A_H(t))}.$$

It is obvious that $A(t)$ is an extension of the operator $A_H(t)$.

Here and in what follows we consider that $D(A(t)) = V$ is independent of t from Lemma 3.2.1.

Consider the initial-value problem of the inhomogeneous second hyperbolic equation

$$\begin{cases} u''(t) + A(t)u(t) = f(t), \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \tag{3.2.2}$$

Put

$$\mathcal{A}(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 & -I \\ A(t) & 0 \end{pmatrix}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} -u_1 \\ A(t)u_0 \end{pmatrix}.$$

Let $U(t) = \binom{u_0(t)}{u_1(t)}$ where $u_1(t) = u_0'(t)$, and let $F(t) = \binom{0}{f(t)}$. Then the equation (3.2.2) can be rewritten by

$$\begin{cases} U'(t) + \mathcal{A}(t)U(t) = F(t) \\ U(0) = U_0, \end{cases} \tag{3.2.3}$$

where $U_0 = \binom{u_0}{u_1}$. We have known that $A_H(t)$ and $A(t)$ generate analytic semigroups in H and V^*, respectively, so the equation (3.2.2) is considered in the space both H and V^*.

Let X be a Banach space. We denoted by $G(X, M, \beta)$ the set of all linear operators A in X such that A generates a C_0-semigroup $\{e^{tA}\}$ with

$$||e^{tA}||_{\mathcal{L}(X)} \leq Me^{\beta t}, \quad 0 \leq t \leq \infty.$$

We write

$$G(X) \equiv \bigcup_{M>0,\, \beta\in\mathbb{R}} G(X, M, \beta).$$

Definition 3.2.1. Let $\{A(t) : 0 \leq t \leq T\}$ be a family of operators in $G(X)$. $\{A(t)\}$ is said to be " stable" with "stability index" M and β if there are $M > 0$ and $\beta \in \mathbb{R}$ suct that

$$||\prod_{j=1}^{k}(A(t_j) + \lambda)^{-1}||_{\mathcal{L}(X)} \leq M(\lambda - \beta)^{-k}, \quad \lambda > \beta$$

for every finite family $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq T, \quad k \in N$.

In the operator product on the left-hand side is time-ordered :

$A(t_j)$ is on the left of $A(t_i)$ if $t_j > t_i$.

Proposition 3.2.1. For each $t \in [0, T]$, let $|| \cdot ||_t$ be a new norm in X equivalent to the original one, depending on t smoothly in the sense that

$$\frac{||x||_t}{||x||_s} \leq \exp(c|t - s|), \;\; x \in X, \;\; s, t \in [0, T].$$

Assume that for each t, $A(t) \in G(X_t, 1, \beta)$, where X_t means the space X with norm $||\cdot||_t$. Then $\{A(t)\}$ is stable, with the stability index $M \equiv \exp(2cT)$ and β with respect to $||\cdot||_t$ for any $t \in [0,T]$(cf. Proposition 4.3.2 of [23]).

Proposition 3.2.2. (Corollary in section 4.4 of [23]) Suppose that $A(t)$ is stable, its domain $D(A(t))(t \geq 0) = V$ is independent of t and $A(t)v$ for each $v \in V$ is strongly continuously differentiable on $[0,T]$. Then there exists a unique function $U(t,s) \in \mathcal{L}(X)$ such that $U(t,s)$ maps V into V, $U(t,s)v$ for each $v \in V$ is strongly continuously differentiable in t and s, and the following results holds :

(a) $U(t,s)$ is strongly continuous in s and t,

$U(s,s) = I$ and $||U(t,s)||_{\mathcal{L}(X)} \leq Me^{\beta(t-s)}$,

(b) $U(t,s) = U(t,r)U(r,s)$ for $s \leq r \leq t$,

(c) $\partial/\partial t\, U(t,s)v = -A(t)U(t,s)v$,

(d) $\partial/\partial s\, U(t,s)v = U(t,s)A(s)v$.

Put $X = (V \times H)^T$, $\widetilde{X} = (H \times V^*)^T$. We define inner product of X and $\widetilde{X}$ by

$$\left(\left(\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}\right)\right)_X = ((u_0, v_0)) + (u_1, v_1),$$

and

$$\left(\begin{pmatrix} f_0 \\ f_1 \end{pmatrix}, \begin{pmatrix} g_0 \\ g_1 \end{pmatrix}\right)_{\widetilde{X}} = (f_0, g_0) + (f_1, g_1)_*,$$

respectively.

We introduce a new inner product $((\ ,\ ))_t$ and norm $||\cdot||_t$ into X as

$$\left(\left(\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}\right)\right)_t = a(t; u_0, v_0) + (u_1, v_1)$$

and

$$\left\|\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}\right\|_t = \{a(t; u_0, u_0) + (u_1, u_1)\}^{\frac{1}{2}}$$

for $\binom{u_0}{u_1}, \binom{v_0}{v_1} \in X$, respectively. Let us introduce a new norm in V^* as follows. For $f_1, g_1 \in V^*$, putting

$$(f_1, g_1)_{*,t} = a(t; A(t)^{-1}f_1, A(t)^{-1}g_1) = (f_1, A(t)^{-1}g_1),$$

it satisfies the inner product properties and its norm is given by

$$||f_1||_{*,t} = (f_1, f_1)_{*,t}^{1/2} = a(t; A(t)^{-1}f_1, A(t)^{-1}f_1)^{1/2} = (f_1, A(t)^{-1}f_1)^{1/2}.$$

It is easily seen that the norm $||\cdot||_{*,t}$ is equivalent to $||\cdot||_*$, i.e, we have

$$\frac{\delta}{\sqrt{c_0}}||\cdot||_{*,t} \le ||\cdot||_* \le \frac{c_0}{\sqrt{\delta}}||\cdot||_{*,t}. \tag{3.2.4}$$

We also introduce an inner product $(\ ,\ )_t$ and norm $|\cdot|_t$ into $\widetilde{X}$ as

$$\left(\begin{pmatrix} f_0 \\ f_1 \end{pmatrix}, \begin{pmatrix} g_0 \\ g_1 \end{pmatrix}\right)_t = (f_0, g_0) + a(t; A(t)^{-1}f_1, A(t)^{-1}g_1)$$

$$= (f_0, g_0) + (f_1, A(t)^{-1}g_1)_{*,t}$$

and

$$\left| \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \right|_t = \left(|f_0|^2 + ||f_1||_{*,t}^2 \right)^{1/2}.$$

The Hilbert spaces defined by the inner products mentioned above denote by X_t and $\widetilde{X}_t$, respectively.

Let $\mathcal{A}_X(t)$ be an operator defined by

$$D(\mathcal{A}_X(t)) = (D(A_H(t)) \times V)^T,$$

$$\mathcal{A}_X(t) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 & -I \\ A_H(t) & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} -u_1 \\ A_H(t)u_0 \end{pmatrix} \in (V \times H)^T = X.$$

In virtue of Lax-Milgram theorem we can also consider as

$$D(\mathcal{A}(t)) = (V \times H)^T = X,$$

$$\mathcal{A}(t) \begin{pmatrix} g_0 \\ g_1 \end{pmatrix} = \begin{pmatrix} 0 & -I \\ A(t) & 0 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \end{pmatrix} = \begin{pmatrix} -g_1 \\ A(t)g_0 \end{pmatrix} \in (H \times V^*)^T = \widetilde{X}.$$

Theorem 3.2.1. The linear operators $\mathcal{A}_X(t)$ and $\mathcal{A}(t)$ mentioned above are the infinitesimal generators of C_0-groups of unitary operators in X_t and $\widetilde{X}_t$, respectively.

Proof. First, we shall prove that $\mathcal{A}_X(t)$ and $\mathcal{A}(t)$ are skew self-adjoint operators on X_t and $\widetilde{X}_t$, respectively. For every $\binom{u_0}{u_1}, \binom{v_0}{v_1} \in D(\mathcal{A}_X(t)) =$

$D(A_H(t)) \times V$, then we have

$$\left(\left(\mathcal{A}_X(t)\begin{pmatrix}u_0\\u_1\end{pmatrix}, \begin{pmatrix}v_0\\v_1\end{pmatrix}\right)\right)_t = \left(\left(\begin{pmatrix}-u_1\\A_H(t)u_0\end{pmatrix}, \begin{pmatrix}v_0\\v_1\end{pmatrix}\right)\right)_t$$

$$= a(t; -u_1, v_0) + (A_H(t)u_0, v_1)$$

$$= -(A_H(t)u_1, v_0) + (A_H(t)u_0, v_1),$$

and

$$\left(\left(\begin{pmatrix}u_0\\u_1\end{pmatrix}, \mathcal{A}_X(t)\begin{pmatrix}v_0\\v_1\end{pmatrix}\right)\right)_t = \left(\left(\begin{pmatrix}u_0\\u_1\end{pmatrix}, \begin{pmatrix}-v_1\\A_H(t)v_0\end{pmatrix}\right)\right)_t$$

$$= -a(t; u_0, v_1) + (u_1, A_H(t)v_0)$$

$$= -(A_H(t)u_0, v_1) + (u_1, A_H(t)v_0).$$

Noting that $\mathcal{A}_X(t)$ is symmetric, we have that which implies that $\mathcal{A}_X^*(t) = -\mathcal{A}_X(t)$, i.e., $i\mathcal{A}_X(t) = (i\mathcal{A}_X(t))^*$, therefore, $i\mathcal{A}_X(t)$ is self adjoint(skew self adjoint). Hence, from Stone's theorem, it follows that $\mathcal{A}_X(t)$ is the infinitesimal generator of a C_0-group of unitary operators on X if and only if $i\mathcal{A}_X(t)$ is self adjoint.

For every $\begin{pmatrix}u_0\\u_1\end{pmatrix}, \begin{pmatrix}v_0\\v_1\end{pmatrix} \in \widetilde{X} = V \times H$, we have also obtained that

$$\left(\mathcal{A}(t)\begin{pmatrix}u_0\\u_1\end{pmatrix}, \begin{pmatrix}v_0\\v_1\end{pmatrix}\right)_t = \left(\begin{pmatrix}-u_1\\A(t)u_0\end{pmatrix}, \begin{pmatrix}v_0\\v_1\end{pmatrix}\right)_t$$

$$= (-u_1, v_0) + (A(t)u_0, v_1)_*$$

$$= -(u_1, v_0) + a(t; A(t)^{-1}A(t)u_0, A(t)^{-1}v_1)$$

$$= -(u_1, v_0) + a(t; u_0, A(t)^{-1}v_1)$$

$$= -(u_1, v_0) + \overline{a(t; A(t)^{-1}v_1, u_0)}$$

$$= -(u_1, v_0) + \overline{(v_1, u_0)} = -(u_1, v_0) + (u_0, v_1)$$

and

$$\left(\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \mathcal{A}(t)\begin{pmatrix} v_0 \\ v_1 \end{pmatrix}\right)_t = \left(\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \begin{pmatrix} -v_1 \\ A(t)v_0 \end{pmatrix}\right)_t$$

$$= (u_0, -v_1) + (u_1, A(t)v_0)_*$$

$$= -(u_0, v_1) + a(t; A(t)^{-1}u_1, A(t)^{-1}A(t)v_0)$$

$$= -(u_0, v_1) + a(t; A(t)^{-1}u_1, v_0) = -(u_0, v_1) + (u_1, v_0).$$

Hence,

$$\left(\mathcal{A}(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}\right)_t = -\left(\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \mathcal{A}(t)\begin{pmatrix} v_0 \\ v_1 \end{pmatrix}\right)_t,$$

so, $\mathcal{A}(t)$ is skew self-adjoint operator on $\widetilde{X}_t$. □

Theorem 3.2.2. Assume the hypotheses as in Theorem 3.2.1. Then $\mathcal{A}_X(t)$ and $\mathcal{A}(t)$ are stable on X and $\widetilde{X}$, respectively.

Proof. In virtue of Theorem 3.2.1, we may consider that

$$\mathcal{A}_X(t) \in G(X_t, 1, \beta) \ (\text{or} \mathcal{A}(t) \in G(\widetilde{X}_t, 1, \beta)).$$

For every $\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in X$, we have

$$\Big|\frac{\|\binom{u_0}{u_1}\|_t^2}{\|\binom{u_0}{u_1}\|_s^2}-1\Big| = \Big|\frac{a(t;u_0,u_0)+|u_1|^2}{a(s;u_0,u_0)+|u_1|^2}-1\Big|$$

$$= \Big|\frac{a(t;u_0,u_0)-a(s;u_0,u_0)}{a(s;u_0,u_0)+|u_1|^2}\Big|$$

$$\le \frac{c_1(t-s)||u_0||^2}{\delta||u_0||^2} = \frac{c_1}{\delta}|t-s|,$$

so that

$$\frac{\|\binom{u_0}{u_1}\|_t^2}{\|\binom{u_0}{u_1}\|_s^2} \le 1+\frac{c_1}{\delta}|t-s| \le e^{c_1|t-s|/\delta}.$$

Therefore $\{\mathcal{A}_X(t)\}$ is stable with the stability index $M = e^{2c_1T/\delta}$ and $\beta = 0$ on X. For $f \in V^*$, we have

$$||f||_{*,t}^2 - ||f||_{*,s}^2 = (f, A(t)^{-1}f) - (f, A(s)^{-1}f)$$

$$= (f, A(t)^{-1}f - A(s)^{-1}f).$$

Put $v = A(t)^{-1}f - A(s)^{-1}f$. From

$$\delta||v||^2 \le a(t;v,v) = a(t;A(t)^{-1}f - A(s)^{-1}f, v)$$

$$= a(t;A(t)^{-1}f,v) - a(t;A(s)^{-1}f,v) + a(s;A(s)^{-1}f,v) - a(s;A(s)^{-1}f,v)$$

$$= (f,v) - a(t;A(s)^{-1}f,v) + a(s;A(s)^{-1}f,v) - (f,v)$$

$$= -a(t;A(s)^{-1}f,v) + a(s;A(s)^{-1}f,v)$$

$$\le c_1|t-s|\cdot||A(s)^{-1}f||\cdot||v||,$$

we have

$$\delta||v|| \le c_1|t-s|\cdot||A(s)^{-1}f|| \le \frac{c_1}{\sqrt{\delta}}|t-s|\cdot a(s;A(s)^{-1}f, A(s)^{-1}f)^{1/2}$$

$$= \frac{c_1}{\sqrt{\delta}}|t-s|\cdot||f||_{*,s}.$$

Therefore, from (3.2.4), it holds that

$$\Big|||f||^2_{*,t} - ||f||^2_{*,s}\Big| = |(f,v)| \le ||f||_*||v||$$

$$\le \frac{c_0}{\sqrt{\delta}}||f||_{*,s}\frac{c_1}{\delta\sqrt{\delta}}|t-s|\cdot||f||_{*,s}$$

$$= \frac{c_0c_1}{\delta^2}|t-s|\cdot||f||^2_{*,s}.$$

Finally, we have

$$\Big|\frac{|\begin{pmatrix} f_0 \\ f_1 \end{pmatrix}|^2_t}{|\begin{pmatrix} f_0 \\ f_1 \end{pmatrix}|^2_s} - 1\Big| = \Big|\frac{|f_0|^2 + ||f_1||^2_{*,t}}{|f_0|^2 + ||f_1||^2_{*,s}} - 1\Big|$$

$$= \Big|\frac{||f_1||^2_{*,t} - ||f_1||^2_{*,s}}{|f_0|^2 + ||f_1||^2_{*,s}}\Big|$$

$$\le \frac{c_0c_1|t-s|/\delta^2\cdot||f_1||^2_{*,s}}{||f_1||^2_{*,s}} = c_0c_1|t-s|/\delta^2,$$

so $\{\mathcal{A}(t)\}$ is stable with index $M = exp(2c_0c_1T/\delta^2)$ and $\beta = 0$ on $\widetilde{X}$. □

In virtue of Theorems 3.2.1 and 3.2.2, we obtain follows the following results from Propositions 3.2.1 and 3.2.2.

Theorem 3.2.3. Let $\mathcal{A}_X(t)$ and $\mathcal{A}(t)$ be the operators mentioned above. Then there exist fundamental solutions $\mathcal{U}_X(t,s)$ and $\mathcal{U}(t,s)$ satisfying (a), (b), (c), and (d) in Proposition 3.2.2 in X and $\widetilde{X}$, respectively.

Proof. For every $\binom{u_0}{u_1} \in D(\mathcal{A}_X(t)) = D(A_H(t)) \times V$, we have

$$\frac{d}{dt}\mathcal{A}_X(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} -u_1 \\ A_H(t)u_0 \end{pmatrix} = \begin{pmatrix} 0 \\ d/dt\, A_H(t)u_0 \end{pmatrix}.$$

From which and $d/dt\ (A_H(t)u, v) = \dot{a}(t; u, v)$, it follows that $d/dt\, \mathcal{A}_X(t)\binom{u_0}{u_1}$ is strongly continuous with respect to t, that is, for each $\binom{u_0}{u_1} \in D(\mathcal{A}_X(t)) = D(A_H(t)) \times V$ (or $\binom{u_0}{u_1} \in X$), $\mathcal{A}_X(t)\binom{u_0}{u_1}$ (or $\mathcal{A}(t)\binom{u_0}{u_1}$, respectively) is strongly continuously differentiable on $[0, T]$. Thus this theorem is from Theorems 3.2.1 and 3.2.2, and Proposition 3.2.2. □

3.3. Semilinear equations of hyperbolic type

First, we consider the existence and regularity of solutions for the following linear inhomogeneous wave equation:

$$\begin{cases} u''(t) + A(t)u(t) = f(t) \\ u(0) = u_0, \quad u'(0) = u_1, \end{cases} \tag{3.3.1}$$

where $A(t)$ satisfies the conditions of the preceding section.

Let $\mathbf{x}(t) = \binom{u(t)}{u'(t)}$ and $F(t) = \binom{0}{f(t)}$. We can show that a solution $\mathbf{x}(t)$ of (3.3.1) is represented by

$$\mathbf{x}(t) = \mathcal{U}(t, 0)\mathbf{x}(0) + \int_0^t \mathcal{U}(t, s)F(s)ds \tag{3.3.2}$$

using the fundamental solution $\mathcal{U}(t,s)$ constructed in Theorem 3.2.3 and Proposition 3.2.2. Indeed, by Proposition 3.2.2, we have

$$\begin{aligned}(\partial/\partial s)\mathcal{U}(t,s)\mathbf{x}(s) &= \mathcal{U}(t,s)\mathbf{x}'(s) + \mathcal{U}(t,s)\mathcal{A}(s)\mathbf{x}(s)\\ &= \mathcal{U}(t,s)(\mathbf{x}'(s) + \mathcal{A}(s)\mathbf{x}(s))\\ &= \mathcal{U}(t,s)F(s),\end{aligned}$$

which, being integrated from 0 to t, yields (3.3.2). Let $T>0$. Define

$$W_T = \{u : u\in L^2(0,T;D(A_H)),\ \dot{u}\in L^2(0,T;V),\ \ddot{u}\in L^2(0,T;H)\},$$

$$||u||_{W_T} = ||u||_{L^2(0,T;D(A_H))} + ||\dot{u}||_{L^2(0,T;V)} + ||\ddot{u}||_{L^2(0,T;H)}$$

and

$$\widetilde{W}_T = \{u : u\in L^2(0,T;V),\ \dot{u}\in L^2(0,T;H),\ \ddot{u}\in L^2(0,T;V^*)\},$$

$$||u||_{\widetilde{W}_T} = ||u||_{L^2(0,T;V)} + ||\dot{u}||_{L^2(0,T;H)} + ||\ddot{u}||_{L^2(0,T;V^*)},$$

where $\dot{u}$ denote the derivative of u in the generalized sense. Since

$$\mathcal{A}(t)^{-1} = \begin{pmatrix} 0 & A(t)^{-1}\\ -I & 0\end{pmatrix} : \widetilde{X}\to X$$

is a bounded operator. It holds $\mathcal{A}(t)\mathcal{U}(t,s)\mathcal{A}(t)^{-1} : \widetilde{X}\to\widetilde{X}$ is bounded and strong continuous jointly in s,t. Therefore, there is a constant $M>0$ such that

$$||\mathcal{U}(t,s)||_{\mathcal{L}(\widetilde{X})}\le M,\ \ ||\mathcal{A}(t)\mathcal{U}(t,s)\mathcal{A}(s)^{-1}||_{\mathcal{L}(\widetilde{X})}\le M. \tag{3.3.3}$$

By the assumption i) of $a(s;u,v)$, it holds that for every $u, v \in V$,

$$|d/ds\ (A(s)u,v)| = |\dot{a}(s;u,v)| \le c_1||u||\,||v||,$$

that is, we have that for every $u \in V$, $s \mapsto d/dsA(s)u$ is strongly continuous in V^* and so, $||d/dsA(s)||_{\mathcal{L}(V,V^*)}$ is bounded on $[0,T]$. Hence, noting that for every $\binom{u_0}{u_1} \in X$,

$$\frac{d}{dt}\mathcal{A}(s)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} -u_1 \\ A(s)u_0 \end{pmatrix} = \begin{pmatrix} 0 \\ d/dt\,A(s)u_0 \end{pmatrix},$$

it follows that $d/ds\,\mathcal{A}(s)\binom{u_0}{u_1}$ is strongly continuous with respect to t in $\widetilde{X}$ and so, $||d/ds\mathcal{A}(s)||_{\mathcal{L}(X,\tilde{X})}$ is bounded on $[0,T]$. Therefore, we may assume that

$$||\frac{d}{ds}\mathcal{A}(s)\mathcal{A}(s)^{-1}||_{\mathcal{L}(\tilde{X})} \le M. \tag{3.3.4}$$

Now we show the energy inequalities for our problem (3.3.1), which is an important role in the proof of the existence of solution and in that of the regularity of solutions.

Theorem 3.3.1. Assume that $f \in C([0,T];V^*) \cap W^{1,2}(0,T;V^*)(T > 0)$ and the initial data $(u_0, u_1) \in V \times H$. Then the solution u of (3.3.1) exists and is unique in

$$u \in \widetilde{W_T} \cap C([0,T];V) \cap C^1((0,T);H).$$

Furthermore, the following energy inequality holds: there exists a constant C_T depending on T such that

$$||u||_{\widetilde{W}_T} \le C_T(||u_0|| + |u_1| + ||f(0)||_* + ||f||_{W^{1,2}(0,T;V^*)}). \tag{3.3.5}$$

If $f \in C([0,T];H) \cap W^{1,2}(0,T;H)$ and $(u_0, u_1) \in D(A_H) \times V$, then the solution u of (3.3.1) exists and is unique in

$$u \in W_T \cap C([0,T];D(A_H)) \cap C^1((0,T);V),$$

satisfying

$$||u||_{W_T} \le C_T(||u_0||_{D(A_H)} + ||u_1|| + |f(0)| + ||f||_{W^{1,2}(0,T;H)}). \tag{3.3.6}$$

Proof. Regarding that the equation (3.3.1) may be considered as an equation in both H and V^*, so now we investigate the consequences of the equation as in $\widetilde{X}$. Since $\{\mathcal{A}_X(t) : 0 \le t \le T\}$ and $\{\mathcal{A}(t) : 0 \le t \le T\}$ are stable on X and $\widetilde{X}$, respectively, in virtue of Theorem 3.2.3, there exists a fundamental solution $\mathcal{U}(t,s)$ of

$$\frac{d}{dt}\begin{pmatrix} w_0(t) \\ w_1(t) \end{pmatrix} + \mathcal{A}(t)\begin{pmatrix} w_0(t) \\ w_1(t) \end{pmatrix} = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

Let $u(t)$ be the solution of the equation

$$\begin{cases} u''(t) + A(t)u(t) = f(t) \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases}$$

Then we put $w_0(t) = u(t)$, $w_1(t) = u'(t)$ and hence obtain

$$\begin{pmatrix} w_0(t) \\ w_1(t) \end{pmatrix} = \mathcal{U}(t,0) \begin{pmatrix} w_0(0) \\ w_1(0) \end{pmatrix} + \int_0^t \mathcal{U}(t,s) \begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds. \tag{3.3.7}$$

From the property (d) in Proposition 3.2.2 , it follows

$$\int_0^t \mathcal{U}(t,s) \begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds = \int_0^t \mathcal{U}(t,s)\mathcal{A}(s)\mathcal{A}(s)^{-1} \begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds$$

$$= \int_0^t \frac{\partial}{\partial s}\mathcal{U}(t,s)\mathcal{A}(s)^{-1} \begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds$$

$$= \mathcal{A}(t)^{-1} \begin{pmatrix} 0 \\ f(t) \end{pmatrix} - \mathcal{U}(t,0)\mathcal{A}(0)^{-1} \begin{pmatrix} 0 \\ f(0) \end{pmatrix}$$

$$- \int_0^t \mathcal{U}(t,s)\mathcal{A}(s)^{-1} \left\{ -\frac{d}{ds}\mathcal{A}(s)\mathcal{A}(s)^{-1} \begin{pmatrix} 0 \\ f(s) \end{pmatrix} + \begin{pmatrix} 0 \\ f'(s) \end{pmatrix} \right\} ds.$$

From which and (3.3.7) we have

$$\mathcal{A}(t) \begin{pmatrix} w_0(t) \\ w_1(t) \end{pmatrix} = \mathcal{A}(t)\mathcal{U}(t,0)\mathcal{A}(0)^{-1}\mathcal{A}(0) \begin{pmatrix} w_0(0) \\ w_1(0) \end{pmatrix} \tag{3.3.8}$$

$$+ \begin{pmatrix} 0 \\ f(t) \end{pmatrix} - \mathcal{A}(t)\mathcal{U}(t,0)\mathcal{A}(0)^{-1} \begin{pmatrix} 0 \\ f(0) \end{pmatrix}$$

$$- \int_0^t \mathcal{A}(t)\mathcal{U}(t,s)\mathcal{A}(s)^{-1} \left\{ -\frac{d}{ds}\mathcal{A}(s)\mathcal{A}(s)^{-1} \begin{pmatrix} 0 \\ f(s) \end{pmatrix} + \begin{pmatrix} 0 \\ f'(s) \end{pmatrix} \right\} ds.$$

Therefore, by (3.3.3) we have that there exists a constant c_1 such that

$$||\mathcal{A}(t) \begin{pmatrix} w_0(t) \\ w_1(t) \end{pmatrix} ||_{\widetilde{X}} \le c_1 \left\{ ||\mathcal{A}(0) \begin{pmatrix} w_0(0) \\ w_1(0) \end{pmatrix} || + || \begin{pmatrix} 0 \\ f(t) \end{pmatrix} || \right. \tag{3.3.9}$$

$$\left. + || \begin{pmatrix} 0 \\ f(0) \end{pmatrix} || + \int_0^t || \begin{pmatrix} 0 \\ f(s) \end{pmatrix} || + || \begin{pmatrix} 0 \\ f'(s) \end{pmatrix} || ds \right\}$$

$$= c_1 \Big\{ || \begin{pmatrix} -w_1(0) \\ A(0)w_0(0) \end{pmatrix} || + ||f(t)||_* + ||f(0)||_*$$

$$+ \int_0^t (||f(s)||_* + ||f'(s)||_*)\, ds \Big\},$$

where $c_1 = \max\{M, 1, M^2\}$. Here, we remark that

$$\left\| \mathcal{A}(0) \begin{pmatrix} w_0(0) \\ w_1(0) \end{pmatrix} \right\|_{\widetilde{X}} = \left\| \begin{pmatrix} -w_1(0) \\ A(0)w_0(0) \end{pmatrix} \right\|_{\widetilde{X}} \tag{3.3.10}$$

$$= (|w_1(0)|^2 + ||A(0)w_0(0)||_*^2)^{\frac{1}{2}} \le \max\{1, c_0\}(||w_0(0)|| + |w_1(0)|),$$

$$||f(t)||_* = ||f(0) + \int_0^t f'(s)ds||_* \le ||f_0||_* + \int_0^t ||f'(s)||_* ds \tag{3.3.11}$$

and

$$\int_0^t ||f(s)||_* ds = \int_0^t ||f(0) + \int_0^s f'(\sigma)d\sigma||_* ds \tag{3.3.12}$$

$$\le t||f(0)||_* + \int_0^t \int_0^s ||f'(\sigma)||_* d\sigma ds = t||f(0)||_* + \int_0^t (t-\sigma)||f'(\sigma)||_* d\sigma.$$

We recall that

$$\left\| \mathcal{A}(t) \begin{pmatrix} w_0(t) \\ w_1(t) \end{pmatrix} \right\|_{\widetilde{X}} \ge \min\{1, \delta\}(||w_0(t)||^2 + |w_1(t)|^2)^{\frac{1}{2}}. \tag{3.3.13}$$

Hence, from (3.3.7)-(3.3.13), it follows that

$$(||w_0(t)||^2 + |w_1(t)|^2)^{1/2} \tag{3.3.14}$$

$$\le c_1/\min\{1, \delta\}\{\max\{1, c_0\}(||w_0(0)|| + |w_1(0)|) + ||f(0)||_* + \int_0^t ||f'(s)||_* ds\}.$$

Therefore, we see that $u \in C([0,T];V) \cap C^1((0,T);H)$(or the continuity of solutions for the equation (3.3.1) is also obtained by using an application of the theory of intermediate spaces(see [18], Vol. I, Theorem 3.1)). Since $w_0(t) = u(t)$, $w_1(t) = u'(t)$ and $u''(t) = -A(t)u(t) + f(t)$, it holds

$$||u''(t)||_* = || - A(t)u(t) + f(t)||_* \le c_0||u(t)|| + ||f(t)||_*.$$

By this and (3.3.14), there exists a constant C_T such that

$$||u||_{\widetilde{W}_T} \le C_T(||u_0|| + |u_1| + ||f(0)||_* + ||f||_{W^{1,2}(0,T;V^*)}).$$

Let $f \in C([0,T];H) \cap W^{1,2}(0,T;H)(T > 0)$ and $(u_0, u_1) \in D(A_H) \times V$. Regarding that the equation (3.3.1) is considered as in X. The proof of (3.3.6) on W_T is completely analogous to the situation on $\widetilde{W}_T$ as in $\widetilde{X}$. □

From now on, by using the properties of the linear inhomogeneous equations, we investigate the regularity of solutions for abstract semilinear second order initial value problem:

$$\begin{cases} u''(t) + A(t)u(t) = G(t, u(t)) + f(t) \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \tag{3.3.15}$$

We assume the following hypotheses on the nonlinear term.

Assumption (G). Let $G : [0,T] \times V \to H$ be a nonlinear mapping such that $t \mapsto G(t, \cdot)$ is continuously differentiable on $[0,T]$ and $u \mapsto G(\cdot, u)$ be Lipschitz continuous on V: there exists constant $L > 0$ such that

$$\sup_{0 \le t \le T} |\partial/\partial t G(t,u)| \le L,$$

$$|G(\cdot,u) - G(\cdot,v)| \le L||u-v||, \quad u,v \in V.$$

In case where $A(t) = A$, by Theorem 2.3.1 (or [23]; Theorem 6.1.3), the equation (3.3.15) has a unique local solution on some interval $[0, T_c)$ for some $T_c \le T$. Even if $A(t)$ depend on t, similar results to that above still hold when the equation (3.3.15) has a fundamental solution, see Remark 6.1.1 of [23]. We shall see that the solution can be extended to $[0, T]$ for $T > 0$. To see this, it is enough to show that u is a solution in $0 < T_c \le T < \infty$, then $u(t)$ is bounded in $0 \le t < T_c$. We start with the following results.

Theorem 3.3.2. Let Assumption (G) be satisfied. Assume that $f \in C([0,T]; V^*) \cap W^{1,2}(0,T;V^*)(T > 0)$ and $(u_0, u_1) \in V \times H$. Then the solution u of the equation (3.3.15) exists and is unique in

$$u \in \widetilde{W}_T \cap C([0,T];V) \cap C^1((0,T);H), \quad T > 0.$$

Furthermore, the following energy inequality holds: there exists a constant C_T depending on T such that

$$||u||_{\widetilde{W}_T} \le C_T(1 + ||u_0|| + |u_1| + ||f(0)||_* + ||f||_{W^{1,2}(0,T;V^*)}). \tag{3.3.16}$$

Proof. Let $u(t)$ be the solution of the following equation:

$$\begin{cases} u''(t) + A(t)u(t) = G(t, u(t)) + f(t) \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases}$$

Put $w_0 = u(t)$, $w_1(t) = u'(t)$. Then, by Theorem 3.2.3, there exists a fundamental solution $\mathcal{U}(t,s)$ of

$$\frac{d}{dt}\begin{pmatrix} w_0(t) \\ w_1(t) \end{pmatrix} + \mathcal{A}(t)\begin{pmatrix} w_0(t) \\ w_1(t) \end{pmatrix} = \begin{pmatrix} 0 \\ G(t, w_0(t)) + f(t) \end{pmatrix},$$

and hence obtain

$$\begin{pmatrix} w_0(t) \\ w_1(t) \end{pmatrix} = \mathcal{U}(t,0)\begin{pmatrix} w_0(0) \\ w_1(0) \end{pmatrix} + \int_0^t \mathcal{U}(t,s)\begin{pmatrix} 0 \\ G(t, w_0(t)) + f(t) \end{pmatrix} ds. \quad (3.3.17)$$

For the estimate of the semilinear case for (3.3.7), from (3.3.17), we have

$$\mathcal{A}(t)\begin{pmatrix} w_0(t) \\ w_1(t) \end{pmatrix} = \mathcal{A}(t)\mathcal{U}(t,0)\mathcal{A}(0)^{-1}\mathcal{A}(0)\begin{pmatrix} w_0(t) \\ w_1(t) \end{pmatrix} \quad (3.3.18)$$

$$+ \begin{pmatrix} 0 \\ G(t, w_0(t)) + f(t) \end{pmatrix} - \mathcal{A}(t)\mathcal{U}(t,0)\mathcal{A}(0)^{-1}\begin{pmatrix} 0 \\ G(0, w_0(0)) + f(0) \end{pmatrix}$$

$$- \int_0^t \mathcal{A}(t)\mathcal{U}(t,s)\mathcal{A}(s)^{-1}\Bigg\{ -\frac{d}{ds}\mathcal{A}(s)\mathcal{A}(s)^{-1}\begin{pmatrix} 0 \\ G(s, w_0(s)) + f(s) \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ \frac{d}{ds}(G(s, w_0(s)) + f(s) \end{pmatrix} \Bigg\} ds.$$

Furthermore, by the similar way to (3.3.9) there exists a constant c_1 such that

$$||\mathcal{A}(t)\begin{pmatrix} w_0(t) \\ w_1(t) \end{pmatrix}|| \leq \quad (3.3.19)$$

$$c_1 \Bigg\{ || \begin{pmatrix} -w_1(0) \\ A(0)w_0(0) \end{pmatrix} || + |G(t, w_0(t)) + f(t)|_* + |G(0, w_0(0)) + f(0)|_*$$

$$+ \int_0^t \left(||G(s, w_0(s)) + f(s)\|_* + ||\frac{d}{ds}G(s, w_0(s)) + f(s)||_* \right) ds \Bigg\}.$$

Now, noting that

$$||d/dsG(s,w_0(s))||_* = ||G_1(s,w_0(s)) + G_2(s,w_0(s))u'(s)||^*$$

$$\leq ||G_1(s,w_0(s))||_* + L||w_0'(s)||$$

where $G_i(i = 1, 2)$ is the partial derivative of G, we have

$$||G(t,w_0(t)||_* = ||G(0,w_0(0)) + \int_0^t \frac{d}{ds}G(s,w_0(s))ds||_* \tag{3.3.20}$$

$$\leq (|G(0,w_0(0)) - G(0,0)| + |G(0,0)| + \int_0^t (|G_1(s,w_0(s))| + L||w_0'(s)||ds)$$

$$\leq L||w_0(0)|| + |G(0,0)| + L\int_0^t (1 + ||w_0'(s)||)ds$$

and

$$\int_0^t ||G(s,w_0(s)||_* ds$$

$$= \int_0^t \left\| G(0,w_0(0)) + \int_0^s \frac{d}{d\sigma}G(\sigma,w_0(\sigma))d\sigma \right\|_* ds \tag{3.3.21}$$

$$\leq t||G(0,w_0(0))||_* + \int_0^t (t-\sigma)\left\| \frac{d}{d\sigma}G(\sigma,w_0(\sigma)) \right\|_* ds$$

$$\leq t||G(0,w_0(0))||_* + L\int_0^t (t-\sigma)(1 + ||w_0'(\sigma)||)d\sigma.$$

Thus, from (3.3.10), (3.3.12), (3.3.18) - (3.3.21) we have that there exists a constant C' depending on T such that

$$(||w_0(t)||^2 + |w_1(t)|^2)^{1/2} \leq C'(1 + ||w_0|| + |w_1| + ||f(0)||_*$$

$$+ ||f||_{W^{1,2}(0,T;V^*)} + \int_0^t (1 + ||w_0'(s)||)ds),$$

noting that $w_1(t) = u'(t)$, which by Gronwall's inequality implies (3.3.16)

□

3.4. Applications

For each $t \in [0,T]$ and $u, v \in H^1(\Omega)$, let us consider the following sesquilinear form:

$$a(t; u, v) = \sum_{i,j=1}^{n} \int_\Omega (a_{ij}(t,x)\frac{\partial u}{\partial x_i}\overline{\frac{\partial v}{\partial x_j}} + c(t,x)u\bar{v})dx$$

where the matrix $(a_{ij}(t,x))$ is uniformly positive definite, i.e., there exists a positive constant δ such that

$$\sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\bar{\xi_j} \geq \delta|\xi|^2$$

for all $x \in \Omega,\ t \in [0,T]$ and for all real vectors ξ. Let

$$a_{ij}, \quad \frac{\partial}{\partial x_j}a_{ij}, \quad \frac{\partial}{\partial t}a_{ij}, \quad \frac{\partial^2}{\partial t \partial x_j}a_{ij}, \quad c \geq 0, \quad \frac{\partial}{\partial t}c$$

be all continuous and bounded on $\Omega \times [0,T]$, and

$$a_{ij}, \quad \frac{\partial}{\partial x_j}a_{ij}, \quad c$$

satisfy uniformly Lipschitz continuity with respect to t. Then there exist constants $c_0, c_1 > 0$ such that

$$|a(t,u,v)| \le c_0||u||||v||$$

$$\left|\frac{d}{dt}a(t,u,v)\right| = \Big|\int_\Omega (\sum_{i,j=1}^n \dot{a}_{ij}(t,x)\frac{\partial u}{\partial x_i}\overline{\frac{\partial v}{\partial x_j}} + \dot{c}(t,x)u\bar{v})dx\Big| \le c_1||u||\cdot||v||$$

and it holds Gårding's inequality ;

$$\begin{aligned} a(t;u,u) &= \int_\Omega (\sum_{i,j=1}^n a_{ij}(t,x)\frac{\partial u}{\partial x_i}\overline{\frac{\partial u}{\partial x_j}} + c(t,x)u\bar{u})dx \\ &\ge \delta\int_\Omega \Big|\frac{\partial u}{\partial x_i}\Big|^2 dx = \delta||u||^2. \end{aligned}$$

Consider the Cauchy problem for the hyperbolic equation:

$$\begin{cases} \frac{\partial^2}{\partial t^2}u(t,x) - \sum_{i,j=1}^n \frac{\partial}{\partial x_j}(a_{ij}(t,x)\frac{\partial u}{\partial x_i}) + c(t,x)u \\ = \int_0^t \sum_{i=1}^n \frac{\partial}{\partial x_i}\sigma_i(s,\nabla u(s,x))ds + h(t), \quad 0 \le t, \quad x \in \Omega, \\ u(t,x) = 0, \quad\quad 0 \le t, \quad x \in \partial\Omega, \\ u(0,x) = u_0(x), \quad \frac{\partial}{\partial t}u(0,x) = u_1(x), \quad x \in \Omega. \end{cases} \tag{3.4.1}$$

Define the operator $A(t)$ by

$$(A(t)u,v) = a(t;u,v) = \int_\Omega (\sum_{i,j=1}^n a_{ij}(t,x)\frac{\partial u}{\partial x_i}\overline{\frac{\partial v}{\partial x_j}} + c(t,x)u\bar{v})dx,$$

$$D(A(t)) = \{u : u \in H^2(\Omega)\cap H_0^1(\Omega)\} = \{u : u \in H^2(\Omega),\ \ u|_{\partial\Omega} = 0\}.$$

The operator $A(t)$ in $L^2(\Omega)$ is defined as the following that for any $v \in H_0^1(\Omega)$ there exists $f \in L^2(\Omega)$ such that

$$a(t; u, v) = (f, v)$$

then, for any $u \in D(A(t))$, $A(t)u = f$ and $A(t)$ is a positive definite self-adjoint operator. Let u be fixed if we consider the functional $H_0^1(\Omega) \ni v \mapsto a(t; u, v)$, this function is a continuous linear. For any $l \in H^{-1}(\Omega)$, it follow that $(l, v) = a(t; u, v)$. We denote that for any $u, v \in H_0^1(\Omega)$

$$a(t; u, v) = (\widetilde{A}(t)u, v),$$

that is, $\widetilde{A}(t)u = l$. The operator $\widetilde{A}(t)$ is one to one mapping from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$. The relation of operators $A(t)$ and $\widetilde{A}(t)$ satisfy the following that

$$D(A(t)) = \{u \in H_0^1(\Omega),\ \ \widetilde{A}(t)u \in L^2(\Omega)\}$$

$$A(t)u = \widetilde{A}(t)u \ \text{ for any } \ u \in D(A(t)).$$

From now on, both $A(t)$ and $\widetilde{A}(t)$ are denoted simply by $A(t)$. For any $u \in D(A(t))$, we define the following that

$$G(t, u(t,x)) = \int_0^t \sum_{i=1}^n \frac{\partial}{\partial x_i} \sigma_i(s, \nabla u(s,x)) ds.$$

Then we treat (3.4.1) as the initial value problem for the abstract second order equations:

$$\begin{cases} u''(t) + A(t)u(t) = G(t, u(t)) + f(t) \\ u(0) = u_0, \qquad u'(0) = u_1. \end{cases} \tag{3.4.2}$$

We assume the following:

Assumption (G1). The partial derivatives $\sigma_i(s,\xi)$, $\partial/\partial t\,\sigma_i(s,\xi)$ and $\partial/\partial \xi_j\,\sigma_i(s,\xi)$ exist and continuous for $i=1,2$, $j=1,2,\cdots,n$, and $\sigma_i(s,\xi)$ satisfies an uniform Lipschitz condition with respect to ξ, that is, there exists a constant $L>0$ such that

$$|\sigma_i(s,\xi)-\sigma_i(s,\widehat{\xi})|\le L|\xi-\widehat{\xi}|$$

where $|\cdot|$ denotes the norm of $L^2(\Omega)$.

Lemma 3.4.1. If Assumption (G1) is satisfied, then the mapping $t\mapsto G(t,\cdot)$ is continuously differentiable on $[0,T]$ and $u\mapsto G(\cdot,u)$ is Lipschitz continuous on V.

Proof. Put

$$g(s,u)=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\sigma_i(s,\nabla u).$$

Then we have $g(s,u)\in H^{-1}(\Omega)$. For each $w\in H_0^1(\Omega)$, we satisfy the following that

$$(g(s,u),w)=-\sum_{i=1}^{n}(\sigma_i(s,\nabla u),\frac{\partial}{\partial x_i}w).$$

The nonlinear term is given by

$$G(t,u)=\int_0^t g(s,u)ds.$$

For any $w \in H_0^1(\Omega)$, if u and $\hat{u}$ belong to $H_0^1(\Omega)$, by Assumption (G1) we obtain

$$|(G(t,u) - G(t,\hat{u})), w| \leq LT||u - \hat{u}|| \, ||w||.$$

□

Now in virtue of Lemma 3.4.1, we can apply the results of Theorem 3.3.2 as follows.

Theorem 3.4.1. Let Assumption (G1) be satisfied. Assume that $f \in C([0,T]; H^{-1}(\Omega)) \cap W^{1,2}(0,T; H^{-1}(\Omega)) (T > 0)$ and $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. Then the solution u of (3.4.1) exists and is unique in

$$u \in \widetilde{W_T} \cap C([0,T]; H_0^1(\Omega)) \cap C^1((0,T); L^2(\Omega)), \quad T > 0$$

where

$$\widetilde{W_T} = L^2(0,T; H_0^1(\Omega)) \cap W^{1,2}(0,T; L^2(\Omega)) \cap W^{2,2}(0,T; H^{-1}(\Omega)).$$

Furthermore, the following energy inequality holds: there exists a constant C_T depending on T such that

$$||u||_{\widetilde{W_T}} \leq C_T(1 + ||u_0|| + |u_1| + ||f(0)||_* + ||f||_{W^{1,2}(0,T;H^{-1}(\Omega))}).$$

Chapter 4
Perturbation results for hyperbolic evolution systems

4.1. Introduction

The purpose of this chapter is to derive a perturbation theory of the following perturbed inhomogeneous second order hyperbolic equation:

$$\begin{cases} u''(t) + (A(t) + B(t))u(t) = f(t) \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \tag{4.1.1}$$

Phillips [22] started the study of properties of C_0-semigroups which are conserved under bounded perturbations, and perturbations of infinitesimal generators of analytic semigroups by a bounded operator is due to Kato [14]. Recently, Belarbi and Benchohra [2] proved the existence of solutions for a perturbed impulsive hyperbolic differential inclusion with variable times under the mixed generalized Lipschitz and Carathéodory's conditions.

Kato [12] was the first to succeed in constructing the fundamental solution of temporally inhomogeneous second hyperbolic equation:

$$\begin{cases} u''(t) + A(t)u(t) = f(t) \\ u(0) = u_0, \quad u'(0) = u_1 \end{cases} \tag{4.1.2}$$

in a Hilbert space H. For more general results see any of a number of source, including [14] and Tanabe [23]. Applications to initial value problem of hyperbolic equations have been referred to Goldstein [7] and Yosida [28] in addition [23]. Typical models can be found in the works of materials with biology, engineering, population models, etc.(see e.t., [27, 5] and the bibliography therein). As the second order nonlinear functional evolutions, Kalsatos and Markov in [15] have analyzed some questions on existence of solutions for functional differential inclusions of second order in time, and in [6] proved them in the case where a damping term is added. In [11] they have studied the wellposedness of solutions and the regularity properties of solutions for the mixed problems for semilinear hyperbolic equations of second order with unbounded principal operators.

In this chapter, in order to give a construction of an evolution system of $A(t) + B(t)$, we will assume general conditions that $A(t)$, for each $t \in [0, T]$, is self adjoint and bounded and $A(t)v$ for each $v \in V$ is strongly continuously differentiable on $[0, T]$.

Let V be a Hilbert space forming a Gelfand triple $V \subset H \subset V^*$ with pivot space H. Recall that

$$\mathcal{A}(t) = \begin{pmatrix} 0 & -I \\ A(t) & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} -u_1 \\ A(t)u_0 \end{pmatrix}, \tag{4.1.3}$$

$$\mathcal{B}(t) = \begin{pmatrix} 0 & 0 \\ B(t) & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 \\ B(t)u_0 \end{pmatrix},$$

for any $\binom{u_0}{u_1} \in X = (V \times H)^T$ (or $\widetilde{X} = (H \times V^*)^T$), our problem can be applied to second order time dependent equations by writing them as first order systems. Consequently, we deal with constructing of the fundamental solution of (4.1.2) explained the arguments in given in [1, 14]. In addition to assumptions of $A(t)$, Tanabe [23] dealt with a singular perturbation of evolution systems in a Banach space X with conditions that $B(t)$ is strongly continuous and there exists a real number λ_0 satisfying $\lambda_0 \in \rho(A(t))$ for all $t \in [0, T]$, such that

$$A(t)B(t)(A(t) - \lambda_0)^{-1} \in \mathcal{L}(X), \tag{4.1.4}$$

where $\mathcal{L}(X)$ denotes the set of all bounded linear operators from X into itself. But in section 4.2, we will give a perturbation approach under the more general conditions that X is a Hilbert space and $B(t)v$ for each $v \in V$ is strongly continuously differentiable on $[0, T]$ instead of (4.1.4) even in special cases of second order equations. In the last section we give an example of a partial functional equation as an application of the preceding result in a mixed problem for hyperbolic case that

$$A(t) = -\sum_{i,j=1}^{n} \frac{\partial u}{\partial x_j}(a_{ij}(t,x)\frac{\partial u}{\partial x_i}), \quad B(t) = \sum_{i=1}^{n} b_i(t,x)\frac{\partial u}{\partial x_i} + c(t,x)u,$$

where the matrix $(a_{ij}(t,x))$ is uniformly positive definite.

4.2. Perturbation for fundamental solutions

Consider the following perturbed inhomogeneous second order hyperbolic equation:

$$\begin{cases} u''(t) + (A(t) + B(t))u(t) = f(t) \\ u(0) = u_0, \quad u'(0) = u_1 \end{cases} \tag{4.2.1}$$

where $A(t)$ satisfies the conditions in section 3.2. From now on, both $A_H(t)$ and $A(t)$ are denoted simply by $A(t)$ without the risk of confusing. Let $B(t)$ be defined on $[0, T]$ as a strongly continuously differentiable operator satisfying

$$B(t)u \in C^1((0,T); H), \quad |B(t)u| \le B|u| \quad \text{for} \quad \text{all} \quad u \in H \tag{4.2.2}$$

for some constant $B > 0$. For $\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in (V \times H)^T = X$, let $\mathcal{B}(t)$ be an operator defined by

$$\mathcal{B}(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -B(t) & 0 \end{pmatrix}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 \\ -B(t)u_0 \end{pmatrix} \in X.$$

Then we have that $\mathcal{B}(t) : H_0^1(\Omega) \times L^2(\Omega) \to H_0^1(\Omega) \times L^2(\Omega)$ is a bounded and strongly continuously differentiable operator with respect to t.

Theorem 4.2.1. Assume that $\{\mathcal{A}(t) : 0 \le t \le T\}$ satisfies the conditions in section 3.2. Assume also that $B(t)$ is defined on $[0, T]$ as a strongly continuously differentiable operator with values in $B(X)$. Then there exists a fundamental solution $\mathcal{W}(t, s)$ of (4.2.1) satisfying the following results: for each $\mathbf{x} \in D(\mathcal{A}(t)) = (D(A(t)) \times V)^T$,

(a) $\mathcal{W}(t, s)$ is strongly continuously in s and t, and $||\mathcal{W}(t, s)|| \le Me^{\beta(t-s)}$,

(b) $\mathcal{W}(s, s) = I$, and $\mathcal{W}(t, s) = \mathcal{W}(t, r)\mathcal{W}(r, s)$ for $s \le r \le t$,

(c) $\partial/\partial t\mathcal{W}(t, s)\mathbf{x} = -(\mathcal{A}(t) + \mathcal{B}(t))\mathcal{W}(t, s)\mathbf{x}$,

(d) $\partial/\partial s\mathcal{W}(t,s)\mathbf{x} = \mathcal{W}(t,s)(\mathcal{A}(t)+\mathcal{B}(t))\mathbf{x}$.

Proof. Let us denote $\mathcal{U}(t,s)$ the evolution fundamental system of $\mathbf{x}'(t) + \mathcal{A}(t)\mathbf{x}(t) = F(t)$ whose existence is proved by Theorem 3.2.3 and 3.3.1. For the sake of simplicity in sense of (3.3.3), we assume that there are constants M_0, M_1 such that

$$||\mathcal{U}(t,s)|| \le M_0, \quad ||\mathcal{A}(t)\mathcal{U}(t,s)\mathcal{A}(s)^{-1}|| \le M_1. \tag{4.2.3}$$

Put

$$\mathcal{W}_0(t,s) = \mathcal{U}(t,s), \quad \mathcal{W}_m(t,s) = -\int_s^t \mathcal{U}(t,\tau)\mathcal{B}(\tau)\mathcal{W}_{m-1}(\tau,s)d\tau, \tag{4.2.4}$$

$$\mathcal{W}(t,s) = \sum_{m=0}^{\infty} \mathcal{W}_m(t,s), \tag{4.2.5}$$

for $m = 1, 2\cdots$. Then we have

$$\mathcal{W}(t,s) = \mathcal{U}(t,s) - \int_s^t \mathcal{U}(t,\tau)\mathcal{B}(\tau)\mathcal{W}(\tau,s)d\tau \tag{4.2.6}$$

and the series on the right hand side of (4.2.5) is strongly convergent uniformly in $0 \le s \le t \le T$. Indeed, by (4.2.5)

$$\int_s^t \mathcal{U}(t,\tau)\mathcal{B}(\tau)\mathcal{W}(\tau,s)d\tau = \int_s^t \mathcal{U}(t,\tau)\mathcal{B}(\tau)\sum_{m=0}^{\infty}\mathcal{W}_m(\tau,s)d\tau$$

$$= \sum_{m=0}^{\infty}\int_s^t \mathcal{U}(t,\tau)\mathcal{B}(\tau)\mathcal{W}_m(\tau,s)d\tau = -\sum_{m=0}^{\infty}\mathcal{W}_{m+1}(t,s) = -\sum_{m=0}^{\infty}\mathcal{W}_m(t,s) + \mathcal{U}(t,s),$$

which yields (4.2.6). From (4.2.2), (4.2.3), it follows, by mathematical induction, that

$$||\mathcal{U}(t,s)|| \leq M_0,$$

$$||\mathcal{W}_m(t,s)|| \leq || -\int_s^t \mathcal{U}(t,\tau)\mathcal{B}(\tau)\mathcal{W}_{m-1}(\tau,s)d\tau|| \leq M_0^{m+1}B^m\frac{(t-s)^m}{m!}.$$

Hence $\sum_0^\infty \mathcal{W}_m(t,s)$ is uniformly convergence.

First, we will show that $\partial/\partial t\,\mathcal{W}_m(t,s)\mathcal{A}(s)^{-1}$ exists and is strongly continuous on $B(X)$ for all $m = 1, 2, \cdots$. From (d) of Theorem 3.2.3 and 3.3.1, we have

$$\mathcal{U}(t,s) = \frac{\partial}{\partial s}\mathcal{U}(t,s)\mathcal{A}(s)^{-1} \tag{4.2.7}$$

and

$$\mathcal{W}_m(t,s)\mathcal{A}(s)^{-1} = -\int_s^t \mathcal{U}(t,\tau)\mathcal{B}(\tau)\mathcal{W}_{m-1}(\tau,s)\mathcal{A}(s)^{-1}d\tau \tag{4.2.8}$$

$$= -\int_s^t \frac{\partial}{\partial\tau}\mathcal{U}(t,\tau)\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau)\mathcal{W}_{m-1}(\tau,s)\mathcal{A}(s)^{-1}d\tau$$

$$= -\mathcal{A}(t)^{-1}\mathcal{B}(t)\mathcal{W}_{m-1}(t,s)\mathcal{A}(s)^{-1} + \mathcal{U}(t,s)\mathcal{A}(s)^{-1}\mathcal{B}(s)\mathcal{W}_{m-1}(s,s)\mathcal{A}(s)^{-1}$$

$$+ \int_s^t \mathcal{U}(t,\tau)\frac{\partial}{\partial\tau}(\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau)\mathcal{W}_{m-1}(\tau,s)\mathcal{A}(s)^{-1})d\tau.$$

Here,

$$\frac{\partial}{\partial\tau}(\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau)\mathcal{W}_{m-1}(\tau,s)\mathcal{A}(s)^{-1}) \tag{4.2.9}$$

$$= \mathcal{A}(\tau)^{-1}(-\dot{\mathcal{A}}(\tau)\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau) + \dot{\mathcal{B}}(\tau))\mathcal{W}_{m-1}(\tau,s)\mathcal{A}(s)^{-1}$$

$$+ \mathcal{A}(\tau)^{-1}\mathcal{B}(\tau)\frac{\partial}{\partial\tau}\mathcal{W}_{m-1}(\tau,s)\mathcal{A}(s)^{-1}.$$

Now we shall show that the right side of (4.2.8) is differentiable with respect to t and therfore $\mathcal{W}(t,s)\mathcal{A}(s)^{-1}$ is differentiable. Noting that

$$\frac{\partial}{\partial t}\mathcal{U}(t,s) = -\mathcal{A}\mathcal{U}(t,s),$$

consider that

$$\frac{\partial}{\partial t}\mathcal{W}_m(t,s)\mathcal{A}(s)^{-1} = -\frac{\partial}{\partial t}(\mathcal{A}(t)^{-1}\mathcal{B}(t)\mathcal{W}_{m-1}(t,s)\mathcal{A}(s)^{-1}) \tag{4.2.10}$$

$$- \mathcal{A}(t)\mathcal{U}(t,s)\mathcal{A}(s)^{-1}\mathcal{B}(s)\mathcal{W}_{m-1}(s,s)\mathcal{A}(s)^{-1} + \frac{\partial}{\partial t}(\mathcal{A}(t)^{-1}\mathcal{B}(t)\mathcal{W}_{m-1}(\tau,s)\mathcal{A}(s)^{-1})$$

$$- \int_s^t \mathcal{A}(t)\mathcal{U}(t,\tau)\frac{\partial}{\partial \tau}(\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau)\mathcal{W}_{m-1}(\tau,s)\mathcal{A}(s)^{-1})d\tau$$

$$= -\mathcal{A}(t)\mathcal{U}(t,s)\mathcal{A}(s)^{-1}\mathcal{B}(s)\mathcal{W}_{m-1}(s,s)\mathcal{A}(s)^{-1}$$

$$- \int_s^t \mathcal{A}(t)\mathcal{U}(t,\tau)\frac{\partial}{\partial \tau}(\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau)\mathcal{W}_{m-1}(\tau,s)\mathcal{A}(s)^{-1})d\tau$$

$$= -\mathcal{A}(t)\mathcal{U}(t,s)\mathcal{A}(s)^{-1}\mathcal{B}(s)\mathcal{W}_{m-1}(s,s)\mathcal{A}(s)^{-1}$$

$$- \int_s^t \mathcal{A}(t)\mathcal{U}(t,\tau)\mathcal{A}(\tau)^{-1}\{-\dot{\mathcal{A}}(\tau)\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau) + \dot{\mathcal{B}}(\tau)\}\mathcal{W}_{m-1}(\tau,s)\mathcal{A}(s)^{-1}d\tau$$

$$- \int_s^t \mathcal{A}(t)\mathcal{U}(t,\tau)\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau)\frac{\partial}{\partial \tau}\mathcal{W}_{m-1}(\tau,s)\mathcal{A}(s)^{-1}d\tau.$$

From (3.3.4) and (4.2.2), we know that $-\dot{\mathcal{A}}(\tau)\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau)+\dot{\mathcal{B}}(\tau)$ is uniformly bounded, and so there exists a contant M_2 such that

$$||\dot{\mathcal{A}}(\tau)\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau) + \dot{\mathcal{B}}(\tau)|| \leq M_2. \tag{4.2.11}$$

If $m = 1$ in (4.2.10), then

$$||\frac{\partial}{\partial t}\mathcal{W}_1(t,s)\mathcal{A}(s)^{-1}||$$

$$\leq M_1B||\mathcal{A}(s)^{-1}|| + \int_s^t M_1M_2M_0||\mathcal{A}(s)^{-1}||d\tau + \int_s^t M_1B||\frac{\partial}{\partial \tau}\mathcal{U}(\tau,s)\mathcal{A}(s)^{-1}||d\tau$$

$$\leq M_1B||\mathcal{A}(s)^{-1}|| + M_1M_2M_0||\mathcal{A}(s)^{-1}||(t-s) + M_1^2B(t-s).$$

If $m \geq 2$, then $\mathcal{W}_{m-1}(s,s) = 0$ by (4.2.4) and hence

$$||\frac{\partial}{\partial t}\mathcal{W}_m(t,s)\mathcal{A}(s)^{-1}|| \leq \int_s^t M_1M_2M_0^mB^{m-1}\frac{(\tau-s)^{m-1}}{(m-1)!}||\mathcal{A}(s)^{-1}||d\tau$$

$$+ \int_s^t M_1B||\frac{\partial}{\partial \tau}\mathcal{W}_{m-1}(\tau,s)\mathcal{A}(s)^{-1}||d\tau$$

$$\leq M_1M_2M_0^mB^{m-1}||\mathcal{A}(s)^{-1}||\frac{(t-s)^m}{m!} + M_1B\int_s^t ||\frac{\partial}{\partial \tau}\mathcal{W}_{m-1}(\tau,s)\mathcal{A}(s)^{-1}||d\tau.$$

By mathematical induction, it satisfies the following that

$$||\frac{\partial}{\partial t}\mathcal{W}_m(t,s)\mathcal{A}(s)^{-1}|| \leq M_1^mB^m||\mathcal{A}(s)^{-1}||\frac{(t-s)^{m-1}}{(m-1)!}$$

$$+ M_1M_2M_0B^{m-1}\sum_{i=0}^{m-1}M_0^{m-1-i}M_1^i||\mathcal{A}(s)^{-1}||\frac{(t-s)^m}{m!} + M_1^{m+1}B^m\frac{(t-s)^m}{m!}.$$

Hence, we have

$$||\frac{\partial}{\partial t}\mathcal{W}_m(t,s)\mathcal{A}(s)^{-1}|| \leq M_1^mB^mm||\mathcal{A}(s)^{-1}||\frac{(t-s)^{m-1}}{m!}$$

$$+ M_1M_2M_0B^{m-1}m\{\max\{M_0,M_1\}\}^{m-1}||\mathcal{A}(s)^{-1}||\frac{(t-s)^m}{m!} + M_1^{m+1}B^m\frac{(t-s)^m}{m!}$$

for all m, so that $\sum_{m=0}^{\infty} ||\partial/\partial t\, \mathcal{W}_m(t,s)\mathcal{A}(s)^{-1}||$ is uniformly convergence. Therefore

$$\frac{\partial}{\partial t}\mathcal{W}(t,s)\mathcal{A}(s)^{-1} = \frac{\partial}{\partial t}\sum_{m=0}^{\infty}\mathcal{W}_m(t,s)\mathcal{A}(s)^{-1}$$

exists and is strongly continuous. Noting that

$$\mathcal{W}(t,s) = \mathcal{U}(t,s) - \int_s^t \mathcal{U}(t,s)\mathcal{B}(\tau)\mathcal{W}(\tau,s)d\tau$$

and $\mathcal{U}(t,s) = \partial/\partial s\,\mathcal{U}_(t,s)\mathcal{A}(s)^{-1}$, it holds

$$\mathcal{W}(t,s)\mathcal{A}(s)^{-1} = \mathcal{U}(t,s)\mathcal{A}(s)^{-1} - \int_s^t \frac{\partial}{\partial\tau}\mathcal{U}(t,\tau)\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau)\mathcal{W}(\tau,s)\mathcal{A}(s)^{-1}d\tau \tag{4.2.12}$$

$$= \mathcal{U}(t,s)\mathcal{A}(s)^{-1} - \mathcal{A}(t)^{-1}\mathcal{B}(t)\mathcal{W}(t,s)\mathcal{A}(s)^{-1} + \mathcal{U}(t,s)\mathcal{A}(s)^{-1}\mathcal{B}(s)\mathcal{A}(s)^{-1}$$

$$+ \int_s^t \mathcal{U}(t,\tau)\frac{\partial}{\partial\tau}(\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau)\mathcal{W}(\tau,s)\mathcal{A}(s)^{-1})d\tau.$$

from which it follows

$$\frac{\partial}{\partial t}\mathcal{W}(t,s)\mathcal{A}(s)^{-1} = -\mathcal{A}(t)\mathcal{U}(t,s)\mathcal{A}(s)^{-1} - \frac{\partial}{\partial t}\mathcal{A}(t)^{-1}\mathcal{B}(t)\mathcal{W}(t,s)\mathcal{A}(s)^{-1} \tag{4.2.13}$$

$$- \mathcal{A}(t)\mathcal{U}(t,s)\mathcal{A}(s)^{-1}\mathcal{B}(s)\mathcal{A}(s)^{-1} + \frac{\partial}{\partial t}(\mathcal{A}(t)^{-1}\mathcal{B}(t)\mathcal{W}(t,s)\mathcal{A}(s)^{-1})$$

$$- \int_s^t \mathcal{A}(t)\mathcal{U}(t,\tau)\frac{\partial}{\partial\tau}(\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau)\mathcal{W}(\tau,s)\mathcal{A}(s)^{-1})d\tau$$

$$= -\mathcal{A}(t)\mathcal{U}(t,s)\mathcal{A}(t)^{-1} - \mathcal{A}(t)\mathcal{U}(t,s)\mathcal{A}(s)^{-1}\mathcal{B}(s)\mathcal{A}(s)^{-1}$$

$$- \int_s^t \mathcal{A}(t)\mathcal{U}(t,\tau)\frac{\partial}{\partial\tau}(\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau)\mathcal{W}(\tau,s)\mathcal{A}(s)^{-1})d\tau.$$

Put $\widetilde{\mathcal{A}(t)} = \mathcal{A}(t) + \mathcal{B}(t)$, Then from (4.2.12) we obtain that

$$\widetilde{\mathcal{A}(t)}\mathcal{W}(t,s)\mathcal{A}(s)^{-1} \tag{4.2.14}$$

$$= \mathcal{A}(t)\mathcal{U}(t,s)\mathcal{A}(s)^{-1} + \mathcal{B}(t)\mathcal{U}(t,s)\mathcal{A}(s)^{-1} - \mathcal{B}(t)\mathcal{W}(t,s)\mathcal{A}(s)^{-1}$$

$$- \mathcal{B}(t)\mathcal{A}(t)^{-1}\mathcal{B}(t)\mathcal{W}(t,s)\mathcal{A}(s)^{-1} + \mathcal{A}(t)\mathcal{U}(t,s)\mathcal{A}(s)^{-1}\mathcal{B}(s)\mathcal{A}(s)^{-1}$$

$$+ \mathcal{B}(t)\mathcal{U}(t,s)\mathcal{A}(s)^{-1}\mathcal{B}(s)\mathcal{A}(s)^{-1} + \int_s^t \mathcal{A}(t)\mathcal{U}(t,\tau)\frac{\partial}{\partial\tau}(\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau)\mathcal{W}(\tau,s)\mathcal{A}(s)^{-1})d\tau$$

$$+ \mathcal{B}(t)\int_s^t \mathcal{U}(t,\tau)\frac{\partial}{\partial\tau}(\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau)\mathcal{W}(\tau,s)\mathcal{A}(s)^{-1})d\tau.$$

Therefore from which and (4.2.13) it follows that

$$\frac{\partial}{\partial t}\mathcal{W}(t,s)\mathcal{A}(s)^{-1} + \widetilde{\mathcal{A}(t)}\mathcal{W}(t,s)\mathcal{A}(s)^{-1}$$

$$= \mathcal{B}(t)\{\mathcal{U}(t,s)\mathcal{A}(s)^{-1} - \mathcal{W}(t,s)\mathcal{A}(s)^{-1} - \mathcal{A}(t)^{-1}\mathcal{B}(t)\mathcal{W}(t,s)\mathcal{A}(s)^{-1}$$

$$+ \mathcal{U}(t,s)\mathcal{A}(s)^{-1}\mathcal{B}(s)\mathcal{A}(s)^{-1} + \int_s^t \mathcal{U}(t,\tau)\frac{\partial}{\partial\tau}(\mathcal{A}(\tau)^{-1}\mathcal{B}(\tau)\mathcal{W}(\tau,s)\mathcal{A}(s)^{-1})d\tau\}.$$

By (4.2.12), the right side of (4.2.14) equals zero. Thus, it is evident that $\mathcal{W}(t,s)\mathbf{x}$ is differentiable in s and t and satisfies

$$\frac{\partial}{\partial t}W(t,s)\mathbf{x} = -(\mathcal{A}(t) + \mathcal{B}(t))W(t,s)\mathbf{x},$$

$$\frac{\partial}{\partial s}W(t,s)\mathbf{x} = W(t,s)(\mathcal{A}(t) + \mathcal{B}(t))\mathbf{x}$$

for each $\mathbf{x} \in D(\mathcal{A}(t)) = (D(A(t)) \times V)^T$ (or $\mathbf{x} \in (V \times H)^T = X$). Hence such an operator valued function $\mathcal{W}(t,s)$ is the fundamental solution of

$\partial/\partial t\, \mathbf{x}(t) + (\mathcal{A}(t) + \mathcal{B}(t))\mathbf{x}(t) = 0.$ □

Remark 4.2.1. Let us assume also that $B(t)$ is defined on $[0, T]$ as a strongly continuously differentiable with values in $B(\widetilde{X})$. Then for each $\mathbf{x} \in (V \times H)^T = X$, there exists a fundamental solution $\mathcal{W}(t, s)$ of (4.3.1) satisfying (a), (b), (c), and (d) in Theorem 4.2.1 in $\widetilde{X}$.

4.3. Mixed problem of hyperbolic equations

Consider the mixed problem for the hyperbolic equation:

$$\begin{cases} \frac{\partial^2}{\partial t^2} u(t,x) - \sum_{i,j=1}^n \frac{\partial}{\partial x_j}(a_{ij}(t,x)\frac{\partial u}{\partial x_i}) + \sum_{i=1}^n b_i(t,x)\frac{\partial u}{\partial x_i} + c(t,x)u \\ = f(t), \qquad\qquad 0 \le t < \infty, \quad x \in \Omega, \\ u(t,x) = 0, \qquad\qquad 0 \le t < \infty, \quad x \in \partial\Omega, \\ u(0,x) = u_0(x), \quad \frac{\partial}{\partial t} u(0,x) = u_1(x), \qquad x \in \Omega. \end{cases} \tag{4.3.1}$$

We deal with the Dirichlet condition's case as follows. The matrix $(a_{ij}(x,t))$ is uniformly positive definite, i.e., there exists a positive constant δ such that

$$\sum_{i,j=1}^n a_{ij}(x,t)\xi_i\bar{\xi}_j \ge \delta|\xi|^2$$

for all $x \in \Omega, \quad t \in [0, T]$ and for all real vectors ξ. Let

$$a_{ij}, \quad \frac{\partial}{\partial x_j} a_{ij}, \frac{\partial}{\partial t} a_{ij}, \quad \frac{\partial^2}{\partial t \partial x_j} a_{ij}, \quad c \ge 0, \quad \frac{\partial}{\partial t} c$$

be all continuous and bounded on $\Omega \times [0, T]$, and

$$a_{ij}, \quad \frac{\partial}{\partial x_j} a_{ij}, \quad c$$

satisfy uniformly Lipschitz's condition with respect to t.

For each $t \in [0, T]$ and $u, v \in H_0^1(\Omega)$, let us consider the following sesquilinear form:

$$a(t; u, v) = \sum_{i,j=1}^{n} \int_\Omega a_{ij}(t, x) \frac{\partial u}{\partial x_i} \frac{\overline{\partial v}}{\partial x_j} dx$$

Then there exist constants $c_0,\ c_1 > 0$ such that

$$|a(t, u, v)| \leq c_0 ||u|| ||v||$$

$$\left| \frac{d}{dt} a(t, u, v) \right| = \left| \int_\Omega \sum_{i,j=1}^{n} \dot{a}_{ij}(t, x) \frac{\partial u}{\partial x_i} \frac{\overline{\partial v}}{\partial x_j} dx \right| \leq c_1 ||u|| \cdot ||v||$$

and it holds Gårding's inequality ;

$$a(t; u, u) = \int_\Omega \sum_{i,j=1}^{n} a_{ij}(t, x) \frac{\partial u}{\partial x_i} \frac{\overline{\partial u}}{\partial x_j} dx \geq \delta \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^2 dx = \delta ||u||^2.$$

Define the operator $A(t)$ by

$$(A(t)u, v) = a(t; u, v) = \int_\Omega \sum_{i,j=1}^{n} a_{ij}(t, x) \frac{\partial u}{\partial x_i} \frac{\overline{\partial v}}{\partial x_j} dx,$$

$$D(A(t)) = \{u : u \in H^2(\Omega) \cap H_0^1(\Omega)\} = \{u : u \in H^2(\Omega),\ \ u|_{\partial\Omega} = 0\}.$$

The operator $A(t)$ in $L^2(\Omega)$ is defined as the following that for any $v \in H_0^1(\Omega)$ there exists $f \in L^2(\Omega)$ such that

$$a(t; u, v) = (f, v)$$

then, for any $u \in D(A(t))$, $A(t)u = f$ and $A(t)$ is a positive definite self-adjoint operator. Let u be fixed if we consider the functional $H_0^1(\Omega) \ni v \mapsto a(t; u, v)$, this function is a continuous linear. For any $l \in H^{-1}(\Omega)$, it follow that $(l, v) = a(t; u, v)$. We denote that for any $u, v \in H_0^1(\Omega)$

$$a(t; u, v) = (\widetilde{A}(t)u, v),$$

that is, $\widetilde{A}(t)u = l$. The operator $\widetilde{A}(t)$ is one to one mapping from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$. The relation of operators $A(t)$ and $\widetilde{A}(t)$ satisfy the following that for any $u \in D(A(t))$

$$D(A(t)) = \{u \in H_0^1(\Omega),\ \ \widetilde{A}(t)u \in L^2(\Omega)\}, \quad A(t)u = \widetilde{A}(t)u.$$

From now on, both $A(t)$ and $\widetilde{A}(t)$ are denoted simply by A. Put

$$D(B(t)) = H_0^1(\Omega), \quad B(t)u = \sum_{i,j=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u,$$

and for $\binom{u_0}{u_1} \in H_0^1(\Omega) \times L^2(\Omega)$,

$$\mathcal{A}(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 & -I \\ A(t) & 0 \end{pmatrix}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix},$$

$$\mathcal{B}(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ B(t) & 0 \end{pmatrix}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 \\ B(t)u_0 \end{pmatrix}.$$

Then $\mathcal{B}(t)$ is a bounded operatror from $X = (H_0^1(\Omega) \times L^2(\Omega))^T$ to itself and strongly continuously differentiable with respect to t. Since

$$|B(t)u_0| \le \max\{(|b_1|, |b_2|, \cdots, |b_n|, |c|\}(\sum |\frac{\partial u_0}{\partial x_i}|^2 + |u_0|^2)^{\frac{1}{2}} \le c||u_0||_{H_0^1(\Omega)},$$

we have

$$\left\| \mathcal{B}(t) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right\|_X \le c \left| \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right|_X (\mathcal{B}(t) \in \mathcal{L}(X)).$$

Then we treat (4.3.1) as the initial value problem for the abstract second order equations

$$\begin{cases} u''(t) + (A(t) + B(t))u(t) = f(t) \\ u(0) = u_0, \qquad u'(0) = u_1. \end{cases} \tag{4.3.2}$$

Now we can apply the results of Theorem 4.2.1 and Remark 4.2.1 as follows.

Theorem 4.3.1. Assume that $\{\mathcal{A}(t) : 0 \le t \le T\}$ is defined as mentioned above and $B(t)$ is defined on $[0, T]$ as a strongly continuously differentiable with values in $\mathcal{L}(L^2(\Omega))$. Let us assume that $f \in C([0, T]; H^{-1}(\Omega)) \cap W^{1,2}(0, T; H^{-1}(\Omega))(T > 0)$ and $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. Then, there exists a fundamental solution $\mathcal{W}(t, s)$ of (4.3.2) satisfying (a), (b), (c), and (d) in Theorem 4.2.1 and the solution u of (4.3.1) exists and is unique in

$$u \in \widetilde{W_T} \cap C([0, T]; H_0^1(\Omega)) \cap C^1((0, T); L^2(\Omega)), \quad T > 0$$

where

$$\widetilde{W}_T = L^2(0,T;H_0^1(\Omega)) \cap W^{1,2}(0,T;L^2(\Omega)) \cap W^{2,2}(0,T;H^{-1}(\Omega)).$$

Furthermore, the following energy inequality holds: there exists a constant C_T depending on T such that

$$||u||_{\widetilde{W}_T} \leq C_T(1 + ||u_0|| + |u_1| + ||f(0)||_* + ||f||_{W^{1,2}(0,T;H^{-1}(\Omega))}).$$

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